Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(2): 3320–3331.
DOI: 10.3934/math.2024162
Received: 20 November 2023
Revised: 19 December 2023
Accepted: 26 December 2023
Published: 04 January 2024

Research article

An inevitable note on bipolar metric spaces

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Abstract: Bipolar metric spaces and related fixed point theorems therein were introduced based on the motivation of measuring the distance between the elements of distinct sets. The question regarding the independence of these results from the analogous results on a fixed point of an induced mapping on a Cartesian product of two sets. We proved that bipolar metric space is metrizable and we presented two different approaches for defining a metric induced by a bipolar metric. Two obtained metric spaces demonstrated the lack of novelty of fixed point theorems for covariant and contravariant contraction.

Keywords: metric; bipolar metric; Banach contraction; Kannan contraction
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

The idea of a bipolar metric was introduced by Mutlu and Gurdal in [9] in 2016 and was motivated by real-life applications and necessity driven by numerous examples where "distance" is measured between elements of different sets. Some of them are the distance between lines and points in Euclidian space, distance between sets and elements of a set, and the distance from a set of planetary bodies to the inverse of visible luminosities of a set of stars.

The question of bipolarity is also investigated in the fuzzy logic through the notion of a bipolar-valued set. Importance of bipolar-valued sets may be foreseen through significant application in decision making. The importance of the concept of bipolarity in the process of decision-making is based on the possibility to overcome previous obstacles in this field, lowering the level of uncertainty, and successfully solving problems in a range of different decision-making processes as testified by several case studies. The relation between the notion of a bipolar fuzzy metric and a bipolar fuzzy set is still an open problem and there are no results of the application of a bipolar metric as can be seen for the bipolar set [14–16].

There are numerous results on fixed point problems in the bipolar metric space or bipolar pseudometric space [2–4, 6, 8–13, 17], along with different extensions of this concept like fuzzy bipolar metric space, bipolar soft metric space, C^*-algebra valued bipolar metric space, and partially ordered bipolar metric space, among others.

We will recall basic definitions and theorems in both metric space and bipolar metric space. For the sake of simplicity, we will use BMS when denoting a bipolar metric space.

Definition 1. *Let X be a nonempty set. A function $d : X \times X \to [0, +\infty)$ is a metric on a set X if it fulfills the following assumptions for any $x, y, z \in X$:*

(d_1) $d(x, x) = 0$;
(d_2) $d(x, y) = d(y, x) = 0$ *implies* $x = y$;
(d_3) $d(x, y) = d(y, x)$;
(d_4) $d(x, z) \leq d(x, y) + d(y, z)$

then the ordered pair (X, d) is a metric space.

The bipolar metric is mimicking the properties of a metric adjusted to the new environment.

Definition 2. *[9] If X and Y are nonempty sets, then a mapping $d : X \times Y \mapsto [0, +\infty)$ satisfying:*

(d_1^*) $d(x, x) = 0$ *for any* $x \in X \cap Y$;
(d_2^*) *if* $d(x, y) = d(y, x) = 0$ *for some* $x \in X$ *and* $y \in Y$, *then* $x = y$;
(d_3^*) $d(x, y) = d(y, x)$ *for all* $x, y \in X \cap Y$;
(d_4^*) $d(x_1, y_1) \leq d(x_1, y_2) + d(x_2, y_1) + d(x_2, y_2)$ *for all* $x_1, x_2 \in X$ *and* $y_1, y_2 \in Y$

is a bipolar metric and a triple (X, Y, d) is a bipolar metric space (BMS).

The bipolar pseudometric is a mapping $d : X \times Y \mapsto [0, +\infty)$ satisfying (d_1^*), (d_3^*), and (d_4^*), assuming that X and Y are nonempty sets and (X, Y, d) is a bipolar pseudometric space (BPMS).

Example 1. *If (X, d) is a metric space (pseudometric space), then (X, X, d) is a bipolar metric space (bipolar pseudometric space).*

The BMS $(X \times Y, d)$ is disjoint if $X \cap Y = \emptyset$, while it is a joint BMS otherwise. The sets X and Y are the left pole and the right pole of a BMS (X, Y, d), respectively.

