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Research article

The a posteriori error estimates of the Ciarlet-Raviart mixed finite element method for the biharmonic eigenvalue problem

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Abstract: The biharmonic equation/eigenvalue problem is one of the fundamental model problems in mathematics and physics and has wide applications. In this paper, for the biharmonic eigenvalue problem, based on the work of Gudi [*Numer. Methods Partial Differ. Equ.*, **27** (2011), 315–328], we study the a posteriori error estimates of the approximate eigenpairs obtained by the Ciarlet-Raviart mixed finite element method. We prove the reliability and efficiency of the error estimator of the approximate eigenfunction and analyze the reliability of the error estimator of the approximate eigenvalues. We also implement the adaptive calculation and exhibit the numerical experiments which show that our method is efficient and can get an approximate solution with high accuracy.

Keywords: the biharmonic eigenvalue; Ciarlet-Raviart mixed method; conforming finite element; a posteriori error estimator; adaptive algorithm
Mathematics Subject Classification: 65N25, 65N30

1. Introduction

The biharmonic equation/eigenvalue problem is a fundamental model in mathematics and physics, and many numerical methods for these problems have been developed. Among these methods, the Ciarlet-Raviart mixed finite element method [1] is popular and classical, and it has been applied to the biharmonic equation (see [2–7], etc.), the biharmonic eigenvalue problem (see [8–12], etc.), and the transmission eigenvalue problem which has a similar structure with the biharmonic eigenvalue problem (see [13, 14], etc.).

In practical calculations, in order to obtain high-precision approximations, a posteriori error estimation and adaptive algorithms have been widely applied (such as those in introductory textbooks [15, 16] and review article [17]). For the biharmonic eigenvalue problem, Li and Yang [18] gave C^0IPG adaptive algorithms. Under the condition that the eigenfunctions u and $v = \Delta u$ have the same regularity, Wang et al. [10] proposed a mixed discontinuous Galerkin (denoted as DG mixed) approximation scheme, and got the residual-based a posteriori error estimator of the approximate eigenpair. Feng et al. [19] proposed the reliable residual-based a posteriori error estimator of the approximate eigenvalue under the condition that the eigenfunction u and $v = \Delta u$ have different regularity. This paper aims to study the a posteriori error estimation and adaptive algorithms of the Ciarlet-Raviart mixed conforming finite element method (denoted as the C-R mixed method) for the biharmonic eigenvalue problem. Discontinuous Galerkin methods are also effective methods for solving the biharmonic eigenvalue problem (see [10,19]) and they have advantages for irregular regions as they preserve local conservative properties and allow hanging nodes in the mesh adaption. But, on the same adaptive mesh without hanging nodes, the C-R mixed method has much fewer degrees of freedom than the DG mixed method. For the biharmonic eigenvalue problem on convex polygons, the C-R mixed method is simple and efficient. However, we have not seen literature on the a posteriori error analysis of this method.

As we know, the finite element method and its error estimates for an eigenvalue problem are based on the finite element method and its error estimates for the corresponding source problem. For the biharmonic equations, Charbonneau et al. [20] explored the residual-based a posteriori error estimate of the C-R mixed method, and Gudi [21] further studied the a posteriori error estimate under the condition that there are no quasi-uniformity assumptions on the triangulation.

In this paper, we extend the a posteriori error analysis of the biharmonic equation in [21] to the eigenvalue problem, prove the reliability and efficiency of the estimator of the approximate eigenfunction, use the error identity (2.15) to study the a posteriori error estimates of the approximate eigenvalues, and analyze the reliability of the error estimator of the approximate eigenvalues. We also implement adaptive computation. Numerical experiments indicate that our method is efficient and can get an approximate solution with high accuracy.

The organization of this paper is as follows. In the next section, we introduce the biharmonic eigenvalue problem and its C-R mixed approximation. In Section 3, we discuss the a posteriori error estimates. Finally, we present some numerical experiments to validate our theoretical results.

In this paper, C represents a generic positive constant independent of the mesh size h, which may not be the same constant in different places. For simplicity, we use the symbol $a \lesssim b$ to mean that $a \leq Cb$.

2. Preliminaries

Consider the biharmonic eigenvalue problem

$$\begin{cases} \Delta^2 u = \lambda u, & in\ \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & on\ \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain with boundary $\partial\Omega$, and ν is the unit outward normal to $\partial\Omega$.

Let $v = \Delta u$. We can rewrite the forth-order problem (2.1) as a system of second-order problems:

$$\begin{cases} -\Delta u + v = 0, & in\ \Omega, \\ \Delta v = \lambda u, & in\ \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & on\ \partial\Omega. \end{cases} \tag{2.2}$$

Multiplying the first and the second equations of (2.2) by test functions ψ and φ, respectively, integrating by parts and using the boundary conditions, we can obtain the following C-R mixed variational form of (2.1): find $(\lambda, u, v) \in \mathbb{R} \times H_0^1(\Omega) \times H^1(\Omega)$ such that $\|u\|_0 = 1$ and

$$(v, \psi) + b(\psi, u) = 0, \quad \forall \psi \in H^1(\Omega), \tag{2.3}$$

$$b(v, \varphi) = -\lambda(u, \varphi), \qquad \forall \varphi \in H_0^1(\Omega), \tag{2.4}$$

where the bilinear forms are defined as follows:

$$(\varphi, \psi) = \int_\Omega \varphi\psi dx, \tag{2.5}$$

$$b(\psi, \varphi) = \int_\Omega \nabla\psi \cdot \nabla\varphi dx. \tag{2.6}$$

In this paper, we assume $D \subseteq \Omega$. Let $H^\rho(D)$ denote the standard Sobolev space on D with norm $\|\cdot\|_{\rho,D}$, seminorm $|\cdot|_{\rho,D}$, and $H^0(D) = L^2(D)$. When $D = \Omega$, $\|\cdot\|_{\rho,\Omega}$ and $|\cdot|_{\rho,\Omega}$ are simply denoted by $\|\cdot\|_\rho$ and $|\cdot|_\rho$, respectively. Let $H^\rho(\partial D)$ denote the Sobolev space on ∂D with norm $\|\cdot\|_{\rho,\partial D}$ and seminorm $|\cdot|_{\rho,\partial D}$.

