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Research article

Compactness of commutators of fractional integral operators on ball Banach function spaces

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Abstract: Let $0 < \alpha < n$ and b be a locally integrable function. In this paper, we obtain the characterization of compactness of the iterated commutator $(T_{\Omega,\alpha})_b^m$ generated by the function b and the fractional integral operator with the homogeneous kernel $T_{\Omega,\alpha}$ on ball Banach function spaces. As applications, we derive the characterization of compactness via the commutator $(T_{\Omega,\alpha})_b^m$ on weighted Lebesgue spaces, and further obtain a necessary and sufficient condition for the compactness of the iterated commutator $(T_\alpha)_b^m$ generated by the function b and the fractional integral operator T_α on Morrey spaces. Moreover, we also show the necessary and sufficient condition for the compactness of the commutator $[b, T_\alpha]$ generated by the function b and the fractional integral operator T_α on variable Lebesgue spaces and mixed Morrey spaces.

Keywords: ball Banach function space; fractional integral operator; commutator; compactness
Mathematics Subject Classification: 42B20, 42B25, 42B35, 47B47

1. Introduction

In 1961, John and Nirenberg [1] introduced the space of functions of bounded mean oscillation $BMO(\mathbb{R}^n)$, which is defined as the set of all locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B|\, dx < \infty,$$

where the supremum is taken over all balls in $\mathbb{R}^n$ and $f_B := \frac{1}{|B|} \int_B f(x)dx$. In 1976, Coifman et al. [2] stated that $b \in BMO(\mathbb{R}^n)$ if and only if the commutator

$$[b, T]f(x) = bTf(x) - T(bf)(x)$$

is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, where T is the classical singular integral operator. In 1978, Uchiyama [3] proved that $b \in BMO(\mathbb{R}^n)$ if and only if the commutator generated by the locally integrable function b and the singular integral operator with the homogeneous kernel is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Later, the characterization of $BMO(\mathbb{R}^n)$ was also established for various function spaces including by Karlovich and Lerner [4] on variable exponent Lebesgue spaces and Di Fazio and Ragusa [5] on Morrey spaces.

In 1978, Uchiyama [3] refined the boundedness results on the commutator $[b, T]$ to compactness. This is achieved by requiring the symbol $b \in CMO(\mathbb{R}^n)$, which is the closure of $C_c^\infty(\mathbb{R}^n)$ in the $BMO(\mathbb{R}^n)$. In 2012, Chen et al. [6] showed that $b \in CMO(\mathbb{R}^n)$ if and only if the commutator generated by the locally integrable function b and the singular integral operator with the homogeneous kernel is compact on the Morrey spaces. Recently, Tao et al. [7] obtained that $b \in CMO(\mathbb{R}^n)$ if and only if the commutator generated by the locally integrable function b and the singular integral operator with the homogeneous kernel is compact on ball Banach function spaces. The purpose of this paper is to prove the characterization of compactness of the iterated commutator generated by the locally integrable function and the fractional integral operator with the homogeneous kernel on ball Banach function spaces.

In this paper, we establish the characterization of compactness of the iterated commutator $(T_{\Omega,\alpha})_b^m$ generated by the locally integrable function b and the fractional integral operator with the homogeneous kernel $T_{\Omega,\alpha}$ on ball Banach function spaces. As applications, we show that $b \in CMO(\mathbb{R}^n)$ if and only if the iterated commutator $(T_{\Omega,\alpha})_b^m$ is compact from $L^p_{\omega^p}(\mathbb{R}^n)$ to $L^q_{\omega^q}(\mathbb{R}^n)$, where $1 < p, q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and ω is a weight, and we obtain that $b \in CMO(\mathbb{R}^n)$ if and only if the iterated commutator $(T_\alpha)_b^m$ generated by the locally integrable function b and the fractional integral operator is compact from $M^p_q(\mathbb{R}^n)$ to $M^s_t(\mathbb{R}^n)$, where $0 < p \le q < \infty$, $1 < t \le s < \infty$, $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{t}{s} = \frac{q}{p}$. Moreover, we obtain that $b \in CMO(\mathbb{R}^n)$ if and only if the commutator $[b, T_\alpha]$ generated by the locally integrable function b and the fractional integral operator is compact from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$, where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$. We also obtain that $b \in CMO(\mathbb{R}^n)$ if and only if the commutator $[b, T_\alpha]$ generated by the locally integrable function b and the fractional integral operator is compact from $\mathcal{M}^{p_0}_{\vec{p}}(\mathbb{R}^n)$ to $\mathcal{M}^{q_0}_{\vec{q}}(\mathbb{R}^n)$, where $\vec{p} = (p_1, \cdots, p_n)$, $\vec{q} = (q_1, \cdots, q_n)$, $\frac{n}{p_0} \le \sum_{j=1}^n \frac{1}{p_j}$, $\frac{n}{q_0} \le \sum_{j=1}^n \frac{1}{q_j}$, $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and $\frac{\vec{p}}{p_0} = \frac{\vec{q}}{q_0}$.

To state our main results, we begin with the definition of the ball Banach function spaces introduced in [8].

The symbol $\mathcal{U}(\mathbb{R}^n)$ is denoted as the set of all measurable functions on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $0 < r < \infty$, let $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$ and

$$\mathbb{B} := \{B(x, r) : x \in \mathbb{R}^n \text{ and } 0 < r < \infty\}. \tag{1.1}$$

Definition 1.1. *A quasi-Banach space $X \subset \mathcal{U}(\mathbb{R}^n)$ is called a ball quasi-Banach function space if it satisfies the following conditions:*

(i) $\|f\|_X = 0$ *implies that* $f = 0$ *almost everywhere;*
(ii) $|g| \le |f|$ *almost everywhere implies that* $\|g\|_X \le \|f\|_X$ *;*
(iii) $0 \le f_m \uparrow f$ *almost everywhere implies that* $\|f_m\|_X \uparrow \|f\|_X$ *;*
(iv) $B \in \mathbb{B}$ *implies that* $\chi_B \in X$*, where* $\mathbb{B}$ *is as in (1.1) .*

The ball quasi-Banach function space X is called the ball Banach function space if the norm of X satisfies the triangle inequality: for all $f, g \in X$,

$$\|f + g\|_X \le \|f\|_X + \|g\|_X \tag{1.2}$$

and, for any $B \in \mathbb{B}$, there exists a positive constant $C_{(B)}$ that is dependent on B, such that, for all $f \in X$,

$$\int_B |f(x)|dx \leq C_{(B)}\|f\|_X.$$

The following notion of the associate space of the ball Banach function space can be found in [9, Chapter 1, Definitions 2.1 and 2.3].

Definition 1.2. *For any a ball Banach function space X, the associate space X′ is defined by*

$$X' := \{f \in \mathcal{U}(\mathbb{R}^n) : \|f\|_{X'} < \infty\},$$

where, for any $f \in \mathcal{U}(\mathbb{R}^n)$*,*

$$\|f\|_{X'} := \sup_{\{g \in X : \|g\|_X = 1\}} \|fg\|_{L^1(\mathbb{R}^n)},$$

and $\|\cdot\|_{X'}$ *is called the associate norm of* $\|\cdot\|_X$ *.*

The theory of commutators plays an important role in harmonic analysis (see, for example, [10–16]) and partial differential equations (see, for example, [17–19]).

Let $0 < \alpha < n$. Recall that the fractional integral operator with the homogeneous kernel is defined by

$$T_{\Omega,\alpha}f(x) := \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy, \tag{1.3}$$

where the function Ω satisfies the following conditions:

$$\Omega(\lambda x') := \Omega(x') \text{ for any } 0 < \lambda < \infty \text{ and } x' \in \mathbb{S}^{n-1}, \tag{1.4}$$

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.5}$$

$$|\Omega(x') - \Omega(y')| \leq |x' - y'| \text{ for any } x', y' \in \mathbb{S}^{n-1}, \tag{1.6}$$

where $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ denotes the unit sphere in $\mathbb{R}^n$ and $d\sigma$ is the area measure on $\mathbb{S}^{n-1}$.

We suppose that $m \in \mathbb{N}$; the iterated commutator of the operator $T_{\Omega,\alpha}$ is defined by

$$(T_{\Omega,\alpha})_b^m(f)(x) := [b, (T_{\Omega,\alpha})_b^{m-1}](f)(x), \quad (T_{\Omega,\alpha})_b^0 f(x) := T_{\Omega,\alpha}(f)(x).$$

Put $T_\alpha := T_{1,\alpha}$, where T_α is a classical fractional integral operator, which is defined by

$$T_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

In 2014, Pérez et al. [20] introduced the iterated commutator generated by the locally integrable function b and the multilinear singular integral operator, and they studied the boundedness of these operators on product Lebesgue spaces. Bényi and Torres [21] studied the compactness of the iterated commutator generated by two locally integrable functions b_1, b_2 and the bilinear singular integral operator on the Lebesgue spaces. Later, Bényi et al. [22] extended the work of Bényi and Torres [21] to weighted Lebesgue spaces. Wang et al. [23] studied the characterization of compactness of the iterated commutator generated by two locally integrable functions b_1, b_2 as well as the bilinear fractional integral operator on the Lebesgue spaces. Hytönen and Lappas [24] studied the compactness of the iterated commutator generated by two locally integrable functions b_1, b_2 and the bilinear singular integral operator, the bilinear fractional integral operator and the bilinear Fourier multiplier on the weighted Lebesgue spaces. Recently, Guo et al. [25] studied the compactness of the iterated commutator generated by the locally integrable function b and the singular integral operator with the homogeneous kernel on the weighted Lebesgue spaces. In this paper, we consider the characterization of compactness of the iterated commutator $(T_{\Omega,\alpha})_b^m$ generated by the locally integrable function b and the fractional integral operator with the homogeneous kernel on ball Banach function spaces.

To show our main results, we need some assumptions.

Let $0 < \alpha < n$. For a locally integrable function f on $\mathbb{R}^n$, the fractional maximal operator $\mathcal{M}_\alpha$ is defined by

$$\mathcal{M}_\alpha(f)(x) := \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/n}} \int_B |f(y)| dy, \tag{1.7}$$

where the supremum is taken over all balls $B \in \mathbb{B}$ containing x. For a locally integrable function f on $\mathbb{R}^n$, the Hardy-Littlewood maximal operator $\mathcal{M}$ is defined by

$$\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls $B \in \mathbb{B}$ containing x.

Assumption 1.1. *Let X be a ball Banach function space. Suppose that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on X and X'.*

Assumption 1.2. *Let X and Y be ball Banach function spaces. Then the following statements are true:*

(i) The operator T_α is bounded from X to Y.

