AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(2): 3349–3368.
DOI: 10.3934/math.2024164
Received: 24 July 2023
Revised: 14 December 2023
Accepted: 19 December 2023
Published: 05 January 2024

Research article

Numerical treatment for time fractional order phytoplankton-toxic phytoplankton-zooplankton system

D. Priyadarsini[1], P. K. Sahu[2], M. Routaray[1] and D. Chalishajar[3,*]

[1] Department of Mathematics, School of Applied Sciences, KIIT University, Odisha-751024, India
[2] Department of Mathematics, Model Degree College, Nayagarh, Odisha-752079, India
[3] Department of Applied Mathematics, Mallory Hall, Virginia Military Institute (VMI), Lexington, VA 24450, USA

* **Correspondence:** Email: dipu17370@gmail.com.

Abstract: The study of time-fractional problems with derivatives in terms of Caputo is a recent area of study in biological models. In this article, fractional differential equations with phytoplankton-toxic phytoplankton-zooplankton (PTPZ) system were solved using the Laplace transform method (LTM), the Adomain decomposition method (ADM), and the differential transform method (DTM). This study demonstrates the good agreement between the results produced by using the specified computational techniques. The numerical results displayed as graphs demonstrate the accuracy of the computational methods. The approaches that have been established are thus quite relevant and suitable for solving nonlinear fractional models. Meanwhile, the impact of changing the fractional order of a time derivative and time t on populations of phytoplankton, toxic-phytoplankton, and zooplankton has been examined using graphical representations. Furthermore, the stability analysis of the LTM approach has been discussed.

Keywords: phytoplankton-toxic phytoplankton-zooplankton system; stability conditions; Laplace transform method; Adomian decomposition method; differential transform method
Mathematics Subject Classification: 26A33, 37N30

1. Introduction

In science, modeling and making the usability of real-world situations have always been essential. The nature of rapid shifts in the plankton population is the most important area of research in marine plankton ecology. The base of the entire aquatic food chain is plankton, phytoplankton organisms which are present in both freshwater and saltwater environments. They sustain all aquatic food chains, produce oxygen after absorbing CO_2 from their environment, and obtain their food through photosynthetic processes, meaning phytoplankton need sunlight for their existence. Primary production, nitrogen cycling, and food webs all depend on phytoplankton. Due to the rapid cell reproduction caused by the increasing phytoplankton population, biomass increases rapidly. This kind of rapid change in phytoplankton population density is known as "bloom". It is difficult to calculate the plankton biomass. Plankton population mathematical modelling has become an essential area of research for both the physical and biological processes involved in plankton ecology. Many other organisms depend on phytoplankton for sustenance. Several species of phytoplankton, referred to as toxic phytoplankton, are capable of producing toxins. By preying on phytoplankton, zooplankton grazers can reduce the phytoplankton population while simultaneously giving other marine animals a plentiful food source.

Numerous researchers have recently investigated the interaction of phytoplankton, toxic phytoplankton, and zooplankton. Roy [1] created a mathematical model to explain how non-toxic and toxic phytoplankton interact when only one nutrient is available. Yunfei et al. [2] developed a model for harvesting phytoplankton and zooplankton. Janga et al. [3] developed and analyzed models of nutrient-plankton interaction with a hazardous chemical that influences the rate of development of phytoplankton, zooplankton, or both trophic levels. Singh et al. [4] investigated the role of virus infection in a basic phytoplankton and zooplankton model. Zhang et al. [5] showed toxin avoidance effects on spatiotemporal pattern selection in a nontoxic phytoplankton-toxic phytoplankton-zooplankton model. Nutrient loss in phytoplankton-nutrient systems was studied and numerically simulated by Dimitrov et al. [6]. Javedi et al. [7] illustrated dynamic analysis of the time-fractional order phytoplankton-toxic phytoplankton-zooplankton system. In [8], Veeresha et al. solved the fractional approach for a mathematical model of the phytoplankton-toxic phytoplankton-zooplankton system with Mittag-Leffler kernel and explained that the system exists and produces a unique solution.

Due to the memory property of fractional derivatives, the theory and applications of fractional calculus (FC) have become extremely useful and important in the modeling of biological processes, applied mathematics, physics, and engineering. Sardar et al. [9] constructed a mosquito-transmitted disease model using fractional differential equations. In [10], Liu and Chen developed a fractional-order competition model for love triangles. Javidi et al. [11] created a fractional-order model for cholera infection. Mahdy et al. [12] developed a numerical approach for solving Emden-Fowler nonlinear equations. Gepreel et al. [13] studied optimum control, signal flow graph, and system electrical circuit realization for nonlinear Anopheles mosquito model.

Recently published literature have demonstrated that fractional differential equations are a valuable tool for modeling. The Laplace transform method (LTM), proposed by Pierre-Simon Laplace in 1986 [14], is the most efficient technique for examining a nonlinear model. For instance, Alharbi [15], suggested the communicable disease model in biological and physical models. Algehyne [16] proposed application of fractional calculus on time dilation in special theory of relativity. Sheikh used Liouville-Caputo fractional derivatives in [17] to evaluate and undertake a quantitative investigation of sediment loss. The authors of [18] developed a fractional order model for cholera infection. More recently researchers have employed fractional differential equations to solve biological models using the Adomain decomposition method (ADM), for instance for the solution of a nonlinear fractional differential equation [19]. Some authors also used the differential transform method (DTM) to solve fractional problems. Ifeyinwa et al. [20] used the differential transformation approach to describe the transmission dynamics of syphilis illness. Kumar et al. [21] used the differential transform technique (DTM)-Pade approximation to establish the analytical approximation for the non-dimensional temperature. Sowmya et al. [22] showed the importance of thermal stress in a convective-radiative annular fin with magnetic field and heat generation: DTM application. Gamaoun et al. [23] applied the α-parameterized DTM method to the energy transfer of a fin wetted with ZnO-SAE 50 nanolubricant. In [24] Routaray et al. solved the fuzzy differential transform method for the solution of the system of fuzzy integro-differential equations arising in the biological model.

In this article, our focus is on numerical solution of the model of phytoplankton-toxic phytoplankton-zooplankton (PTPZ) system proposed by [25] in Caputo sense:

$$\begin{aligned} {}_0^C D_t^{\alpha} P(t) &= r_1 P\Big(1 - \frac{P}{H_1}\Big) - aPT - cPZ \\ {}_0^C D_t^{\alpha} T(t) &= r_2 T\Big(1 - \frac{T}{H_2}\Big) - bPT - TZ \\ {}_0^C D_t^{\alpha} Z(t) &= ePZ - TZ - mZ \end{aligned} \tag{1.1}$$

where $\alpha \in (0, 1]$ is the order of the fractional derivative, $P(t)$ is the density of the phytoplankton population, $T(t)$ is the density of the toxic phytoplankton population, and $Z(t)$ is the density of the zooplankton population at any instant in time t. In model (1.1), r_1 is the intrinsic growth rate and H_1 is the phytoplankton population's environmental carrying capacity. The term a reflects the functional response of zooplankton to phytoplankton grazing, while c denotes the biomass conversion ratio. r_2 is the intrinsic growth rate, H_2 is the environmental carrying capacity of the toxic-phytoplankton population, and b is the zooplankton prediction rate, where the parameters e and m denote the rate of zooplankton growth and toxic production, respectively. $P_0(t) = P(0)$; $T_0(t) = T(0)$; $Z_0(t) = Z(0)$ are the initial conditions for the above system.

In many applications fractional calculus provide more accurate models of the physical systems than ordinary calculus do. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. In the fractional model (1.1), the fractional derivatives indicate the speed at which the solution goes to the equilibrium. A fractional derivative has numerous definitions of order $\alpha > 0$. The Riemann-Liouville and Caputo definitions are the two most widely employed. The Caputo definition of the fractional derivative is very useful in the time domain studies, because the initial conditions for the fractional order differential equations with the Caputo derivatives can be given in the same manner as the ordinary differential equations with a known physical interpretation. Here we collect the well-known definitions of Caputo's fractional derivatives.

