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Research article

Robustness analysis of Cohen-Grossberg neural network with piecewise constant argument and stochastic disturbances

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Abstract: Robustness of neural networks has been a hot topic in recent years. This paper mainly studies the robustness of the global exponential stability of Cohen-Grossberg neural networks with a piecewise constant argument and stochastic disturbances, and discusses the problem of whether the Cohen-Grossberg neural networks can still maintain global exponential stability under the perturbation of the piecewise constant argument and stochastic disturbances. By using stochastic analysis theory and inequality techniques, the interval length of the piecewise constant argument and the upper bound of the noise intensity are derived by solving transcendental equations. In the end, we offer several examples to illustrate the efficacy of the findings.

Keywords: Cohen-Grossberg neural network; robustness; piecewise constant argument; stochastic disturbances
Mathematics Subject Classification: 34D20

1. Introduction

Due to its parallel processing ability, high fault tolerance, adaptive ability, neural networks (NNs) have considerable application prospects in signal processing, automatic control, and artificial intelligence, which has gradually attracted great attention to NNs [1–4]. Research on NNs have yielded multiple excellent results up to this point [5–16]. For instance, the literature [5–7] discuss the stability of NNs, and [8, 9] are about the robustness analysis of NNs. The literature [11–14] introduce the synchronization problems of NNs, which are fundamental to the application of NNs. The Cohen-Grossberg neural network (CGNN) is an important continuous-time feedback neural network. Analysis of the dynamic behavior of CGNNs was first started in 1983 [17], and further studied in [18]. The CGNN model includes well-known models from disciplines such as population biology and neurobiology. Moreover, the CGNN is more general as it includes a Hopfield neural network (HNN) and a cellular neural network (CNN) as its special cases [19–23]. The CGNN attracted wide attention when it was first proposed, and quickly became one of the hottest topics in the field of research at that time. With the passage of time and in-depth research, the CGNN model has been further improved and expanded, leading to numerous breakthrough achievements in various fields such as stability and periodicity [24–31]. For example, the stability of NNs is studied in [24–26], the periodic solution problem of NNs is mainly discussed in [27, 28], and the synchronization problem is explored in [29–31].

Stochastic disturbances (SDs) in nervous systems often arise from the stochastic processes involved in neural transmission. SDs can lead to unstable output results and even cause errors in NNs. Therefore, it is necessary to consider the impact of SDs on the dynamic behavior of NNs. In recent years, many results on the stability analysis of CGNNs with SDs have been proposed [32–34].

In the implementation of CGNNs, time delay phenomenon is almost unavoidable, which may lead to instability or poor performance of CGNNs. Systems with piecewise constant argument (PCA) are a generalization of time-delay systems. Cooke and Wiener introduced the notion of differential equations for PCA (EQPCA) in [35]. The concept of EQPCA was extended in [36–39]. With the development and continuous improvement of EQPCA, PCA systems have attracted the interests of many researchers. Some new stability conditions of PCA systems were obtained in [40–43], and PCA systems have been successfully applied in many fields, such as biomedicine, mechanical engineering, physics, aerodynamic engineering, and other fields.

Successful applications of NNs greatly depend on understanding the intrinsic dynamic behavior and characteristics of NNs. It is necessary to comprehensively and deeply analyze the dynamic behavior of NNs. In the field of control theory, robustness is an essential topic of study. Many authors have conducted in-depth investigations into the robustness of NNs in [8–10, 44–46]. The researchers of [8] investigated the robustness of random disturbances and time delays on the global exponential stability (GES) of recurrent neural networks (RNNs). The robustness of RNNs was described by finding upper bounds for these parameters. Subsequently, Shen et al. studied the robustness of the GES of nonlinear systems with time delays and random disturbances in [44]. The researchers of [9] discussed robustness for connection weight matrices RNNs with time-varying delayed. The researchers of [45] examined the robustness of hybrid stochastic NNs with neutral terms and time-varying delay. The robustness of the GES of nonlinear systems with a deviation argument and SDs was primarily investigated in [10], where the upper bound of the noise intensity and the interval length of deviating function were estimated. Robustness of bidirectional associative memory neural network (BAMNN) with neutral terms and time delays was studied in [46]. It is worth noting that there are many articles on analyzing the stability of CGNNs, but few results on the robustness of CGNNs caused by PCA and SDs.

Based on the above discussion, the aim of this article is to analyze the robustness of the GES of CGNNs with PCA and SDs. In this case, the stability of perturbed CGNNs is generally affected by the strengths of PCA and SDs. If PCA, SDs, or both are small enough, then the disturbed CGNN can still be stable. However, if the interval length of the deviation function or noise intensity exceeds a certain limit, the originally stable CGNN may become unstable. It would be interesting to determine this "certain limit". This paper applies stochastic analysis theory and inequality techniques to establish robustness results of the GES of CGNNs in the presence of PCA and SDs, and directly quantifies the PCA and SD levels of stable systems. That is, we estimate the upper bounds of PCA and SDs by solving transcendental equations, and further characterize the robustness of CGNNs with PCA and

SDs. The following are this paper's primary works:

(1) This paper studies the robustness of the GES of CGNNs with PCA and SDs. The research contents of literature [8, 9] are about the robustness of RNNs, and the CGNN studied in this paper is a more general NN. However, the existing results of the robustness analysis for the GES of CGNNs are not common. Therefore, it is crucial to study the robustness of CGNNs with PCA and SDs.

(2) In this paper, the effects of SDs and PCA on the stability of CGNNs are discussed. Robustness results of CGNNs with PCA and SDs are derived by applying stochastic analysis theory and inequality techniques, and the upper bounds of PCA and SDs are estimated by solving transcendental equations.

(3) The existence of the amplification function of CGNNs provides a challenge for exploring the robustness of the GES for CGNNs in this paper. How to solve the influence of the amplification function on the study of CGNNs is a problem. In previous studies, it is found that we can apply a hypothetical condition to the amplification function. Therefore, the amplification functions are directly replaced by the upper and lower bounds of the amplification function in the operation process of this paper, so as to solve the influence of the amplification function on the CGNNs model.

Finally, based on our findings in this article, if both PCA and SDs are below the upper bounds obtained here, the disturbed CGNNs will remain exponentially stable.