Assume that $\mathcal{J}_h = \{\kappa\}$ is a family of regular triangulation of Ω (see [2]). Let h_κ be the diameter of κ and $h = \max\{h_\kappa : \kappa \in \mathcal{J}_h\}$. The set of interior edges in $\mathcal{J}_h$ is denoted by Γ_I and the set of boundary edges is denoted by Γ_B. Set $\Gamma = \Gamma_I \cup \Gamma_B$. Denote the length of any edge $e \in \Gamma$ by $|e|$. For any $e \in \Gamma_I$ and $e = \partial\kappa^+ \bigcap \partial\kappa^-$, the jump of the derivative of $\psi \in V_h$ on e is defined as

$$[\frac{\partial\psi}{\partial\nu}] = \frac{\partial\psi^+}{\partial\nu} - \frac{\partial\psi^-}{\partial\nu}$$

where ν denotes a unit normal vector on e, which is directed outward from κ^+; for $e \in \Gamma_B = \partial\kappa \bigcap \partial\Omega$,

$$[\frac{\partial\psi}{\partial\nu}] = -\frac{\partial\psi}{\partial\nu}$$

where ν denotes a unit normal vector directed outward from the boundary $\partial\Omega$.

Define the finite element spaces as

$$V_h^0 = \{\varphi \in H_0^1(\Omega) : \varphi|_\kappa \in P_m(\kappa), \forall\kappa \in \mathcal{J}_h\},$$
$$V_h = \{\psi \in H^1(\Omega) : \psi|_\kappa \in P_m(\kappa), \forall\kappa \in \mathcal{J}_h\},$$

where $P_m(\kappa)$ is the space of polynomials of degree $\leq m$ ($m \geqslant 2$).

Define the broken Sobolev space

$$H^2(\Omega, \mathcal{J}_h) = \{\psi \in H_0^1(\Omega) : \psi|_\kappa \in H^2(\kappa), \kappa \in \mathcal{J}_h\}$$

with the mesh-dependent norm

$$|||\psi|||^2 = \sum_{\kappa\in\mathcal{J}_h} ||\Delta\psi||_{0,\kappa}^2 + \sum_{e\in\Gamma}\int_e \frac{1}{|e|}[\frac{\partial\psi}{\partial\nu}]^2 ds.$$

Define the following norm on product space $W = L^2(\Omega)\times H^2(\Omega,\mathcal{J}_h)$ as

$$||(\chi,\varphi)||_W = (||\chi||_0^2 + |||\varphi|||^2)^{\frac{1}{2}},\ \chi\in L^2(\Omega)\ and\ \varphi\in H^2(\Omega,\mathcal{J}_h).$$

Based on the mixed formulation (2.3) and (2.4), we can get the C-R mixed finite element approximation: find $(\lambda_h,u_h,v_h)\in\mathbb{R}\times V_h^0\times V_h$, $||u_h||_0 = 1$, such that

$$(v_h,\psi_h) + b(\psi_h,u_h) = 0,\quad \forall\psi_h\in V_h, \tag{2.7}$$

$$b(v_h,\varphi_h) = -\lambda_h(u_h,\varphi_h),\quad \forall\varphi_h\in V_h^0. \tag{2.8}$$

Consider the following fourth-order problem:

$$\begin{cases} -\Delta\omega+\varphi = 0, & in\ \Omega,\\ \Delta\varphi = g, & in\ \Omega,\\ \omega = 0, & on\ \partial\Omega,\\ \nabla\omega\cdot\nu = 0, & on\ \partial\Omega. \end{cases} \tag{2.9}$$

We assume the following regularity assumption is valid:

For given $g\in L^2(\Omega)$, there is a unique solution $(\omega,\varphi)\in H_0^2(\Omega)\times H^1(\Omega)$ to the problem (2.9) satisfying the following elliptic regularity estimate:

$$||\omega||_4 + ||\varphi||_2 \lesssim ||g||_0. \tag{2.10}$$

When Ω is a smooth domain, (2.10) is valid. However, when $\Omega\subset\mathbb{R}^2$ is a bounded convex domain, Grisvard [22] only stated that $\Delta^2 : H^3(\Omega)\to H^{-1}(\Omega)$ is isomorphic, and Blum et al. [23] stated that (2.10) is true if the maximum interior angle of Ω is less than $126.283696\cdots$. This assumption is made only to reduce the technical complexity of the error analysis.

Let λ and λ_h be the kth eigenvalue of (2.3), (2.4) and (2.7), (2.8), respectively. The algebraic multiplicity of λ is q, $\lambda = \lambda_k = \lambda_{k+1} = ... = \lambda_{k+q-1}$. Let V_λ denote the space spanned by all eigenfunctions corresponding to λ, and let $V_\lambda(h)$ denote the space spanned by all eigenfunctions corresponding to the eigenvalues $\lambda_{j,h}$ that converge to λ.

Lemma 2.1. Let λ be the kth eigenvalue of (2.3) and (2.4), $V_\lambda\subset H^{m+1}(\Omega)$, and (λ_h,v_h,u_h) be the kth eigenpair of (2.7) and (2.8) with $||u_h||_0 = 1$, then there exists an eigenfunction (v,u) corresponding to λ, such that $||u||_0 = 1$ and

$$|\lambda_h-\lambda| \lesssim h^{2m-2}, \tag{2.11}$$

$$||v-v_h||_0 \lesssim h^{m-1}, \tag{2.12}$$

$$||u-u_h||_0 \lesssim h^{m+\varepsilon}, \tag{2.13}$$

$$||u-u_h||_1 \lesssim h^m \tag{2.14}$$

where $\varepsilon = 0$ when $m = 2$ and $\varepsilon = 1$ when $m \geqslant 3$. Let $u \in V_\lambda$ and $\|u\|_0 = 1$, then there exists $u_h \in V_\lambda(h)$ such that $\|u - u_h\|_1 \lesssim h^m$.

Proof. We know that (2.11), (2.12) and (2.14) are valid from Theorem 11.4 in [8]. We obtain the conclusion (2.13) from [4].