The question that arises is how the bipolar metric defined on $X \times Y$ may be transferred to the poles forming a metric space and what is a general answer to the metrizability problem.

Definition 3. *If (X, Y, d) is a BPMS, then the functions $d_X : X \times X \mapsto \mathbb{R}$ and $d_Y : Y \times Y \mapsto \mathbb{R}$ defined by*

$$d_M(x_1, x_2) = \sup_{y \in Y} |d(x_1, y) - d(x_2, y)|, \ \ x_1, x_2 \in X,$$
$$d_N(y_1, y_2) = \sup_{x \in X} |d(x, y_1) - d(x, y_2)|, \ \ y_1, y_2 \in Y$$

are inner pseudo-metrics on X and Y, respectively, induced by a bipolar metric d.

It is shown in [9] that the inner pseudo-metrics are pseudo-metrics on X and Y, respectively. In the case of BMS, d_X and d_Y are metrics on X and Y, respectively, which was implicitly proven in [9] through the embedding theorem. Meanwhile in the case of a joint BPMS (BMS) (X, Y, d), the pseudo-metric (metric) d_M and bipolar pseudo-metric (metric) d coincide on $X \times (X \cap Y)$ as well as d_N and d agree on $(X \cap Y) \times Y$.

First, we collect some results from [9] regarding topological properties, convergence, and continuity of a BMS. The definitions will be presented in the setting of BMS, but the same approach is used in the case of BPMS.

If (X, Y, d) is a BMS, then a sequence $((x_n, y_n)) \subseteq X \times Y$ is a bisequence in a BMS (X, Y, d). The idea behind the convergence of a bisequence $((x_n, y_n))$ in a bipolar metric space $(X \times Y, D)$ relies on the convergence of a sequence $(x_n) \subseteq X$ to some $y \in Y$ under which we consider the usual metric convergence, i.e., that for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, y) < \varepsilon$ for any $n \geq n_0$, and, analogously, on the convergence of a sequence $(y_n) \subseteq Y$ to some $x \in X$ assuming that, for any $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$, fulfilling $d(x, y_n) < \varepsilon$ for any $n \geq m_0$.

Definition 4. *[9] If (X, Y, d) is a BMS, then*

(i) A bisequence $((x_n, y_n)) \subseteq X \times Y$ is convergent in (X, Y, d) if both (x_n) and (y_n) converge.
(ii) If (x_n) and (y_n) both converge to the same point $z \in X \cap Y$, then a bisequence $((x_n, y_n))$ is a biconvergent bisequence in (X, Y, d).

Definition 5. *[9] If (X, Y, d) is a BMS and $(x_n, y_n) \subseteq X \times Y$ is a bisequence, then it is a Cauchy bisequence if for any $\varepsilon > 0$, there exists some $n_0 \in \mathbb{N}$ fulfilling $d(x_n, y_m) < \varepsilon$ for any $n, m \geq n_0$.*

To clarify the question of biconvergence, note that the bisequence $((x_n, y_n))$ converges to some $(z, z) \in X \times Y$ for some $z \in X \cap Y$, if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n, m \geq n_0$ we have $d(x_n, z) < \varepsilon$ and $d(z, y_n) < \varepsilon$.
The relation between Cauchy property and convergence is obtained.

Theorem 1. *[9] A biconvergent bisequence in a BMS is a Cauchy bisequence.*

It is important to notice that if there exists a convergent Cauchy bisequence, then the bipolar metric space must be joint. Also, observe that from Theorem 1 it could be deduced that convergence of a sequence does not necessarily imply Cauchiness which is not substantiated by any example in [9]. As usual, completeness is defined according to the convergence of a Cauchy sequence; in this case, a bisequence.

Definition 6. *[9] A BMS (X, Y, d) is complete if any Cauchy bisequence in (X, Y, d) is convergent.*

It is important to mention that the definition of complete BMS corresponds to the definition of a complete metric space (X, d) in the case when we observe its bipolar equivalent (X, X, d).