Here is how we structure the rest: Section 2 introduces the system and the preliminaries we need in later. Section 3 to Section 4 give the main results, and we discuss the influence of SDs and PCA on the GES of CGNNs. Finally, some numerical examples are given to prove the validity of the results.

Notations: Let $R^+ = [0, \infty)$ and $N = \{1, 2, ...\}$, where R^n is defined as the n-dimensional Euclidean space. Denote $\|\phi\| = \sum_{i=1}^{n} |\phi_i|$ for a vector ϕ, and for a matrix K, $\|K\| = \max_{1\leq j\leq n} \sum_{i=1}^{n} |k_{ij}|$. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., the filtration is right continuous and contains all P-null sets. The scalar Brownian motion $\omega(t)$ is defined in the probability space, and E stands for the mathematical expectation operator about the probability measure P. Fix two real-value sequences $\{\theta_i\}$, $\{\vartheta_i\}$, $i \in N$, such that $\theta_i < \theta_{i+1}$, $\theta_i < \vartheta_i < \theta_{i+1}$ for all $i \in N$ with $\theta_i \longrightarrow +\infty$ as $i \longrightarrow +\infty$.

2. Model formulation and preliminaries

Consider the CGNN model:

$$\begin{aligned} \dot{e}_i(t) &= d_i(e_i(t))[-c_i(e_i(t)) + \sum_{j=1}^{n} k_{ij} g_j(e_j(t)) + u_i], \qquad (2.1) \\ e(t_0) &= e_0, i = 1, ..., n \end{aligned}$$

where n refers to the number of units, t_0 and e_0 are the initial values of CGNN (2.1), $e(t) = (e_1(t), ..., e_n(t))^T$ is the state of the ith unit at time t, $d_i(\cdot)$ is an amplification function, $c_i(\cdot)$ is an appropriately behaved function that makes the solutions of CGNN (2.1) be bounded, $g_i(\cdot)$ is an activation function, k_{ij} is the connection strength between cell i and j, and u_i is a constant representing the external input.

Assume $e^* = (e_1^*, ..., e_n^*)$ is an equilibrium point of CGNN (2.1), and we translate the equilibrium point to the orign. Let $h(t) = e(t) - e^*$, then the model (2.1) can be converted into:

$$\dot{h}_i(t) = a_i(h_i(t))[-b_i(h_i(t)) + \sum_{j=1}^{n} k_{ij} f_j(h_j(t))], \tag{2.2}$$
$$h(t_0) = h_0, i = 1, 2, ..., n$$

where $h_0 = e_0 - e^*$,$a_i(h_i(t)) = d_i(h_i(t) + e_i^*)$, $b_i(h_i(t)) = c_i(h_i(t) + e_i^*) - c_i(e_i^*)$,$f_i(h_i(t)) = g_i(h_i(t) + e_i^*) - g_i(e_i^*)$. The origin is obviously an equilibrium point of CGNN (2.2). Then, the stability of e^* is the same as the stability for the origin of (2.2).

Next, we give some assumptions:

Assumption 1. Functions $f_i(\cdot)$ satisfy the Lipschitz condition

$$|f_i(h) - f_i(l)| \leq F_i|h - l|, \forall h, l \in R \tag{2.3}$$

with $f(0) = 0$, $i = 1, ...n$, where F_i are known constants.

Assumption 2. The functions $a_i(\cdot)$ are continuous and bounded, and there exist constants $\underline{\mu} > 0$ and $\bar{\mu} > 0$ such that

$$\underline{\mu} \leq a_i(h) \leq \bar{\mu}, \forall h \in R, i = 1, 2, ..., n.$$

Assumption 3. For $b_i(\cdot)$, there exist constants $B_i > 0$, $i = 1, 2, ..., n$ such that

$$\frac{b_i(h) - b_i(l)}{h - l} \leq B_i, \forall h, l \in R, h \neq l.$$

From Assumption 1, for any initial value t_0, h_0, CGNN (2.2) has a unique state $h(t; t_0, h_0)$ on $t \geq t_0$. Now we give the definition for the GES of (2.2).

Definition 1. *CGNN (2.2) is globally exponentially stable if for* $\forall t_0 \in R^+$, $h_0 \in R^n$, $t > t_0$, *there are positive constants* α *and* ν *such that*

$$||h(t; t_0, h_0)|| \leq \alpha ||h(t_0)|| \exp(-\nu(t - t_0))$$

where $h(t; t_0, h_0)$ *is the state of CGNN (2.2).*

3. Effects of random disturbances

Consider the SCGNN perturbed by SDs:

$$\mathrm{d}l_i(t) = \{a_i(l_i(t))[-b_i(l_i(t)) + \sum_{j=1}^{n} k_{ij} f_j(l_j(t))]\}\mathrm{d}t + \sigma l_i(t)\mathrm{d}\omega(t), \tag{3.1}$$
$$l(t_0) = l_0 = h_0 \in R^n, i = 1, 2, ..., n$$

where $a_i(\cdot)$, $b_i(\cdot)$, $f_i(\cdot)$, and k_{ij} are same as in (2.2). σ is the noise intensity, and $\omega(t)$ is a scalar Brownian motion defined in $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$.

Obviously, if Assumption 1 holds, for $\forall t_0 \in R^+$, $h_0 \in R^n$, SCGNN (3.1) has a unique state $l(t; t_0, h_0)$ on $t > t_0$, and $l = 0$ is the equilibrium point of (3.1).

Definition 2. *[47] SCGNN (3.1) is almost surely globally exponentially stable (ASGES), if for* $\forall t_0 \in R^+$, $h_0 \in R^n$, *the Lyapunov exponent*

$$\lim_{t\longrightarrow\infty}\sup\left(\frac{\ln|l(t;t_0,h_0)|}{t}\right) < 0$$

almost surely.