Lemma 2.2. Suppose (λ, u, v) and (λ_h, u_h, v_h) are the eigenpairs of (2.3), (2.4) and (2.7), (2.8), respectively. Then

$$\lambda_h - \lambda = \frac{(v_h - v, v_h - v) + 2b(v_h - v, u_h - u)}{-(u_h, u_h)} + \lambda \frac{(u_h - u, u_h - u)}{-(u_h, u_h)}. \tag{2.15}$$

Proof. By (2.3) and (2.4) we deduce that

$$\begin{aligned}
&(v_h - v, v_h - v) + 2b(v_h - v, u_h - u) + \lambda(u_h - u, u_h - u)\\
=&(v_h, v_h) + b(v_h, u_h) + b(v_h, u_h) + \lambda(u_h, u_h) - ((v, v_h - v) + b(v_h - v, u) + b(v, u_h - u)\\
&+ \lambda(u, u_h - u)) - ((v_h, v) + b(v, u_h) + b(v_h, u) + \lambda(u_h, u))\\
=&(v_h, v_h) + 2b(v_h, u_h) + \lambda(u_h, u_h).
\end{aligned} \tag{2.16}$$

By (2.7) and (2.8) we have

$$\lambda_h = \frac{(v_h, v_h) + 2b(v_h, u_h)}{-(u_h, u_h)}.$$

Then, dividing by $-(u_h, u_h)$ on both sides of (2.16), we obtain (2.15).

To discuss the error estimates, we state some results on the approximation properties of interpolation in [24] without proof, which will play a crucial role in our analysis.

Lemma 2.3. For any $\phi \in H_0^2(\Omega)$, let $\phi_h \in V_h$ be the Lagrange interpolant of ϕ. Then, for any $\kappa \in \mathcal{J}_h$, there exists a positive constant C which is independent of h such that

$$\|\phi - \phi_h\|_{0,\kappa} \leq C h_\kappa^2 \|\phi\|_{2,\kappa}, \tag{2.17}$$

$$\|\phi - \phi_h\|_{0,\partial\kappa} \leq C h_\kappa^{\frac{3}{2}} \|\phi\|_{2,\kappa}. \tag{2.18}$$

Denote the piecewise (element-wise) Laplacian of $v \in V_h$ by $\Delta_h v$.

Lemma 2.4. For all $q_h \in V_h$ there exists a positive constant C independent of h such that

$$\|\triangle_h (q_h - E_h q_h)\|_{0,\Omega}^2 \leq C \sum_{e \in \Gamma} \int_e \frac{1}{|e|} [\frac{\partial q_h}{\partial \nu}]^2 ds, \tag{2.19}$$

where $E_h : V_h \to \widetilde{V_h} \subset H_0^2(\Omega)$ is a recovery operator defined as in [21], $\widetilde{V_h}$ is a Hsieh-Clough-Tocher (HCT) finite element space associated with $\mathcal{J}_h$.

Proof. Charbonneau et al. [20] and Gudi [21] proved the above conclusion for $m = 2$ and 3. From Lemma 1 in [25], we know the above conclusions are valid for $m \geq 2$.

3. A posteriori error estimates

Based on the a posteriori error analysis of the source problem corresponding to the biharmonic eigenvalue problem (2.1) in [21], the local estimator can be defined as follows:

For $\kappa \in \mathcal{J}_h$,

$$\eta_\kappa^2 = h_\kappa^4 \|\lambda_h u_h - \Delta_h v_h\|_{0,\kappa}^2 + \|v_h - \triangle_h u_h\|_{0,\kappa}^2;$$

for $e \in \Gamma_I$,

$$\eta_{1,e}^2 = |e|^3 \|[\frac{\partial v_h}{\partial \nu}]\|_{0,e}^2;$$

and for $e \in \Gamma$

$$\eta_{2,e}^2 = \frac{1}{|e|} \|[\frac{\partial u_h}{\partial \nu}]\|_{0,e}^2.$$

Let

$$\eta_h(\kappa)^2 = \eta_\kappa^2 + \frac{1}{2} \sum_{e \subset \partial\kappa, e \in \Gamma_I} (\eta_{1,e}^2 + \eta_{2,e}^2) + \sum_{e \subset \partial\kappa, e \in \Gamma_B} \eta_{2,e}^2,$$

and

$$\eta_h^2(\Omega) = \sum_{\kappa \in \mathcal{J}_h} \eta_h(\kappa)^2.$$

We can get the following theorem.

Theorem 3.1. Let (λ, u, v) and (λ_h, u_h, v_h) be the kth eigenpairs of (2.3), (2.4) and (2.7), (2.8), respectively. Then it holds that

$$\|(\Delta u - v_h, u - u_h)\|_W^2 \lesssim \eta_h^2(\Omega) + \|\lambda u - \lambda_h u_h\|_0^2. \tag{3.1}$$

Proof. From the definitions of the norm $\|\cdot\|_W$ and $|||\cdot|||$, we know that

$$\|(\Delta u - v_h, u - u_h)\|_W^2 = \|\Delta u - v_h\|_0^2 + |||u - u_h|||^2, \tag{3.2}$$

$$|||u - u_h|||^2 = \sum_{\kappa \in \mathcal{J}_h} \|\Delta_h (u - u_h)\|_{0,\kappa}^2 + \sum_{e \in \Gamma} \int_e \frac{1}{|e|} [\frac{\partial (u - u_h)}{\partial \nu}]^2 ds. \tag{3.3}$$

Now we estimate $|||u - u_h|||$. Since $[\frac{\partial u}{\partial \nu}] = 0$ on e, we have

$$\sum_{e \in \Gamma} \int_e \frac{1}{|e|} [\frac{\partial (u - u_h)}{\partial \nu}]^2 ds = \sum_{e \in \Gamma} \int_e \frac{1}{|e|} [\frac{\partial u_h}{\partial \nu}]^2 ds. \tag{3.4}$$

Using the triangle inequality and Lemma 2.4 we obtain

$$\begin{aligned} \|\Delta_h (u - u_h)\|_0 &\le \|\Delta_h (u - E_h u_h)\|_0 + \|\Delta_h (E_h u_h - u_h)\|_0 \\ &\lesssim \|\Delta_h (u - E_h u_h)\|_0 + (\sum_{e \in \Gamma} \int_e \frac{1}{|e|} [\frac{\partial u_h}{\partial \nu}]^2 ds)^{\frac{1}{2}}. \end{aligned} \tag{3.5}$$