Definition 1. (Fractional derivative in terms of Caputo) The Caputo fractional derivative is a mathematical concept that generalizes the concept of a derivative to non-integer orders. It is particularly useful in the field of fractional calculus, which deals with derivatives and integrals of non-integer order. The Caputo derivative is named after Italian mathematician Michele Caputo, who introduced it in the 1960s. Suppose $f(t)$ is k-times continuously differentiable function and $f^{(k)}(t)$ is integrable in $[a, t]$. For $t \in [a, b]$, the Caputo fractional derivative of order α for a function $f(t)$ is defined as

$${}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(k - \alpha)} \int_a^t \frac{f^{(k)}(\tau)}{(t - \tau)^{\alpha + 1 - k}} d\tau \tag{1.2}$$

where α is the order of the fractional derivative and can be any positive real number, τ is an integrable variable, a is a lower limit of integration and can be any real number, $\Gamma(.)$ refers to the Gamma function, $t > a$, and k is a positive integer such that $k - 1 < \alpha < k$.

In this study, we studied a system with complex behavior that exemplified the food chain system for zooplankton, toxic phytoplankton, and, especially, phytoplankton with a reduced predation rate at high toxic-phytoplankton density. When we do not have an exact solution for the system, the test for the correctness of the solution is critical. This model has been solved by many authors for positive integer order derivatives, but only the equilibrium and stability have been discussed in those papers. In general, the complexity of a dynamic system will always be determined by the initial conditions, which could affect every component of the system. In the present work, the authors have solved the model using three techniques: Laplace transform method (LTM), differential transform method (DTM), and Adomain decomposition method (ADM) for fractional order derivative to verify how the behavior of the solution is changing for different values of α. The LTM's stability analysis is described here. The authors solve this system using three methods and compare themselves to manifest that all the methods give very similar results with less computational time.

This research helps young researchers and readers understand the novelty of the proposed strategy for investigating the numerical solution of complicated nature of real-world problems. Capturing the essence of an unpredictable system with parametric plots, in particular, plays an important role, and it also creates interests in many young researchers due to its attractive visual appearance. This work can be investigated using polynomial-based numerical methods.

The rest of the paper is organized as follows: Section 1 highlights the importance of the PTPZ system. It also mentions the main objectives or goals of the research, as well as any background information required for understanding the next parts. The Section 2 details the suggested numerical technique for approximating solutions to the PTPZ system using three numerical methods: LTM, ADM, and DTM. The second component will go over solving methods and stability analysis of the LTM method. It also explains the step-by-step procedure of the numerical method. The proposed methods are used to solve fractional order PTPZ system about their accuracy and applicability in Section 3. The contributions of the work are summarized in Section 4, which highlights the achievements of the proposed method and identifies possible directions for further study. This is followed by references in the section titled "References".

2. Solving methods

2.1. Laplace transform method

Laplace transform to Caputo fractional order derivative gives us

$$\mathcal{L}\left\{{}_0^C D_t^{\alpha}\{f(x)\}\right\} = \lambda^{\alpha} F(\lambda) - \sum_{n=0}^{k-1} \lambda^{\alpha-n-1} f^{(n)}(0). \tag{2.1}$$

Using the Laplace transform given by Eq (2.1) and the inverse Laplace transform on both sides of Eq (1.1), we have the following solutions

$$P(t) = P(0) + \mathcal{L}^{-1}\left\{\frac{1}{S^{\gamma}}\mathcal{L}\left\{r_1 P(t)(1-\frac{P(t)}{H_1}) - aP(t)T(t) - cP(t)Z(t)\right\}(s)\right\}(t)$$
$$T(t) = T(0) + \mathcal{L}^{-1}\left\{\frac{1}{S^{\gamma}}\mathcal{L}\left\{r_2 T(t)(1-\frac{T(t)}{H_2}) - bP(t)T(t) - T(t)Z(t)\right\}(s)\right\}(t) \tag{2.2}$$
$$Z(t) = Z(0) + \mathcal{L}^{-1}\left\{\frac{1}{S^{\gamma}}\mathcal{L}\left\{eP(t)Z(t) - T(t)Z(t) - mZ(t)\right\}(s)\right\}(t).$$

Then, the following iterative formula is proposed

$$P_n(t) = P(0) + \mathcal{L}^{-1}\left\{\frac{1}{S^{\gamma}}\mathcal{L}\left\{r_1 P_{n-1}(1-\frac{P_{n-1}}{H_1}) - aP_{n-1}(t) T_{n-1}(t) - cP_{n-1}(t)Z_{n-1}(t)\right\}(s)\right\}(t)$$
$$T_n(t) = T(0) + \mathcal{L}^{-1}\left\{\frac{1}{S^{\gamma}}\mathcal{L}\left\{r_2 T_{n-1}(t)(1-\frac{T_{n-1}(t)}{H_2}) - bP_{n-1}(t) T_{n-1}(t) - T_{n-1}(t)Z_{n-1}(t)\right\}(s)\right\}(t) \tag{2.3}$$
$$Z_n(t) = Z(0) + \mathcal{L}^{-1}\left\{\frac{1}{S^{\gamma}}\mathcal{L}\left\{eP_{n-1}(t)Z_{n-1}(t) - T_{n-1}(t)Z_{n-1}(t) Z_{n-1}(t) - mZ_{n-1}\right\}(s)\right\}(t).$$

The approximate solution is assumed to be obtained as a limit where n tends to infinity

$$P(t) = \lim_{n\to\infty} P_{(n)}(t)$$
$$T(t) = \lim_{n\to\infty} T_{(n)}(t) \tag{2.4}$$
$$Z(t) = \lim_{n\to\infty} Z_{(n)}(t).$$

2.1.1. Stability analysis for LTM method

Theorem 3.1. We show that the recursive approach defined above is stable.
Proof. We make the following assumptions. p, q, and r are three positive constants such that for all

$$0 \le t \le T \le \infty$$
$$||P(t)|| < p; ||T(t)|| < q; ||Z(t)|| < r. \tag{2.5}$$

Now we consider a subset of $C_2\left((a,b)\,(0,T)\right)$ defined by

$$[k]=\left\{n:(a,b)(0,T)\to[k],\frac{1}{\Gamma(\alpha)}\int(t-\eta)^{\alpha-1}v(\eta)u(\eta)d\eta<\infty\right\}. \tag{2.6}$$

Now consider the following operator defined as

$$\Theta(P,T,Z)=\begin{cases} r_1P(1-\frac{P}{H_1})-aPT-cP,\\ r_2T(1-\frac{T}{H_2})-bPT-TZ,\\ \quad ePZ-TZ-mZ.\end{cases} \tag{2.7}$$

Then,

$$=\begin{cases} \langle\Theta(P,T,Z)-\Theta(P_1,T_1,Z_1),(P-P_1,T-T_1,Z-Z_1)\rangle,\\ \langle r_1(P(t)-P_1(t))(1-\frac{(P(t)-P_1(t))}{H_1})-a(P(t)-P_1(t))(T(t)-T_1(t))\\ \quad -c(P(t)-P_1(t))(Z-Z_1(t))\rangle,\\ \langle r_2(T(t)-T_1(t))(1-\frac{(T(t)-T_1(t))}{H_2})-b(P(t)-P_1(t))(T(t)-T_1(t))\\ \quad -(T(t)-T_1(t))(Z-Z_1(t))\rangle,\\ \langle e(P(t)-P_1(t))(Z-Z_1(t))-(T(t)-T_1(t))(Z-Z_1(t))\\ \quad -m(Z-Z_1(t))\rangle\end{cases} \tag{2.8}$$

where

$$\begin{aligned} P(t)&\neq P_1(t)\\ T(t)&\neq T_1(t)\\ Z(t)&\neq Z_1(t). \end{aligned}$$