SCGNN (3.1) is mean square globally exponentially stable (MSGES), if for $\forall t_0 \in R^+$, $l_0 \in R^n$, *the Lyapunov exponent*

$$\lim_{t\longrightarrow\infty}\sup\left(\frac{\ln(E|l(t;t_0,l_0)|^2)}{t}\right) < 0$$

where $l(t;t_0,l_0)$ *is the state of SCGNN (3.1).*

Remark 1. *From Definition 2, it is clear that if SCGNN (3.1) is ASGES, then SCGNN (3.1) is MSGES, but not the contrary. It is worth noting that when A1 is true and SCGNN (3.1) is MSGES, then SCGNN (3.1) is also ASGES (see [47]).*

Assumption 4.

$$\begin{aligned}&[16\Delta(2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2+(\bar{\mu}-\underline{\mu})^2\|B\|^2)]\alpha^2/\nu\times\exp\{2\Delta[16\Delta(\bar{\mu}^2\|B\|^2+\bar{\mu}^2\|K\|^2F^2\\&+2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2)]\}+2\alpha^2\exp\{-2\nu\Delta\}<1.\end{aligned}$$

Theorem 1. *Assume A1–A4 holds, and CGNN (2.2) is globally exponentially stable, SCGNN (3.1) is MSGES and also ASGES if* $|\sigma| < \bar{\sigma}$, *where* $\bar{\sigma}$ *is a unique positive solution of the transcendental equation*

$$\begin{aligned}&[16\Delta(2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2+(\bar{\mu}-\underline{\mu})^2\|B\|^2)+4\sigma^2]\alpha^2/\nu\times\exp\{2\Delta[16\Delta(\bar{\mu}^2\|B\|^2\\&+\bar{\mu}^2\|K\|^2F^2+2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2)+4\sigma^2]\}+2\alpha^2\exp\{-2\nu\Delta\}=1\end{aligned}\tag{3.2}$$

and $\triangle > ln(2\alpha^2)/2\nu > 0$.

Proof. Let $h(t;t_0,h_0) \equiv h(t)$, $l(t;t_0,l_0) \equiv l(t)$. From (2.2) and (3.1), for $t \geq t_0$,

$$\begin{aligned}&h_i(t)-l_i(t)\\&=\int_{t_0}^{t}\Big\{a_i(h_i(s))\Big[-b_i(h_i(s))+\sum_{j=1}^{n}k_{ij}f_j(h_j(s))\Big]-a_i(l_i(s))\Big[-b_i(l_i(s))\\&\quad+\sum_{j=1}^{n}k_{ij}f_j(l_j(s))\Big]\Big\}\mathrm{d}s-\int_{t_0}^{t}\sigma l_i(s)\mathrm{d}\omega(s).\end{aligned}$$

When $t \leq t_0 + 2\Delta$, by the Cauchy-Schwarz inequality, A1 and the GES of (2.2),

$$\begin{aligned}&E\|h(t)-l(t)\|^2\\&\leq 2E\sum_{i=1}^{n}\Big|\int_{t_0}^{t}\Big\{a_i(h_i(s))\Big[-b_i(h_i(s))+\sum_{j=1}^{n}k_{ij}f_j(h_j(s))\Big]-a_i(l_i(s))\Big[-b_i(l_i(s))\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}k_{ij}f_j(l_j(s))\Big]\Big\}\mathrm{d}s\Big|^2+2E\sum_{i=1}^{n}\Big|\int_{t_0}^{t}\sigma l_i(s)\mathrm{d}\omega(s)\Big|^2\\
=&2E\sum_{i=1}^{n}\Big|\int_{t_0}^{t}\Big\{[a_i(l_i(s))b_i(l_i(s))-a_i(h_i(s))b_i(h_i(s))]+\sum_{j=1}^{n}k_{ij}[a_i(h_i(s))f_j(h_j(s))\\
&-a_i(l_i(s))f_j(l_j(s))]\Big\}\mathrm{d}s\Big|^2+2E\sum_{i=1}^{n}\Big|\int_{t_0}^{t}\sigma l_i(s)\mathrm{d}\omega(s)\Big|^2\\
\le&2E\sum_{i=1}^{n}\Big|\int_{t_0}^{t}\Big\{[\bar{\mu}b_i(l_i(s))-\underline{\mu}b_i(h_i(s))]+\sum_{j=1}^{n}k_{ij}[\bar{\mu}f_j(h_j(s))-\underline{\mu}f_j(l_j(s))]\Big\}\mathrm{d}s\Big|^2\\
&+2E\sum_{i=1}^{n}\int_{t_0}^{t}|\sigma l_i(s)|^2\mathrm{d}s\\
\le&2E\sum_{i=1}^{n}\Big\{\int_{t_0}^{t}\Big[\bar{\mu}B_i|l_i(s)-h_i(s)|+(\bar{\mu}-\underline{\mu})B_i|h_i(s)|+\sum_{j=1}^{n}|k_{ij}|(\bar{\mu}F_j|h_j(s)-l_j(s)|\\
&+(\bar{\mu}-\underline{\mu})F_j|l_j(s)|)\Big]\mathrm{d}s\Big\}^2+2E\sum_{i=1}^{n}\int_{t_0}^{t}|\sigma l_i(s)|^2\mathrm{d}s\\
\le&16\Delta\Big[\bar{\mu}^2\|B\|^2\int_{t_0}^{t}E\|h(s)-l(s)\|^2\mathrm{d}s+(\bar{\mu}-\underline{\mu})^2\|B\|^2\int_{t_0}^{t}E\|h(s)\|^2\mathrm{d}s+\bar{\mu}^2\|K\|^2F^2\\
&\times\int_{t_0}^{t}E\|h(s)-l(s)\|^2\mathrm{d}s+(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2\int_{t_0}^{t}E\|l(s)\|^2\mathrm{d}s\Big]+4\sigma^2\int_{t_0}^{t}E\|h(s)\\
&-l(s)\|^2\mathrm{d}s+4\sigma^2\int_{t_0}^{t}E\|h(s)\|^2\mathrm{d}s\\
\le&16\Delta\Big[(\bar{\mu}^2\|B\|^2+\bar{\mu}^2\|K\|^2F^2)\int_{t_0}^{t}E\|h(s)-l(s)\|^2\mathrm{d}s+2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2\int_{t_0}^{t}E\|h(s)\\
&-l(s)\|^2\mathrm{d}s+2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2\int_{t_0}^{t}E\|h(s)\|^2\mathrm{d}s+(\bar{\mu}-\underline{\mu})^2\|B\|^2\int_{t_0}^{t}E\|h(s)\|^2\mathrm{d}s\Big]\\
&+4\sigma^2\int_{t_0}^{t}E\|h(s)-l(s)\|^2\mathrm{d}s+4\sigma^2\int_{t_0}^{t}E\|h(s)\|^2\mathrm{d}s\\
\le&\Big[16\Delta(\bar{\mu}^2\|B\|^2+\bar{\mu}^2\|K\|^2F^2+2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2)+4\sigma^2\Big]\int_{t_0}^{t}E\|h(s)-l(s)\|^2\mathrm{d}s\\
&+\Big[16\Delta(2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2+(\bar{\mu}-\underline{\mu})^2\|B\|^2)+4\sigma^2\Big]\int_{t_0}^{t}E\|h(s)\|^2\mathrm{d}s\\
\le&\Big[16\Delta(\bar{\mu}^2\|B\|^2+\bar{\mu}^2\|K\|^2F^2+2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2)+4\sigma^2\Big]\int_{t_0}^{t}E\|h(s)-l(s)\|^2\mathrm{d}s\\
&+\Big[8\Delta(2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2+(\bar{\mu}-\underline{\mu})^2\|B\|^2)+2\sigma^2\Big]\alpha^2\|h(t_0)\|^2/\nu.
\end{aligned}
$$