Note that by the dual argument we have

$$\|\Delta (u - E_h u_h)\|_0 = \sup_{\phi \in H_0^2(\Omega) \backslash \{0\}} \frac{(\Delta (u - E_h u_h), \Delta \phi)}{\|\Delta \phi\|_0}. \tag{3.6}$$

Let $\phi \in H_0^2(\Omega)$. Then

$$(\Delta(u - E_h u_h), \Delta\phi) = (\Delta u - v_h, \Delta\phi) + (v_h - \Delta E_h u_h, \Delta\phi). \tag{3.7}$$

Let $\phi_h \in V_h^0$ be the Lagrange interpolant of ϕ, then we can deduce that

$$\begin{aligned}(\triangle u - v_h, \triangle\phi) &= (\Delta u, \Delta\phi) - (v_h, \Delta\phi)\\ &= (\lambda u, \phi) + (\nabla v_h, \nabla\phi)\\ &= (\lambda u, \phi) - (\lambda_h u_h, \phi_h) + (\nabla v_h, \nabla(\phi - \phi_h))\\ &= (\lambda u, \phi) - (\lambda_h u_h, \phi_h - \phi) - (\lambda_h u_h, \phi) + (\nabla v_h, \nabla(\phi - \phi_h))\\ &= (\lambda u - \lambda_h u_h, \phi) + (\lambda_h u_h, \phi - \phi_h) + (\nabla v_h, \nabla(\phi - \phi_h))\\ &= \sum_{\kappa\in\mathcal{J}_h}\int_\kappa(\lambda_h u_h - \Delta v_h)(\phi - \phi_h)dx + \sum_{e\in\Gamma_I}\int_e[\frac{\partial v_h}{\partial\nu}](\phi - \phi_h)ds + (\lambda u - \lambda_h u_h, \phi).\end{aligned} \tag{3.8}$$

Using the Cauchy-Schwarz inequality and Lemma 2.3, we know

$$\begin{aligned}|\sum_{\kappa\in\mathcal{J}_h}\int_\kappa(\lambda_h u_h - \Delta v_h)(\phi - \phi_h)dx| &\lesssim (\sum_{T\in\mathcal{J}_h} h_\kappa^4\|\lambda_h u_h - \Delta v_h\|_{0,\kappa}^2)^{\frac{1}{2}}|\phi|_2\\ &\lesssim (\sum_{\kappa\in\mathcal{J}_h} h_\kappa^4\|\lambda_h u_h - \Delta v_h\|_{0,\kappa}^2)^{\frac{1}{2}}\|\Delta\phi\|_0,\end{aligned} \tag{3.9}$$

$$\begin{aligned}|\sum_{e\in\Gamma_I}\int_e[\frac{\partial v_h}{\partial\nu}](\phi - \phi_h)ds| &\lesssim (\sum_{e\in\Gamma_I}\int_e|e|^3[\frac{\partial v_h}{\partial\nu}]^2ds)^{\frac{1}{2}}|\phi|_2\\ &\lesssim (\sum_{e\in\Gamma_I}\int_e|e|^3[\frac{\partial v_h}{\partial\nu}]^2ds)^{\frac{1}{2}}\|\Delta\phi\|_0\end{aligned} \tag{3.10}$$

and

$$|(\lambda u - \lambda_h u_h, \phi)| \leq \|\lambda u - \lambda_h u_h\|_0\|\phi\|_0. \tag{3.11}$$

Substituting (3.9)–(3.11) into (3.8), we obtain

$$|(\Delta u - v_h, \Delta\phi)| \lesssim((\sum_{\kappa\in\mathcal{J}_h} h_\kappa^4\|\lambda_h u_h - \Delta v_h\|_{0,\kappa}^2)^{\frac{1}{2}} + (\sum_{e\in\Gamma_I}\int_e|e|^3[\frac{\partial v_h}{\partial\nu}]^2ds)^{\frac{1}{2}} + \|\lambda u - \lambda_h u_h\|_0)\|\Delta\phi\|_0. \tag{3.12}$$

Using the triangle inequality and Lemma 2.4, we obtain

$$\begin{aligned}|(v_h - \Delta E_h u_h, \Delta\phi)| &\leq (\|v_h - \Delta_h u_h\|_0 + \|\Delta_h(u_h - E_h u_h)\|_0)\|\Delta\phi\|_0\\ &\lesssim (\|v_h - \Delta_h u_h\|_0 + (\sum_{e\in\Gamma}\int_e\frac{1}{|e|}[\frac{\partial u_h}{\partial\nu}]^2ds)^{\frac{1}{2}})\|\Delta\phi\|_0.\end{aligned} \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.7), and using (3.6), we deduce

$$\|\Delta(u - E_h u_h)\|_0 \lesssim (\sum_{e\in\mathcal{J}_h} h_\kappa^4\|\lambda_h u_h - \Delta v_h\|_{0,\kappa}^2)^{\frac{1}{2}} + (\sum_{e\in\Gamma_I}\int_e|e|^3[\frac{\partial v_h}{\partial\nu}]^2ds)^{\frac{1}{2}}$$

$$+\|v_h-\Delta_h u_h\|_0+(\sum_{e\in\Gamma}\int_e\frac{1}{|e|}[\frac{\partial u_h}{\partial\nu}]^2ds)^{\frac{1}{2}}+\|\lambda u-\lambda_h u_h\|_0. \tag{3.14}$$

Then, from (3.3)–(3.5) and (3.14), we can get

$$|||u-u_h|||^2\lesssim\eta_h^2(\Omega)+\|\lambda u-\lambda_h u_h\|_0^2.$$

Using the triangle inequality (3.5) and (3.14), we obtain

$$\begin{aligned}\|\Delta u-v_h\|_0^2&\lesssim\|\Delta_h(u-u_h)\|_0^2+\|\Delta_h u_h-v_h\|_0^2\\&\lesssim\eta_h^2(\Omega)+\|\lambda u-\lambda_h u_h\|_0^2.\end{aligned}$$

The proof is complete.

The following theorem gives the error bounds for the approximate eigenvalue.

Theorem 3.2. Let (λ,u,v) and (λ_h,u_h,v_h) be the kth eigenpairs of (2.3), (2.4) and (2.7), (2.8), respectively. Then it holds that

$$|\lambda-\lambda_h|\lesssim\eta_h^2(\Omega)+\lambda\|u_h-u\|_0^2+\sum_\kappa\sum_{j=0}^{1}h_\kappa^{2j}\|I_hv-v\|_{j,\kappa}^2 \tag{3.15}$$

where $I_hv\in V_h$ is the Lagrange interpolant of v.

