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Research article

Periodic solutions of a class of non-autonomous second-order discrete Hamiltonian systems

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Abstract: In this paper, in the view of control functions, the existence of periodic solutions of the following second-order discrete Hamiltonian system

$$\triangle^2 u(n-1) = \nabla F(n, u(n)), \quad n \in \mathbb{Z}$$

with a generalized sublinear condition is further explored.

Keywords: periodic solution; control function; the least action principle; saddle point theorem
Mathematics Subject Classification: 35B10, 39A23

1. Introduction

In this paper, we consider the non-autonomous discrete Hamiltonian system

$$\triangle^2 u(n-1) = \nabla F(n, u(n)), \quad n \in \mathbb{Z}, \quad u(n) \in \mathbb{R}^N, \tag{1.1}$$

where $\triangle u(n) = u(n+1) - u(n)$, $\triangle^2 u(n) = \triangle(\triangle u(n))$. The gradient with respect to the second variable x of $F(n, x)$ is denoted by $\nabla F(n, x)$. $\mathbb{Z}$ is the set of integers and $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+ = \mathbb{R} \cap [0, +\infty)$. For each $a, b \in \mathbb{Z}$ with $a \leq b$, we define $\mathbb{Z}[a, b] := \mathbb{Z} \cap [a, b]$. We suppose the following condition always holds for $F(n, x)$.

(A) For any fixed $n \in \mathbb{Z}$, $F(n, \cdot) \in C^1(\mathbb{R}^N, \mathbb{R})$, and for any $(n, x) \in \mathbb{Z} \times \mathbb{R}^N$, $F(n+T, x) = F(n, x)$, where T is a positive integer.

(1.1) is the discretization system of the second-order Hamiltonian system:

$$\ddot{u}(t) = \nabla F(t, u(t)), \quad a.e.\ t \in [0, T]. \tag{1.2}$$

In 1978, Rabinowitz [1] investigated the periodic solutions of system (1.2) with $F(t,\cdot)$ being superquadratic at the origin and at infinity by establishing an appropriate variational structure. Later, Rabinowitz et al. further developed a series of the critical point theories and introduced the variational method in their celebrated work [2, 3]. Since then, a considerable effort has been devoted to the study of system (1.2) based on various constraints on the nonlinear term, such as the sublinear nonlinearity [4, 5], the subconvex condition [6–8], the superquadratic condition [9, 10], the asymptotically linear condition [11], and the control function condition [12–14].

In 2003, such a powerful tool was firstly applied to discrete system (1.1) by Guo and Yu. Then, some interesting existence results have also been obtained for discrete system (1.1) with different nonlinear conditions (see [15–19]). In the case that $\nabla F(n,x)$ was bounded, Guo and Yu [16] succeeded to show that system (1.1) had at least one periodic solution. Considering the case that $\nabla F(n,x)$ is unbounded, specifically when $\nabla F(n,x)$ is α-sublinear, i.e.,

$$|\nabla F(n,x)| \le f(n)\,|x|^{\alpha} + g(n) \quad \forall\,(n,x) \in \mathbb{Z}\,[1,T] \times \mathbb{R}^N \tag{1.3}$$

where $f, g : \mathbb{Z}\,[1,T] \to \mathbb{R}^+$ and $\alpha \in [0,1)$, Xue and Tang [18] established some solvability conditions by using minimax techniques in critical point theory under the condition

$$\lim_{|x|\to\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) = +\infty \tag{1.4}$$

or

$$\lim_{|x|\to\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) = -\infty. \tag{1.5}$$

When $\alpha = 1$, inequality (1.3) becomes

$$|\nabla F(n,x)| \le f(n)\,|x| + g(n) \tag{1.6}$$

where $\nabla F(n,x)$ is said to be linear. This case has been investigated by Tang and Zhang [19] with condition (1.4) or (1.5) generalized to

$$\liminf_{|x|\to\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) > B_1 \tag{1.7}$$

or

$$\limsup_{|x|\to\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) < -B_2 \tag{1.8}$$

where B_1 and B_2 are two positive constants and $\alpha \in [0,1]$ here.

In 2010, for system (1.2), Zhang and Wang in [12, 13] developed a new technique to deal with such a problem by establishing a class of control functions. By replacing $|x|^{\alpha}$ in assumption (1.3) with a control function $h\,(|x|)$, they further generalized some previous results in [4–8]. Moreover, the idea, which is an effective approach to unify cases $\alpha \in [0,1)$ and $\alpha = 1$, has recently been expanded in [14].

However, the nonlinear term with a control function condition for discrete system (1.1) is considerably rare and it seems no similar results have been obtained for this case. Driven by references [12–14], in this paper, the main purpose is to develop some new existence results for discrete system (1.1) with a generalized sublinear condition in the view of control functions. To this end, we start below with establishing the class of control functions.

Definition 1.1. We define $\mathcal{H}$ to be the set of functions $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ with the following properties:

(i) $h(s) \leq h(t) \quad \forall s \leq t,\ s, t \in \mathbb{R}^+$;

(ii) $h(s+t) \leq C(h(s) + h(t)) \quad \forall s, t \in \mathbb{R}^+, C > 0$ is a constant;

(iii) $\lim_{s \to +\infty} h(s) = +\infty$;

(iv) $\lim_{s \to +\infty} \frac{h(s)}{s} = K$, $K \geq 0$ is a constant.