By applying norm and absolute value on both sides, we get

$$\langle\Theta(P,T,Z)-\Theta(P_1,T_1,Z_1),(P-P_1,T-T_1,Z-Z_1))\rangle \tag{2.9}$$

$$<\begin{cases} \left\{\begin{array}{c}\left\|\frac{r_1(P(t)-P_1(t))(1-\frac{(P(t)-P_1(t))}{H_1})}{(P(t)-P_1(t))}\right\|\\ -\left\|\frac{a(P(t)-P_1(t))(T(t)-T_1(t))}{(P(t)-P_1(t))}\right\|\\ -\left\|\frac{c(P(t)-P_1(t))(Z(t)-Z_1(t))}{(P(t)-P_1(t))}\right\|\end{array}\right\}\|P(t)-P_1(t)\|^2,\\ \left\{\begin{array}{c}\left\|\frac{r_2(T(t)-T_1(t))(1-\frac{(T(t)-T_1(t))}{H_1})}{(T(t)-T_1(t))}\right\|\\ -\left\|\frac{b(P(t)-P_1(t))(T(t)-T_1(t))}{(T(t)-T_1(t))}\right\|-\frac{(T(t)-T_1(t))(Z(t)-Z_1(t))}{(T(t)-T_1(t))}\end{array}\right\}\|T(t)-T_1(t)\|^2,\\ \left\{\begin{array}{c}\left\|\frac{e(P(t)-P_1(t))(Z(t)-Z_1(t)))}{(Z(t)-Z_1(t))}\right\|\\ -\left\|\frac{(T(t)-T_1(t))(Z-Z_1(t))}{(Z(t)-Z_1(t))}\right\|-\left\|\frac{m(Z-Z_1(t))}{(Z-Z_1(t))}\right\|\end{array}\right\}\|Z(t)-Z_1(t)\|^2\end{cases} \tag{2.10}$$

$$<\begin{cases} \left\{\begin{array}{c}\left\|r_1(1-\frac{(P(t)-P_1(t))}{H_1})\right\|-\|a(T(t)-T_1(t))(P(t)-P_1(t))\|\\ -\|c(Z(t)-Z_1(t))\|\end{array}\right\}\|P(t)-P_1(t)\|^2\\ \left\{\begin{array}{c}\left\|r_2(1-\frac{(T(t)-T_1(t))}{H_1})\right\|\\ -\|b(P(t)-P_1(t))\|-\|(Z(t)-Z_1(t))\|\|\end{array}\right\}\|T(t)-T_1(t)\|^2\\ \left\{\|e(P(t)-P_1(t))\|-\|(T(t)-T_1(t))\|-m\right\}\|Z(t)-Z_1(t)\|^2\end{cases}$$

where

$$\langle \Theta(P,T,Z) - \Theta(P_1,T_1,Z_1)\rangle < \begin{cases} A\,\|P(t) - P_1(t))\|^2 \\ B\,\|T(t) - T_1(t))\|^2 \\ C\,\|Z(t) - Z_1(t))\|^2 \end{cases} \tag{2.11}$$

with

$$\begin{aligned} A &= \left\| r_1(1 - \frac{(P(t) - P_1(t))}{H_1}) \right\| - \|a(T(t) - T_1(t))(P(t) - P_1(t))\|\, \|{-c}(Z(t) - Z_1(t))\| \\ B &= \left\| r_2(1 - \frac{(T(t) - T_1(t))}{H_1}) \right\| - \|b(P(t) - P_1(t))\| - \|(Z(t) - Z_1(t))\| \\ C &= \|e(P(t) - P_1(t))\| - \|(T(t) - T_1(t))\| - m. \end{aligned}$$

Also, if we consider non-null vector (x_1, x_2, x_3), then we obtain

$$\langle \Theta(P,T,Z) - \Theta(P_1,T_1,Z_1)\rangle < \begin{cases} A\,\|P(t) - P_1(t))\|\,\|P(t)\| \\ B\,\|T(t) - T_1(t))\|\,\|T(t)\| \\ C\,\|Z(t) - Z_1(t))\|\,\|Z(t)\| \end{cases}. \tag{2.12}$$

Thus, we conclude that the iterative method is stable.

2.2. *Adomain decomposition method*

The ADM is a popular method for solving differential equations. The method is a powerful technique that offers quick algorithms for analytic approximate solutions and numerical simulations in applied sciences and engineering. The fractional ADM is used to solve the PTPZ system.

2.2.1. Description of the method:

We can use the ADM to solve the nonlinear system of Eq (1.1) using the above initial condition and the given parameters.

Step 1: Consider the fractional order differential equation

$${}_0^C D_t^{\alpha} s(t) + Ls(t) + Ns(t) = w(t) \tag{2.13}$$

where ${}_0^C D_t^{\alpha}$ is the Caputo fractional derivative, L is the linear differential operator, and N is the non linear differential operator.

Equation (2.13) can be written as

$${}_0^C D_t^{\alpha} s(t) = w(t) - Lu(t) - Nu(t). \tag{2.14}$$

Step 2: Applying the Caputo fractional integral to both sides of Eq (2.14), one can obtain

$$s(t) = s(0) +^C I^{\alpha}(w - Ls - Ns). \tag{2.15}$$

Step 3: According to the ADM, the solution s can be examined into an infinite number of components.

$$s = \sum_{n=0}^{\infty} s_n \tag{2.16}$$

and the nonlinear term becomes

$$Ns = \sum_{n=0}^{\infty} \mathcal{A}_n \tag{2.17}$$

where A_n Adomian polynomials can be expressed as

$$\mathcal{A}_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N[\sum_{n=0}^{\infty} \lambda^n s_n]\right]_{\lambda=0} \tag{2.18}$$

$$\sum_{n=0}^{\infty} s_n =^C I^{\alpha}(w - L(\sum_{n=0}^{\infty} s_n) - \sum_{n=0}^{\infty} \mathcal{A}_n). \tag{2.19}$$

Step 4: After substituting the initial condition $s(0)$ in Eq (2.19), we can find $s_0, s_1, s_2, \cdots$
Step 5: Then, the approximate solution becomes

$$s(t) = s_n + \sum_{n=0}^{\infty} s_n. \tag{2.20}$$

Now, utilizing the basic definition of the ADM, we apply the fractional ADM approach to solve the PTPZ system (1.1) and we get

$$\begin{aligned} L_1(P) = r_1 P, N_1(P) &= -r_1 P\Big(\frac{P}{H_1}\Big) - aPT - cPZ \\ L_2(T) = r_2 T, N_2(T) &= -r_2 P\Big(\frac{P}{H_1}\Big) - bPT - TZ \\ L_3(Z) = mZ, N_2(T) &= ePZ - TZ \end{aligned}$$

where P_0, T_0, Z_0 be the initial conditions

$$\begin{aligned} \mathcal{A}_n &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N_1(P)[\sum_{i=0}^{n} \lambda^n P_n, \sum_{i=0}^{n} \lambda^n T_n, \sum_{i=0}^{n} \lambda^n Z_n]\right]_{\lambda=0} \\ \mathcal{B}_n &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N_2(T)[\sum_{i=0}^{n} \lambda^n P_n, \sum_{i=0}^{n} \lambda^n T_n, \sum_{i=0}^{n} \lambda^n Z_n]\right]_{\lambda=0} \\ C_n &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N_3(Z)[\sum_{i=0}^{n} \lambda^n P_n, \sum_{i=0}^{n} \lambda^n T_n, \sum_{i=0}^{n} \lambda^n Z_n]\right]_{\lambda=0} \end{aligned}$$

$$\begin{aligned} P_{n+1} &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} L_1(P(n), T(n), Z(n)) d\theta + \mathcal{A}_n(t) \\ T_{n+1} &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} L_2(P(n), T(n), Z(n)) d\theta + \mathcal{B}_n(t) \\ Z_{n+1} &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} L_3(P(n), T(n), Z(n)) d\theta + C_n(t). \end{aligned} \tag{2.21}$$

2.3. *Differential transform method*

Zhou [40] was the first to use the differential transform approach in electrical circuit analysis to solve linear and nonlinear initial value problems. This DTM approach yields a polynomial-based analytical answer. For large orders, the Taylor series technique is computationally time-consuming. The differential transform is an iterative method for solving ordinary or partial differential equations with analytic Taylor series.