Proof. From (2.3) and (2.7), we get

$$(v_h-v,\psi_h)+b(\psi_h,u_h-u)=0,\quad\forall\psi_h\in V_h.$$

Thus, using (2.15) and integrating by parts, we deduce that

$$\begin{aligned}|\lambda-\lambda_h|&=|-2(I_hv-v,\triangle_h(u_h-u))+2(v_h-v,I_hv-v)-(v_h-v,v_h-v)\\&\quad+\lambda(u_h-u,u_h-u)+2\sum_{e\in\Gamma}\int_e[\frac{\partial(u_h-u)}{\partial\nu}](I_hv-v)ds|\\&\lesssim2\sum_{\kappa\in\mathcal{T}_h}\|I_hv-v\|_{0,\kappa}\|\triangle_h(u_h-u)\|_{0,\kappa}+2\sum_{\kappa\in\mathcal{T}_h}\|v-v_h\|_{0,\kappa}\|I_hv-v\|_{0,\kappa}+\|v-v_h\|_0^2\\&\quad+\lambda\|u_h-u\|_0^2+2\sum_{e\in\Gamma}\frac{1}{|e|^{\frac{1}{2}}}\|[\frac{\partial(u_h-u)}{\partial\nu}]\|_{0,e}|e|^{\frac{1}{2}}\|I_hv-v\|_{0,e}\\&\lesssim\sum_{\kappa\in\mathcal{T}_h}\|I_hv-v\|_{0,\kappa}^2+\sum_{\kappa\in\mathcal{T}_h}\|\triangle_hu_h-v_h\|_{0,\kappa}^2+\sum_{\kappa\in\mathcal{T}_h}\|\triangle u-v_h\|_{0,\kappa}^2+\sum_{\kappa\in\mathcal{T}_h}\|I_hv-v\|_{0,\kappa}^2\\&\quad+\|v-v_h\|_0^2+\lambda\|u_h-u\|_0^2+\sum_{e\in\Gamma}\frac{1}{|e|}\|[\frac{\partial u_h}{\partial\nu}]\|_{0,e}^2+\sum_{\kappa\in\mathcal{T}_h}|h_\kappa|^2\|I_hv-v\|_{1,\kappa}^2.\end{aligned} \tag{3.16}$$

Using the definition of norm $\|\cdot\|_W$ and (3.1), we can get (3.15). The proof is complete.

Now, based on [16, 21] we study the efficiency of the error estimator.

Let e represent a common edge shared by the two elements κ^+ and κ^-, and denote $\omega_e=\kappa^+\cup\kappa^-$.

Theorem 3.3. Let (λ,u,v) and (λ_h,u_h,v_h) be the kth eigenpairs of (2.3), (2.4) and (2.7), (2.8), respectively. Then it holds that

$$h_\kappa^2\|\lambda_hu_h-\Delta v_h\|_{0,\kappa}\lesssim\|\Delta u-v_h\|_{0,\kappa}+h_\kappa^2\|\lambda_hu_h-\lambda u\|_{0,\kappa}, \tag{3.17}$$

$$\int_e |e|^3 [\frac{\partial v_h}{\partial \nu}]^2 ds \lesssim \|\Delta u - v_h\|_{0,\omega_e}^2 + |e|^4 \|\lambda u - \lambda_h u_h\|_{0,\omega_e}^2, \tag{3.18}$$

$$\eta_h^2(\Omega) \lesssim \|(\Delta u - v_h, u - u_h)\|_W^2 + \sum_{\kappa \in \mathcal{J}_h} h_\kappa^4 \|\lambda u - \lambda_h u_h\|_{0,\kappa}^2. \tag{3.19}$$

Proof. Using bubble function techniques (see [16,21]), we first estimate (3.17).

Let $b_\kappa \in H_0^2(\kappa)$ be a bubble polynomial defined on κ. Then

$$\|\lambda_h u_h - \Delta v_h\|_{0,\kappa} \lesssim \|b_\kappa^{\frac{1}{2}}(\lambda_h u_h - \Delta v_h)\|_{0,\kappa}$$
$$\|b_\kappa(\lambda_h u_h - \Delta v_h)\|_{0,\kappa} \lesssim \|\lambda_h u_h - \Delta v_h\|_{0,\kappa}.$$

Let $\phi = b_\kappa(\lambda_h u_h - \Delta v_h)$. Then

$$\|\lambda_h u_h - \Delta v_h\|_{0,\kappa}^2 \lesssim \int_\kappa b_\kappa (\lambda_h u_h - \Delta v_h)^2 dx = \int_\kappa (\lambda_h u_h - \Delta v_h)\phi dx.$$

Integrating by parts twice and using the inverse inequality, we get

$$\begin{aligned}\int_\kappa (\lambda_h u_h - \Delta v_h)\phi dx &= \int_\kappa \Delta^2 u \phi dx - \int_\kappa \Delta v_h \phi dx + \int_\kappa (\lambda_h u_h - \lambda u)\phi dx \\ &= \int_\kappa \Delta u \Delta\phi dx - \int_\kappa v_h \Delta\phi dx + \int_\kappa (\lambda_h u_h - \lambda u)\phi dx \\ &\lesssim h_\kappa^{-2}\|\Delta u - v_h\|_{0,\kappa}\|\phi\|_{0,\kappa} + \|\lambda_h u_h - \lambda u\|_{0,\kappa}\|\phi\|_{0,\kappa}.\end{aligned}$$

Combining the above three estimates, we get (3.17).

In the proof of Lemma 3.3 in [21], let $f = \lambda_h u_h$, then we can get (3.18).

It is clear that

$$\sum_{\kappa \in \mathcal{J}_h} \int_e \frac{1}{|e|} [\frac{\partial u_h}{\partial \nu}]^2 ds = \sum_{\kappa \in \mathcal{J}_h} \int_e \frac{1}{|e|} [\frac{\partial (u - u_h)}{\partial \nu}]^2 ds, \tag{3.20}$$

and using (3.17), (3.18) and the definition of norm $\|\cdot\|_W$, we can get (3.19). The proof is complete.