Remark 1.2. The fourth property here unifies both cases $\alpha \in [0, 1)$ and $\alpha = 1$, avoiding the inconvenience of discussing them separately.

In this paper, we always assume that $h \in \mathcal{H}$. The main results are stated below.

Theorem 1.3. *Suppose F satisfies condition (A) and the following conditions:*

(A_1) *There exist $f, g : \mathbb{Z}[1, T] \to \mathbb{R}^+$ such that*

$$|\nabla F(n, x)| \leq f(n) h(|x|) + g(n) \qquad \forall (n, x) \in \mathbb{Z}[1, T] \times \mathbb{R}^N;$$

(A_2) $R_0 := \frac{1}{4} - \frac{(T^2-1)CK}{6T} \sum_{n=1}^{T} f(n) > 0$, *where $f(n)$ is defined in (A_1) and C, K are the constants defined in Definition 1.1 for the function $h \in \mathcal{H}$;*

(A_3) *There exist $N_1 > 0$ and $\eta_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\eta_1(s) - R_1 h^2(s) \to +\infty$ as $s \to +\infty$, where $R_1 := \frac{2C^2(T^2-1)}{3T} \left(\sum_{n=1}^{T} f(n) \right)^2$, such that*

$$\sum_{n=1}^{T} F(n, x) \geq \eta_1(|x|) \qquad \forall |x| \geq N_1.$$

Then system (1.1) has at least one T-periodic solution.

Theorem 1.4. *Suppose F satisfies conditions (A), (A_1), (A_2) and*

(A_4) *There exist $N_2 > 0$ and $\eta_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\eta_2(s) - \left(\frac{R_1}{R_0} + R_1 \right) h^2(s) \to +\infty$ as $s \to +\infty$, such that*

$$\sum_{n=1}^{T} F(n, x) \leq -\eta_2(|x|) \qquad \forall |x| \geq N_2.$$

Then system (1.1) has at least one T-periodic solution.

Remark 1.5. The remarks below are easy to obtain.

(1) As a result, if $K = 0$, it is always the case that (A_2) is true.

(2) It is clear from (A_3), (A_4), and Definition 1.1 that both $\eta_1(s)$ and $\eta_2(s) \to +\infty$ as s $\to +\infty$.

(3) See Example 4.4 in Section 4, for instance, where the function $F(n, x)$ satisfies the conditions of Theorem 1.3 but not the cases in [18, 19].

The following corollaries can be seen as special cases of Theorems 1.3 and 1.4.

Corollary 1.6. *Suppose F satisfies conditions* (A)*, (1.3) with* $\alpha \in (0, 1)$*, and*

(A_3^*) $\liminf_{|x|\to+\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n, x) > \frac{8\left(T^2-1\right)}{3T} \left(\sum_{n=1}^{T} f(n)\right)^2.$

Then system (1.1) has at least one T-periodic solution.

Corollary 1.7. *Suppose F satisfies conditions* (A)*, (1.6),* (A_3^*) *with* $\alpha = 1$*, and*

(A_2^*) $\frac{1}{4} - \frac{T^2-1}{3T} \sum_{n=1}^{T} f(n) > 0.$

Then system (1.1) has at least one T-periodic solution.

Corollary 1.8. *Suppose F satisfies conditions* (A)*, (1.3) with* $\alpha \in (0, 1)$*, and*

(A_4^*) $\limsup_{|x|\to+\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n, x) < -\frac{40\left(T^2-1\right)}{3T} \left(\sum_{n=1}^{T} f(n)\right)^2.$

Then system (1.1) has at least one T-periodic solution.

Corollary 1.9. *Suppose F satisfies conditions* (A)*, (1.6),* (A_2^*)*, and*

(A_4^{**}) $\limsup_{|x|\to+\infty} |x|^{-2} \sum_{n=1}^{T} F(n, x) < -\frac{8R_2(T^2-1)}{3T} \left(\sum_{n=1}^{T} f(n)\right)^2$*, where* $R_2 := 1 + \frac{1}{\frac{1}{4} - \frac{T^2-1}{3T} \sum_{n=1}^{T} f(n)}$.

Then system (1.1) has at least one T-periodic solution.

A straightforward calculation implies that Theorems 1.1–1.4 in [19] are covered by Corollaries 1.6–1.9, respectively, and the above results are improvements in some sense of those in [18, 19].

2. Preliminaries

Let

$$H_T = \left\{u : \mathbb{Z} \to \mathbb{R}^N | u(n+T) = u(n), n \in \mathbb{Z}\right\},$$

which is a Hilbert space with inner product

$$\langle u, v\rangle = \sum_{n=1}^{T} [(\triangle u(n), \triangle v(n)) + (u(n), v(n))] \qquad \forall u, v \in H_T$$

and norm

$$\|u\| = \left(\sum_{n=1}^{T} \left[|\triangle u(n)|^2 + |u(n)|^2\right]\right)^{\frac{1}{2}} \qquad \forall u \in H_T$$

where $(\cdot, \cdot)$ and $|\cdot|$ are the inner product and the norm defined in $\mathbb{R}^N$. It is clear that H_T is a finite dimensional space.