2.3.1. Description of the method

To solve a nonlinear system of equations, we can utilize the DTM approach. The DTM approach can be used to turn the original function into a transformed form.

$$
\begin{aligned}
m(x) &= n(x) \pm p(x) \rightarrow m_\alpha(k) = N_\alpha(k) \pm P_\alpha(k) \\
m(x) &= D^{\alpha}_{x_0}(x) \rightarrow m_\alpha(k) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}.
\end{aligned}
$$

We apply the fractional differential transform method to solve the PTPZ system. Using the basic definitions of the fractional one-dimensional differential transform and the associated transform of Eq (1.1), we obtain the following system for $h = 0, 1, 2, \cdots$:

$$
\begin{cases}
P(h+1) = \frac{\Gamma(\alpha h+1)}{\Gamma((\alpha h+1)+1)}[r_1 P_\alpha(k) - \frac{r_1}{H_1}(P_\alpha(k))^2 - a \sum_{l=0}^{k} P_\alpha(l) \\
\qquad T_\alpha(k-l) - c \sum_{l=0}^{k} P_\alpha(l) Z_\alpha(k-l)] \\
T(h+1) = \frac{\Gamma(\alpha h+1)}{\Gamma((\alpha h+1)+1)}[r_2 T_\alpha(k) - \frac{r_1}{H_1}(T_\alpha(k))^2 - b \sum_{l=0}^{k} P_\alpha(l) \\
\qquad T_\alpha(k-l) - \sum_{l=0}^{k} T_\alpha(l) Z_\alpha(k-l)] \\
Z(h+1) = \frac{\Gamma(\alpha h+1)}{\Gamma((\alpha h+1)+1)}[e \sum_{l=0}^{k} P_\alpha(l) Z_\alpha(k-l) - \sum_{l=0}^{k} T_\alpha(l) \\
\qquad Z_\alpha(k-l) - m Z_\alpha(k)].
\end{cases}
$$

3. Illustrative examples

Consider the fractional order PTPZ system

$$
\begin{aligned}
{}_0^C D_t^{\alpha} P(t) &= r_1 P(1 - \frac{P}{H_1}) - aPT - cPZ \\
{}_0^C D_t^{\alpha} T(t) &= r_2 T(1 - \frac{T}{H_2}) - bPT - TZ \\
{}_0^C D_t^{\alpha} Z(t) &= ePZ - TZ - mZ.
\end{aligned}
\tag{3.1}
$$

Using the initial conditions and parameters listed below, we obtain $P(0) = 0.2, T(0) = 0.1, Z(0) = 0.1, r_1 = 0.08, r_2 = 0.22, a = 0.1, b = 0.8, c = 1.35, e = 0.63, m = 0.8, H_2 = 0.13, H_1 = 1$ The required equation is then solved using one of three numerical approaches.

3.1. *LTM*

The solution to Eq (3.1) will be illustrated by applying the LTM for $\alpha = 1$. Thus, the solution comes out as:

$$
\begin{aligned}
P(t) =& 0.2 - 0.016200000000000006t - 0.0006480000000000002t^2 \\
&+ 0.0005665771384615386t^3 + 0.000011331542769230774t^4 \\
&+ 5.925436681846158 \times 10^{-6}t^5 - 4.183228561449681 \times 10^{-6}t^6;
\end{aligned}
$$

$$
\begin{aligned}
T(t) =& 0.1 - 0.020923076923076923t + 0.002301538461538461 6t^2 \\
&- 0.00006996478834774692t^3 + 3.848063359126079 \times 10^{-6}t^4 \\
&- 0.000013346399828129273t^5 + 1.8739294893357527 \times 10^{-7}t^6 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
Z(t) =& 0.1 + 0.0826t + 0.03334496538461538 5t^2 + 0.00910609442504931t^3 \\
&+ 0.0018173552495334 36t^4 + 0.0003765145417350532 5t^5 + \\
&3.7302622658569283637t^6 + \cdots
\end{aligned}
$$

3.2. ADM

After solving Eq (3.1) with the ADM for special case $\alpha = 1$, the solution becomes:

$$
\begin{aligned}
P(t) =& 0.2 - 0.016200000000000006t - 0.010156069230769234t^2 - \\
&0.0022094140688362924t^3 - 0.0001304323778414389 5t^4 + \\
&0.00008140810965506208t^5 + 0.00003405337232607953 4t^6 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
T(t) =& 0.1 - 0.020923076923076923t + 0.00047728994082840t^2 \\
&- 0.0007056276349567592t^3 + 9.684641495565019 \times 10^{-6}t^4 \\
&- 6.9670189216597284 \times 10^{-6}t^5 + 3.75550650323141 \times 10^{-6}t^6 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
Z(t) =& 0.1 + 0.0826t + 0.03464965384615385t^2 + 0.00960609442504931t^3 \\
&+ 0.0019173552495334 36t^4 + 0.0002765145417350532 5t^5 + \\
&0.000026226585692836 37t^6 + \cdots
\end{aligned}
$$

3.3. DTM

Using the DTM approach, the solution for the specific case $\alpha = 1$ and the initial conditions $P_0, T_0, Z_0 = 0.2, 0.1, 0.1$ is as follows:

$$
\begin{aligned}
P(t) =& 0.2 - 0.016200000000000059t - 0.01015606925674534t^2 - \\
&0.002209414068786754t^3 - 0.0001389763778414389 5t^4 + \\
&0.00008140810567850620 8t^5 + 0.00003405337876079534t^6 + \cdots
\end{aligned}
$$

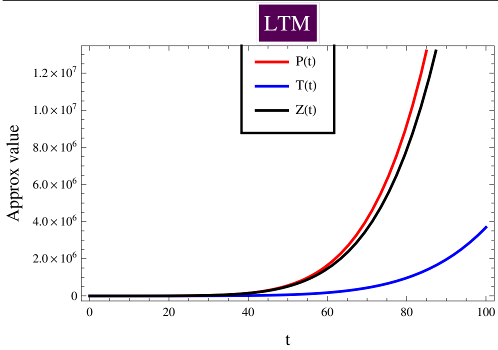
$$
\begin{aligned}
T(t) =& 0.1 - 0.02092305692307577t + 0.00047728994566684 0t^2 \\
&- 0.0007056276349565678t^3 + 9.684641495566754 \times 10^{-6}t^4 \\
&- 6.9670189456597284 \times 10^{-6}t^5 + 3.75550656323141 \times 10^{-6}t^6 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
Z(t) =& 0.1 + 0.0826t + 0.03464945584615385t^2 + 0.00960609442233493 1t^3 \\
&+ 0.0019173553395334 36t^4 + 0.0002765145417330532 5t^5 + \\
&0.000026226533692836 37t^6 + \cdots
\end{aligned}
$$