Remark 3.1. From Lemma 2.1, we know that $\|u_h - u\|_0$ is a higher-order term than $\|\Delta u - v_h\|_0$. And, interpolation theory shows that the estimate of the error $\sum_\kappa \sum_{j=0}^{2} h_\kappa^{2j} \|I_h v - v\|_{j,\kappa}^2$ is optimal with respect to h, so we can expect to get

$$\sum_\kappa \sum_{j=0}^{2} h_\kappa^{2j} \|I_h v - v\|_{j,\kappa}^2 \lesssim \|\Delta u - v_h\|_0^2. \tag{3.21}$$

So, substituting (3.21) into (3.15), we obtain

$$|\lambda - \lambda_h| \lesssim \eta_h^2(\Omega) + \lambda \|u_h - u\|_0^2. \tag{3.22}$$

Therefore, the estimator $\eta_h^2(\Omega)$ of the eigenvalue error $|\lambda_h - \lambda|$ is reliable up to the higher-order term $\lambda \|u_h - u\|_0^2$.

4. Numerical experiments

In this section, we will present some numerical results to validate our theoretical analysis. We calculate the smallest eigenvalue of the biharmonic eigenvalue problem on adaptive meshes in three domains: the unit square $\Omega_S = (0,1)^2$, the regular hexagon Ω_H with side length of 1, and the L-shaped domain $\Omega_L = (-\frac{1}{2}, \frac{1}{2})^2/[0, \frac{1}{2})\times(-\frac{1}{2}, 0]$. For Ω_S, we choose the reference value $\lambda_1 \approx 1294.93397959171$ (see [26]), and take the reference value $\lambda_1 \approx 163.59756815825$ in Ω_H and $\lambda_1 \approx 6703.6047044786$ in Ω_L (see [19]).

The computations are implemented according to the following algorithm, and for Ω_S our calculations refer to Algorithm 2 in [18] when the P4 element is used. All computations are easily realized under the packages of the FEM [27, 28].

The adaptive algorithm of the mixed conforming finite element method:
Choose the parameter $0 < \theta < 1$.
Step 1. Pick any initial mesh $\mathcal{J}_{h_0}$ with initial mesh size h_0.
Step 2. Solve (2.7)-(2.8) on $\mathcal{J}_{h_0}$ for discrete solution $(\lambda_{h_0}, u_{h_0}, v_{h_0})$.
Step 3. Let iterations $l = 0$.
Step 4. Compute the local estimator $\eta_{h_l}(\kappa)$.
Step 5. Construct $\widehat{\mathcal{J}_{h_l}} \subset \mathcal{J}_{h_l}$ by **Marking Strategy E** and parameter θ.
Step 6. Refine $\mathcal{J}_{h_l}$ to get a new mesh $\mathcal{J}_{h_{l+1}}$ by procedure **REFINE**.
Step 7. Solve (2.7)-(2.8) on $\mathcal{J}_{h_{l+1}}$ for discrete solution $(\lambda_{h_{l+1}}, u_{h_{l+1}}, v_{h_{l+1}})$.
Step 8. Let $l \Leftarrow l + 1$ and go to Step 4.

Marking Strategy E:
Step 1. Construct a minimal $\widehat{\mathcal{J}_{h_l}} \subset \mathcal{J}_{h_l}$ by selecting some elements in $\mathcal{J}_{h_l}$ such that

$$\sum_{\kappa\in\widehat{\mathcal{J}_{h_l}}} \eta_{h_l}^2(\kappa) \geq \theta\eta_{h_l}^2(\Omega).$$

Step 2. Mark all elements in $\widehat{\mathcal{J}_{h_l}}$.

The value of θ is set to 0.5. The results computed by the adaptive algorithm with P2, P3 and P4 elements in Ω_S, Ω_H and Ω_L are listed in Tables 1–3, respectively. We also depict the curves of absolute error $|\lambda_h - \lambda_1|$ in the three domains in Figures 1–3 and show the adaptive meshes obtained by P2, P3 and P4 elements in Figures 4–6.

For Ω_S, from Table 1 we can obverse that the approximate eigenvalues of high accuracy can be obtained when using higher degree polynomials. From Table 4, compared with the results obtained by the DG mixed method in [19], we can conclude that with the same degree of freedom, using the mixed conforming finite element method can achieve higher accuracy. And, compared with the results calculated in [11], we can conclude that with the same degree of freedom, the approximations obtained by the adaptive algorithm with P3 element have higher precision than those computed by the C-R mixed method with P3 element on uniform meshes.

Table 1. The smallest eigenvalue using P2, P3 and P4 elements in Ω_S.

m	l	Dof	λ_h	$Error$
2	4	1688	1295.55311799145	6.1914E-01
	7	6308	1294.96737945769	3.3400E-02
	8	10020	1294.94080090246	6.8213E-03
	14	392486	1294.93399037708	1.0785E-05
	15	731622	1294.93398365798	4.0663E-06
3	3	2378	1294.93953450880	5.5549E-03
	6	4868	1294.93734355155	8.1186E-04
	9	15590	1294.93400416261	2.4571E-05
	13	70640	1294.93397953709	5.4620E-08
	14	110612	1294.93397957360	1.8110E-08
	15	166268	1294.93397958965	2.0600E-09
4	5	4402	1294.93400398026	2.4389E-05
	6	17122	1294.93398001229	4.2058E-07
	8	20614	1294.93397969179	1.0008E-07
	11	39726	1294.93397963481	4.3100E-08
	12	45326	1294.93397959210	3.8995E-10
	13	55910	1294.93397958163	1.0080E-08
	14	71082	1294.93397959395	2.2399E-09

Table 2. The smallest eigenvalue using P2, P3 and P4 elements in Ω_H.

m	l	Dof	λ_h	$Error$
2	3	1004	163.63563344085	3.8065E-02
	7	3160	163.61867333594	2.1105E-02
	13	65862	163.59758215821	1.4000E-05
	14	120242	163.59757290575	4.7475E-06
	15	223442	163.59756998769	1.8294E-06
3	3	1688	163.59829409370	7.2594E-04
	9	7790	163.59767457327	1.0642E-04
	12	13148	163.59757702759	9.6994E-05
	15	35216	163.59756843072	2.7247E-07
	17	65954	163.59756822596	6.7710E-08
	19	120422	163.59756817386	1.5610E-08
	20	179708	163.59756817021	1.1960E-08
4	9	4734	163.59757299916	4.8409E-06
	11	6826	163.59756994482	1.7866E-06
	13	9198	163.59756856556	4.0731E-07
	14	11174	163.59756846936	3.1111E-07
	15	12778	163.59756846485	3.0660E-07
	16	15670	163.59756819556	3.7310E-08