The functional φ on H_T defined by

$$\varphi(u) = \frac{1}{2}\sum_{n=1}^{T} |\triangle u(n)|^2 + \sum_{n=1}^{T} F(n, u(n)) \tag{2.1}$$

is continuously differentiable according to assumption (A). Additionally, one has

$$\left\langle \varphi^{'}(u), v\right\rangle = \sum_{n=1}^{T} [(\triangle u(n), \triangle v(n)) + (\nabla F(n, u(n)), v(n))] \qquad \forall u, v \in H_T. \tag{2.2}$$

With analysis as in [18], we can infer

$$-\sum_{n=1}^{T} \left(\triangle^2 u(n-1), v(n)\right) = \sum_{n=1}^{T} (\triangle u(n), \triangle v(n)) \quad \forall u, v \in H_T. \tag{2.3}$$

Combining both (2.2) and (2.3), we find that $\left\langle \varphi^{'}(u), v\right\rangle = 0$ for all $v \in H_T$, if and only if

$$\triangle^2 u(n-1) = \nabla F(n, u(n)), \quad n \in \mathbb{Z}.$$

As a result, a T-periodic solution of system (1.1) is a critical point of the functional φ in H_T.

We first introduce two lemmas which will be used in the proofs.

Lemma 2.1. *(Lemma 2.1 in [19]) Suppose $u \in H_T$ and $\sum_{n=1}^{T} u(n) = 0$, then*

$$\sum_{n=1}^{T} |u(n)|^2 \leq \frac{1}{4\sin^2\frac{\pi}{T}} \sum_{n=1}^{T} |\triangle u(n)|^2$$

and

$$\|u\|_\infty^2 \leq \frac{T^2 - 1}{6T} \sum_{n=1}^{T} |\triangle u(n)|^2,$$

where $\|u\|_\infty := \max_{n\in\mathbb{Z}[1,T]} |u(n)|$. The first inequality is known as the discrete Wirtinger's inequality, and the second inequality is known as the discrete Sobolev's inequality.

For $u \in H_T$, let $\bar{u} = (1/T)\sum_{n=1}^{T} u(n)$ and $\tilde{u}(n) = u(n) - \bar{u}$. Then we obtain

$$\sum_{n=1}^{T} |\tilde{u}(n)|^2 \leq \frac{1}{4\sin^2\frac{\pi}{T}} \sum_{n=1}^{T} |\triangle u(n)|^2 \tag{2.4}$$

and

$$\|\tilde{u}\|_\infty^2 \leq \frac{T^2-1}{6T} \sum_{n=1}^{T} |\triangle u(n)|^2 . \tag{2.5}$$

Lemma 2.2. *(Theorem 4.6 in [2]) Suppose Y is a real Banach space with $Y = Y_1 \oplus Y_2$, where $Y_1 \subset Y$ is finite-dimensional. Let $\varphi \in C^1(Y, \mathbb{R})$ satisfy the (PS) condition, if*

(i) There exist $\delta \in \mathbb{R}$ and a bounded neighborhood $\Omega \subset Y_1$ of the origin such that $\varphi |_{\partial\Omega} \leq \delta$.

(ii) There exists $\lambda \in \mathbb{R}$ with $\lambda > \delta$ such that $\varphi |_{Y_2} \geq \lambda$.

Then φ has a critical value $b \geq \lambda$. Additionally, b can be described as

$$b = \inf_{g\in\Gamma} \max_{u\in\bar{\Omega}} \varphi(g(u)) \tag{2.6}$$

where

$$\Gamma = \left\{ g \in C\left(\bar{\Omega}, Y\right) \mid g = id \text{ on } \partial\Omega \right\}.$$

3. Proofs of theorems

Let $C_i (i = 1, 2, 3 \ldots)$ denote various positive constants.

Proof of Theorem 1.3. It follows from (A_1), (2.5), the Young inequality, and the properties of $h(s)$ that

$$\begin{aligned}
&\left| \sum_{n=1}^{T} [F(n, u(n)) - F(n, \bar{u})] \right| \\
\leq& \sum_{n=1}^{T} \int_0^1 |\nabla F(n, \bar{u} + s\tilde{u}(n))| \, |\tilde{u}(n)| \, ds \\
\leq& \sum_{n=1}^{T} \int_0^1 f(n) h(|\bar{u} + s\tilde{u}(n)|) \, |\tilde{u}(n)| \, ds + \sum_{n=1}^{T} \int_0^1 g(n) \, |\tilde{u}(n)| \, ds \\
\leq& \sum_{n=1}^{T} f(n) C[h(|\bar{u}|) + h(|\tilde{u}(n)|)] \, |\tilde{u}(n)| + \|\tilde{u}\|_\infty \sum_{n=1}^{T} g(n) \\
\leq& C[h(|\bar{u}|) + h(\|\tilde{u}\|_\infty)] \, \|\tilde{u}\|_\infty \sum_{n=1}^{T} f(n) + \|\tilde{u}\|_\infty \sum_{n=1}^{T} g(n) \\
\leq& C\left[\frac{3T}{2C(T^2-1)} \|\tilde{u}\|_\infty^2 + \frac{2C(T^2-1)}{3T} \left(\sum_{n=1}^{T} f(n) \right)^2 h^2(|\bar{u}|) \right] + Ch(\|\tilde{u}\|_\infty) \, \|\tilde{u}\|_\infty \sum_{n=1}^{T} f(n)
\end{aligned}$$