When $t_0+\Delta\le t\le t_0+2\Delta$, from the Gronwall inequality, we get

$$E\|h(t)-l(t)\|^2$$

$$\leq\Big[8\Delta(2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2+(\bar{\mu}-\underline{\mu})^2\|B\|^2)+2\sigma^2\Big]\alpha^2\|h(t_0)\|^2/\nu\exp\{[16\Delta(\bar{\mu}^2\|B\|^2$$
$$+\bar{\mu}^2\|K\|^2F^2+2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2)+4\sigma^2](t-t_0)\}$$
$$\leq\Big[8\Delta(2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2+(\bar{\mu}-\underline{\mu})^2\|B\|^2)+2\sigma^2\Big]\alpha^2/\nu\exp\{2\Delta[16\Delta(\bar{\mu}^2\|B\|^2$$
$$+\bar{\mu}^2\|K\|^2F^2+2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2)+4\sigma^2]\}\sup_{t_0\leq s\leq t_0+\Delta}E\|l(s)\|^2 \tag{3.3}$$

and from (3.3) and the GES of (2.2), we obtain

$$E\|l(t)\|^2$$
$$\leq\Big[16\Delta(2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2+(\bar{\mu}-\underline{\mu})^2\|B\|^2)+4\sigma^2\Big]\alpha^2/\nu\exp\{2\Delta[16\Delta(\bar{\mu}^2\|B\|^2+\bar{\mu}^2\|K\|^2F^2$$
$$+2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2)+4\sigma^2]\}\sup_{t_0\leq s\leq t_0+\Delta}E\|l(s)\|^2+2\alpha^2\|h(t_0)\|^2\exp\{-2\nu(t-t_0)\}$$
$$\leq\Big\{[16\Delta(2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2+(\bar{\mu}-\underline{\mu})^2\|B\|^2)+4\sigma^2]\alpha^2/\nu\exp\{2\Delta[16\Delta(\bar{\mu}^2\|B\|^2+\bar{\mu}^2\|K\|^2F^2$$
$$+2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2)+4\sigma^2]\}+2\alpha^2\exp\{-2\nu\Delta\}\Big\}\sup_{t_0\leq s\leq t_0+\Delta}E\|l(s)\|^2. \tag{3.4}$$

Let $Q(\sigma)=[16\Delta(2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2+(\bar{\mu}-\underline{\mu})^2\|B\|^2)+4\sigma^2]\alpha^2/\nu\exp\{2\Delta[16\Delta(\bar{\mu}^2\|B\|^2+\bar{\mu}^2\|K\|^2F^2+2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2)+4\sigma^2]\}+2\alpha^2\exp\{-2\nu\Delta\}$. It is easy to deduce that $Q(\sigma)$ is strictly increasing for σ. According to A4, we get $Q(0)<1$, so there must be a positive constant $\bar{\sigma}$ such that $Q(\bar{\sigma})=1$. That is, there exists a positive constant $\sigma\in(0,\bar{\sigma})$ such that $Q(\sigma)<1$. Then, for $|\sigma|\leq\bar{\sigma}$, we get

$$[16\Delta(2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2+(\bar{\mu}-\underline{\mu})^2\|B\|^2)+4\sigma^2]\alpha^2/\nu\exp\{2\Delta[16\Delta(\bar{\mu}^2\|B\|^2$$
$$+\bar{\mu}^2\|K\|^2F^2+2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2)+4\sigma^2]\}+2\alpha^2\exp\{-2\nu\Delta\}<1.$$

Let

$$\varphi=-\ln\Big\{[16\Delta(2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2+(\bar{\mu}-\underline{\mu})^2\|B\|^2)+4\sigma^2]\alpha^2/\nu\times\exp\{2\Delta[16\Delta(\bar{\mu}^2\|B\|^2$$
$$+\bar{\mu}^2\|K\|^2F^2+2(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2)+4\sigma^2]\}+2\alpha^2\exp\{-2\nu\Delta\}\Big\}/\Delta$$

so $\varphi>0$, and from (3.4), we have

$$\sup_{t_0+\Delta\leq t\leq t_0+2\Delta}E\|l(t)\|^2\leq\exp(-\varphi\Delta)\Big(\sup_{t_0\leq t\leq t_0+\Delta}E\|l(t)\|^2\Big). \tag{3.5}$$