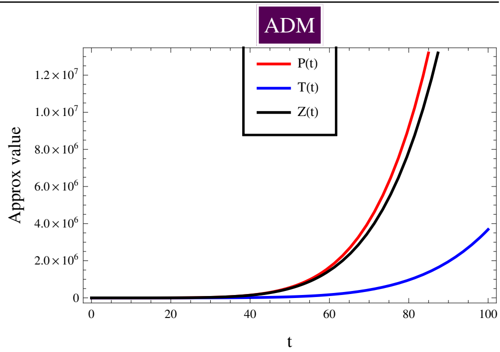
4. Results and discussion

The aim of the numerical simulation is to observe the dynamics of the phytoplankton, toxic-phytoplankton, and zooplankton graphically. The numerical simulation uses the initial values, $P(0), T(0), Z(0) = 0.2, 0.1, 0.1$. In this section, we describe numerical simulations of the model given by Eq (1.1) using the above numerical methods. We investigate the dynamical behavior of the model with fractional order $\alpha = 0.65, 0.95, 1$. The parameter values in the simulations are $r_1 = 0.08, r_2 = 0.22, a = 0.1, b = 0.8, c = 1.35, e = 0.63, m = 0.8, H_2 = 0.13, H_1 = 1$. Figure 1 displays the numerical simulation of our model's specific solutions as a function of time for $\alpha = 1$, for the behavior of phytoplankton plants, toxic phytoplankton, and zooplankton for LTM, ADM, and DTM. Figure 2 illustrates the actions of phytoplankton plants for various values of $\alpha = 0.65, 0.95, 1$. Figure 3 depicts the dynamics of toxic phytoplankton. Figure 4 demonstrates the activity of zooplankton using the PTPZ model using the LTM, ADM, and DTM for different values of $\alpha = 0.05, 0.1.0.15$. Figure 5 illustrates the actions of phytoplankton plants for various values of $\alpha = 0.05, 0.1.0.15$. Figure 6 shows the dynamics of toxic phytoplankton. Figure 7 depicts the activity of zooplankton using the PTPZ model using the LTM, ADM, and DTM for different values of $\alpha = 0.05, 0.1.0.15$. Table 1 shows the approximate phytoplankton plant values for various approaches and $\alpha = 0.65, 0.95, 1$ values. Table 2 displays the approximate toxic-phytoplankton plant values for several approaches and $\alpha = 1, 0.65, 0.95$ values. Table 3 shows the approximate zooplankton plant values for several approaches and $\alpha = 1, 0.65, 0.95$ values. From the figures, the dynamic behavior shows that the growth of the populations of phytoplankton, toxic phytoplankton, and zooplankton are converging to equilibrium point for decreasing the values of α. It is clear that the approximate solutions depend continuously on the fractional derivative α. Table 4 displays the CPU time used for solving the PTPZ system for $M = 5$. The computer used for obtaining the results in this paper is an ASUS laptop with Processor (CPU) Ryzen 5 (3500U), and RAM(8GB), using (Mathematica Software).

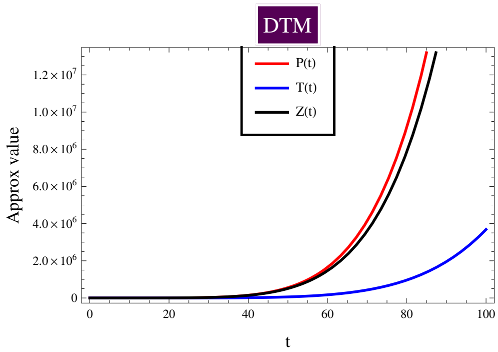
The numerical results in this case indicated that the approximate solutions for different α are similar using these three numerical methods. This accuracy gives us high confidence in the validity of this problem and reveals an excellent agreement of engineering accuracy. According to the results, all three offered methods gave rise to exact solutions. According to this viewpoint, the parameter α is significant in computational approaches and can be utilized to acquire fresh perspectives into generalized biological models.



(a) *LTM*

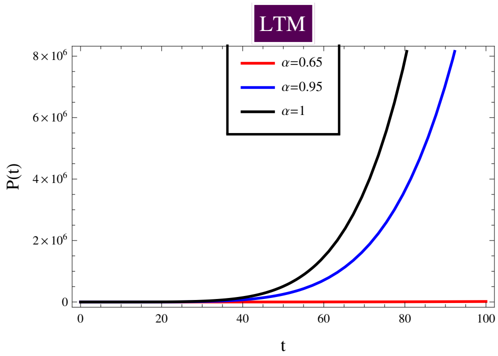


(b) *ADM*

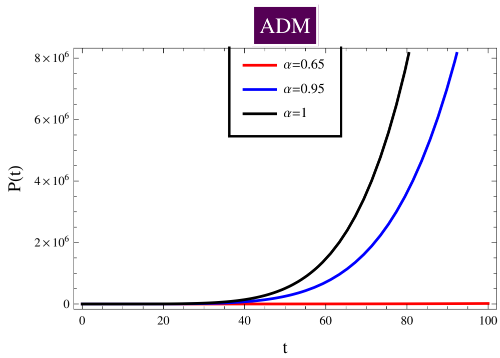


(c) *DTM*

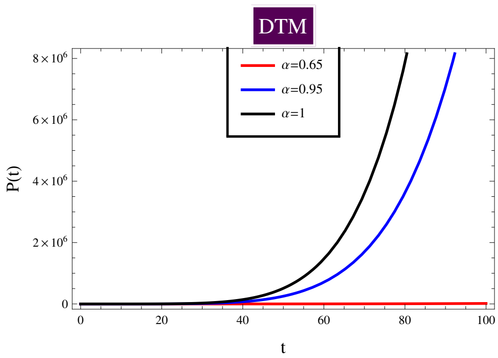
Figure 1. Aproximate value for $\alpha = 1$.



(a) *LTM*

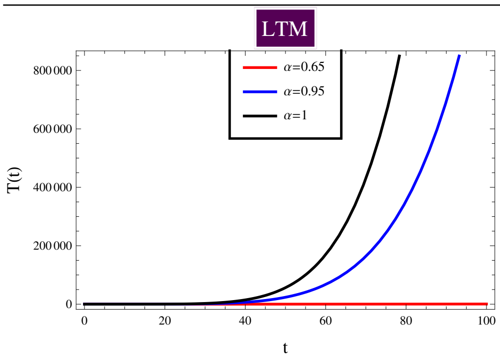


(b) *ADM*

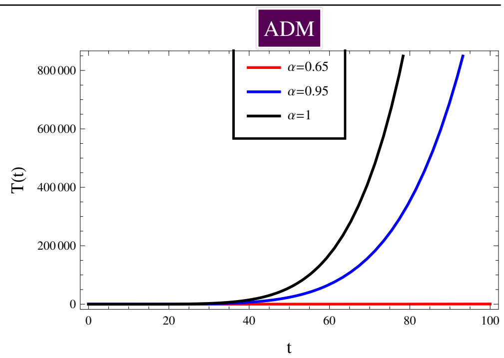


(c) *DTM*

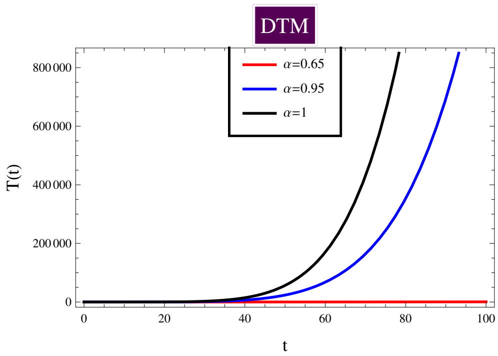
Figure 2. Aproximate value of phytoplankton plant for $\alpha = 0.65, 0.95, 1$.