(ii) Let χ_B be a characteristic function on the ball B. For any ball B, then

$$\frac{\|\chi_B\|_X \|\chi_B\|_{Y'}}{|B|^{1+\alpha/n}} \lesssim 1.$$

(iii) The iterated commutator of the operator $T_{\Omega,\alpha}$ is bounded from X to Y, that is,

$$\|(T_{\Omega,\alpha})_b^m\|_Y \lesssim \|b_i\|_{BMO}^m \|f\|_X.$$

We also need the condition of the L^∞-Dini condition (see, for example, [26]).

Definition 1.3. *A function $\Omega \in L^\infty(\mathbb{S}^{n-1})$ is said to satisfy the L^∞-Dini condition if*

$$\int_0^1 \frac{\omega_\infty(\tau)}{\tau} d\tau < \infty, \tag{1.8}$$

where, for any $0 < \tau < 1$,

$$\omega_\infty(\tau) := \sup_{\{x,y \in \mathbb{S}^{n-1} : |x-y| < \tau\}} |\Omega(x) - \Omega(y)|.$$

Theorem 1.1. *Let $0 < \alpha < n$. Let X and Y be ball Banach function spaces satisfying Assumptions 1.1, 1.2 (i) and (iii) and $T_{\Omega,\alpha}$ be the fractional integral operator with the homogeneous kernel Ω, where $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfies (1.3), (1.4) and (1.8). If $b \in CMO(\mathbb{R}^n)$, then $(T_{\Omega,\alpha})_b^m$ is compact from X to Y.*

Theorem 1.2. *Let $0 < \alpha < n$ and $b \in L^1_{\rm loc}(\mathbb{R}^n)$. Let X and Y be ball Banach function spaces satisfying Assumptions 1.1 and 1.2(i) and (ii). Let $T_{\Omega,\alpha}$ be the fractional integral operator with the homogeneous kernel Ω, where $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfies that there exists an open set $\Gamma \subset \mathbb{S}^{n-1}$ such that Ω never vanishes and never changes sign on Γ. If $(T_{\Omega,\alpha})_b^m$ is compact from X to Y, then $b \in CMO(\mathbb{R}^n)$.*

Corollary 1.1. *Let $0 < \alpha < n$ and $b \in L^1_{\rm loc}(\mathbb{R}^n)$. Let X and Y be ball Banach function spaces satisfying Assumptions 1.1 and 1.2 and $T_{\Omega,\alpha}$ be the fractional integral operator with the homogeneous kernel Ω, where Ω satisfies (1.3), (1.4) and (1.5). Then $(T_{\Omega,\alpha})_b^m$ is compact from X to Y if and only if $b \in CMO(\mathbb{R}^n)$.*

We end this section by stating some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$. We always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. We also use $C_{(\alpha,\beta,\ldots)}$ to denote a positive constant that is dependent on the indicated parameters $\alpha, \beta, \ldots$. The symbol $f \lesssim g$ means that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$. If E is a subset of $\mathbb{R}^n$, we denote by χ_E its characteristic function and by E^c the set $\mathbb{R}^n \backslash E$. Furthermore, for any $t \in (0, \infty)$ and any ball $B := B(x, r)$ in $\mathbb{R}^n$, with $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, we let $tB := B(x, tr)$. For any $q \in [1, \infty]$, we denote by q' its conjugate exponent, namely, $\frac{1}{q} + \frac{1}{q'} = 1$.

2. Preliminaries

We present some necessary lemmas and notions in this section, which is very important to prove our main results.

For any $f \in \mathcal{U}(\mathbb{R}^n)$, the non-increasing rearrangement is defined by

$$f^*(t) := \inf\{\zeta > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \zeta\}| < t\}, \quad 0 < t < \infty,$$

for any $f \in L^1_{\rm loc}(\mathbb{R}^n)$, $0 < \lambda < 1$ and $B \subset \mathbb{R}^n$, the local mean oscillation of f on B is defined by

$$a_\lambda(f; B) := \inf_{c\in\mathbb{R}} ((f - c)\chi_B)^*(\lambda|B|).$$

Lemma 2.1. *[27, Lemma 2.1] Let $0 < \lambda \le \frac{1}{2}$. For any real-valued function b, we write*

$$\|b\|_{BMO_\lambda} := \sup_{B\subset\mathbb{R}^n} a_\lambda(b; B).$$

Then there exists a positive constant C such that

$$C^{-1}\|b\|_{BMO} \le \|b\|_{BMO_\lambda} \le C\|b\|_{BMO}.$$

Lemma 2.2. *[25, Theorem 3.3] Let $0 < \lambda < \frac{1}{2}$ and $b \in BMO(\mathbb{R}^n)$. Then $b \in CMO(\mathbb{R}^n)$ if and only if the function b satisfies the following conditions:*

(i) $\lim_{r\to 0} \sup_{|B|=r} a_\lambda(b; B) = 0$;

(ii) $\lim_{r\to\infty} \sup_{|B|=r} a_\lambda(b; B) = 0$;

(iii) $\lim_{d\to\infty} \sup_{B\cap B(0,d)=\varnothing} a_\lambda(b; B) = 0$.

Lemma 2.3. *[28, Theorem 1.2] Let X be the ball Banach function space such that $\mathcal{M}$ is bounded on X'. For any $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, we denote*

$$\|b\|_{BMO_X} := \sup_{B\subset\mathbb{R}^n} \frac{1}{\|\chi_B\|_X}\|(b - b_B)\chi_B\|_X.$$

Then there exists a positive constant C such that, for all $b \in BMO(\mathbb{R}^n)$,

$$C^{-1}\|b\|_{BMO} \le C\|b\|_{BMO_X} \le C\|b\|_{BMO}.$$

Lemma 2.4. *[7, Lemma 2.6] Let X be a ball quasi-Banach function space satisfying the triangle inequality as in (1.2). If $f \in X$ and $g \in X'$, then f and g are integrable and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le \|f\|_X\|g\|_{X'}.$$

Lemma 2.5. *[8, Definition 2.6] Let $0 < p < \infty$ and X be a ball Banach function space. The p-convexification X^p of X is defined by*

$$X^p := \{f \in \mathcal{U}(\mathbb{R}^n) : |f|^p \in X\}$$

equipped with the quasi-norm $\|f\|_{X^p} = \||f|^p\|_X^{\frac{1}{p}}$.

For any $0 < \theta < \infty$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the powered Hardy-Littlewood maximal operator $\mathcal{M}^{(\theta)}$ is defined by

$$\mathcal{M}^{(\theta)}(f)(x) := \left\{\mathcal{M}\left(|f|^\theta\right)(x)\right\}^{\frac{1}{\theta}}. \tag{2.1}$$

Lemma 2.6. *[29, Remark 2.19] Let $0 < \theta < \infty$ and X be a ball quasi-Banach function space. Assume that there exists a positive constant C such that, for any $f \in \mathcal{U}(\mathbb{R}^n)$,*

$$\|\mathcal{M}^{(\theta)}(f)\|_X \le C\|f\|_X.$$

Then there exists a positive constant C_0 such that, for any ball $B \in \mathbb{B}$ and $1 \le \beta < \infty$,

$$\|\chi_{\beta B}\|_X \le C_0\beta^{\frac{n}{\theta}}\|\chi_B\|_X, \tag{2.2}$$

where the positive constant C_0 is independent of $B \in \mathbb{B}$ and β .

Next, let us recall the following lemma introduced by Tao et al.in [7, Theorem 3.6], which is a sufficient condition for subsets of ball Banach function spaces to be totally bounded and a generalization in ball Banach function spaces of the well-known Fréchet-Kolmogorov theorem in $L^p(\mathbb{R}^n)$ with $1 \le p < \infty$.

Lemma 2.7. *Let X be a ball Banach function space. Then a subset $\mathcal{A}$ of X is totally bounded if the set $\mathcal{A}$ satisfies the following conditions:*

(i) $\mathcal{A}$ *is bounded, namely,*

$$\sup_{f\in\mathcal{A}} \|f\|_X < \infty;$$

(ii) For any given $\epsilon \in (0,\infty)$*, there exists a positive constant* N *such that, for any* $f \in \mathcal{A}$*,*

$$\|f\chi_{\{x\in\mathbb{R}^n:|x|>N\}}\|_X < \epsilon;$$

(iii) $\mathcal{A}$ *is uniformly vanishes equicontinuous, namely, for any given* $\epsilon \in (0,\infty)$*, there exists a positive constant* ρ *such that, for any* $f \in \mathcal{A}$ *and* $z \in \mathbb{R}^n$ *with* $|z| \in [0,\rho)$*,*

$$\|f(\cdot + z) - f(\cdot)\|_X < \epsilon.$$

Conversely, assume that X satisfies the following additional asumptions that $C_c(\mathbb{R}^n)$ is dense in X and that, for any $f \in X$ and $y \in \mathbb{R}^n$,

$$\|f\|_X = \|f(\cdot + y)\|_X.$$

If a subset $\mathcal{A}$ of X is totally bounded, then $\mathcal{A}$ satisfies (i)–(iii) of Lemma 2.7.

Lemma 2.8. *[7, Proposition 3.8] If* X *is a ball Banach function space with an absolutely continuous norm, then* $C_c(\mathbb{R}^n)$ *is dense in* X*.*

The following lemma can be seen in [27, Proposition 4.1].