$$
\begin{aligned}
&+ C_1\left(\sum_{n=1}^{T} |\triangle u(n)|^2\right)^{\frac{1}{2}}\\
\leq&\frac{1}{4}\sum_{n=1}^{T} |\triangle u(n)|^2+\frac{2C^2(T^2-1)}{3T}\left(\sum_{n=1}^{T} f(n)\right)^2 h^2(|\bar{u}|)+C(K\|\tilde{u}\|_\infty+C_2)\|\tilde{u}\|_\infty\sum_{n=1}^{T} f(n)\\
&+ C_1\left(\sum_{n=1}^{T} |\triangle u(n)|^2\right)^{\frac{1}{2}}\\
\leq&\frac{1}{4}\sum_{n=1}^{T} |\triangle u(n)|^2+\frac{2C^2(T^2-1)}{3T}\left(\sum_{n=1}^{T} f(n)\right)^2 h^2(|\bar{u}|)+\frac{(T^2-1)CK}{6T}\sum_{n=1}^{T} f(n)\sum_{n=1}^{T} |\triangle u(n)|^2\\
&+ C_3\left(\sum_{n=1}^{T} |\triangle u(n)|^2\right)^{\frac{1}{2}}\\
=&\left(\frac{1}{4}+\frac{(T^2-1)CK}{6T}\sum_{n=1}^{T} f(n)\right)\sum_{n=1}^{T} |\triangle u(n)|^2+C_3\left(\sum_{n=1}^{T} |\triangle u(n)|^2\right)^{\frac{1}{2}}+R_1h^2(|\bar{u}|). \qquad (3.1)
\end{aligned}
$$

Integrating (2.1) and (3.1), then for any $u \in H_T$, we infer

$$
\begin{aligned}
\varphi(u) =&\frac{1}{2}\sum_{n=1}^{T} |\triangle u(n)|^2+\sum_{n=1}^{T}[F(n,u(n))-F(n,\bar{u})]+\sum_{n=1}^{T} F(n,\bar{u})\\
\geq&\frac{1}{2}\sum_{n=1}^{T} |\triangle u(n)|^2-\left(\frac{1}{4}+\frac{(T^2-1)CK}{6T}\sum_{n=1}^{T} f(n)\right)\sum_{n=1}^{T} |\triangle u(n)|^2-C_3\left(\sum_{n=1}^{T} |\triangle u(n)|^2\right)^{\frac{1}{2}}\\
&-R_1h^2(|\bar{u}|)+\sum_{n=1}^{T} F(n,\bar{u})\\
=&\underbrace{R_0\sum_{n=1}^{T} |\triangle u(n)|^2-C_3\left(\sum_{n=1}^{T} |\triangle u(n)|^2\right)^{\frac{1}{2}}}_{I_1}+\underbrace{\sum_{n=1}^{T} F(n,\bar{u})-R_1h^2(|\bar{u}|)}_{I_2}\\
=&I_1+I_2. \qquad (3.2)
\end{aligned}
$$

We continue the proof with the following three cases.

Case 1. $\sum_{n=1}^{T} |\triangle u(n)|^2 \to +\infty$ and $|\bar{u}|$ is bounded. From (3.2), (A_2), and assumption (A), one has

$$
\begin{aligned}
&I_1 \to +\infty \quad as \ \sum_{n=1}^{T} |\triangle u(n)|^2 \to +\infty, \ and\\
&I_2 \ is \ bounded.
\end{aligned}
$$

Case 2. $\sum_{n=1}^{T} |\triangle u(n)|^2$ is bounded and $|\bar{u}| \to +\infty$. With the assistance of (A_3), we obtain

$$\begin{aligned} &I_1 \textit{ is bounded, and} \\ &I_2 \geq \eta_1(|\bar{u}|) - R_1 h^2(|\bar{u}|) \to +\infty \ \ as \ \ |\bar{u}| \to +\infty. \end{aligned}$$

Case 3. $\sum_{n=1}^{T} |\triangle u(n)|^2 \to +\infty$ and $|\bar{u}| \to +\infty$. Considering the analysis of the previous two situations, one can infer

$$\begin{aligned} &I_1 \to +\infty \quad as \ \ \sum_{n=1}^{T} |\triangle u(n)|^2 \to +\infty, \ and \\ &I_2 \to +\infty \quad as \ \ |\bar{u}| \to +\infty. \end{aligned}$$

As $\|u\| \to +\infty$ if and only if $\left(|\bar{u}|^2 + \sum_{n=1}^{T} |\triangle u(n)|^2\right)^{\frac{1}{2}} \to +\infty$. Then, inequality (3.2) and Cases 1–3 imply

$$\varphi(u) \to +\infty \ as \ \ \|u\| \to +\infty.$$

Since H_T is a finite dimensional space, by applying the least action principle to the functional $\varphi(u)$ on H_T, we come to the conclusion that system (1.1) has at least one T-periodic solution which is the minimizer of the functional $\varphi(u)$ on the space H_T. □