(a) LTM

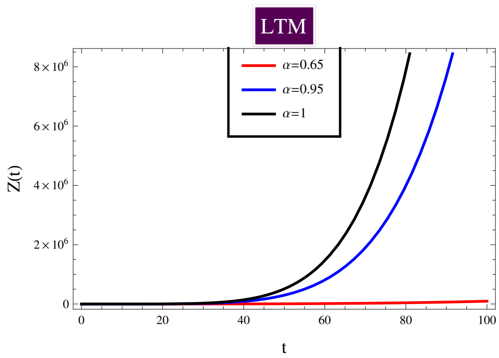


(b) ADM

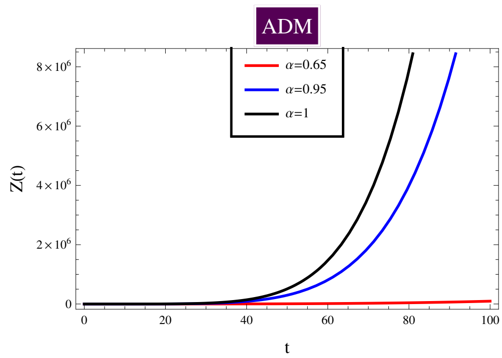


(c) DTM

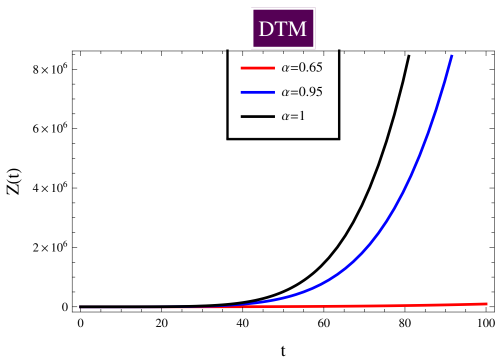
Figure 3. Aproximate value of toxic-phytoplankton for $\alpha = 0.65, 0.95, 1$.



(a) LTM

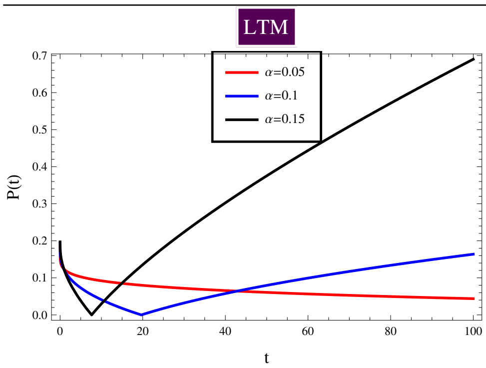


(b) ADM

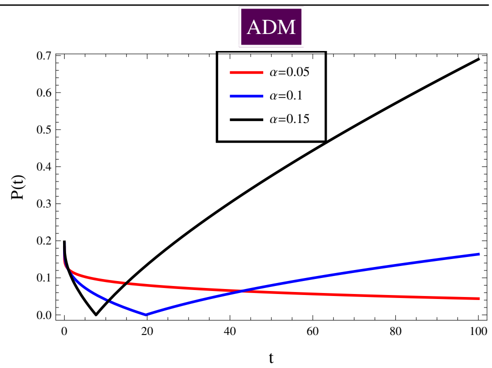


(c) DTM

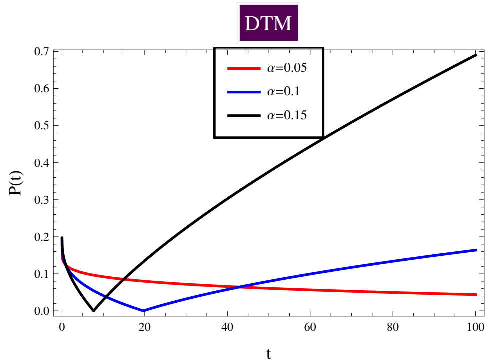
Figure 4. Aproximate value of zooplankton for $\alpha = 0.65, 0.95, 1$.



(a) *LTM*

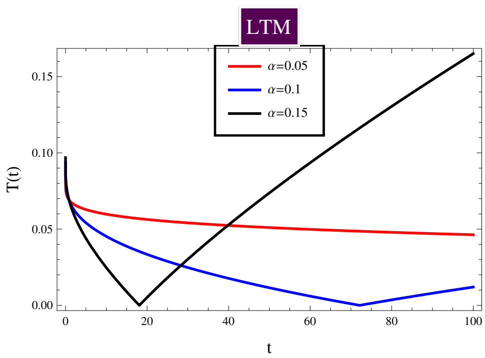


(b) *ADM*

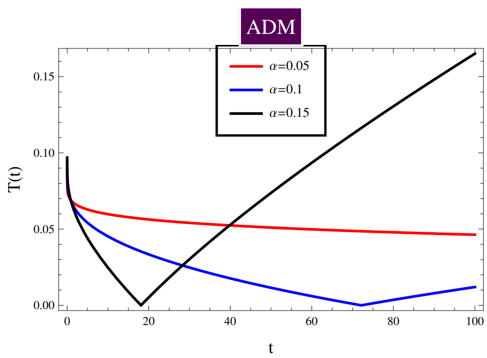


(c) *DTM*

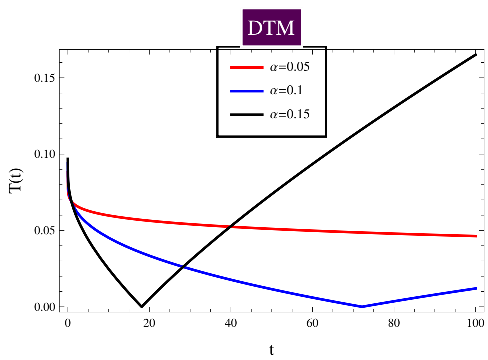
Figure 5. Aproximate value of phytoplankton plant for $\alpha = 0.05, 0.1, 0.15$.



(a) *LTM*

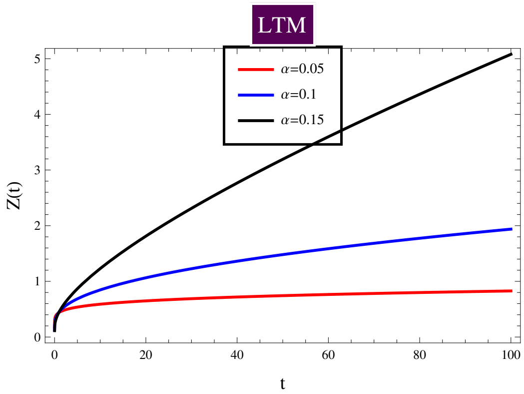


(b) *ADM*

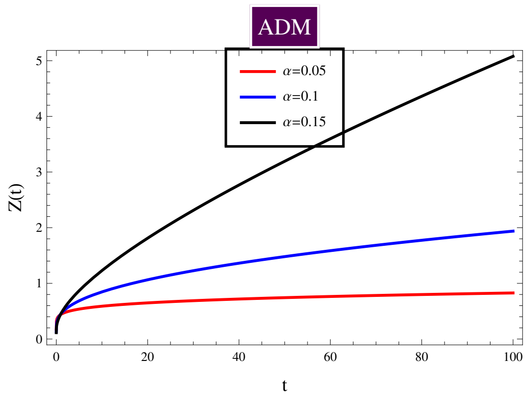


(c) *DTM*

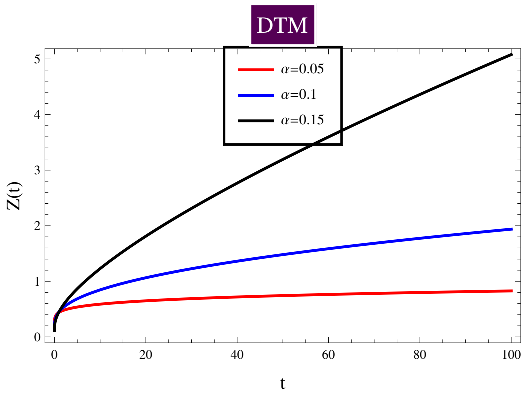
Figure 6. Aproximate value of toxic-phytoplankton for $\alpha = 0.05, 0.1, 0.15$.