Lemma 2.9. *Let* $\lambda \in (0,1)$ *and* $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$*. Let* $\Omega \in L^\infty(\mathbb{S}^{n-1})$ *satisfy that there exists an open set* $\Gamma \subset \mathbb{S}^{n-1}$ *such that* Ω *never vanishes and never changes sign on* Γ*. There exist* $\epsilon_0 > 0$ *and* $k_0 > 10\sqrt{n}$ *depending only on* Ω *and* n *such that, for any ball* $B(x_0, r_0) \subset \mathbb{R}^n$ *with* $x_0 \in \mathbb{R}^n$ *and* $r_0 \in (0,\infty)$*, there exist another ball* $B(x_1, r_0)$ *and measurable sets* $E \subset B(x_0, r_0)$ *with* $|E| = \frac{\lambda}{2}|B(x_0, r_0)|$ *as well as* $F \subset B(x_1, r_0)$ *with* $|x_0 - x_1| = 2k_0 r_0$ *and* $|F| = \frac{\lambda}{2}|B(x_1, r_0)|$ *and* $G \subset E \times F$ *with* $|G| \geq \frac{\lambda}{8}|B(x_0, r_0)|^2$ *such that they satisfy the following properties:*

(i) for any $x \in E$ *and* $y \in F$*,* $a_\lambda(b; B) \leq |b(x) - b(y)|$*;*
(ii) $\Omega\left(\frac{x-y}{|x-y|}\right)$ *and* $b(x) - b(y)$ *do not change sign on* $E \times F$*;*
(iii) for any $(x,y) \in G$*,* $\left|\Omega(\frac{x-y}{|x-y|})\right| \geq \epsilon_0$*.*

3. Proof of Theorem 1.1

In this section, we first recall the following smooth truncated technique in [30] (see also [7, 31]). Let $\varphi \in C^\infty([0,\infty))$ satisfy

$$0 \leq \varphi \leq 1 \quad \text{and} \quad \varphi(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}], \\ 0, & x \in [1,\infty]. \end{cases}$$

Let $0 < \alpha < n$ and $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfy (1.4), (1.5) and the L^∞-Dini condition. For any $\epsilon \in (0,\infty)$ and any $x, y \in \mathbb{R}^n$, define $K_\epsilon(x,y) := \frac{\Omega(x-y)}{|x-y|^{n-\alpha}}\left[1 - \varphi\left(\frac{|x-y|}{\epsilon}\right)\right]$. Let X be the ball Banach function space satisfying Assumption 1.1. Using [7, Lemma 2.12] and [32, Lemma 7.4.5], we know that, for any $f \in X, \epsilon \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$T^{(\epsilon)}_{\Omega,\alpha} f(x) := \int_{\mathbb{R}^n} K_\epsilon(x,y) f(y)dy < \infty.$$

Proposition 3.1. *Let $0 < \alpha < n$ and $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfy (1.4), (1.5) and (1.8). Then, for any given $\epsilon \in (0,\infty)$, K_ϵ satisfies the following smoothness condition: there exists a positive constant C which is independent of ϵ, x, y and z for any $x, y, z \in \mathbb{R}^n$ with $|z| \le \frac{|x-y|}{2}$ such that*

$$|K_\epsilon(x,y) - K_\epsilon(x+z,y)| \le C\left[\frac{1}{|x-y|^{n-\alpha}}\omega_\infty\left(\frac{4|z|}{|x-y|}\right) + \frac{|z|}{|x-y|^{n-\alpha+1}}\right].$$

Proof. For any $x, y, z \in \mathbb{R}^n$ with $|z| \le \frac{|x-y|}{2}$, applying (1.4), we know that

$$|\Omega(x-y) - \Omega(x+z-y)| = \left|\Omega\left(\frac{x-y}{|x-y|}\right) - \Omega\left(\frac{x+\xi-y}{|x+z-y|}\right)\right| \le \omega_\infty\left(\frac{4|z|}{|x-y|}\right).$$

Using the mean value theorem, for any $x, y, z \in \mathbb{R}^n$ with $|z| \le \frac{|x-y|}{2}$, we have

$$\begin{aligned}
|\, K_\epsilon(x,y) - K_\epsilon(x+z,y)\,| &\le \left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x+z-y)}{|x+z-y|^{n-\alpha}}\right|\left|1-\varphi\left(\frac{|x-y|}{\epsilon}\right)\right| \\
&\quad + \left|\frac{\Omega(x+z-y)}{|x+z-y|^{n-\alpha}}\right|\left|\varphi\left(\frac{|x+z-y|}{\epsilon}\right) - \varphi\left(\frac{|x-y|}{\epsilon}\right)\right| \\
&\lesssim \left|\frac{\Omega(x-y)-\Omega(x+z-y)}{|x-y|^{n-\alpha}}\right| + \left|\frac{1}{|x+z-y|^{n-\alpha}} - \frac{1}{|x-y|^{n-\alpha}}\right| \\
&\quad + \frac{\|\varphi'\|_{L^\infty(\mathbb{R}_+)}}{|x+z-y|^{n-\alpha}}\left|\frac{|x+z-y|}{\epsilon} - \frac{|x-y|}{\epsilon}\right|\chi_{\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n:\frac{1}{3}\epsilon\le|x-y|\le2\epsilon\}}(x,y) \\
&\lesssim \frac{1}{|x-y|^{n-\alpha}}\omega_\infty\left(\frac{4|z|}{|x-y|}\right) + \frac{|z|}{|x-y|^{n-\alpha+1}} \\
&\quad + \frac{1}{\epsilon}\frac{|z|}{|x+z-y|^{n-\alpha}}\chi_{\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n:\frac{1}{3}\epsilon\le|x-y|\le2\epsilon\}}(x,y) \\
&\lesssim \frac{1}{|x-y|^{n-\alpha}}\omega_\infty\left(\frac{4|z|}{|x-y|}\right) + \frac{|z|}{|x-y|^{n-\alpha+1}},
\end{aligned}$$

Thus, the proof of Proposition 3.1 is complete.

Proposition 3.2. *Let $0 < \alpha < n$ and $b \in C_c^\infty(\mathbb{R}^n)$. Let X and Y be ball Banach function spaces and $T_{\Omega,\alpha}$ be the fractional integral operator with homogeneous kernel Ω, where $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfies (1.3), (1.4) and (1.8). Then there exists a positive constant C such that, for any $\epsilon > 0$, $f \in X$ and $x \in \mathbb{R}^n$,*

$$\left|(T^{(\epsilon)}_{\Omega,\alpha})^m_b(f)(x) - (T_{\Omega,\alpha})^m_b(f)(x)\right| \le C\epsilon\|\nabla b\|^m_{L^\infty(\mathbb{R}^n)}\mathcal{M}_\alpha(f)(x).$$

Moreover, if $\mathcal{M}_\alpha$ is bounded from X to Y, then

$$\lim_{\epsilon\to0^+}\|(T^{(\epsilon)}_{\Omega,\alpha})^m_b - (T_{\Omega,\alpha})^m_b\|_{X\to Y} = 0.$$

Proof. For $f \in X$ and $x \in \mathbb{R}^n$, using the mean value theorem, we have

$$\left|(T^{(\epsilon)}_{\Omega,\alpha})^m_b(f)(x) - (T_{\Omega,\alpha})^m_b(f)(x)\right| \le \left|\int_{\{y\in\mathbb{R}^n:|x-y|<\epsilon\}} |b(x)-b(y)|^m\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}\varphi\left(\frac{|x-y|}{\epsilon}\right)f(y)dy\right|$$

$$
\begin{aligned}
&\leq \int_{\{y\in\mathbb{R}^n:|x-y|<\epsilon\}} |b(x)-b(y)|^m \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|dy \\
&\leq \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\|\nabla b\|^m_{L^\infty(\mathbb{R}^n)} \sum_{j=0}^{\infty} \int_{\{y\in\mathbb{R}^n:\frac{\epsilon}{2^{j+1}}<|x-y|\leq\frac{\epsilon}{2^j}\}} |x-y|^m \frac{|f(y)|}{|x-y|^{n-\alpha}}dy \\
&\leq \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\|\nabla b\|^m_{L^\infty(\mathbb{R}^n)} \sum_{j=0}^{\infty} \frac{\epsilon}{2^{mj}}\cdot\frac{1}{|2^{-j}\epsilon|^{n-\alpha}} \int_{\{y\in\mathbb{R}^n:|x-y|\leq\frac{\epsilon}{2^j}\}} |f(y)|dy \\
&\lesssim \epsilon\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\|\nabla b\|^m_{L^\infty(\mathbb{R}^n)} \sum_{j=0}^{\infty} \frac{1}{2^{mj}}\mathcal{M}_\alpha(f)(x) \\
&\lesssim \epsilon\|\nabla b\|^m_{L^\infty(\mathbb{R}^n)}\mathcal{M}_\alpha(f)(x).
\end{aligned}
$$

Applying Assumption 1.2(i), we have

$$
\|(T^{(\epsilon)}_{\Omega,\alpha})^m_b(f)-(T_{\Omega,\alpha})^m_b(f)\|_Y \lesssim \epsilon\|\mathcal{M}_\alpha(f)\|_Y \lesssim \epsilon\|f\|_X,
$$

which implies that $\lim_{\epsilon\to0^+}\|(T^{(\epsilon)}_{\Omega,\alpha})^m_b-(T_{\Omega,\alpha})^m_b\|_{X\to Y}=0$. The proof of Proposition 3.2 is complete.

Proof of Theorem 1.1. Let $b\in \mathrm{CMO}(\mathbb{R}^n)$. We know that, for any given $\kappa\in(0,\infty)$, there exists a $b^{(\kappa)}\in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ such that $\|b-b^{(\kappa)}\|_{\mathrm{BMO}(\mathbb{R}^n)}<\kappa$. Then, by the boundedness of $(T_{\Omega,\alpha})^m_{b-b^{(\kappa)}}$ from X to Y, we obtain the following, for any given $\kappa\in(0,\infty)$ and for any $f\in X$;

$$
\left\|(T_{\Omega,\alpha})^m_b(f)-(T_{\Omega,\alpha})^m_{b^{(\kappa)}}(f)\right\|_Y=\left\|(T_{\Omega,\alpha})^m_{b-b^{(\kappa)}}(f)\right\|_Y\lesssim\left\|b-b^{(\kappa)}\right\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_X\lesssim\kappa^m\|f\|_X.
$$

By Proposition 3.2, it suffices to show that, for any $b\in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and any $\epsilon\in(0,\infty)$ small enough, the operator $(T^{(\epsilon)}_{\Omega,\alpha})^m_b$ is compact from X to Y. Thus, we only need to prove that

$$
A_\epsilon := \left\{(T^{(\epsilon)}_{\Omega,\alpha})^m_b(f) : \|f\|_X\leq 1\right\}
$$

satisfies (i)–(iii) of Lemma 2.7 respectively.

For $\epsilon<1$, applying Proposition 3.2 and Assumption 1.2(iii), we have

$$
\|(T^{(\epsilon)}_{\Omega,\alpha})^m_b(f)\|_Y\leq\|(T_{\Omega,\alpha})^m_b(f)\|_Y+\|(T_{\Omega,\alpha})^m_b(f)-(T^{(\epsilon)}_{\Omega,\alpha})^m_b(f)\|_Y\lesssim(1+\epsilon)\|f\|_X\leq2\|f\|_X.
$$

This proves that A_ϵ satisfies (i) of Lemma 2.7.