Proof of Theorem 1.4. We first verify that the (PS) condition holds. Under the assumption that the (PS) sequence $\{u_k\} \subset H_T$ satisfies $\varphi'(u_k) \to 0 \ as \ k \to +\infty$ and $\{\varphi(u_k)\}$ is bounded, as in (3.1), we have for all k that

$$\begin{aligned} &\left|\sum_{n=1}^{T} (\nabla F(n, u_k(n)), \tilde{u}_k(n))\right| \\ \leq &\left(\frac{1}{4} + \frac{(T^2-1)CK}{6T} \sum_{n=1}^{T} f(n)\right) \sum_{n=1}^{T} |\triangle u_k(n)|^2 + C_3 \left(\sum_{n=1}^{T} |\triangle u_k(n)|^2\right)^{\frac{1}{2}} + R_1 h^2(|\bar{u}_k|). \end{aligned} \tag{3.3}$$

This implies that

$$\begin{aligned} \|\tilde{u}_k\| \geq &\left\langle \varphi'(u_k), \tilde{u}_k \right\rangle \\ = &\sum_{n=1}^{T} |\triangle u_k(n)|^2 + \sum_{n=1}^{T} (\nabla F(n, u_k(n)), \tilde{u}_k(n)) \\ \geq &\sum_{n=1}^{T} |\triangle u_k(n)|^2 - \left(\frac{1}{4} + \frac{(T^2-1)CK}{6T} \sum_{n=1}^{T} f(n)\right) \sum_{n=1}^{T} |\triangle u_k(n)|^2 \\ &- C_3 \left(\sum_{n=1}^{T} |\triangle u_k(n)|^2\right)^{\frac{1}{2}} - R_1 h^2(|\bar{u}_k|) \end{aligned}$$

$$
\begin{aligned}
&=\left(\frac{3}{4}-\frac{(T^2-1)CK}{6T}\sum_{n=1}^{T}f(n)\right)\sum_{n=1}^{T}\ |\triangle u_k(n)|^2\\
&\quad -C_3\left(\sum_{n=1}^{T}\ |\triangle u_k(n)|^2\right)^{\frac{1}{2}}-R_1h^2(|\bar{u}_k|)
\end{aligned}
\tag{3.4}
$$

for all large k. (2.4) implies that

$$
\|\tilde{u}_k\|\leq\left(1+\frac{1}{4\sin^2\frac{\pi}{T}}\right)^{\frac{1}{2}}\ \left(\sum_{n=1}^{T}|\triangle u_k(n)|^2\right)^{\frac{1}{2}}. \tag{3.5}
$$

Combining (3.4) and (3.5), we get for all large k that

$$
\begin{aligned}
R_1h^2(|\bar{u}_k|)\geq&\left(\frac{3}{4}-\frac{(T^2-1)CK}{6T}\sum_{n=1}^{T}f(n)\right)\sum_{n=1}^{T}\ |\triangle u_k(n)|^2-C_4\left(\sum_{n=1}^{T}\ |\triangle u_k(n)|^2\right)^{\frac{1}{2}}\\
=&\left(\frac{1}{4}-\frac{(T^2-1)CK}{6T}\sum_{n=1}^{T}f(n)\right)\sum_{n=1}^{T}\ |\triangle u_k(n)|^2+\frac{1}{2}\sum_{n=1}^{T}\ |\triangle u_k(n)|^2\\
&-C_4\left(\sum_{n=1}^{T}\ |\triangle u_k(n)|^2\right)^{\frac{1}{2}}\\
\geq&R_0\sum_{n=1}^{T}|\triangle u_k(n)|^2-C_5,
\end{aligned}
\tag{3.6}
$$

where

$$
C_5=\max_{s\in[0,+\infty)}\left\{-\frac{1}{2}s^2+C_4s\right\}.
$$

It is clear that $C_5>0$. It follows from (3.6) that

$$
\sum_{n=1}^{T}|\triangle u_k(n)|^2\leq\frac{R_1}{R_0}h^2(|\bar{u}_k|)+C_6 \tag{3.7}
$$

for all large k.

If $\{|\bar{u}_k|\}$ is unbounded, one may assume that $|\bar{u}_k|\to+\infty$ as $k\to+\infty$. Given (3.1), (3.7), and (A_4), as well as the fact that $h(s)\to+\infty$ as $s\to+\infty$ for large k we can obtain

$$
\begin{aligned}
\varphi(u_k)=&\frac{1}{2}\sum_{n=1}^{T}\ |\triangle u_k(n)|^2+\sum_{n=1}^{T}[F(n,u_k(n))-F(n,\bar{u}_k)]+\sum_{n=1}^{T}F(n,\bar{u}_k)\\
\leq&\left(\frac{3}{4}+\frac{(T^2-1)CK}{6T}\sum_{n=1}^{T}f(n)\right)\sum_{n=1}^{T}\ |\triangle u_k(n)|^2+C_3\left(\sum_{n=1}^{T}\ |\triangle u_k(n)|^2\right)^{\frac{1}{2}}\\
&+R_1h^2(|\bar{u}_k|)+\sum_{n=1}^{T}F(n,\bar{u}_k)
\end{aligned}
$$

$$\leq\left(\frac{3}{4}+\frac{(T^2-1)CK}{6T}\sum_{n=1}^{T}f(n)\right)\left(\frac{R_1}{R_0}h^2(|\bar{u}_k|)+C_6\right)+C_3\left(\frac{R_1}{R_0}h^2(|\bar{u}_k|)+C_6\right)^{\frac{1}{2}}$$
$$+R_1h^2(|\bar{u}_k|)-\eta_2\left(|\bar{u}_k|\right)$$
$$\leq\left(\frac{R_1}{R_0}+R_1\right)h^2(|\bar{u}_k|)-\eta_2\left(|\bar{u}_k|\right)+C_8\to-\infty,\ as\ k\to+\infty. \tag{3.8}$$