Theorem 2. *[9] A convergent Cauchy bisequence is biconvergent in a BMS.*

Recall that a self-mapping T on a metric space (X, d) is a contraction if there exists $q \in [0, 1)$ such that the inequality

$$d(Tx, Ty) \leq qd(x, y) \tag{1.1}$$

holds for all $x, y \in X$. Famous Banach contraction principle presents a highlight of the metric fixed point theory with numerous applications, extensions, and generalizations.

Theorem 3. *[1] If (X, d) is a complete metric space and $T : X \mapsto X$ is a contraction on X for some contractive constant $q \in [0, 1)$, then a mapping T has a unique fixed point $x^* \in X$, and for arbitrary initial point $x \in X$, the iterative sequence $(T^n x)$ converges to the fixed point x^*.*

It is possible to observe two different concepts of contractive mappings in BMS: Covariant and contravariant mappings, as presented in [9]. These concepts present an analog of a contraction in the setting of BMS.

Covariant mapping assumes that $T : (X, Y, d) \rightrightarrows (X, Y, d)$ is such that $T(X) \subseteq X$ and $T(Y) \subseteq Y$, while for contravariant mappings $T : (X, Y, d) \rightleftarrows (X, Y, d)$ implies $T(X) \subseteq Y$ and $T(Y) \subseteq X$. In both cases, the domain and codomain of a mapping T is indeed $X \cup Y$.

Definition 7. *[9] If (X, Y, d) is a BMS, a covariant mapping $T : (X, Y, d) \rightrightarrows (X, Y, d)$ is a covariant contraction on (X, Y, d) if there exists some $q \in [0, 1)$ such that the inequality*

$$d(Tx, Ty) \leq qd(x, y)$$

holds for all $x \in X$ and all $y \in Y$.

Definition 8. *If (X, Y, d) is a BMS, a contravariant mapping $T : (X, Y, d) \rightleftarrows (X, Y, d)$ is a contravariant contraction if there exists some $q \in [0, 1)$ such that the inequality*

$$d(Ty, Tx) \leq qd(x, y)$$

holds for all $x \in X$ and all $y \in Y$.

Fixed point for both covariant and contravariant contraction on a complete BMS exists and is unique, which is the main result of [9]. In the sequel, covariant contraction mapping will be denoted as cocontraction, while contravariant contraction mapping will be contracontraction.

The main aim of this manuscript is to prove that fixed point results for both covariant and contravariant contractive mappings are a direct corollary of the Banach fixed point theorem after proper metrization of a complete bipolar metric space is done.

2. Metrizability of a bipolar metric space

We intend to prove that this approach of introducing a new concept of BMS is unnecessary since those results are easily derived from the usual theorems in metric space concerning existence and uniqueness of a fixed point.

Theorem 4. *If (X, Y, d) is a BMS, then $(X \times Y, D)$ is a metric space where $D : (X \times Y) \times (X \times Y) \mapsto [0, +\infty)$ is defined by*

$$D((x_1, y_1), (x_2, y_2)) = \begin{cases} \sum\limits_{i,j \in \{1,2\}} d(x_i, y_j), & x_1 \neq x_2 \text{ or } y_1 \neq y_2 \\ 0, & x_1 = x_2 \text{ and } y_1 = y_2 \end{cases} \tag{2.1}$$

for any $(x_1, y_1), (x_2, y_2) \in X \times Y$.

Proof. Assume that (X, Y, d) is a BMS and that $D : (X \times Y) \times (X \times Y) \mapsto [0, +\infty)$ is defined by (2.1). Evidently, D is a well-defined function since $d(x_i, y_j) \geq 0$, where $i, j = \{1, 2\}$ for any $(x_1, y_1), (x_2, y_2) \in X \times Y$.

(d_1) If $(x, y) \in X \times Y$, then $D((x, y), (x, y)) = 0$ by the Definition 2.1.