For (ii) of Lemma 2.7, we suppose that there exists a positive constant R_0 such that $\mathrm{supp}(b)\subset B(0,R_0)$. For any $y\in B(0,R_0)$ and $x\in\mathbb{R}^n$ with $|x|\in(2R_0,\infty)$, we have that $|x-y|\sim|x|$. Thus, for any $f\in\mathcal{A}$, $x\in\mathbb{R}^n$ with $|x|\in(2R_0,\infty)$ and $\Omega\in L^\infty(\mathbb{S}^{n-1})$, using Lemma 2.6, we have

$$
\begin{aligned}
|(T^{(\epsilon)}_{\Omega,\alpha})^m_b(f)(x)| &\leq \int_{\mathbb{R}^n}|b(x)-b(y)|^m\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|dy \\
&\lesssim \int_{B(0,R_0)}\frac{|f(y)|}{|x-y|^{n-\alpha}}dy \\
&\lesssim \frac{1}{|x|^{n-\alpha}}\|f\|_X\|\chi_{B(0,R_0)}\|_{X'}
\end{aligned}
$$

$$\lesssim \frac{1}{|x|^{n-\alpha}}.$$

Applying Lemma 2.6 and [8, Lemma 2.15], we deduce that there exists $\eta \in (1, \infty)$ such that

$$\begin{aligned}
\|(T^{(\epsilon)}_{\Omega,\alpha})^m_b(f)\chi_{\{x\in\mathbb{R}^n:|x|>N\}}\|_Y &\lesssim \sum_{j=0}^{\infty} \|\frac{1}{|\cdot|^{n-\alpha}}\chi_{\{x\in\mathbb{R}^n:2^jN\le|x|<2^{j+1}N\}}\|_Y \\
&\lesssim \sum_{j=0}^{\infty} \frac{\|\chi_{\{x\in\mathbb{R}^n:2^jN\le|x|<2^{j+1}N\}}\|_Y}{(2^jN)^{n-\alpha}} \\
&\lesssim \sum_{j=0}^{\infty} \frac{1}{(2^jN)^{n-\alpha-n/\eta}} \\
&\lesssim \frac{1}{N^{n-\alpha-n/\eta}}.
\end{aligned}$$

Thus, A_ϵ satisfies the condition (ii) of Lemma 2.7.

Next, we prove that A_ϵ satisfies (iii) of Lemma 2.7. For any $f \in \mathcal{A}$ and $z \in \mathbb{R}^n\backslash\{0\}$ with $|z| \le \frac{\epsilon}{8}$, we see that

$$\begin{aligned}
&(T^{(\epsilon)}_{\Omega,\alpha})^m_b(f)(x+z) - (T^{(\epsilon)}_{\Omega,\alpha})^m_b(f)(x) \\
&\le \int_{\mathbb{R}^n} (b(x)-b(y))^m K_\epsilon(x,y)f(y)dy - \int_{\mathbb{R}^n} (b(x+z)-b(y))^m K_\epsilon(x+z,y)f(y)dy \\
&\le \int_{\{y\in\mathbb{R}^n:|x-y|>\epsilon/4\}} (b(x+z)-b(y))^m \left(K_\epsilon(x+z,y)-K_\epsilon(x,y)\right) f(y)dy \\
&+ \int_{\{y\in\mathbb{R}^n:|x-y|>\epsilon/4\}} \left((b(x+z)-b(y))^m - (b(x)-b(y))^m\right) K_\epsilon(x,y)f(y)dy \\
&=: I_1(x,z) + I_2(x,z).
\end{aligned}$$

For $I_1(x,z)$, applying Proposition 3.1, if $|z| \le \frac{|x-y|}{2}$, we have

$$\begin{aligned}
I_1(x,z) &\le \int_{|x-y|>\epsilon/4} |b(x+z)-b(y)|^m \left|K_\epsilon(x+z,y)-K_\epsilon(x,y)\right| |f(y)|dy \\
&\lesssim \int_{\{y\in\mathbb{R}^n:|x-y|>\epsilon/4\}} |b(x+z)-b(y)|^m \omega_\infty\left(\frac{4|z|}{|x-y|}\right)\frac{1}{|x-y|^{n-\alpha}}|f(y)|dy \\
&+ \int_{\{y\in\mathbb{R}^n:|x-y|>\epsilon/4\}} |b(x+z)-b(y)|^m \frac{1}{|x-y|}\frac{1}{|x-y|^{n-\alpha}}|f(y)|dy \\
&=: I_{11}(x,z) + I_{12}(x,z).
\end{aligned}$$

For $I_{11}(x,z)$, by the L^∞-Dini condition, we obtain that

$$\begin{aligned}
I_{11}(x,z) &= \int_{\{y\in\mathbb{R}^n:|x-y|>\epsilon/4\}} |b(x+z)-b(y)|^m \omega_\infty\left(\frac{4|z|}{|x-y|}\right)\frac{1}{|x-y|^{n-\alpha}}|f(y)|dy \\
&\lesssim |z|^m\|\nabla b\|^m_{L^\infty(\mathbb{R}^n)} \sum_{j=0}^{\infty} \omega_\infty\left(\frac{|z|}{2^{j-2}\epsilon}\right)\int_{\{y\in\mathbb{R}^n:2^{j-2}\epsilon<|x-y|<2^{j-1}\epsilon\}} \frac{1}{|x-y|^{n-\alpha}}|f(y)|dy
\end{aligned}$$

$$
\begin{aligned}
&\lesssim |z|^m \|\nabla b\|_{L^\infty(\mathbb{R}^n)}^m \sum_{j=0}^{\infty} \omega_\infty\left(\frac{|z|}{2^{j-2}\epsilon}\right) \frac{1}{|2^{j-1}\epsilon|^{n-\alpha}} \int_{\{y\in\mathbb{R}^n:|x-y|<2^{j-1}\epsilon\}} |f(y)|dy \\
&\leq |z|^m \|\nabla b\|_{L^\infty(\mathbb{R}^n)}^m \sum_{j=0}^{\infty} \omega_\infty\left(\frac{|z|}{2^{j-2}\epsilon}\right) \mathcal{M}_\alpha(f)(x) \\
&\lesssim |z|^m \|\nabla b\|_{L^\infty(\mathbb{R}^n)}^m \mathcal{M}_\alpha(f)(x) \sum_{j=0}^{\infty} \omega_\infty\left(\frac{|z|}{2^{j-2}\epsilon}\right) \int_{2^{-(j+1)}}^{2^{-j}} \frac{dt}{t} \\
&\leq |z|^m \|\nabla b\|_{L^\infty(\mathbb{R}^n)}^m \mathcal{M}_\alpha(f)(x) \int_0^1 \omega\left(\frac{8|z|}{\epsilon} t\right) \frac{dt}{t} \\
&\leq |z|^m \|\nabla b\|_{L^\infty(\mathbb{R}^n)}^m \mathcal{M}_\alpha(f)(x) \int_0^{\frac{8|z|}{\epsilon}} \omega(t) \frac{dt}{t}.
\end{aligned}
$$

For $I_{12}(x,z)$, by the mean value theorem,

$$
I_{12}(x,z) \lesssim |z|^m \|\nabla b\|_{L^\infty(\mathbb{R}^n)}^m \sum_{j=1}^{\infty} \frac{1}{|2^{j-2}\epsilon|} \frac{1}{|2^{j-1}\epsilon|^{n-\alpha}} \int_{\{y\in\mathbb{R}^n:|x-y|<2^{j-1}\epsilon\}} |f(y)|dy \lesssim |z|^m \mathcal{M}_\alpha(f)(x).
$$

Applying Assumption 1.2(i) and the L^∞-Dini condition, we obtain that

$$
\|I_1(\cdot,z)\|_Y \lesssim |z|^m \cdot \|\mathcal{M}_\alpha f\|_Y \lesssim |z|^m \cdot \|f\|_X \lesssim |z|^m.
$$

Next, we write

$$
\begin{aligned}
|(b(x+z)-b(y))^m - (b(x)-b(y))^m| &= |(b(x+z)-b(x)+b(x)-b(y))^m - (b(x)-b(y))^m| \\
&= \sum_{j=1}^{m} C_m^i (b(x+z)-b(x))^j (b(x)-b(y))^{m-j} \\
&= \sum_{j=1}^{m} C_m^i (b(x+z)-b(x))^j \sum_{i=0}^{m-j} C_{m-j}^i b(x)^i b(y)^{m-j-i}.
\end{aligned}
$$

Thus, for $I_2(x,z)$, we have

$$
\begin{aligned}
I_2(x,z) \leq &\sum_{j=1}^{m} C_m^i |b(x+z)-b(x)|^j \sum_{i=0}^{m-j} C_{m-j}^i |b(x)|^i \left| \int_{|x-y|>\epsilon/2} b(y)^{m-j-i} \left[K_\epsilon(x,y) - \frac{\Omega(x-y)}{|x-y|^{n-\alpha}}\right] f(y)dy \right| \\
&+ \sum_{j=1}^{m} C_m^i |b(x+z)-b(x)|^j \sum_{i=0}^{m-j} C_{m-j}^i |b(x)|^i \left| \int_{|x-y|>\epsilon/2} b(y)^{m-j-i} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy \right| \\
&\lesssim |z| \sum_{i=0}^{m-j} C_{m-j}^i \left| \int_{\{y\in\mathbb{R}^n:\epsilon\geq|x-y|\geq\epsilon/2\}} b(y)^{m-j-i} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy + \int_{|x-y|>\epsilon/2} b(y)^{m-j-i} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy \right| \\
&\lesssim |z| \sum_{i=0}^{m-j} C_{m-j}^i \left[\mathcal{M}_\alpha(b^{m-j-i}f)(x) + T_\alpha(b^{m-j-i}f)(x)\right].
\end{aligned}
$$

Assumption 1.2(i) yields

$$\|I_2(\cdot,z)\|_Y \lesssim |z|\cdot \sum_{i=0}^{m-j} C_{m-j}^i \left\|\mathcal{M}_\alpha(b^{m-j-i}f)(x)+T_\alpha(b^{m-j-i}f)(x)\right\|_Y \lesssim |z|\cdot\|f\|_X \lesssim |z|.$$

Combining the estimates of I_1 and I_2, we have

$$\lim_{|z|\to 0^+} \|(T_{\Omega,\alpha}^{(\epsilon)})_b^m(f)(\cdot+z)-(T_{\Omega,\alpha}^{(\epsilon)})_b^m(f)(\cdot)\|_Y = 0,$$

which implies the condition (iii) of Lemma 2.7. Thus, the iterated commutator $(T_{\Omega,\alpha}^{(\epsilon)})_b^m$ is compact from X to Y for any given $b\in C_c^\infty(\mathbb{R}^n)$ and $\epsilon\in(0,\infty)$. The proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.2

In this section, we first show the lower and upper estimates of the iterated commutator of the fractional integral operator $(T_{\Omega,\alpha})_b^m$. Furthermore, we give the proof of Theorem 1.2.

Now, we begin to show the lower estimate of the iterated commutator of the fractional integral operator $(T_{\Omega,\alpha})_b^m$.