From the existence and uniqueness of the state of SCGNN (3.1), for any $\varpi=1,...,n$, when $t\geq t_0+(\varpi-1)\Delta$,

$$l(t;t_0,h_0)=l(t;t_0+(\varpi-1)\Delta,l(t_0+(\varpi-1)\Delta;t_0,h_0)). \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$\sup_{t_0+\varpi\Delta\leq t\leq t_0+(\varpi+1)\Delta}E\|l(t)\|^2$$

$$
\begin{aligned}
&\leq \exp(-\varphi\Delta) \sup_{t_0+(\varpi-1)\Delta\leq t\leq t_0+(\varpi-1)\Delta+\Delta} E\|l(t;t_0,h_0)\|^2 \\
&\leq ... \\
&\leq \exp(-\varphi\varpi\Delta) \sup_{t_0\leq t\leq t_0+\Delta} E\|l(t;t_0,h_0)\|^2 \\
&= \tau\exp(-\varphi\varpi\Delta)
\end{aligned}
$$

where $\tau = \sup_{t_0\leq t\leq t_0+\Delta} E\|l(t;t_0,h_0)\|^2$. Hence, for all $t > t_0 + \Delta$, there exists an integer $\varpi > 0$ such that $t_0 + \varpi\Delta \leq t \leq t_0 + (\varpi + 1)\Delta$, and then

$$E\|l(t;t_0,h_0)\|^2 \leq \tau\exp(-\varphi t + \varphi t_0 + \varphi\Delta) = (\tau\exp(\varphi\Delta))\exp(-\varphi(t - t_0)).$$

For $t_0 \leq t \leq t_0 + \Delta$, the above formula also holds. So, SCGNN (3.1) is MSGES and also ASGES.

□

4. Effects of piecewise constant argument

Consider the model of CGNN with PCA:

$$
\begin{aligned}
&\dot{l}_i(t) = a_i(l_i(t))[-b_i(l_i(t)) + \sum_{j=1}^{n} k_{ij}f_j(l_j(t)) + \sum_{j=1}^{n} w_{ij}f_j(l_j(\varrho(t)))], \\
&l(t_0) = l_0 = h_0, i = 1, ..., n
\end{aligned}
\tag{4.1}
$$

where $\varrho(t) = \vartheta_k$, when $t \in [\theta_k, \theta_{k+1}]$, $k \in N$. $\varrho(t)$ is an identification function, $a_i(\cdot)$, $b_i(\cdot)$, k_{ij}, $f_j(\cdot)$ are as the same as in (2.2).

Model (4.1) is a hybrid system. Fix $k \in N$, and on the interval $[\theta_k, \theta_{k+1})$, if $\theta_k \leq t < \vartheta_k$ holds for the argument t, i.e., $t < \varrho(t)$, then (4.1) is an advanced system. Similarly, if $\vartheta_k \leq t < \theta_{k+1}$, (4.1) is a delayed system. System (4.1) is deviated if the argument is advanced or delayed.

For each initial conditions t_0 and l_0, (4.1) has a unique state $l(t;t_0,l_0)$ and clearly has the trivial state $l = 0$.

The model of the CGNN without PCA is as follows:

$$
\begin{aligned}
&\dot{h}_i(t) = a_i(h_i(t))[-b_i(h_i(t)) + \sum_{j=1}^{n} k_{ij}f_j(h_j(t)) + \sum_{j=1}^{n} w_{ij}f_j(h_j(t))], \\
&h(t_0) = h_0, i = 1, 2, ..., n.
\end{aligned}
\tag{4.2}
$$

Assumption 5. There exists a constant $\theta > 0$ such that $\theta_{k+1} - \theta_k \leq \theta$, $k \in N$.
Assumption 6. $\theta[\bar{\mu}F\|W\| + (\bar{\mu}\|B\| + \bar{\mu}F\|K\|)(1 + \theta\bar{\mu}F\|W|)\exp(\theta(\bar{\mu}\|B\| + \bar{\mu}F\|K\|))] < 1$.
Assumption 7.

$$
\begin{aligned}
&\alpha\exp(-\nu\Delta) + [(\bar{\mu} - \underline{\mu})\|B\| + (\bar{\mu} - \underline{\mu})\|K\|F + 2\bar{\mu}\|W\|F]\alpha/\nu\exp\{2\Delta[\bar{\mu}\|B\| + 2\bar{\mu}\|K\|F \\
&+ 3\bar{\mu}\|W\|F - \underline{\mu}\|K\|F]\} < 1.
\end{aligned}
$$

Lemma 1. *Let A1–A5 hold, and $l(t)$ be a solution of system (4.1). For all $t \in R^+$, the inequality*

$$\|l(\varrho(t))\| \leq \lambda\|l(t)\| \tag{4.3}$$

holds, where $\lambda = \left\{1 - \theta[\bar{\mu}F\|W\| + (\bar{\mu}\|B\| + \bar{\mu}F\|K\|)(1 + \theta\bar{\mu}F\|W\|)\exp(\theta(\bar{\mu}\|B\| + \bar{\mu}F\|K\|))]\right\}^{-1}$.

Proof. Fix $k \in N$, for any $t \in [\theta_k, \theta_{k+1})$, we have

$$l_i(t) = l_i(\vartheta_k) + \int_{\vartheta_k}^{t} a_i(l_i(s))[-b_i(l_i(s)) + \sum_{j=1}^{n} k_{ij} f_j(l_j(s)) + \sum_{j=1}^{n} w_{ij} f_j(l_j(\vartheta_k))]\mathrm{d}s.$$

From A1–A4, both sides take absolute values, and adding them together, we have

$$\begin{aligned}\|l(t)\| \le & \|l(\vartheta_k)\| + \sum_{i=1}^{n} \left| \int_{\vartheta_k}^{t} a_i(l_i(s))[-b_i(l_i(s)) + \sum_{j=1}^{n} k_{ij} f_j(l_j(s)) + \sum_{j=1}^{n} w_{ij} f_j(l_j(\vartheta_k))]\mathrm{d}s \right| \\ \le & \|l(\vartheta_k)\| + \bar{\mu}\|B\| \int_{\vartheta_k}^{t} \|l(s)\|\mathrm{d}s + \bar{\mu}F\|K\| \int_{\vartheta_k}^{t} \|l(s)\|\mathrm{d}s + \bar{\mu}F\|W\| \int_{\vartheta_k}^{t} \|l(\vartheta_k)\|\mathrm{d}s \\ \le & \|l(\vartheta_k)\| + (\bar{\mu}\|B\| + \bar{\mu}F\|K\|) \int_{\vartheta_k}^{t} \|l(s)\|\mathrm{d}s + \theta\bar{\mu}\|W\|F\|l(\vartheta_k)\| \\ = & (1 + \theta\bar{\mu}F\|W\|)\|l(\vartheta_k)\| + (\bar{\mu}\|B\| + \bar{\mu}F\|K\|) \int_{\vartheta_k}^{t} \|l(s)\|\mathrm{d}s\end{aligned}$$

and from the Gronwall inequality, we obtain

$$\|l(t)\| \le (1 + \theta\bar{\mu}F\|W\|)\|l(\vartheta_k)\| \exp\{\theta(\bar{\mu}\|B\| + \bar{\mu}F\|K\|)\}.$$