(a) LTM



(b) ADM



(c) DTM

Figure 7. Aproximate value of zooplankton for $\alpha = 0.05, 0.1, 0.15$.

Table 1. Numerical results of phytoplankton plants for $\alpha = 0.65, 0.95, 1$.

α	t	**DTM**	**ADM**	**LTM**
1	0.1	0.19827621769852	0.19827621745652	0.19827621456952
	0.2	0.19633590258642	0.19633590156992	0.19633590158642
	0.3	0.19416546573372	0.19416546573586	0.19416544569372
0.65	0.1	0.19500976189226	0.19500976189568	0.19500976256226
	0.2	0.19117644432775	0.19117644432789	0.19117644432126
	0.3	0.18733494309661	0.18855494309661	0.18733494390876
0.95	0.1	0.19800123535421	0.18945035253321	0.18945035253347
	0.2	0.19586627058323	0.19586622596500	0.19586627576567
	0.3	0.19351510113475	0.19351512698475	0.19351511236475

Table 2. Numerical results of toxic-phytoplankton plants for $\alpha = 0.65, 0.95, 1$.

α	t	**DTM**	**ADM**	**LTM**
1	0.1	0.09791176526401	0.09791176058951	0.09791176048201
	0.2	0.09582884469827	0.09582884465685	0.09582884469845
	0.3	0.0937445678511	0.09374245932511	0.09374712652511
0.65	0.1	0.09480925450322	0.09480925455467	0.09480925678322
	0.2	0.09182562363828	0.09182562363786	0.09182562367543
	0.3	0.14107716308949	0.14117716308949	0.14108896308949
0.95	0.1	0.09760954744007	0.09791561107829	0.09843126047825
	0.2	0.09538735364362	0.09512345364362	0.09538234364382
	0.3	0.09322183268611	0.09322183268611	0.09356434258734

Table 3. Numerical results of zooplankton for $\alpha = 0.65, 0.95, 1$.

α	t	**DTM**	**ADM**	**LTM**
1	0.1	0.10861629715978	0.10861629715978	0.10861629715978
	0.2	0.11798599284080	0.11798599284080	0.11798599209780
	0.3	0.12817405502123	0.12817405502456	0.12817405456266
0.65	0.1	0.12388995574050	0.12388995698059	0.12388995574050
	0.2	0.14107265808941	0.14107716308949	0.14107716304569
	0.3	0.15787192318989	0.15787192318981	0.15787192318982
0.95	0.1	0.10995231212813	0.10995451212813	0.10995231245698
	0.2	0.12017725114568	0.12017725124569	0.12017725121258
	0.3	0.13110908139799	0.13110908139800	0.13110908139800

Table 4. CPU time for $M = 5$.

LTM	DTM	ADM
0.246 /s	0.243/s	0.289/s

5. Conclusions

The study of nonlinear systems is always an important area in research due to their essence of understanding the behavior of real-world problems and models. In this paper, we solve the time fractional PTPZ system numerically using three numerical methods (LTM, ADM, and DTM) to solve the time fractional PTPZ system, which deals with the impact of toxic phytoplankton on zooplankton and phytoplankton in the sea. We have applied the Laplace transform method to show that the method is stable for the required system and the stability analysis of ADM and DTM can be demonstrated in the same way. The research demonstrates that the three techniques involve less computational work and provide quantitative, similar outcomes. From the figures, the dynamic behavior shows that the growth of the populations of phytoplankton, toxic phytoplankton, and zooplankton converges to equilibrium point for decreasing the values of α. The approximate solutions $P(t)$, $T(t)$, and $Z(t)$ are displayed in figures show that the concentrations of phytoplankton, toxic-phytoplankton, and zooplankton all reach their equilibrium values as time passes. An important feature of the fractional-order model is that it controls the speed at which the solution to equilibrium is reached.

These types of studies can help us to investigate more interesting consequences of the system with specific parameters and assumptions related to initial conditions. Furthermore, it opens the door for innovation in the concept of examining and predicting more properties and behaviors of the corresponding systems and mathematical models that exemplify real-world problems. Finally, we observe that the PTPZ system can exhibit complex dynamical behavior. Furthermore, the proposed method could be used in a broader class of biological systems, such as mathematical modeling of infectious disease dynamics, as well as other key fields of study including economics, finance, and engineering.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Authors are grateful to the anonymous referees and editor for their constructive comments to strengthen this manuscript.

Conflict of interest

The authors does not have any conflicts of interest.

References

1. S. Roy, The coevolution of two phytoplankton species on a single resource: Allelopathy as a pseudo-mixotrophy, *Theor. Popul. Biol.*, **75** (2009), 68–75. https://doi.org/10.1016/j.tpb.2008.11.003
2. Y. Lv, Y. Pei, S. Gao, C. Li, Harvesting of a phytoplankton-zooplankton model, *Nonlinear Anal.-Real*, **11** (2010), 3608–3619. https://doi.org/10.1016/j.nonrwa.2010.01.007
3. S. J. Jang, J. Baglama, J. Rick, Nutrient-phytoplankton-zooplankton models with a toxin, *Math. Comput. Model.*, **43** (2006), 105–118. https://doi.org/10.1016/j.mcm.2005.09.030
4. B. K. Singh, J. Chattopadhyay, S. Sinha, The role of virus infection in a simple phytoplankton zooplankton system, *J. Theor. Biol.*, **231** (2004), 153–166. https://doi.org/10.1016/j.jtbi.2004.06.010
5. F. Zhang, J. Sun, W. Tian, Spatiotemporal pattern selection in a nontoxic-phytoplankton-toxic-phytoplankton-zooplankton model with toxin avoidance effects, *Appl. Math. Comput.*, **423** (2022), 127007. https://doi.org/10.1016/j.amc.2022.127007
6. D. T. Dimitrov, H. V. Kojouharov, Analysis and numerical simulation of phytoplankton-nutrient systems with nutrient loss, *Math. Comput. Simul.*, **70** (2005), 33–43. https://doi.org/10.1016/j.matcom.2005.03.001

7. M. Javidi, B. Ahmad, Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton-zooplankton system, *Ecol. Model.*, **318** (2015), 8–18. https://doi.org/10.1016/j.ecolmodel.2015.06.016

8. P. Veeresha, L. Akinyemi, Fractional approach for a mathematical model of Phytoplankton-toxic phytoplankton-zooplankton system with Mittag-Leffler kernel, *Int. J. Biomath.*, **16** (2023), 2250090. https://doi.org/10.1142/S1793524522500905

9. T. Sardar, S. Rana, S. Bhattacharya, K. Al-Khaled, J. Chattopadhyay, A generic model for a single strain mosquito-transmitted disease with memory on the host and the vector, *Math. Biosci.*, **263** (2015), 18–36. https://doi.org/10.1016/j.mbs.2015.01.009

10. W. Liu, K. Chen, Chaotic behavior in a new fractional-order love triangle system with competition, *J. Appl. Anal. Comput.*, **5** (2015), 103–113. https://doi.org/10.11948/2015009

11. M. Javidi, B. Ahmad, A study of a fractional-order cholera model, *Appl. Math. Inform. Sci.*, **8** (2014), 2195. https://doi.org/10.12785/amis/080513

12. A. M. S. Mahdy, A numerical method for solving the nonlinear equations of Emden-Fowler models, *J. Ocean Eng. Sci.*, 2022. https://doi.org/10.1016/j.joes.2022.04.019