(d_2) Assuming that $(x_1, y_1) \neq (x_2, y_2)$ and $D((x_1, y_1), (x_2, y_2)) = 0$ and observing that

$$D((x_1, y_1), (x_2, y_2)) \geq d(x_i, y_j) \geq 0$$

for $i, j = \overline{1, 2}$ along with (d_2^*) further imply that $x_1 = y_1 = y_2 = x_2$; hence, $(x_1, y_1) = (x_2, y_2)$ in the case of a joint BMS and we obtain a contradiction. If the BMS is disjoint, then $D((x_1, y_1), (x_2, y_2)) = 0$ only in the case of $(x_1, y_1) = (x_2, y_2)$ due to the previous estimations as $X \cap Y = \emptyset$.

Thus, $D((x_1, y_1), (x_2, y_2)) = 0$ implies $(x_1, y_1) = (x_2, y_2)$ is not related to the structure of $X \cap Y$.

(d_3) For arbitrary $(x_1, y_1), (x_2, y_2) \in X \times Y$, we have

$$\begin{aligned} D((x_1, y_1), (x_2, y_2)) &= d(x_1, y_2) + d(x_2, y_1) + d(x_1, y_1) + d(x_2, y_2) \\ &= d(x_2, y_1) + d(x_1, y_2) + d(x_2, y_2) + d(x_1, y_1) \\ &= D((x_2, y_2), (x_1, y_1)) \end{aligned}$$

concluding that D is a symmetric function on $X \times Y$ thanks to (d_3^*).

(d_4) Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$ be arbitrary, then

$$\begin{aligned} D((x_1, y_1), (x_2, y_2)) &= d(x_1, y_2) + d(x_2, y_1) + d(x_1, y_1) + d(x_2, y_2) \\ &\leq d(x_1, y_3) + d(x_3, y_2) + d(x_3, y_3) + d(x_2, y_3) \\ &\quad + d(x_3, y_1) + d(x_3, y_3) + d(x_1, y_1) + d(x_2, y_2) \\ &= D((x_1, y_1), (x_3, y_3)) + D((x_3, y_3), (x_2, y_2)) \end{aligned}$$

so the triangle inequality holds in $(X \times Y, D)$. Consequently, $(X \times Y, D)$ is a metric space. □

In this manner, each BMS is associated to the properly defined metric space defined as in Theorem 4. What still remains open is a question of the transfer of completeness.

Theorem 5. *If (X, Y, d) is a complete BMS, then $(X \times Y, D)$ is a complete metric space where $D : (X \times Y) \times (X \times Y) \mapsto [0, +\infty)$ is determined by (2.1).*

Proof. If (X, Y, d) is a complete BMS, then $(X \times Y, D)$ is a metric space due to Theorem 4, assuming that $D : (X \times Y) \times (X \times Y) \mapsto [0, +\infty)$ is defined by (2.1).
Assume that $((x_n, y_n)) \subseteq X \times Y$ is a Cauchy sequence in a metric space (X, Y, D), meaning that for arbitrary $\varepsilon > 0$, there exists some $n_0 \in \mathbb{N}$ such that $D((x_n, y_n), (x_m, y_m)) < \varepsilon$ for any integers $n, m \geq n_0$. Equivalently,

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_m) + d(x_m, y_n) + d(x_n, y_n) + d(x_n, y_n) \\ &= D((x_n, y_n), (x_m, y_m)) \\ &< \varepsilon \end{aligned}$$

and $((x_n, y_n))$ is a Cauchy bisequence in a complete BMS (X, Y, d). Further, based on Theorem 2 and the assumption of completeness, the bisequence $((x_n, y_n))$ is biconvergent. So, there exists a limit $z \in X \cap Y$ of both (x_n) and (y_n) such that, for an $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n, z) < \frac{\varepsilon}{3} \text{ and } d(y_m, z) < \frac{\varepsilon}{3}$$

for any $n, m \geq n_1$. Additionally, let $n_2 \in \mathbb{N}$ be such that $d(x_n, y_m) < \frac{\varepsilon}{3}$ for any $n, m \geq n_2$.
If $m_0 = \max\{n_1, n_2\}$, then for any $n \geq m_0$, it follows that

$$\begin{aligned} D((x_n, y_n), (z, z)) &= d(x_n, z) + d(z, y_n) + d(x_n, y_n) + d(z, z) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

leading to the conclusion that $((x_n, y_n))$ is converging to (z, z) in a metric space $(X \times Y, D)$. Therefore, the metric space $(X \times Y, D)$ is a complete metric space. □

The following corollary is easily deduced from the presented proof of Theorem 5.