Proposition 4.1. *Let $0<\alpha<n$ and $b\in L^1_{loc}(\mathbb{R}^n)$. Let X and Y be ball Banach function spaces satisfying Assumption 1.2(ii) and $T_{\Omega,\alpha}$ be the fractional integral operator with the homogeneous kernel Ω, where $\Omega\in L^\infty(\mathbb{S}^{n-1})$ satisfies that there exists an open set $\Gamma\subset\mathbb{S}^{n-1}$ such that Ω never vanishes and never changes sign on Γ. Let $B:=B(x_0,r_0)$ and k_0, ϵ_0, G, E and F be as in Lemma 2.9. Then there exists a positive constant C that is independent of B and just depends on α, λ, k_0, ϵ_0 and n such that, for any measurable set $U\subset\mathbb{R}^n$ with $|U|\le\frac{\lambda}{8}|B(x_0,r_0)|$,*

$$\|(T_{\Omega,\alpha})_b^m(\chi_F)\chi_{E\backslash U}\|_Y \ge Ca_\lambda(b;B)^m\|\chi_F\|_X.$$

Proof. Applying Lemma 2.9, we have

$$\begin{aligned}
a_\lambda(b;B)^m|2(k_0+1))B|^{-1+\frac{\alpha}{n}}|(E\backslash U\times F)\cap G| &\le \frac{1}{\epsilon_0}\int_{E\backslash U}\left|\int_F \frac{(b(x)-b(y))^m}{|x-y|^{n-\alpha}}\Omega(\frac{x-y}{|x-y|})dy\right|dx\\
&\le \frac{1}{\epsilon_0}\int_{E\backslash U}|(T_{\Omega,\alpha})_b^m(\chi_F)(x)|dx\\
&\le \frac{1}{\epsilon_0}\|(T_{\Omega,\alpha})_b^m(\chi_F)\chi_{E\backslash U}\|_Y\cdot\|\chi_{E\backslash U}\|_{Y'}.
\end{aligned}$$

And using the facts that $|U|\le\frac{\lambda}{8}|B(x_0,r_0)|$ and $|F|=\frac{\lambda}{2}|B(x_1,r_0)|$, $|G|\ge\frac{\lambda}{8}|B(x_0,r_0)|^2$, we see

$$\begin{aligned}
|(E\backslash U\times F)\cap G| &\ge |G|-|U||F|\\
&\ge \frac{\lambda}{8}|B(x_0,r_0)|^2-\frac{\lambda}{8}|B(x_0,r_0)|\cdot\frac{\lambda}{2}|B(x_1,r_0)|\\
&= \frac{\lambda}{16}|B(x_0,r_0)|^2.
\end{aligned}$$

We apply Assumption 1.2(ii) and obtain that

$$\begin{aligned} a_\lambda(b;B)^m\|\chi_F\|_X &\lesssim \frac{\|\chi_F\|_X\|\chi_{E\setminus U}\|_{Y'}}{|2(k_0+1)B|^{-1+\alpha/n}\cdot|B|^2}\cdot\|(T_{\Omega,\alpha})^m_b(\chi_F)\chi_{E\setminus U}\|_Y \\ &\lesssim \frac{\|\chi_F\|_X\|\chi_{E\setminus U}\|_{Y'}}{|2(k_0+1)B|^{1+\alpha/n}}\cdot\|(T_{\Omega,\alpha})^m_b(\chi_F)\chi_{E\setminus U}\|_Y \\ &\lesssim \frac{\|\chi_{2(k_0+1)B}\|_X\|\chi_{2(k_0+1)B}\|_{Y'}}{|2(k_0+1)B|^{1+\alpha/n}}\cdot\|(T_{\Omega,\alpha})^m_b(\chi_F)\chi_{E\setminus U}\|_Y \\ &\lesssim \|(T_{\Omega,\alpha})^m_b(\chi_F)\chi_{E\setminus U}\|_Y. \end{aligned}$$

The proof of Proposition 4.1 is complete.

Obviously, we have the following corollary.

Corollary 4.1. *Let $0<\alpha<n$ and $b\in L^1_{loc}(\mathbb{R}^n)$. Let X and Y be ball Banach function spaces satisfying Assumption 1.2(ii) and $T_{\Omega,\alpha}$ be the fractional integral operator with the homogeneous kernel Ω, where $\Omega\in L^\infty(\mathbb{S}^{n-1})$ satisfies that there exists an open set $\Gamma\subset\mathbb{S}^{n-1}$ such that Ω never vanishes and never changes sign on Γ. If $(T_{\Omega,\alpha})^m_b$ is bounded from X to Y, then $b\in BMO(\mathbb{R}^n)$.*

Next, we give the upper estimate of the commutator $(T_{\Omega,\alpha})^m_b$.

Proposition 4.2. *Let $0<\alpha<n$ and $b\in BMO(\mathbb{R}^n)$. Let X and Y be ball Banach function spaces satisfying Assumption 1.1 and Assumption 1.2(i) and $T_{\Omega,\alpha}$ be the fractional integral operator with the homogeneous kernel Ω, where $\Omega\in L^\infty(\mathbb{S}^{n-1})$ satisfies that there exists an open set $\Gamma\subset\mathbb{S}^{n-1}$ such that Ω never vanishes and never changes sign on Γ. Let $B=B(x_0,r_0)$ and F and k_0 be as in Lemma 2.9. Then, there exist positive constants d_0 satisfying $d_0<d<\infty$, δ and C that is independent on d, B, d_0 and k_0 such that,*

$$\|(T_{\Omega,\alpha})^m_b(\chi_F)\chi_{2^{d+1}B\setminus 2^dB}\|_Y\le C2^{-\delta dn}d\|b\|^m_{BMO(\mathbb{R}^n)}\|\chi_F\|.$$

Proof. Let $B=B(x_0,r_0)$, $B(x_1,r_0)$, ϵ_0, k_0, G, E, and F be as in Lemma 2.9. Taking $d_0>0$ such that $2^{d_0}\in(4k_0,\infty)$, we get the following for any $x\in 2^{d+1}B\setminus 2^dB$ and $y\in F\subset B(x_1,r_0)$:

$$|x-y|\sim 2^dr_0.$$

By Lemma 2.4, for any $x\in 2^{d+1}B\setminus 2^dB$, we have

$$\begin{aligned} |(T_{\Omega,\alpha})^m_b(\chi_F)(x)| &= \left|\int_F (b(x)-b(y))^m\frac{\Omega(\frac{x-y}{|x-y|})}{|x-y|^{n-\alpha}}dy\right| \\ &\le \int_F |b(x)-b_{B(x_1,r_0)}+b_{B(x_1,r_0)}-b(y)|^m\frac{\Omega(\frac{x-y}{|x-y|})}{|x-y|^{n-\alpha}}dy \\ &\le \sum_{i=0}^m C^i_m|b(x)-b_{B(x_1,r_0)}|^{m-i}\int_F|b_{B(x_1,r_0)}-b(y)|^i\frac{\Omega(\frac{x-y}{|x-y|})}{|x-y|^{n-\alpha}}dy \\ &\le \sum_{i=0}^m C^i_m|b(x)-b_{B(x_1,r_0)}|^{m-i}\frac{\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}}{|2^dB(x_1,r_0)|^{1-\alpha/n}}\int_F|b_{B(x_1,r_0)}-b(y)|^idy \end{aligned}$$

$$\lesssim \sum_{i=0}^{m} C_m^i |b(x) - b_{B(x_1,r_0)}|^{m-i} \frac{1}{|2^d B(x_1,r_0)|^{1-\alpha/n}} \| |b_{B(x_1,r_0)} - b|^i \chi_F \|_{X'} \|\chi_F\|_X.$$

Applying Lemma 2.3 and the fact that $F \subset B(x_1, r_0)$, we have

$$\begin{aligned} \| |b_{B(x_1,r_0)} - b|^i \chi_F \|_{X'} &\lesssim \| |b_{B(x_1,r_0)} - b|^i \chi_{B(x_1,r_0)} \|_{X'} \\ &= \| |b_{B(x_1,r_0)} - b| \chi_{B(x_1,r_0)} \|_{(X')^i}^i \\ &\leq \|b\|_{BMO}^i \|\chi_{B(x_1,r_0)}\|_{X'}. \end{aligned}$$

Hence

$$|(T_{\Omega,\alpha})_b^m(\chi_F)(x)| \lesssim \sum_{i=0}^{m} C_m^i |b(x) - b_{B(x_1,r_0)}|^{m-i} \frac{\|\chi_{B(x_1,r_0)}\|_{X'} \|\chi_F\|_X}{|2^d B(x_1,r_0)|^{1-\alpha/n}} \|b\|_{BMO}^i. \tag{4.1}$$

Let $\kappa \in \{2, 4, 6, \cdots\}$, depending only on k_0 such that $x_0 \in 2^\kappa B(x_1, r_0)$. Thus, for any $y \in B(x_0, 2^{d+1} r_0)$, we have

$$|y - x_1| \leq |y - x_0| + |x_0 - x_1| \leq 2^{d+1} r_0 + 2^\kappa r_0 \leq 2^{d+\kappa} r_0,$$

which implies that

$$B(x_0, 2^{d+1} r_0) \subset B(x_1, 2^{d+\kappa} r_0).$$

Thus, we have

$$|b_{B(x_1,r_0)} - b_{2^{d+\kappa} B(x_1,r_0)}| \leq (d + \kappa) 2^n \|b\|_{BMO}.$$

Using Lemma 2.6 and [8, Lemma2.15], we can see that

$$\|\chi_{\beta B}\|_Y \lesssim \beta^n \|\chi_B\|_Y.$$

Applying Lemma 2.8, we have

$$\begin{aligned} \| |b(x) - b_{B(x_1,r_0)}|^{m-i} \chi_{2^{d+1} B \setminus 2^d B} \|_Y &\leq \| |b(x) - b_{B(x_1,r_0)}|^{m-i} \chi_{2^{d+\kappa} B(x_1,r_0)} \|_Y \\ &\leq 2^{nd/\eta} d \|b\|_{BMO}^{m-i} \|\chi_{B(x_1,r_0)}\|_Y. \end{aligned} \tag{4.2}$$

By Assumption 1.2(i), we obtain that

$$\frac{\|\chi_B\|_Y \|\chi_B\|_{X'}}{|B|^{1-\frac{\alpha}{n}}} \lesssim 1. \tag{4.3}$$

In fact, it is easy to see that

$$\|\chi_{\{x \in \mathbb{R}^n : M_\alpha f(x) \geq \gamma\}}\|_Y \leq \frac{1}{\gamma} \|\mathcal{M}_\alpha f\|_Y \leq \frac{C}{\gamma} \|f\|_X.$$