Similarly, for $t \in [\theta_k, \theta_{k+1})$, we have

$$\|l(\vartheta_k)\| \le \|l(t)\| + \bar{\mu}F\|W\| \int_{\vartheta_k}^{t} \|l(\vartheta_k)\|\mathrm{d}s + \theta(\bar{\mu}\|B\| + \bar{\mu}F\|K\|)\|l(s)\|.$$

That is,

$$\begin{aligned}\|l(\vartheta_k)\| \le & \|l(t)\| + \theta[\bar{\mu}F\|W\| + (\bar{\mu}\|B\| + \bar{\mu}F\|K\|)(1 + \theta\bar{\mu}F\|W\|) \exp\{\theta(\bar{\mu}\|B\| \\ & + \bar{\mu}F\|K\|)\}]\|l(\vartheta_k)\|.\end{aligned}$$

For $t \in [\theta_k, \theta_{k+1})$,

$$\begin{aligned}\|l(\vartheta_k)| \le & \{1 - \theta[\bar{\mu}F\|W\| + (\bar{\mu}\|B\| + \bar{\mu}F\|K\|)(1 + \theta\bar{\mu}F\|W|) \exp(\theta(\bar{\mu}\|B\| \\ & + \bar{\mu}F\|K\|))]\}^{-1}\|l(t)\|.\end{aligned}$$

The above formula holds for all $t \in R^+$ due to the arbitrariness of t and k.

□

We discuss the robustness for the GES of CGNNs with PCA in Theorem 2.

Theorem 2. *Suppose A1–A7 hold, and the CGNN (4.2) is globally exponentially stable. CGNN (4.1) is globally exponentially stable if* $\theta < \min(\frac{\Delta}{2}, \bar{\theta}, \bar{\bar{\theta}})$, *where* $\bar{\theta}$ *is the unique positive solution* $\hat{x}$ *of Eq (4.4):*

$$\begin{aligned}&\alpha \exp(-\nu(\Delta - \hat{x})) + [(\bar{\mu} - \underline{\mu})\|B\| + (\bar{\mu} - \underline{\mu})\|K\|F + \bar{\mu}\|W\|F + \bar{\mu}\|W\|F(1 - \hat{x}(\bar{\mu}F\|W\| + (\bar{\mu}\|B\| \\ &+ \bar{\mu}F\|K\|)(1 + \hat{x}\bar{\mu}F\|W\|) \exp\{\hat{x}(\bar{\mu}\|B\| + \bar{\mu}F\|K\|)\}))]\alpha/\nu \exp\{2\Delta[\bar{\mu}\|B\| + 2\bar{\mu}\|K\|F \\ &+ 2\bar{\mu}\|W\|F - \underline{\mu}\|K\|F + \bar{\mu}\|W\|F(1 - \hat{x}(\bar{\mu}F\|W\| + (\bar{\mu}\|B\| + \bar{\mu}F\|K\|)(1 + \hat{x}\bar{\mu}F\|W\|)\end{aligned}$$

$$\times \exp\{\hat{x}(\bar{\mu}\|B\| + \bar{\mu}F\|K\|)\}))]\} = 1, \tag{4.4}$$

and $\bar{\bar{\theta}}$ *is a unique positive solution* $\check{x}$ *of Eq (4.5)*

$$\check{x}[\bar{\mu}F\|W\| + (\bar{\mu}\|B\| + \bar{\mu}F\|K\|)(1 + \check{x}\bar{\mu}F\|W\|)\exp\{(\bar{\mu}\|B\| + \bar{\mu}F\|K\|)\check{x}\}] = 1 \tag{4.5}$$

and $\Delta > \frac{ln(\alpha)}{\nu} > 0$.

Proof. For convenience, we write $h(t; t_0, h_0) \equiv h(t)$, $l(t; t_0, l_0) \equiv l(t)$.

Combined with (4.1), (4.2), and Lemma 1, $\forall t \geq t_0 > 0$,

$$\begin{aligned}
&\|h(t) - l(t)\| \\
\leq &\sum_{i=1}^{n} \Big| \int_{t_0}^{t} \Big\{ (a_i(l_i(s))b_i(l_i(s)) - a_i(h_i(s))b_i(h_i(s))) + \sum_{j=1}^{n} k_{ij}[a_i(h_i(s))f_j(h_j(s)) - a_i(l_i(s))f_j(l_j(s))] \\
&+ \sum_{j=1}^{n} w_{ij}[a_i(h_i(s))f_j(h_j(s)) - a_i(l_i(s))f_j(l_j(\varrho(s)))] \Big\} \mathrm{d}s \Big| \\
\leq &\sum_{i=1}^{n} \Big| \int_{t_0}^{t} \Big\{ (\bar{\mu}b_i(l_i(s)) - \underline{\mu}b_i(h_i(s))) + \sum_{j=1}^{n} k_{ij}[\bar{\mu}f_j(h_j(s)) - \underline{\mu}f_j(l_j(s))] + \sum_{j=1}^{n} w_{ij}[\bar{\mu}f_j(h_j(s)) \\
&- \underline{\mu}f_j(l_j(\varrho(s)))] \Big\} \mathrm{d}s \Big| \\
\leq &\sum_{i=1}^{n} \int_{t_0}^{t} \Big\{ \bar{\mu}B_i|l_i(s) - h_i(s)| + (\bar{\mu} - \underline{\mu})B_i|h_i(s)| + \sum_{j=1}^{n} |k_{ij}|\bar{\mu}F_j|h_j(s) - l_j(s)| + \sum_{j=1}^{n} |k_{ij}|(\bar{\mu} - \underline{\mu})F_j|l_j(s)| \\
&+ \sum_{j=1}^{n} |w_{ij}|\bar{\mu}F_j|h_j(s) - l_j(s)| + \sum_{j=1}^{n} |w_{ij}|\bar{\mu}F_j|l_j(s)| + \sum_{j=1}^{n} |w_{ij}|\bar{\mu}F_j|l_j(\varrho(s))| \Big\} \mathrm{d}s \\
\leq &(\bar{\mu}\|B\| + \bar{\mu}\|K\|F + \bar{\mu}\|W\|F) \int_{t_0}^{t} \|h(s) - l(s)\| \mathrm{d}s + (\bar{\mu} - \underline{\mu})\|B\| \int_{t_0}^{t} \|h(s)\| \mathrm{d}s \\
&+ [(\bar{\mu} - \underline{\mu})\|K\|F + \bar{\mu}\|W\|F + \bar{\mu}\lambda\|W\|F] \int_{t_0}^{t} \|l(s)\| \mathrm{d}s \\
\leq &[\bar{\mu}\|B\| + 2\bar{\mu}\|K\|F + 2\bar{\mu}\|W\|F - \underline{\mu}\|K\|F + \bar{\mu}\lambda\|W\|F] \int_{t_0}^{t} \|h(s) - l(s)\| \mathrm{d}s \\
&+ [(\bar{\mu} - \underline{\mu})\|B\| + (\bar{\mu} - \underline{\mu})\|K\|F + \bar{\mu}\|W\|F + \bar{\mu}\lambda\|W\|F] \int_{t_0}^{t} \|h(s)\| \mathrm{d}s.
\end{aligned}$$