Corollary 6. *If (X, Y, d) in a BMS and $(X \times Y, D)$ is an associated metric space as in Theorem 4, then any Cauchy sequence $((x_n, y_n)) \subseteq X \times Y$ in a metric space $(X \times Y, D)$ is a Cauchy bisequence in a BMS (X, Y, d).*

Proof. As proven in the part of the proof of Theorem 5. □

Corollary 7. *If (X, Y, d) is a complete BMS and $(X \times Y, D)$ is an associated metric space as in Theorem 4, then any convergent sequence $((x_n, y_n)) \subseteq X \times Y$ in a metric space $(X \times Y, D)$ is biconvergent in a BMS (X, Y, d).*

Proof. If (X, Y, d) is a complete BMS and $(X \times Y, D)$ is an associated metric space as in Theorem 4, then by Theorem 5 it is a complete metric space. Assume that $((x_n, y_n)) \subseteq X \times Y$ is a convergent sequence in a metric space $(X \times Y, D)$, then it is necessarily a Cauchy sequence in $(X \times Y, D)$ and, due to Corollary 6, it is a Cauchy bisequence in (X, Y, d).

As (X, Y, d) is a complete BMS, the sequence $((x_n, y_n))$ is a convergent Cauchy bisequence in a BMS and Theorem 2 yields biconvergency of a bisequence $((x_n, y_n))$ in a BMS (X, Y, d). □

2.1. Covariant contraction

As mentioned, all fixed point theorems in the setting of BMSs may be reduced in this way on their analogues in associated metric space as described in Theorem 4. We intend to present a proof of the main result of [9] through the Banach contraction principle showing that this set of fixed point results is a direct corollary of Banach and, analogously, the Kannan fixed point theorem. Generally, the same approach is applicable for any contractive mapping in a complete BMS and related fixed point theorem in metric space.

We will take into the consideration cocontraction, showing that it induces a contraction on a complete metric space $(X \times Y, D)$ whose existence is guaranteed by Theorems 4 and 5.

Theorem 8. *[9] If (X, Y, d) is a complete BMS and a mapping $T : (X, Y, d) \rightrightarrows (X, Y, d)$ is a cocontraction, then the mapping $T : X \cup Y \mapsto X \cup Y$ possesses a unique fixed point in $X \cup Y$.*

Proof. Observe a cocontraction T on a complete BMS (X, Y, d) with a contractive constant $q \in [0, 1)$ and define a mapping

$$S(x, y) = (Tx, Ty)$$

for any $x \in X$ and $y \in Y$. For arbitrary pairs $(x_1, y_1), (x_2, y_2) \in X \times Y$:

$$\begin{aligned} D(S(x_1, y_1), S(x_2, y_2)) &= D((Tx_1, Ty_1), (Tx_2, Ty_2)) \\ &= d(Tx_1, Ty_2) + d(Tx_2, Ty_1) + d(Tx_1, Ty_1) + d(Tx_2, Ty_2) \\ &\leq q\,(d(x_1, y_2) + d(x_2, y_1) + d(x_1, y_1) + d(x_2, y_2)) \\ &= qD((x_1, y_1), (x_2, y_2)) \end{aligned}$$

assuming that $S(x_1, y_1) \neq S(x_2, y_2)$, since we have applied the definition of a metric D in that manner. Evidently, the inequality holds anyway since $S(x_1, y_1) = S(x_2, y_2)$ implies $D(S(x_1, y_1), S(x_2, y_2)) = 0$ and, yet again,

$$D(S(x_1, y_1), S(x_2, y_2)) \leq qD((x_1, y_1), (x_2, y_2)).$$

Accordingly, S is a contraction on a complete metric space $(X \times Y, D)$ and, consequently, it possesses a single fixed point $(x^*, y^*) \in X \times Y$, and any sequence of a form $(S^n(x, y))$ for arbitrary initial point $(x, y) \in X \times Y$ converges to the fixed point as $n \to +\infty$, with respect to the metric D. Having in mind that the sequence $(S^n(x, y)) = ((T^n x, T^n y))$ is a convergent bisequence in a complete BMS because of Corollary 7, thus biconvergent, we conclude that $x^* = y^* \in X \cap Y$ is a unique fixed point of a mapping T. □

Obviously, Theorem 8 is a direct corollary of the Banach contraction principle.