This contradicts the fact that $\varphi(u_k)$ is bounded. Therefore, by (3.7), $\{u_k\}$ is bounded. As a result, the (PS) condition can be verified since H_T is finite-dimensional.

The following are the only requirements for using the saddle point theorem.

(a) For $u\in\mathbb{R}^N\subset H_T$, it has $\varphi(u)\to-\infty$ as $|u|\to+\infty$;

(b) For $u\in\tilde{H}_T:=\{u\in H_T|\bar{u}=0\}$, it has $\varphi(u)\to+\infty$ as $\|u\|\to+\infty$.

In fact, by (A_4) and Remark 1.5(2), we conclude that

$$\varphi(u)=\sum_{n=1}^{T}F(n,u)\leq-\eta_2(|u|)\to-\infty\ in\ \mathbb{R}^N.$$

Thus, (a) is confirmed.

For condition (b) above, arguing as in (3.1), we get

$$\left|\sum_{n=1}^{T}[F(n,u(n))-F(n,0)]\right|$$
$$\leq\left(\frac{1}{4}+\frac{\left(T^2-1\right)CK}{6T}\sum_{n=1}^{T}f(n)\right)\sum_{n=1}^{T}\ |\triangle u(n)|^2+C_3\left(\sum_{n=1}^{T}|\triangle u(n)|^2\right)^{\frac{1}{2}}+C_9 \tag{3.9}$$

for all $u\in\tilde{H}_T$. (3.9) implies that

$$\varphi(u)=\frac{1}{2}\sum_{n=1}^{T}\ |\triangle u(n)|^2+\sum_{n=1}^{T}[F(n,u(n))-F(n,0)]+\sum_{n=1}^{T}F(n,0)$$
$$\geq\left(\frac{1}{4}-\frac{\left(T^2-1\right)CK}{6T}\sum_{n=1}^{T}f(n)\right)\sum_{n=1}^{T}\ |\triangle u(n)|^2-C_3\left(\sum_{n=1}^{T}|\triangle u(n)|^2\right)^{\frac{1}{2}}$$
$$+\sum_{n=1}^{T}F(n,0)-C_9. \tag{3.10}$$

By (2.4), in $\tilde{H}_T$, one has

$$\|u\|\to+\infty\Leftrightarrow\left(\sum_{n=1}^{T}\ |\triangle u(n)|^2\right)^{\frac{1}{2}}\to+\infty.$$

Therefore, by (A_2) and (3.10), we get $\varphi(u)\to+\infty$ as $\|u\|\to+\infty$ in $\tilde{H}_T$, that is, (b) also holds. Consequently, by using the saddle point theorem (Lemma 2.2), we come to the conclusion that system (1.1) has at least one T-periodic solution. □

We will now provide a thorough proof for Corollaries 1.6–1.9.

Proof of Corollary 1.6. For the application of Theorem 1.3, we should define the control function h. Let

$$h(s) = s^{\alpha},\ \alpha \in (0,1),\ C = 2,\ K = 0,$$
$$R_0 = \frac{1}{4},\ R_1 = \frac{8\left(T^2-1\right)}{3T}\left(\sum_{n=1}^{T} f(n)\right)^2.$$

It is easy to check that both (A_1) and (A_2) hold. For condition (A_3), one can derive from (A_3^*) that there exist $N_1 > 0,\ \varepsilon_1 > 0$ such that

$$\sum_{n=1}^{T} F(n,x) \geq (R_1 + \varepsilon_1)\,|x|^{2\alpha} \qquad \forall\, |x| \geq N_1.$$

Now, one can take $\eta_1(s) = (R_1 + \varepsilon_1)\, s^{2\alpha}$ in (A_3), and Corollary 1.6 is verified through Theorem 1.3. □

Proof of Corollary 1.7. Let

$$h(s) = s,\ C = 2,\ K = 1,$$
$$R_0 = \frac{1}{4} - \frac{T^2-1}{3T}\sum_{n=1}^{T} f(n),\ R_1 = \frac{8\left(T^2-1\right)}{3T}\left(\sum_{n=1}^{T} f(n)\right)^2.$$

In this case (A_2) is (A_2^*), and (A_3^*) implies (A_3), which is shown in the proof of Corollary 1.6. Therefore, all conditions of Theorem 1.3 hold and Corollary 1.7 is verified. □

Proof of Corollary 1.8. Similar to the above, let

$$h(s) = s^{\alpha},\ \alpha \in (0,1),\ C = 2,\ K = 0,$$

$$R_0 = \frac{1}{4},\ -\left(\frac{R_1}{R_0} + R_1\right) = -\frac{40(T^2-1)}{3T}\left(\sum_{n=1}^{T} f(n)\right)^2.$$