Table 3. The smallest eigenvalue using P2, P3 and P4 elements in Ω_L.

m	l	Dof	λ_h	$Error$
2	5	1112	6709.12631054012	5.5216E+00
	13	4288	6705.19344965942	1.5887E+00
	16	8888	6704.12267974168	5.1798E-01
	20	17050	6703.75315736491	1.4845E-01
	21	18864	6703.73737923773	1.3267E-01
	22	20764	6703.71676073157	1.1206E-01
3	10	1988	6699.01003534454	4.5947E+00
	23	6812	6703.70738462775	1.0268E-01
	27	13682	6703.61272707405	8.0226E-03
	28	17834	6703.60693637842	2.2319E-03
	29	22142	6703.60592592628	1.2214E-03
	30	27698	6703.60534928621	6.4481E-04
	31	36884	6703.60491084803	2.0637E-04
4	3	2026	6673.41764738391	3.0187E+01
	12	4130	6701.92626113286	1.6784E+00
	21	7090	6703.55885779365	4.5847E-02
	26	8718	6703.60033078041	4.3737E-03
	27	9034	6703.60178462851	2.9199E-03
	28	9394	6703.60411046150	5.9402E-04

Table 4. The smallest eigenvalue using P2, P3 and P4 elements in Ω_S, Ω_H and Ω_L by the C-R mixed method and DG mixed method.

m	$Method$	Ω_S		Ω_H		Ω_L	
		Dof	λ_h	Dof	λ_h	Dof	λ_h
2	mixed	10020	1294.94080	65862	163.59758	17050	6703.75316
	DG mixed	10368	1295.73547	63672	163.61795	17712	6707.69651
3	mixed	70640	1294.93398	35216	163.59757	17834	6703.60694
	DG mixed	79740	1294.93441	39340	163.59781	17640	6702.29878
4	mixed	20614	1294.93398	9198	163.59757	8718	6703.60033
	DG mixed	20400	1294.93399	9510	163.59752	8850	6700.01769

Figure 1 shows that the error curves are approximately parallel to the line with slope −2, −3 and −4, and the algorithm can achieve the optimal convergence order $O(dof^{-2})$, $O(dof^{-3})$ and $O(dof^{-4})$ when P2, P3 and P4 elements are used, respectively. This means that the results obtained in numerical experiments have higher order convergence than theoretical analysis, and we think the reason is that $\Delta u \in H^2(\Omega)$ when $u \in H^4(\Omega)$, thus the regularity of $v = \Delta u$ is underestimated in the theoretical analysis of the C-R mixed method.

For Ω_H and Ω_L, we can observe similar conclusions. Although we only analyze the C-R mixed method for convex or smooth domains, we also implement adaptive calculations in the L-shaped domain, and the results in Table 3 and Figure 3 indicate that our method is still convergent.

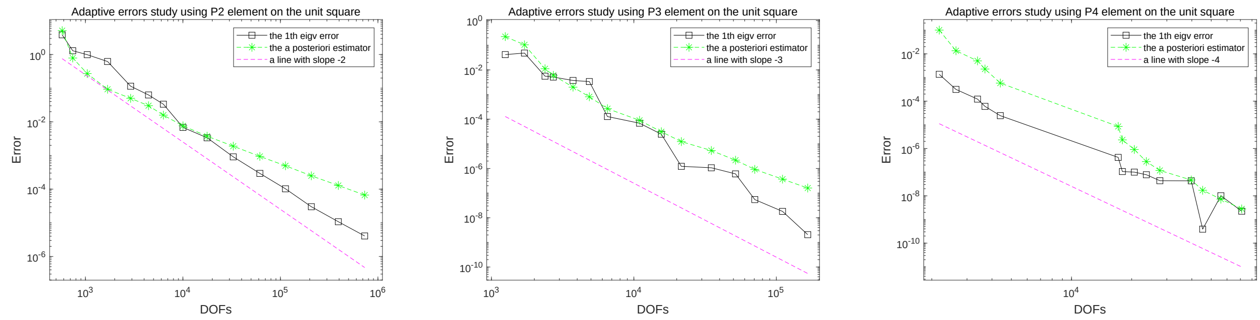

Figure 1. Error curves for the smallest eigenvalue in Ω_S by P2, P3 and P4 elements.

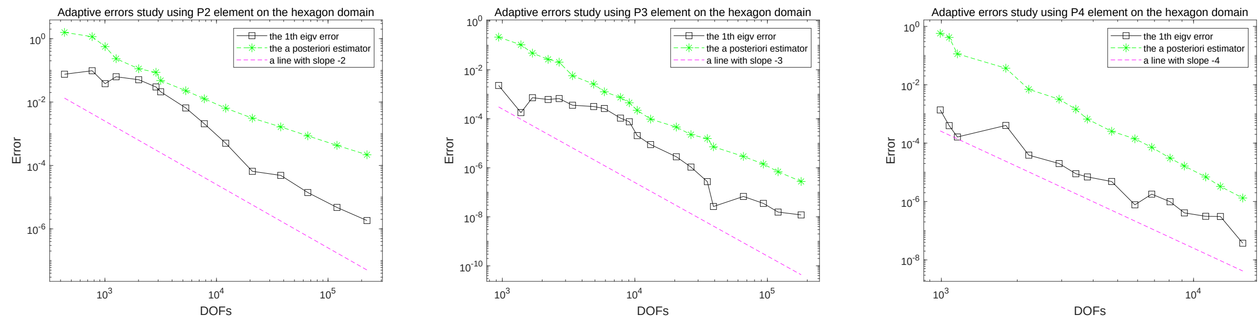

Figure 2. Error curves for the smallest eigenvalue in Ω_H by P2, P3 and P4 elements.