In view of the GES of (4.2), for any $t \geq t_0 \geq 0$,

$$\begin{aligned}
&\|h(t) - l(t)\| \\
\leq &[\bar{\mu}\|B\| + 2\bar{\mu}\|K\|F + 2\bar{\mu}\|W\|F - \underline{\mu}|K\|F + \bar{\mu}\lambda\|W\|F] \int_{t_0}^{t} \|h(s) - l(s)\| \mathrm{d}s + [(\bar{\mu} - \underline{\mu})\|B\| \\
&+ (\bar{\mu} - \underline{\mu})\|K\|F + \bar{\mu}\|W\|F + \bar{\mu}\lambda\|W\|F]\alpha\|h(t_0)\|/\nu.
\end{aligned} \tag{4.6}$$

Applying the Gronwall-Bellman lemma to (4.6), for $t_0 + \theta \leq t \leq t_0 + 2\Delta$,

$$\|h(t) - l(t)\|$$

$$\leq [(\bar{\mu}-\underline{\mu})\|B\| + (\bar{\mu}-\underline{\mu})\|K\|F + \bar{\mu}\|W\|F + \bar{\mu}\lambda\|W\|F]\alpha\|h(t_0)\|/\nu \times \exp\{2\Delta[\bar{\mu}\|B\| + 2\bar{\mu}\|K\|F + 2\bar{\mu}\|W\|F - \underline{\mu}\|K\|F + \bar{\mu}\lambda\|W\|F]\}$$

then

$$\begin{aligned}&\|l(t)\| \\ \leq &\|h(t)\| + [(\bar{\mu}-\underline{\mu})\|B\| + (\bar{\mu}-\underline{\mu})\|K\|F + \bar{\mu}\|W\|F + \bar{\mu}\lambda\|W\|F] \times \alpha\|h(t_0)\|/\nu \times \exp\{2\Delta[\bar{\mu}\|B\| \\ &+ 2\bar{\mu}\|K\|F + 2\bar{\mu}\|W\|F - \underline{\mu}\|K\|F + \bar{\mu}\lambda\|W\|F]\}. \end{aligned} \tag{4.7}$$

Note that $\theta < \min\{\frac{\Delta}{2}, \bar{\theta}\}$, and from the GES of (4.2) and (4.7), for $t_0 - \theta + \Delta \leq t \leq t_0 - \theta + 2\Delta$, we have

$$\begin{aligned}&\|l(t)\| \\ \leq &\alpha\|h(t_0)\|\exp\{-\nu(t-t_0)\} + [(\bar{\mu}-\underline{\mu})\|B\| + (\bar{\mu}-\underline{\mu})\|K\|F + \bar{\mu}\|W\|F + \bar{\mu}\lambda\|W\|F]\alpha\|h(t_0)\|/\nu \\ &\times \exp\{2\Delta[\bar{\mu}\|B\| + 2\bar{\mu}\|K\|F + 2\bar{\mu}\|W\|F - \underline{\mu}\|K\|F + \bar{\mu}\lambda\|W\|F]\} \\ \leq &\Big\{\alpha\exp\{-\nu(\Delta-\theta)\} + [(\bar{\mu}-\underline{\mu})\|B\| + (\bar{\mu}-\underline{\mu})\|K\|F + \bar{\mu}\|W\|F + \bar{\mu}\lambda\|W\|F]\alpha/\nu \\ &\times \exp\{2\Delta[\bar{\mu}\|B\| + 2\bar{\mu}\|K\|F + 2\bar{\mu}\|W\|F - \underline{\mu}\|K\|F + \bar{\mu}\lambda\|W\|F]\}\Big\}\|l_0\|.\end{aligned}$$

Let $P(\theta) = \alpha\exp\{-\nu(\Delta-\theta)\} + [(\bar{\mu}-\underline{\mu})\|B\| + (\bar{\mu}-\underline{\mu})\|K\|F + \bar{\mu}\|W\|F + \bar{\mu}\lambda\|W\|F]\alpha/\nu\exp\{2\Delta[\bar{\mu}\|B\| + 2\bar{\mu}\|K\|F + 2\bar{\mu}\|W\|F - \underline{\mu}\|K\|F + \bar{\mu}\lambda\|W\|F]\}$, $R(\theta) = \theta[\bar{\mu}F\|W\| + (\bar{\mu}\|B\| + \bar{\mu}F\|K\|)(1+\theta\bar{\mu}F\|W\|)\exp\{(\bar{\mu}\|B\| + \bar{\mu}F\|K\|)\theta\}]$. Obviously, $R(\theta)$ is strictly increasing for θ, so there must exist $\bar{\theta}$ such that $R(\bar{\theta}) = 1$. In addition, $P(\theta)$ is also strictly increasing for θ, and from A7 we have $P(0) < 1$. Thus, there must be a positive constant $\bar{\bar{\theta}}$ such that $P(\bar{\bar{\theta}}) = 1$. Since $P(\theta)$ is also increasing for θ on the interval $(0, \bar{\theta})$, therefore, we can know that $P(\theta) < 1$ when $\theta < \bar{\theta}$. That is, we know that $P(\theta) < 1$ when $\theta < \min\{\frac{\Delta}{2}, \bar{\theta}, \bar{\bar{\theta}}\}$.