From (A_4^*), there exist $N_2 > 0,\ \varepsilon_2 > 0$ such that

$$\sum_{n=1}^{T} F(n,x) \leq -\left[\left(\frac{R_1}{R_0} + R_1\right) + \varepsilon_2\right]|x|^{2\alpha} \qquad \forall\, |x| \geq N_2.$$

Take $\eta_2(s) = \left[\left(\frac{R_1}{R_0} + R_1\right) + \varepsilon_2\right] s^{2\alpha}$, then Corollary 1.8 can be verified through Theorem 1.4. □

Proof of Corollary 1.9. Let

$$h(s) = s,\ C = 2,\ K = 1,\ R_0 = \frac{1}{4} - \frac{T^2-1}{3T}\sum_{n=1}^{T} f(n),$$

$$-\left(\frac{R_1}{R_0}+R_1\right)=-\left(1+\frac{1}{\frac{1}{4}-\frac{T^2-1}{3T}\sum_{n=1}^{T}f(n)}\right)\frac{8(T^2-1)}{3T}\left(\sum_{n=1}^{T}f(n)\right)^2.$$

Similar to the process in Corollary 1.8, we can show that both (A_2) and (A_4) are true, so Corollary 1.9 can be verified through Theorem 1.4. □

4. Examples

In this section, we will illustrate our findings by providing some concrete examples.

Example 4.1. ($\nabla F(n,x)$ *is* α-sublinear, $\alpha\in(0,1)$). Here, we consider the function

$$F(n,x)=(n-T)\,|x|^{7/4}+(h(n),x)\quad \forall x\in\mathbb{R}^N,\ n\in\mathbb{Z}\,[1,T]$$

where $h:\mathbb{Z}\,[1,T]\to\mathbb{R}^N$ satisfies $h(n+T)=h(n)\forall n\in\mathbb{Z}$. It is easy to obtain

$$|\nabla F(n,x)|\leq\frac{7}{4}\,|n-T|\,|x|^{\frac{3}{4}}+|h(n)|$$

implying that $\nabla F(n,x)$ is α-sublinear with $\alpha=3/4$. Next, we verify that Theorem 1.4 can be used here. Actually, let

$$C=2,\ K=0,\ f(n)=\frac{7}{4}\,|n-T|\,,$$

$$h(s)=s^{\frac{3}{4}},\ \eta_2(s)=\frac{T(T-1)}{2}s^{7/4}-\sum_{n=1}^{T}|h(n)|\,s$$

then

$$R_0=\frac{1}{4}>0,\ R_1=\frac{8(T^2-1)}{3T}\left(\sum_{n=1}^{T}f(n)\right)^2$$

and for all $T>1$, there exists $N_2>0$ such that

$$\begin{aligned}\sum_{n=1}^{T}F(n,x)&=\sum_{n=1}^{T}(n-T)\,|x|^{7/4}+\sum_{n=1}^{T}(h(n),x)\\&\leq-\left[\frac{T(T-1)}{2}\,|x|^{7/4}-\sum_{n=1}^{T}|h(n)|\,|x|\right]\\&=-\,\eta_2(|x|)\quad\forall\,|x|\geq N_2.\end{aligned}$$

In addition, we have

$$\eta_2(s)-\left(\frac{R_1}{R_0}+R_1\right)h^2(s)=\frac{T(T-1)}{2}s^{7/4}-\sum_{n=1}^{T}|h(n)|\,s-5R_1s^{3/2}\to+\infty\ \ as\ \ s\to+\infty.$$

These show that the three conditions of Theorem 1.4 are achieved.

Example 4.2. ($\nabla F(n,x)$ is weak linear, i.e., $\nabla F(n,x)$ grows less than $|x|$ at infinity but more than $|x|^{\alpha}$, $\alpha \in [0,1)$). Here, we give the function

$$F(n,x) = (T-n)\frac{|x|^2}{\ln\left(100+|x|^2\right)} \quad \forall x \in \mathbb{R}^N,\ n \in \mathbb{Z}[1,T].$$

We can obtain

$$\begin{aligned}|\nabla F(n,x)| \leq& |T-n|\left[\frac{2|x|\ln\left(100+|x|^2\right)+2|x|\frac{|x|^2}{100+|x|^2}}{\ln^2\left(100+|x|^2\right)}\right]\\ \leq& 4|T-n|\frac{|x|}{\ln\left(100+|x|^2\right)}\end{aligned}$$

for all large $x \in \mathbb{R}^N$. The weak linearity of $\nabla F(n,x)$ is evident, and the results of [18, 19] cannot be used to solve problem (1.1) with the function $F(n,x)$ in this example. We confirm that this case can be handled by Theorem 1.3. In fact, take

$$C = 2,\ K = 0,\ f(n) = 4|T-n|,$$

$$h(s) = \frac{s}{\ln(100+s^2)},\ \eta_1(s) = \frac{T(T-1)}{2}\frac{s^2}{\ln(100+s^2)}$$

then

$$R_0 = \frac{1}{4} > 0,\ R_1 = \frac{8(T^2-1)}{3T}\left(\sum_{n=1}^{T} f(n)\right)^2,$$

$$\sum_{n=1}^{T} F(n,x) = \frac{|x|^2}{\ln\left(100+|x|^2\right)}\sum_{n=1}^{T}(T-n) = \frac{T(T-1)}{2}\frac{|x|^2}{\ln\left(100+|x|^2\right)} \geq \eta_1(|x|) \quad \forall x \in \mathbb{R}^N$$

and

$$\eta_1(s) - R_1 h^2(s) = \frac{T(T-1)}{2}\frac{s^2}{\ln(100+s^2)} - R_1\frac{s^2}{\ln^2(100+s^2)} \to +\infty \ as\ s \to +\infty.$$

Therefore, all the conditions of Theorem 1.3 are achieved.