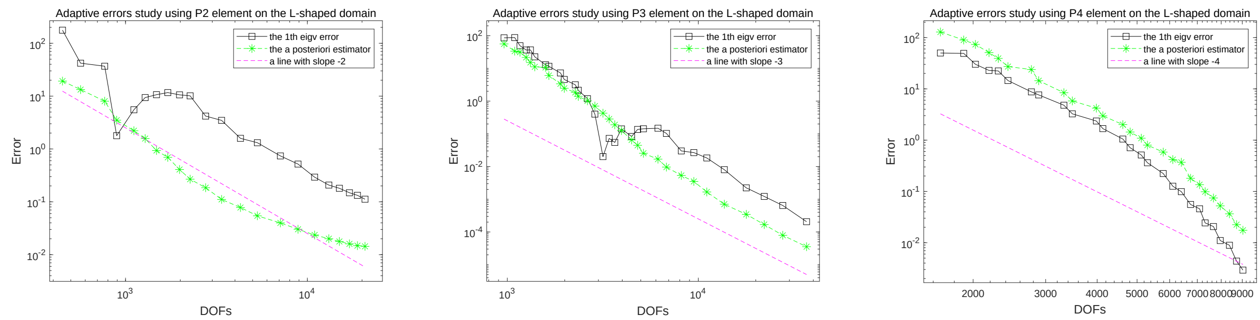

Figure 3. Error curves for the smallest eigenvalue in Ω_L by P2, P3 and P4 elements.

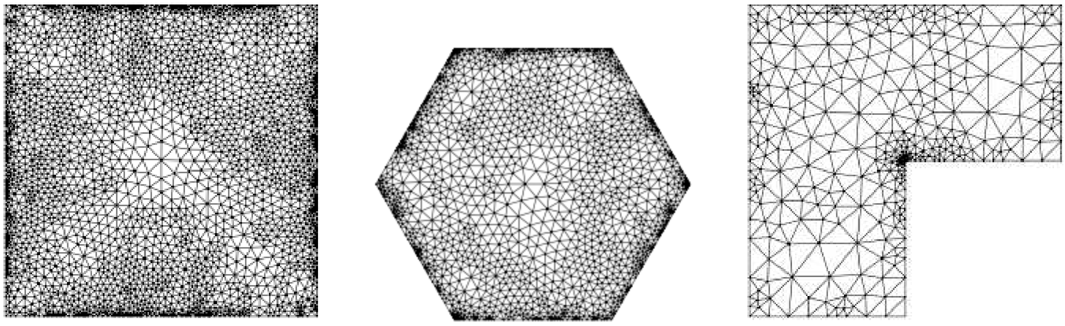

Figure 4. Adaptive mesh in Ω_S, Ω_H and Ω_L by P2 element.

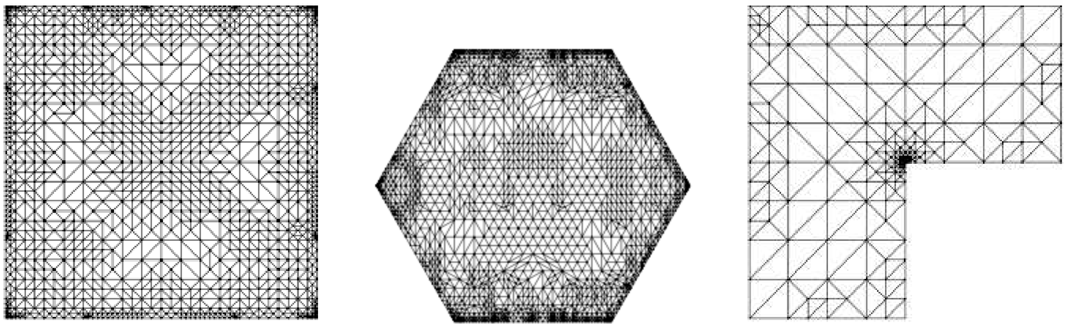

Figure 5. Adaptive mesh in Ω_S, Ω_H and Ω_L by P3 element.

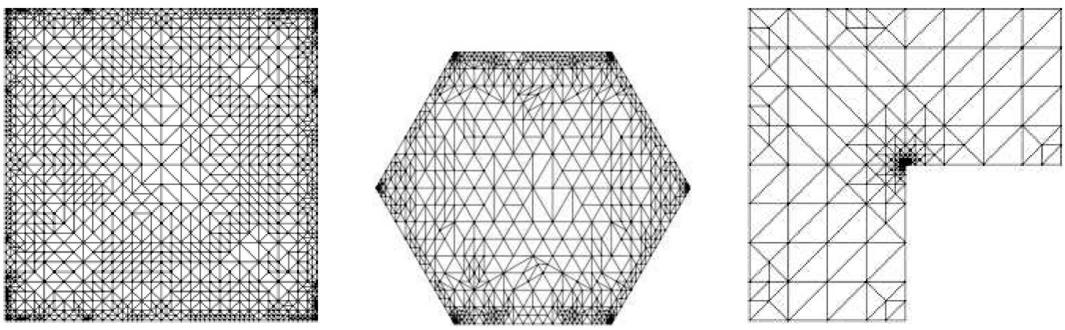

Figure 6. Adaptive mesh in Ω_S, Ω_H and Ω_L by P4 element.

Remark 4.1. There are usually two ways to determine when to terminate the iteration. One is by the error estimator. The adaptive procedure will continue until the error estimator is less than a prefixed tolerance. The other is by the difference between adjacent two or several iterations. When the difference is less than a prefixed tolerance, the iteration will be terminated. However, in this paper, since our error estimator is not asymptotically accurate and the error curves fluctuate, we judge whether the calculation result is accurate by observing the changing trend of the error.

5. Conclusions

In this paper, we study the a posteriori error estimates and adaptive calculation of the C-R mixed method for the biharmonic eigenvalue problem on convex polygon domains. We propose a posteriori error estimators, prove the reliability and efficiency of the error estimator of the approximate eigenfunction, and analyze the reliability of the error estimator of the approximate eigenvalues. Numerical experiments confirm our theoretical analysis and indicate that our adaptive algorithm is efficient. Meanwhile, the results in Table 3 and Figure 3 show that the C-R mixed method in adaptive fashion is convergent and efficient on nonconvex domains. It is a challenging and valuable work to prove the convergence of C-R mixed method on nonconvex domains.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that this work does not have any conflicts of interest.

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