3. Another approach on a metrizability of a bipolar metric space

Since we are aiming to consider contracontraction in a complete BMS, we will present another idea for metrizability of BMS.

We state obvious remarks regarding fixed point(s) of a contravariant mapping. Furthermore, it is essential to observe only joint BMS for that purpose.

Lemma 9. *If (X, Y, d) is a BMS and $T : (X, Y, d) \rightleftarrows (X, Y, d)$ is a caontravariant mapping, then the set of fixed points of a mapping T is a subset of $X \cap Y$.*

Proof. Recall that if $T : X \cup Y \mapsto X \cup Y$ is a contravariant mapping in a BMS (X, Y, d), then $T(X) \subseteq Y$ and $T(Y) \subseteq X$. Thus, if $Tz = z$ for some $z \in X \cup Y$, then

$$z \in X \cap T(X) \subseteq X \cap Y \text{ or } z \in Y \cap T(Y) \subseteq X \cap Y$$

□

Hence, the focus will be on set $X \cap Y$ instead of on the whole $X \cup Y$ (or substituted by $X \times Y$).

Theorem 10. *If* (X, Y, d) *is a joint BMS and* $d^* : (X \cap Y) \times (X \cap Y) \mapsto [0, +\infty)$ *is a mapping defined with*

$$d^*(z_1, z_2) = d(z_1, z_2) \tag{3.1}$$

for any $z_1, z_2 \in X \cap Y$*, then* $(X \cap Y, d^*)$ *is a metric space.*

Proof. Suppose that (X, Y, d) is a joint BMS and $d^* : (X \cap Y) \times (X \cap Y) \mapsto [0, +\infty)$ is defined with (3.1). Obviously, d^* is well-defined. We will comment on the fulfillment of (d_1)–(d_4) for d^* which will be valid due to fulfillment of (d_1^*)–(d_4^*) for a bipolar metric d.

(d_1) If $z \in X \cap Y$, then $d^*(z, z) = d(z, z) = 0$.

(d_2) Assume that $d^*(z_1, z_2) = 0$ for some $z_1, z_2 \in X \cap Y$, then $d(z_1, z_2) = 0$ implying $z_1 = z_2$.

(d_3) If $z_1, z_2 \in X \cap Y$ are arbitrary, then

$$d^*(z_1, z_2) = d(z_1, z_2) = d(z_2, z_1) = d^*(z_2, z_1)$$

according to (d_3^*).

(d_4) For some $z_1.z_2, z_3 \in X \cap Y$, we get:

$$\begin{aligned} d^*(z_1, z_2) &= d(z_1, z_2) \\ &\leq d(z_1, z_3) + d(z_3, z_2) + d(z_3, z_3) \\ &= d(z_1, z_3) + d(z_3, z_2) \end{aligned}$$

so the triangle inequality holds on $X \cap Y$.
Hence, $(X \cap Y, d^*)$ is a metric space. □

Remark 1. *The major part of the proof of Theorem 10 is deducible from the fact that* d^* *may be observed as a restriction of a bipolar metric* d *on a set* $(X \cap Y) \times (X \cap Y)$*, while* (d_4) *needs to be anyway additionally commented as it is necessary to apply* (d_4^*) *for* (z_1, z_2) *and* (z_3, z_3) *to acquire triangle inequality for* d^**.*

Theorem 11. *Let* (X, Y, d) *be a BMS and* $d^* : (X \cap Y) \times (X \cap Y) \mapsto [0, +\infty)$ *is a mapping defined by (3.1). If* (X, Y, d) *is a complete BMS, then* $(X \cap Y, d^*)$ *is a complete metric space.*

Proof. Assume that (X, Y, d) is a complete BMS and let $d^* : (X \cap Y) \times (X \cap Y) \mapsto [0, +\infty)$ be defined by (3.1). Theorem 10 asserts that $(X \cap Y, d^*)$ is a metric space.