Example 4.3. ($\nabla F(n,x)$ is linear). Here, we give the function

$$F(n,x) = \frac{T-n}{100}|x|^2 \quad \forall x \in \mathbb{R}^N,\ n \in \mathbb{Z}[1,T].$$

It is clear to see that

$$|\nabla F(n,x)| \leq \frac{|T-n|}{50}|x|$$

which suggests that $\nabla F(n,x)$ is linear. Choosing

$$C = 2,\ K = 1,\ f(n) = \frac{|T-n|}{50},$$

$$h(s) = s, \ \eta_1(s) = \frac{T(T-1)}{200}s^2$$

then if T ∈ {2, 3}, we have

$$R_0 = \frac{1}{4} - \frac{T^2-1}{3T}\sum_{n=1}^{T} f(n) > 0,$$

$$R_1 = \frac{8\left(T^2-1\right)}{3T}\left(\sum_{n=1}^{T} f(n)\right)^2 < \frac{T(T-1)}{200}.$$

Moreover, we have

$$\sum_{n=1}^{T} F(n.x) = |x|^2 \sum_{n=1}^{T} \frac{T-n}{100} = \frac{T(T-1)}{200}|x|^2 \geq \eta_1(|x|) \quad \forall x \in \mathbb{R}^N,$$

$$\eta_1(s) - R_1 h^2(s) = \frac{T(T-1)}{200}s^2 - R_1 s^2 \to +\infty \ \ as \ \ s \to +\infty.$$

These show that the three conditions of Theorem 1.3 hold.

Example 4.4. ($F(n, x)$ does not satisfy (A_3^*)). Here, we give the function

$$F(n, x) = \left(\frac{T+1}{2} - n\right)|x|^{7/4} + (2T-n)|x|^{3/2} + \ln\left(1 + |x|^2\right) \quad \forall x \in \mathbb{R}^N, \ n \in \mathbb{Z}[1, T].$$

One may easily confirm that

$$\begin{aligned}|\nabla F(n, x)| \leq& \frac{7}{8}|T+1-2n||x|^{3/4} + \frac{3}{2}|2T-n||x|^{1/2} + \frac{2|x|}{1+|x|^2} \\ \leq& \frac{7}{8}(|T+1-2n| + \varepsilon)|x|^{3/4} + \frac{9T^3}{\varepsilon^2} + 1\end{aligned}$$

where $\varepsilon > 0$. As can be shown from the above inequality, $\nabla F(n, x)$ is α-sublinear with $\alpha = 3/4$. Let

$$C = 2, \ K = 0, \ f(n) = \frac{7}{8}(|T+1-2n| + \varepsilon),$$

$$h(s) = s^{3/4}, \ \eta_1(s) = \frac{T(3T-1)}{2}s^{3/2} + T\ln\left(1+s^2\right)$$

then $R_0 = 1/4 > 0$ and we can choose $\varepsilon > 0$ such that

$$R_1 = \frac{8\left(T^2-1\right)}{3T}\left(\sum_{n=1}^{T} f(n)\right)^2 = \frac{T(3T-1)}{2}.$$

Additionally, one has

$$\sum_{n=1}^{T} F(n, x) = \frac{T(3T-1)}{2}|x|^{3/2} + T\ln\left(1+|x|^2\right) \geq \eta_1(|x|) \quad \forall x \in \mathbb{R}^N,$$

$$\eta_1(s) - R_1 h^2(s) = \frac{T(3T-1)}{2} s^{3/2} + T\ln\left(1+s^2\right) - R_1 s^{3/2}$$
$$= T\ln\left(1+s^2\right) \to +\infty \;\; as \;\; s \to +\infty.$$

Therefore, by Theorem 1.3, system (1.1) has at least one solution that is T-periodic. However, since F(n,x) does not satisfy (A_3^*), for

$$\liminf_{|x|\to+\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) = \frac{T(3T-1)}{2}$$

[18, 19] can not be applied in this case.

5. Conclusions

This paper is concerned with the existence of periodic solutions for a class of non-autonomous discrete Hamiltonian systems. In general, the study of non-autonomous Hamiltonian systems on discrete cases is less carried out than the analogous analysis on continuous cases. The main feature of our problem is that two main theorems and several corollaries for a non-autonomous discrete Hamiltonian systems. Here, using a kind of control function argument together with the least action principle and the saddle point theorem, we show that the problem admits at least one T-periodic solution. We also point out that our hypotheses here are more general under sublinear conditions.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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