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. For almost every $x \in \mathbb{R}^n$ and $|f|_B \chi_B(x) = \frac{1}{|B|} \int_B |f(y)| dy \cdot \chi_B(x) \leq |B|^{-\frac{\alpha}{n}} \mathcal{M}_\alpha(f\chi_B)(x)$, we obtain that $\mathcal{M}_\alpha(f\chi_B) > \gamma$ for almost every $x \in B$ and $\gamma := \frac{1}{2} |f|_B |B|^{\frac{\alpha}{n}}$. Thus, we have

$$|f|_B \|\chi_B\|_Y \leq |f|_B \|\chi_{\{x \in \mathbb{R}^n : M_\alpha f(x) \geq \gamma\}}\|_Y \leq |f|_B \cdot C\gamma^{-1} \|f\chi_B\|_X = 2C|B|^{-\frac{\alpha}{n}} \|f\chi_B\|_X.$$

Further, we obtain that

$$\frac{1}{|B|^{1-\alpha/n}} \|\chi_B\|_Y \|\chi_B\|_{X'} = \frac{1}{|B|^{1-\alpha/n}} \|\chi_B\|_Y \sup \left\{ \int_B |g(x)| dx : \|g\|_X \leq 1 \right\}$$

$$
\begin{aligned}
&= \sup\left\{|B|^{\alpha/n}|g|_B\|\chi_B\|_Y : \|g\|_X \le 1\right\}\\
&\le \sup\{2C\|g\chi_B\|_X : \|g\|_X \le 1\}\\
&\lesssim 1.
\end{aligned}
$$

Using (4.1)–(4.3), we have

$$
\begin{aligned}
\|(T_{\Omega,\alpha})_b^m(\chi_F)\|_Y &\lesssim \sum_{i=0}^{m} C_m^i \||b(x)-b_{B(x_1,r_0)}|^{m-i}\chi_{2^{d+1}B\setminus 2^d B}\|_Y \frac{\|\chi_{B(x_1,r_0)}\|_{X'}\|\chi_F\|_X}{|2^d B(x_1,r_0)|^{1-\alpha/n}}\|b\|_{BMO}^i\\
&\le \sum_{i=0}^{m} C_m^i 2^{nd} d\|b\|_{BMO}^{m-i}\|\chi_{B(x_1,r_0)}\|_Y \frac{\|\chi_{B(x_1,r_0)}\|_{X'}\|\chi_F\|_X}{|2^d B(x_1,r_0)|^{1-\alpha/n}}\|b\|_{BMO}^i\\
&\lesssim 2^{nd} d\|b\|_{BMO}^m \frac{\|\chi_{B(x_1,r_0)}\|_{X'}\|\chi_{B(x_1,r_0)}\|_Y}{|2^d B(x_1,r_0)|^{1-\alpha/n}}\|\chi_F\|_X\\
&\lesssim 2^{nd(\frac{\alpha}{n}-1)} d\|b\|_{BMO}^m\|\chi_F\|_X.
\end{aligned}
$$

Let $\delta := 1-\frac{\alpha}{n} > 0$. Then we complete the proof of Proposition 4.2.

Proof of Theorem 1.2. By Corollary 4.1, we conclude that $b \in BMO(\mathbb{R}^n)$; then, without loss of generality, we can assume that $\|b\|_{BMO(\mathbb{R}^n)} = 1$. To show that $b \in CMO(\mathbb{R}^n)$, we use a contradiction argument via Lemma 2.2. Observe that, if $b \notin CMO(\mathbb{R}^n)$, then b does not satisfy at least one condition among (i), (ii) and (iii) of Lemma 2.2. To finish the proof of this theorem, we only need to show that, if b does not satisfy at least one condition among (i), (ii) and (iii) of Lemma 2.2, then $(T_{\Omega,\alpha})_b^m$ is not compact from X to Y. We prove this by three cases on b as follows.

Case 1: Suppose that b does not satisfy (i) of Lemma 2.2. Then, there exist a constant $\epsilon_0 \in (0,1)$ and a sequence of balls $\{B_j\}_{j\in\mathbb{N}}$ with $|B_j| \to 0$ as $j \to \infty$ such that, for any $j \in \mathbb{N}$,

$$a_\lambda(b; B_j) \ge \epsilon_0, \tag{4.4}$$

where $\lambda \in (0, 1/2)$. For any given ball $B := B(x_0, r_0)$, let E and F be the set mentioned in Lemma 2.9,

$$f := \|\chi_F\|_X^{-1}\chi_F$$

and $2C_0 := C_{(\lambda,k_0,\varepsilon_0,n)}$ be as in Proposition 4.1. Then, by Proposition 4.1, we conclude that, for any measurable set $U \subset \mathbb{R}^n$ with $|U| \le \frac{\lambda}{8}|B|$,

$$\|(T_{\Omega,\alpha})_b^m(f)\chi_{E\setminus U}\|_Y \ge 2C_0 a_\lambda(b; B)^m. \tag{4.5}$$

For such chosen C_0 and ϵ_0, by Proposition 4.2, there exists a positive constant d_0 such that

$$\|(T_{\Omega,\alpha})_b^m(f)\chi_{\mathbb{R}^n\setminus 2^{d_0}B}\|_Y \le \sum_{k=0}^{\infty}\|(T_{\Omega,\alpha})_b^m(f)\chi_{2^{d_0+k+1}B\setminus 2^{d_0+k}B}\|_Y \le C_0\epsilon_0^m. \tag{4.6}$$

Take a subsequence of balls $\{B_j\}_{j\in\mathbb{N}}$, still denoted by $\{B_j\}_{j\in\mathbb{N}}$, such that, for any $j \in \mathbb{N}$,

$$\frac{|B_j|}{|B_{j+1}|} \le \min\left\{\frac{\lambda^2}{64}, 2^{-2d_0 n}\right\}.$$

Let $B_j^* := (|B_j|/|B_{j-1}|)^{1/2n} B_j$ for any $j \in \mathbb{N}$ and $j \geq 2$. It is easy to see that, for any $j \in \mathbb{N}$ and $j \geq 2$,

$$\left(\frac{|B_j|}{|B_{j-1}|}\right)^{\frac{1}{2n}} \geq 2^{d_0} \text{ and } |B_j^*| \leq \frac{\lambda}{8}|B_{j-1}|.$$

From this and the monotonicity of $\{B_j\}_{j\in\mathbb{N}}$, we deduce that, for any integers k and j with $k > j \geq 2$,

$$2^{d_0} B_k \subset B_k^* \text{ and } |B_k^*| \leq \frac{\lambda}{8}|B_{k-1}| \leq \frac{\lambda}{8}|B_j|. \tag{4.7}$$

Now, for any $j \in \mathbb{N}$, let E_j and F_j be the sets associated with B_j as in Lemma 2.9 with B replaced by B_j, and

$$f_j := \|\chi_{F_j}\|_X^{-1} \chi_{F_j}.$$

Then, for any integers k and j with $k > j \geq 2$, by (4.4), (4.5), (4.6) and (4.7), we conclude that

$$\|(T_{\Omega,\alpha})_b^m(f_j)\chi_{E_j\setminus B_k^*}\|_Y \geq 2C_0 a_\lambda(b; B_j)^m \geq 2C_0\epsilon_0^m$$

and

$$\|(T_{\Omega,\alpha})_b^m(f_k)\chi_{E_j\setminus B_k^*}\|_Y \leq \|(T_{\Omega,\alpha})_b^m(f_k)\chi_{\mathbb{R}^n\setminus 2^{d_0}B_k}\|_Y \leq C_0\epsilon_0^m,$$

which further implies that

$$\begin{aligned}\|(T_{\Omega,\alpha})_b^m(f_j) - (T_{\Omega,\alpha})_b^m(f_k)\|_Y &\geq \|\{(T_{\Omega,\alpha})_b^m(f_j) - (T_{\Omega,\alpha})_b^m(f_k)\}\chi_{E_j\setminus B_k^*}\|_Y \\ &\geq \|(T_{\Omega,\alpha})_b^m(f_j)\chi_{E_j\setminus B_k^*}\|_Y - \|(T_{\Omega,\alpha})_b^m(f_k)\chi_{E_j\setminus B_k^*}\|_Y \\ &\geq C_0\epsilon_0^m.\end{aligned}$$

Therefore, $\{(T_{\Omega,\alpha})_b^m(f_j)\}_{j\in\mathbb{N}}$ is not relatively compact from X to Y, which leads to a contradiction with the compactness of $(T_{\Omega,\alpha})_b^m$ from X to Y. This shows that b satisfies (i) of Lemma 2.2, which is the desired conclusion.

Case 2: Suppose that b does not satisfy (ii) of Lemma 2.2. In this case, similarly to the above Case 1, there exist a constant $\epsilon_0 \in (0, 1)$ and a sequence of balls $\{B_j\}_{j\in\mathbb{N}}$ with $|B_j| \to \infty$ as $j \to \infty$ such that, for any $j \in \mathbb{N}$,

$$a_\lambda(b; B_j) \geq \epsilon_0 \quad \text{and} \quad \frac{|B_j|}{|B_{j+1}|} \leq \min\left\{\frac{\lambda^2}{64}, 2^{-2d_0 n}\right\},$$

where C_0 and d_0 are as in Case 1 such that (4.5) and (4.6) hold true. For any $j \in \mathbb{N}$, let E_j, F_j and f_j be as in Case 1 and $B_j^* := (|B_j|/|B_{j-1}|)^{1/2n} B_j$ for any $j \geq 2$. It is easy to see that, for any integers k and j with $k > j \geq 2$,

$$2^{d_0} B_k \subset B_k^* \text{ and } |B_k^*| \leq \frac{\lambda}{8}|B_j|.$$

Using a method similar to that used in Case 1, we conclude that

$$\|(T_{\Omega,\alpha})_b^m(f_j) - (T_{\Omega,\alpha})_b^m(f_k)\|_Y \geq C_0\epsilon_0^m;$$

hence $\{(T_{\Omega,\alpha})_b^m(f_j)\}_{j\in\mathbb{N}}$ is not relatively compact from X to Y, which is a contradiction. This shows that b satisfies (ii) of Lemma 2.2, which is also the desired conclusion.