Moreover, let $(z_n) \subseteq X \cap Y$ be a Cauchy sequence in a metric space $(X \cap Y, d^*)$. Consequently, for any $\varepsilon > 0$, there exists some $n_0 \in \mathbb{N}$ such that $d^*(z_n, z_m) < \varepsilon$ for all $n, m \geq n_0$. Equivalently, $d(z_n, z_m) < \varepsilon$ for any $n, m \geq n_0$ and $((z_n, z_n))$ is a Cauchy bisequence in a complete BMS (X, Y, d). Theorem 2 allows us to conclude that the bisequence is biconvergent.

Therefore, there must exist some $z \in X \cap Y$ satisfying that both the left and the right sequence converge to z. Meaning, there exists some $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, we get $d(z_k, z) < \varepsilon$. Taking into the account the definition of a metric d^*, it is evident that the sequence (z_n) converges in $(X \cap Y, d^*)$ to the same limit z. □

Theorems 10 and 11 affirm the existence of a unique fixed point for any contraction on a complete metric space $(X \cap Y, d^*)$ with the presumption of completeness of a BMS (X, Y, d). Moreover, supported by Lemma 9, it leads to the conclusion that the second main result of [9] concerning the fixed point problem for contracontraction is a direct corollary of the Banach theorem.

Theorem 12. *[9] If (X, Y, d) is a complete BMS and $T : (X, Y, d) \rightleftarrows (X, Y, d)$ is a contracontraction, then the mapping $T : X \cup Y \mapsto X \cup Y$ has a unique fixed point.*

Proof. Assume that (X, Y, d) is a complete BMS and a contravariant mapping $T : (X, Y, d) \rightleftarrows (X, Y, d)$ is a contraction for some contractive constant $q \in [0, 1)$. Lemma 9 claims that, if it exists, a fixed point of T must be precisely in $X \cap Y$. Hence, if $S = T \restriction_{X \cap Y}$, then $Fix(S) = Fix(T)$, and problem of existence and uniqueness of a fixed point of a mapping T is now reduced to the fixed point existence and uniqueness problem for a mapping S. Note that in the same time we can observe S as a restriction and a reduction of a mapping T, $S : X \cap Y \mapsto X \cap Y$ since $S(X \cap Y) \subseteq S(X) \cap S(Y) = Y \cap X$.

Observing a remark concerning restriction and Theorem 10 along with Theorem 11, it follows that $(X \cap Y, d^*)$ is a complete metric space and S a self-mapping on $X \cap Y$. Also, for arbitrary $z_1, z_2 \in X \cap Y$:

$$\begin{aligned} d^*(Sz_1, Sz_2) &= d^*(Tz_1, Tz_2) \\ &= d(Tz_1, Tz_2) \\ &\leq qd(z_1, z_2) \\ &= qd^*(z_1, z_2) \end{aligned}$$

and, implicitly, S is a contraction on a complete metric space. Banach fixed point theorem claims the existence and uniqueness of a fixed point of a mapping S, which further gives the existence and uniqueness of a fixed point for a mapping T. □

Remark 2. *Analogously to the first main result of [9], we may conclude that the second main results concerning the fixed point problem of a contracontraction is a direct corollary of a Banach fixed point theorem.*

The same approach may be used for any contravariant contractive mapping, like contravariant Kannan contraction.

Theorem 13. *[5] If (X, d) is a complete metric space and $T : X \to X$ is a mapping such that there exists a constant $k \in [0, \frac{1}{2})$ fulfilling*

$$d(Tx, Ty) \leq k\,(d(x, Tx) + d(y, Ty)) \tag{3.2}$$

for all $x, y \in X$, then T possesses a unique fixed point in X.

The result regarding uniqueness of a fixed point for a Kannan contracontraction on a complete BMS is given in [9].