13. K. A. Gepreel, M. Higazy, A. M. S. Mahdy, Optimal control, signal flow graph, and system electronic circuit realization for nonlinear Anopheles mosquito model, *Int. J. Mod. Phys. C*, **31** (2020), 2050130. https://doi.org/10.1142/S0129183120501302

14. P. A. Lynn, P. A. Lynn, The Laplace transform and the z-transform, *Electron. Signal. Syst.*, 1986, 225–272.

15. W. Alharbi, A. Shater, A. Ebaid, C. Cattani, M. Areshi, M. Jalal, et al., Communicable disease model in view of fractional calculus, *AIMS Math.*, **8** (2023), 10033–10048. https://doi.org/10.3934/math.2023508

16. E. A. Algehyne, M. S. Aldhabani, M. Areshi, E. R. El-Zahar, A. Ebaid, H. K. Al-Jeaid, A proposed application of fractional calculus on time dilation in special theory of relativity, *Mathematics*, **11** (2023), 3343. https://doi.org/10.3390/ math11153343

17. N. A. Sheikh, M. Jamil, D. L. C. Ching, I. Khan, M. Usman, K. S. Nisar, A generalized model for quantitative analysis of sediments loss: A Caputo time fractional model, *J. King Saud Univ.-Sci.*, **33** (2021), 101179. https://doi.org/10.1016/j.jksus.2020.09.006

18. D. Baleanu, F. A. Ghassabzade, J. J. Nieto, A. Jajarmi, On a new and generalized fractional model for a real cholera outbreak, *Alex. Eng. J.*, **61** (2022), 9175–9186. https://doi.org/10.1016/j.aej.2022.02.054

19. S. S. Ray, R. K. Bera, An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method, *Appl. Math. Comput.*, **167** (2005), 561–571. https://doi.org/10.1016/j.amc.2004.07.020

20. M. H. Ifeyinwa, Mathematical modeling of the transmission dynamics of syphilis disease using differential transformation method, *Math. Model. Appl.*, **5** (2020), 47–54. https://doi.org/10.11648/J.MMA.20200502.11

21. R. V. Kumar, I. E. Sarris, G. Sowmya, A. Abdulrahman, Iterative solutions for the nonlinear heat transfer equation of a convective-radiative annular fin with power law temperature-dependent thermal properties, *Symmetry*, **15** (2023), 1204. https://doi.org/10.3390/sym15061204

22. G. Sowmya, F. Gamaoun, A. Abdulrahman, R. S. V. Kumar, B. C. Prasannakumara, Significance of thermal stress in a convective-radiative annular fin with magnetic field and heat generation: Application of DTM and MRPSM, *Propuls. Power Res.*, **11** (2022), 527–543. https://doi.org/10.1016/j.jppr.2022.11.001

23. F. Gamaoun, N. M. Said, R. Makki, R. V. Kumar, G. Sowmya, B. C. Prasannakumara, et al., Energy transfer of a fin wetted with ZnO-SAE 50 nanolubricant: An application of α-parameterized differential transform method, *Case Stud. Therm. Eng.*, **40** (2022), 102501. https://doi.org/10.1016/j.csite.2022.102501

24. M. Routaray, P. K. Sahu, D. N. Chalishajar, The fuzzy differential transform method for the solution of the system of fuzzy integro-differential equations arising in biological model, *Mathematics*, **11** (2023), 3840. https://doi.org/10.3390/ math11183840

25. M. Banerjee, E. Venturino, A phytoplankton-toxic phytoplankton-zooplankton model, *Ecol. Complex.*, **8** (2011), 239–248.

26. E. Ahmed, A. S. Elgazzar, On fractional order differential equations model for nonlocal epidemics, *Physica A*, **379** (2007), 607–614. https://doi.org/10.1016/j.physa.2007.01.010

27. J. E. Truscott, J. Brindley, Ocean plankton populations as excitable media, *B. Math. Biol.*, **56** (1994), 981–998. https://doi.org/10.1016/S0092-8240(05)80300-3

28. T. Gao, X. Meng, Stability and Hopf bifurcation of a delayed diffusive phytoplankton-zooplankton-fish model with refuge and two functional responses, *AIMS Math.*, **8** (2023), 8867–8901. https://doi.org/10.3934/math.2023445

29. S. Pleumpreedaporn, C. Pleumpreedaporn, J. Kongson, C. Thaiprayoon, J. Alzabut, W. Sudsutad, Dynamical analysis of nutrient-phytoplankton-zooplankton model with viral disease in phytoplankton species under Atangana-Baleanu-Caputo derivative, *Mathematics*, **10** (2022), 1578. https://doi.org/10.3390/math10091578

30. J. Yang, S. Yuan, Dynamics of a toxic producing phytoplankton-zooplankton model with three-dimensional patch, *Appl. Math. Lett.*, **118** (2021), 107146. https://doi.org/10.1016/j.aml.2021.107146

31. H. Wang, M. Liu, Stationary distribution of a stochastic hybrid phytoplankton-zooplankton model with toxin-producing phytoplankton, *Appl. Math. Lett.*, **101** (2020), 106077. https://doi.org/10.1016/j.aml.2019.106077

32. K. Agnihotri, H. Kaur, Optimal control of harvesting effort in a phytoplankton-zooplankton model with infected zooplankton under the influence of toxicity, *Math. Comput. Simul.*, **190** (2021), 946–964. https://doi.org/10.1016/j.matcom.2021.06.022

33. Z. Chen, Z. Tian, S. Zhang, C. Wei, The stationary distribution and ergodicity of a stochastic phytoplankton-zooplankton model with toxin-producing phytoplankton under regime switching, *Physica A*, **537** (2020), 122728. https://doi.org/10.1016/j.physa.2019.122728

34. S. N. Raw, S. R. Sahu, Strong stability with impact of maturation delay and diffusion on a toxin producing phytoplankton-zooplankton model, *Math. Comput. Simul.*, **210** (2023), 547–570. https://doi.org/10.1016/j.matcom.2023.03.023

35. A. M. Edwards, Adding detritus to a nutrient-phytoplankton-zooplankton model: A dynamical-systems approach, *J. Plankton Res.*, **23** (2001), 389–413. https://doi.org/10.1093/plankt/23.4.389

36. R. R. Sarkar, S. Pal, J. Chattopadhyay, Role of two toxin-producing plankton and their effect on phytoplankton-zooplankton system-a mathematical study supported by experimental findings, *BioSystems*, **80** (2005), 11–23. https://doi.org/10.1016/j.biosystems.2004.09.029

37. S. Chakraborty, J. Chattopadhyay, Nutrient-phytoplankton-zooplankton dynamics in the presence of additional food source—A mathematical study, *J. Biol. Syst.*, **16** (2008), 547–564. https://doi.org/10.1142/S0218339008002654

38. R. Pal, D. Basu, M. Banerjee, Modelling of phytoplankton allelopathy with Monod-Haldane-type functional response—A mathematical study, *Biosystems*, **95** (2009), 243–253. https://doi.org/10.1016/j.biosystems.2008.11.002

39. M. Caputo, Elasticita e dissipazione, *Zanichelli*, 1969.

40. J. K. Zhou, *Differential transformation and its applications for electrical circuits*, Huazhong University Press, Wuhan, 1986.

41. G. Sowmya, R. S. V. Kumar, Y. Banu, Thermal performance of a longitudinal fin under the influence of magnetic field using Sumudu transform method with pade approximant (STM-PA), *J. Appl. Math. Mech.*, 2023, e202100526. https://doi.org/10.1002/zamm.202100526

42. A. M. S. Mahdy, K. A. Gepreel, K. Lotfy, A. A. El-Bary, A numerical method for solving the Rubella ailment disease model, *Int. J. Mod. Phys. C*, **32** (2021), 2150097. https://doi.org/10.1142/S0129183121500972

43. I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier, 1998.

AIMS Press