Case 3: Suppose that b does not satisfy (iii) of Lemma 2.2. In this case, there exist a constant $\epsilon_0 \in (0, 1)$ and a sequence of balls $\{B_j\}_{j\in\mathbb{N}}$ such that, for any $j \in \mathbb{N}$,

$$a_\lambda(b; B_j) \geq \epsilon_0. \tag{4.8}$$

From this and Cases 1 and 2, we deduce that there exist a constant $d_1 \in [d_0, \infty)$ with d_0 as in Lemma 2.9 and a subsequence of balls $\{B_j\}_{j\in\mathbb{N}}$, still denoted by $\{B_j\}_{j\in\mathbb{N}}$, such that

$$|B_j| \sim 1, \quad \forall j \in \mathbb{N},$$

and

$$2^{d_1} B_i \cap 2^{d_1} B_j = \emptyset, \quad \forall i \neq j.$$

For any $j \in \mathbb{N}$, let E_j, F_j, f_j and C_0 be as in Case 1. Notice that, for any positive integers k and j,

$$\left(2^{d_0} B_k \cap E_j\right) \subset \left(2^{d_1} B_k \cap 2^{d_1} B_j\right) = \emptyset.$$

By this, Proposition 4.1 with $U := \emptyset$ and (4.8), we conclude that, for any positive integers k and j,

$$\|(T_{\Omega,\alpha})_b^m(f_j)\chi_{E_j\setminus 2^{d_0}B_k}\|_Y = \|(T_{\Omega,\alpha})_b^m(f_j)\chi_{E_j}\|_Y \geq 2C_0 a_\lambda(b; B)^m \geq 2C_0\epsilon_0^m. \tag{4.9}$$

Moreover, by Proposition 4.2, we deduce that, for any positive integers k and j,

$$\|(T_{\Omega,\alpha})_b^m(f_k)\chi_{E_j\setminus 2^{d_0}B_k}\|_Y \leq \|(T_{\Omega,\alpha})_b^m(f_k)\chi_{\mathbb{R}^n\setminus 2^{d_0}B_k}\|_Y \leq C_0\epsilon_0^m. \tag{4.10}$$

Combining (4.9) and (4.10), we obtain

$$\begin{aligned}\|(T_{\Omega,\alpha})_b^m(f_j) - (T_{\Omega,\alpha})_b^m(f_k)\|_Y &\geq \|\{(T_{\Omega,\alpha})_b^m(f_j) - (T_{\Omega,\alpha})_b^m(f_k)\}\chi_{E_j\setminus 2^{d_0}B_k}\|_Y \\ &\geq \|(T_{\Omega,\alpha})_b^m(f_j)\chi_{E_j\setminus 2^{d_0}B_k}\|_Y - \|(T_{\Omega,\alpha})_b^m(f_k)\chi_{E_j\setminus 2^{d_0}B_k}\|_Y \\ &\geq C_0\epsilon_0^m;\end{aligned}$$

hence $\{(T_{\Omega,\alpha})_b^m(f_j)\}_{j\in\mathbb{N}}$ is not relatively compact from X to Y, which is a contradiction. This shows that b satisfies (iii) of Lemma 2.2, which completes the proof of Theorem 1.2.

5. Applications

5.1. Weighted Lebesgue spaces

We begin this section with the definition of Muckenhoupt weights $A_p(\mathbb{R}^n)$. A weight will always mean a positive function which is locally integrable. Also, for a weight ω and a measurable set E, we define $\omega(E) := \int_E \omega(y)dy$.

Definition 5.1. *For $1 < p < \infty$, a weight ω is said to be of class $A_p(\mathbb{R}^n)$ if*

$$\sup_{B\subset\mathbb{R}^n}\left(\frac{1}{|B|}\int_B \omega(x)dx\right)^{\frac{1}{p}}\left(\frac{1}{|B|}\int_B \omega(x)^{1-p'}dx\right)^{\frac{1}{p'}} < \infty.$$

Definition 5.2. *For $1 < p, q < \infty$, a weight ω is said to be of class $A_{p,q}(\mathbb{R}^n)$ if*

$$\sup_{B\subset\mathbb{R}^n}\left(\frac{1}{|B|}\int_B \omega(x)^q dx\right)^{\frac{1}{q}}\left(\frac{1}{|B|}\int_B \omega(x)^{-p'} dx\right)^{\frac{1}{p'}} < \infty.$$

Next, let us recall the weighted Lebesgue spaces which are defined as follows.

Definition 5.3. *Let $1 < p < \infty$ and ω be a weight. The weighted Lebesgue space $L^p_\omega(\mathbb{R}^n)$ denotes the set of all locally integrable functions f on $\mathbb{R}^n$ such that*

$$\|f\|_{L^p_\omega(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx\right)^{1/p} < \infty.$$

Theorem 5.1. *Let $0 < \alpha < n$, $1 < p, q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let $\omega \in A_{p,q}(\mathbb{R}^n)$ and $T_{\Omega,\alpha}$ be a fractional integral operator with the homogeneous kernel Ω, where $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfies (1.3), (1.4) and (1.8). If $b \in CMO(\mathbb{R}^n)$, then $(T_{\Omega,\alpha})^m_b$ is compact from $L^p_{\omega^p}(\mathbb{R}^n)$ to $L^q_{\omega^q}(\mathbb{R}^n)$.*

Theorem 5.2. *Let $0 < \alpha < n$, $1 < p, q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let $\omega \in A_{p,q}(\mathbb{R}^n)$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $T_{\Omega,\alpha}$ be a fractional integral operator with the homogeneous kernel Ω, where $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfies that there exists an open set $\Gamma \subset \mathbb{S}^{n-1}$ such that Ω never vanishes and never changes sign on Γ. If $(T_{\Omega,\alpha})^m_b$ is compact from $L^p_{\omega^p}(\mathbb{R}^n)$ to $L^q_{\omega^q}(\mathbb{R}^n)$, then $b \in CMO(\mathbb{R}^n)$.*

Corollary 5.1. *Let $0 < \alpha < n$, $1 < p, q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let $\omega \in A_{p,q}(\mathbb{R}^n)$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $T_{\Omega,\alpha}$ be a fractional integral operator with the homogeneous kernel Ω, where Ω satisfies (1.3)–(1.5). Then $(T_{\Omega,\alpha})^m_b$ is compact from $L^p_{\omega^p}(\mathbb{R}^n)$ to $L^q_{\omega^q}(\mathbb{R}^n)$ if and only if $b \in CMO(\mathbb{R}^n)$.*

Proof. Let $X := L^p_{\omega^p}(\mathbb{R}^n)$ and $\omega \in A_{p,q}(\mathbb{R}^n)$. From [33], we then get that $\omega^q \in A_q(\mathbb{R}^n)$ and $\omega^{-q'} \in A_{q'}(\mathbb{R}^n)$. By the fact that $\mathcal{M}$ is bounded on $L^q_{\omega^q}(\mathbb{R}^n)$ and X', where $X' = L^{q'}_{\omega^{-q'}}(\mathbb{R}^n)$ in [34, Theorem 3.1], $T_{\Omega,\alpha} : L^p_{\omega^p}(\mathbb{R}^n) \to L^q_{\omega^q}(\mathbb{R}^n)$ in [35, Theorem 1] and the iterated commutator $(T_{\Omega,\alpha})^m_b$ is bounded from $L^p_{\omega^p}(\mathbb{R}^n)$ to $L^q_{\omega^q}(\mathbb{R}^n)$ in [36, Theorem 1] for $1 < p, q < \infty$, $0 < \alpha < n$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\omega \in A_{p,q}(\mathbb{R}^n)$, we then use Hölder's inequality to obtain that

$$\begin{aligned}\frac{\|\chi_B\|_{Y'}\|\chi_B\|_X}{|B|^{1+\alpha/n}} &= \frac{\|\chi_B\|_{L^{q'}_{\omega^{-q'}}}\|\chi_B\|_{L^p_{\omega^p}}}{|B|^{1+\alpha/n}}\\ &= \left(\frac{1}{|B|}\int_B \omega(x)^{-q'}dx\right)^{1/q'}\left(\frac{1}{|B|}\int_B \omega(x)^p dx\right)^{1/p}\\ &\le \left(\frac{1}{|B|}\int_B \omega(x)^{-p'}dx\right)^{1/p'}\left(\frac{1}{|B|}\int_B \omega(x)^q dx\right)^{1/q}\\ &\lesssim 1.\end{aligned}$$

Thus, Theorems 1.1 and 1.2 and Corollary 1.1 are true with X replaced by $L^p_{\omega^p}(\mathbb{R}^n)$ and Y replaced by $L^q_{\omega^q}(\mathbb{R}^n)$.

5.2. Morrey spaces

Recall that the definition of the Morrey space $M^q_p(\mathbb{R}^n)$ holds for $0 < p \le q < \infty$ and was introduced by Morrey in [39].

Definition 5.4. *Let $0 < p \le q < \infty$. The Morrey space $M_p^q(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that*

$$\|f\|_{M_p^q(\mathbb{R}^n)} := \sup_{B\in\mathbb{B}} |B|^{1/p-1/r}\|f\|_{L^r(B)} < \infty,$$

where $\mathbb{B}$ is as in (1.1).

Theorem 5.3. *Let $0 < \alpha < n$, $1 < p \le q < \infty$, $1 < t \le s < \infty$, $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{t}{s} = \frac{q}{p}$. Let $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and T_α be a fractional integral operator. Then $(T_\alpha)_b^m$ is compact from $M_q^p(\mathbb{R}^n)$ to $M_t^s(\mathbb{R}^n)$ if and only if $b \in CMO(\mathbb{R}^n)$.*

Proof. We know that the Morrey space is the ball Banach function space in [8, P. 86]. Moreover, $[b, T_\alpha]$ is bounded from $M_q^p(\mathbb{R}^n)$ to $M_t^s(\mathbb{R}^n)$ in [37, Theorem 3.1] for $1 < p \le q < \infty$, $1 < t \le s < \infty$, $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{t}{s} = \frac{q}{p}$, and $(T_\alpha)_b^m$ is bounded from $M_q^p(\mathbb{R}^n)$ to $M_t^s(\mathbb{R}^n)$ in [40, Corollary 3] for $1 < p \le q < \infty$, $1 < t \le s < \infty$, $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{t}{s} = \frac{q}{p}$. It is easy to calculate that

$$\frac{\|\chi_B\|_X\|\chi_B\|_{Y'}}{|B|^{1+\alpha/n}} = \frac{\|\chi_B\|_{M_q^p}\|\chi_B\|_{(M_t^s)'}}{|B|^{1+\alpha/n}} \lesssim \frac{|B|^{1/p+1/s'}}{|B|^{1+\alpha/n}} \lesssim 1.$$

Thus, using Corollary 1.1, we complete the proof of Theorem 5.3.

5.3. Variable Lebesgue spaces

In this section, we apply our results on variable Lebesgue spaces with $X = L^{p(\cdot)}(\mathbb{R}^n)$, $Y = L^{q(\cdot)}(\mathbb{R}^n)$ and $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$. We write $p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x)$ and $p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x)$. Recall the definition of the variable Lebesgue spaces.