Theorem 14. *[9] If (X, Y, d) is a complete BMS and $T : X \to X$ is a mapping such that there exists a constant $k \in [0, \frac{1}{2})$ fulfilling*

$$d(Ty, Tx) \leq k\,(d(x, Tx) + d(Ty, y)) \tag{3.3}$$

for all $x \in X, y \in Y$, then T possesses a unique fixed point.

Proof. Under the assumption that (X, Y, d) is a complete BMS, we have that $(X \cap Y, d^*)$ is a complete metric space due to Theorems 10 and 11. Moreover, if $S : X \cap Y \mapsto X \cap Y$ is defined by $Sz = Tz$ for any $z \in X \cap Y$, then $Fix(S) = Fix(T)$. A mapping S is a Kannan contraction on $(X \cap Y, d^*)$ since

$$d^*(Sz_1, Sz_2) = d^*(Tz_1, Tz_2)$$

$$
\begin{aligned}
&= d(Tz_1, TZ_2)\\
&\leq k\,(d(z_1, Tz_1) + d(z_2, Tz_2))\\
&= k\,(d(z_1, Tz_1) + d(z_2, Tz_2))\\
&= k\,(d(z_1, Sz_1) + d(z_2, Sz_2))\\
&= k\,(d^*(z_1, Sz_1) + d^*(z_2, Sz_2))
\end{aligned}
$$

for any $z_1, z_2 \in X \cap Y$. Consequently, S has a unique fixed point in $X \cap Y$ by Theorem 13. Further, T has a unique fixed point. □

The example included in [9] regarding contravariant mapping concerns a Kannan contraction.

Example 2. *Denote with X the class of all singletons of* $\mathbb{R}$ *and with Y the class of all nonempty compact subsets of* $\mathbb{R}$*. Define* $d : X \times Y \to \mathbb{R}$ *as*

$$d(\{x\}, A) = |x - \inf(A)| + |x - \sup(A)|$$

for any $\{x\} \in X$ *and any* $A \in Y$.
Evidently, the triple (X, Y, d) *is a complete BMS.*
Observe the contravariant mapping $T : (X, Y, d) \rightleftarrows (X, Y, d)$*, defined as*

$$TA = \left\{\frac{\inf(A) + \sup(A) + 6}{8}\right\}$$

for any $A \in X \cup Y$*. A mapping T is a Kannan mapping on a BMS* (X, Y, d) *for* $k = \frac{1}{3}$*, meaning that*

$$d(TA, T\{x\}) \leq \frac{1}{3}\,(d(\{x\}, T\{x\}) + d(TA, A))$$

for any $\{x\} \in X$ *and* $A \in Y$.
Note that $X \cap Y = X$ *and* $d(\{x\}, \{y\}) = 2|x - y|$ *for any* $\{x\}, \{y\} \in X$ *is a metric on X (more precisely,* d^* *as its restriction is a metric on X). As for the case of Banach contraction, T is a Kannan mapping on* (X, d^*)*, it possesses a fixed point by Theorem 13, and it is unique.*

4. Conclusions

Among numerous extensions and generalizations of the concept of metric space, it is important to make a clear distinction among those who present true scientific novelty and meticulously analyze the topological properties of introduced generalization and examine its metrizability. Otherwise, those concepts become further incorrectly utilized as a preferred setting for various types of contractions and related fixed-point theorems.

The purpose of this manuscript was to gather the main fixed point results in the setting of BMS and to investigate their relation with analogous fixed point results in metric space obtaining equivalence as a final result. The equivalence was obtained based on several different approaches to the metrizability of BMS presented therein. A similar approach may be used for the generalizations of a BMS like soft BMS, bipolar fuzzy metric space, and bipolar R-metric space, among others.

Consequently, those results did not deliver any scientific novelty and in the case of their application, like in solving an integral and differential equation, the same results are acquired if the metric analog is applied.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The second author is supported by Ministry of Education, Science and Technological Development, Republic of Serbia, no. 451-03-47/2023-01/200124.

Conflict of interest

Authors declare that there is no conflict of interest.

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