Definition 5.5. *Let $p(\cdot) : \mathbb{R}^n \mapsto [0, \infty)$ be a measurable function. Then the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that*

$$\|f\|_{L^{p(\cdot)}} := \inf\left\{\lambda \in (0, \infty) : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\} < \infty.$$

Lemma 5.1. *[38, Theorem 1.1] Let $p(\cdot) : \mathbb{R}^n \mapsto [0, \infty)$ be a measurable function satisfying that*

$$|p(x) - p(y)| \le C\frac{1}{-\log(|x-y|)} \quad \text{if } |x-y| \le \frac{1}{2}, \tag{5.1}$$

and

$$|p(x) - p(y)| \le C\frac{1}{\log(e + |x|)} \quad \text{if } |x| \le |y|; \tag{5.2}$$

then $\mathcal{M}$ is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ and $L^{p(\cdot)'}(\mathbb{R}^n)$.

Lemma 5.2. *[41, Lemma 2.5] Let $p(x)$ satisfy (5.1) and (5.2), and let it satisfy that $1 < p_- \le p_+ < \infty$. Then,*

$$\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \sim \begin{cases} |B|^{\frac{1}{p(x)}} & \text{if } |B| \le 2^n \text{ and } x \in B; \\ |B|^{\frac{1}{p(\infty)}} & \text{if } |B| \ge 1, \end{cases}$$

where $p(\infty) = \lim_{x\to\infty} p(x)$.

Theorem 5.4. *Let* $0 < \alpha < min\{n, n/p_+\}$*,* $1 < p_- \leq p_+ < \infty$ *and* $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$*. Let* $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and* T_α *be a fractional integral operator. Then* $[b, T_\alpha]$ *is compact from* $L^{p(\cdot)}(\mathbb{R}^n)$ *to* $L^{q(\cdot)}(\mathbb{R}^n)$ *if and only if* $b \in CMO(\mathbb{R}^n)$*.*

Proof. If $1 < p_- \leq p_+ < \infty$, we know that the space $L^{p(\cdot)}(\mathbb{R}^n)$ is a ball Banach function space in [8]. If $p(x)$ satisfies (5.1) and (5.2), $1 < p_- \leq p_+ < \infty$ and $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$, then $q(x)$ also satisfies (5.1) and (5.2), $1 < q_- \leq q_+ < \infty$. Let $0 < \alpha < n/p_+$. We know that if $p(x)$ satisfies (5.1) and (5.2) and $1 < p_- \leq p_+ < \infty$, T_α is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}$ in [42, theorem 2] for $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$ and $[b, T_\alpha]$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}$ in [37, Theorem 3.1] for $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$. If $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$, then $\frac{1}{p(x)} + \frac{1}{q(x)'} = 1 + \frac{\alpha}{n}$ and $\frac{1}{p(\infty)} + \frac{1}{q(\infty)'} = 1 + \frac{\alpha}{n}$. Thus, using Lemma 5.2, we have

$$\frac{\|\chi_B\|_X \|\chi_B\|_{Y'}}{|B|^{1+\alpha/n}} = \frac{\|\chi_B\|_{L^{p(\cdot)}} \|\chi_B\|_{L^{q(\cdot)'}}}{|B|^{1+\alpha/n}} \lesssim 1.$$

Applying Corollary 1.1, we have the desired result.

5.4. Mixed Morrey spaces

Let us begin with the definition of the mixed-norm Lebesgue spaces.

Definition 5.6. *Let* $\vec{p} := (p_1, \ldots, p_n) \in (0, \infty]^n$*. The mixed-norm Lebesgue space* $L^{\vec{p}}(\mathbb{R}^n)$ *is defined to be the set of all measurable functions* f *on* $\mathbb{R}^n$ *such that*

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} |f(x_1, \ldots, x_n)|^{p_1} \, dx_1 \right]^{\frac{p_2}{p_1}} \cdots dx_n \right\}^{\frac{1}{p_n}} < \infty$$

with the usual modifications made when $p_i = \infty$ *for some* $i \in \{1, \ldots, n\}$*.*

Next, we recall the definition of the mixed Morrey spaces. In 2019, Nogayama [43, 44] first introduced the mixed Morrey space $\mathcal{M}^{p_0}_{\vec{p}}(\mathbb{R}^n)$, which is defined as follows.

Definition 5.7. *Let* $\vec{p} = (p_1, p_2, \cdots, p_n) \in (1, \infty)^n$ *and* $p_0 \in (1, \infty)$ *satisfy*

$$\frac{n}{p_0} \leq \sum_{j=1}^{n} \frac{1}{p_j}.$$

The mixed Morrey space $\mathcal{M}^{p_0}_{\vec{p}}(\mathbb{R}^n)$ *is defined to be the set of all measurable functions* f *such that*

$$\|f\|_{\mathcal{M}^{p_0}_{\vec{p}}} := \sup_Q |Q|^{\frac{1}{p_0} - \frac{1}{n}(\sum_{j=1}^{n} \frac{1}{p_j})} \|f\chi_Q\|_{L_{\vec{p}}} < \infty.$$

Let $\vec{p} = (p_1, p_2, \cdots, p_n) \in (1, \infty)^n$ and $p_0 \in (1, \infty)$ satisfy

$$\frac{n}{p_0} \leq \sum_{j=1}^{n} \frac{1}{p_j}.$$

The mixed Morrey space is a ball Banach function space in [45, Remark 2.9]. Moreover, the space $\mathcal{B}^{p_0'}_{\vec{p}'}(\mathbb{R}^n)$ is the associate space of the mixed Morrey space $\mathcal{M}^{p_0}_{\vec{p}}(\mathbb{R}^n)$ and $\mathcal{M}$ is bounded in $\mathcal{B}^{q_0'}_{\vec{q}'}(\mathbb{R}^n)$.

Definition 5.8. *Let $\vec{p} = (p_1, p_2, \cdots, p_n) \in (1,\infty)^n$, $p_0 \in (1,\infty)$ and $\frac{n}{p_0} \le \sum_{i=1}^n \frac{1}{p_i}$. A measurable function $b(x)$ is said to be a $(p_0', \vec{p}\,')$-block if there exists a cube Q such that*

$$supp\ b \subset Q,\ \|b\|_{L^{\vec{p}\,'}} \le |Q|^{\frac{1}{n}\sum_{i=1}^n \frac{1}{p_i} - \frac{1}{p_0}}.$$

The block spaces $\mathcal{B}^{p_0'}_{\vec{p}\,'}(\mathbb{R}^n)$ denote the measurable function set of $f = \sum_{i=1}^\infty \lambda_i b_i(x)$, where $\{\lambda_i\}_{i=1}^\infty \in \ell^1$ and b_i is a $(p_0', \vec{p}\,')$-block for any i. The norm $\|f\|_{\mathcal{B}^{p_0'}_{\vec{p}\,'}(\mathbb{R}^n)}$ for $f \in \mathcal{B}^{p_0'}_{\vec{p}\,'}(\mathbb{R}^n)$ is defined as

$$\|f\|_{\mathcal{B}^{p_0'}_{\vec{p}\,'}} = \inf\left\{\|\{\lambda_i\}_{i=1}^\infty\|_{\ell^1} : f = \sum_{i=1}^\infty \lambda_i b_i(x), \{\lambda_i\}_{i=1}^\infty \in \ell^1, b_i \text{ is a } (p_0', \vec{p}\,') - \text{block for any } i\right\}.$$

Theorem 5.5. *Let $0 < \alpha < n$, $1 < p_0, q_0 < \infty$, $1 < \vec{p}, \vec{q} < \infty$, $\frac{n}{p_0} \le \sum_{j=1}^n \frac{1}{p_j}$, $\frac{n}{q_0} \le \sum_{j=1}^n \frac{1}{q_j}$, $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and $\frac{\vec{p}}{p_0} = \frac{\vec{q}}{q_0}$. Let $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and T_α be a fractional integral operator. Then $[b, T_\alpha]$ is compact from $\mathcal{M}^{p_0}_{\vec{p}}(\mathbb{R}^n)$ to $\mathcal{M}^{q_0}_{\vec{q}}(\mathbb{R}^n)$ if and only if $b \in CMO(\mathbb{R}^n)$.*

Proof. Given T_α is bounded from $\mathcal{M}^{p_0}_{\vec{p}}(\mathbb{R}^n)$ to $\mathcal{M}^{q_0}_{\vec{q}}(\mathbb{R}^n)$ in [43, Theorem 1.11] for $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and $\frac{\vec{p}}{p_0} = \frac{\vec{q}}{q_0}$ and $[b, T_\alpha]$ is bounded from $\mathcal{M}^{p_0}_{\vec{p}}(\mathbb{R}^n)$ to $\mathcal{M}^{q_0}_{\vec{q}}(\mathbb{R}^n)$ in [44, Theorem 1.2] for $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and $\frac{\vec{p}}{p_0} = \frac{\vec{q}}{q_0}$, by [44, Example 2.8], it is easy to see that

$$\frac{\|\chi_B\|_X \|\chi_B\|_{Y'}}{|B|^{1+\alpha/n}} = \frac{\|\chi_B\|_{\mathcal{M}^{p_0}_{\vec{p}}} \|\chi_B\|_{(\mathcal{M}^{q_0}_{\vec{q}})'}}{|B|^{1+\alpha/n}} \lesssim \frac{|B|^{1/p_0 + 1/q_0'}}{|B|^{1+\alpha/n}} \lesssim 1.$$

Thus, using Corollary 1.1, we finish the proof of Theorem 5.4.

6. Conclusions

In this work, we establish the characterization of compactness of the iterated commutator $(T_{\Omega,\alpha})^m_b$ generated by the locally integrable function b and the fractional integral operator with the homogeneous kernel $T_{\Omega,\alpha}$ on ball Banach function spaces. As applications, we show that $b \in CMO(\mathbb{R}^n)$ if and only if the iterated commutator $(T_{\Omega,\alpha})^m_b$ is compact from $L^p_{\omega^p}(\mathbb{R}^n)$ to $L^q_{\omega^q}(\mathbb{R}^n)$ and we obtain that $b \in CMO(\mathbb{R}^n)$ if and only if the iterated commutator $(T_\alpha)^m_b$ generated by the locally integrable function b and the fractional integral operator is compact from $M^p_q(\mathbb{R}^n)$ to $M^s_t(\mathbb{R}^n)$. Moreover, we obtain that $b \in CMO(\mathbb{R}^n)$ if and only if the commutator $[b, T_\alpha]$ generated by the locally integrable function b and the fractional integral operator is compact from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$. We also have that $b \in CMO(\mathbb{R}^n)$ if and only if the commutator $[b, T_\alpha]$ generated by the locally integrable function b and the fractional integral operator is compact from $\mathcal{M}^{p_0}_{\vec{p}}(\mathbb{R}^n)$ to $\mathcal{M}^{q_0}_{\vec{q}}(\mathbb{R}^n)$.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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