Letting

$$\begin{aligned}\iota = -\ln\Big\{&\alpha\exp\{-\nu(\Delta-\theta)\} + [(\bar{\mu}-\underline{\mu})\|B\| + (\bar{\mu}-\underline{\mu})\|K\|F + \bar{\mu}\|W\|F + \bar{\mu}\lambda\|W\|F]\alpha/\nu \\ &\times \exp\{2\Delta[\bar{\mu}\|B\| + 2\bar{\mu}\|K\|F + 2\bar{\mu}\|W\|F - \underline{\mu}\|K\|F + \bar{\mu}\lambda\|W\|F]\}\Big\}/\Delta\end{aligned}$$

we get

$$\|l(t)\| \leq \exp\{-\iota\Delta\}\|l_0\|. \tag{4.8}$$

Considering the uniqueness of the solution of CGNN (4.2), for a positive integer β, we have

$$l(t; t_0, l_0) = l(t; t_0 + (\beta-1)\Delta, l(t_0 + (\beta-1)\Delta; t_0, l_0)). \tag{4.9}$$

Then, from (4.8) and (4.9), for $t \geq t_0 - \theta + \beta\Delta$,

$$\begin{aligned}&\|l(t; t_0, l_0)\| \\ &\leq \exp\{-\iota\Delta\}\|l(t_0 + (\beta-1)\Delta; t_0, l_0)\|\end{aligned}$$

$$\leq ...$$
$$\leq \exp\{-\beta\iota\Delta\}||l_0||$$

and so, for any $t > t_0 - \theta + \Delta$, there exists an integer $\beta > 0$ such that $t_0 - \theta + (\beta - 1)\Delta \leq t \leq t_0 - \theta + \beta\Delta$,

$$||l(t; t_0, l_0)|| \leq \exp\{-\iota(t - t_0)\} \exp\{\iota(\Delta - \theta)\}||l_0||. \tag{4.10}$$

Clearly, (4.10) also holds for $t_0 \leq t \leq t_0 - \theta + \Delta$. That is, CGNN (4.1) is globally exponentially stable.

□

Remark 2. *Compared to other systems, systems with PCA are considered a hybrid system. Through addressing the transcendental equation, we can derive an upper bound on the interval length of PCA. If the interval length of PCA is less than* $\min(\Delta/2, \bar{\theta}, \bar{\bar{\theta}})$*, then the perturbed CGNN can maintain a stable state.*

5. Effects of piecewise constant argument and random disturbances

Consider the impacts of both SDs and PCA together on the GES of CGNNs:

$$\begin{aligned} \mathrm{d}l_i(t) =& \{a_i(l_i(t))[-b_i(l_i(t)) + \sum_{j=1}^{n} k_{ij} f_j(l_j(t)) + \sum_{j=1}^{n} w_{ij} f_j(l_j(\varrho(t)))]\}\mathrm{d}t + \sigma l_i(t)\mathrm{d}\omega(t), \\ l(t_0) =& l_0, t \geq t_0 \geq 0, i = 1, 2, ..., n \end{aligned} \tag{5.1}$$

where $\varrho(t) = \vartheta_k$, when $t \in [\theta_k, \theta_{k+1}]$, $k \in N$. $\varrho(t)$ is an identification function, σ is the noise intensity, and $a_i(\cdot)$, $b_i(\cdot)$, k_{ij} and $f_j(\cdot)$ are as the same as in (2.2).

SPCGNN (5.1) is a hybrid system in a stochastic environment. Fix $k \in N$, and on the interval $[\theta_k, \theta_{k+1})$, if $\theta_k \leq t < \vartheta_k$ holds for argument t, (5.1) is an advanced system. Similarly, if $\vartheta_k \leq t < \theta_{k+1}$, SPCGNN (5.1) is a delayed system.

SPCGNN (5.1) can be viewed as the perturbed system of the following model:

$$\begin{aligned} \mathrm{d}h_i(t) =& \{a_i(h_i(t))[-b_i(h_i(t)) + \sum_{j=1}^{n} k_{ij} f_j(h_j(t)) + \sum_{j=1}^{n} w_{ij} f_j(h_j(t))]\}\mathrm{d}t, \\ h(t_0) =& h_0 = l_0, t \geq t_0 \geq 0, i = 1, 2, ..., n. \end{aligned} \tag{5.2}$$

It is evident that $l = 0$ is a trivial state of SPCGNN (5.1) and $h = 0$ is a trivial state of CGNN (5.2). We suppose that SPCGNN (5.1) has a unique state $l(t; t_0, l_0)$ for initial conditions t_0 and l_0 .

Now, the stability definition of SPCGNN (5.1) is described as follows.

Definition 3. *SPCGNN (5.1) is MSGES if, for any* $t_0 \in R^+, l_0 \in R^n$*, there exist constants* $\alpha > 0$ *and* $\nu > 0$ *such that*

$$E||l(t; t_0, l_0)||^2 \leq \alpha||l_0||^2 \exp(-\nu(t - t_0)), \forall t \geq t_0 \geq 0.$$

Assumption 8. $9\theta^2\bar{\mu}^2F^2||W||^2 + 9\theta(3\theta\bar{\mu}^2||B||^2 + 3\theta\bar{\mu}^2F^2||K||^2 + \sigma^2)(1 + 3\theta^2\bar{\mu}^2F^2||W||^2) \times \exp\{3\theta(3\theta\bar{\mu}^2||B||^2 + 3\theta\bar{\mu}^2F^2||K||^2 + \sigma^2)\} < 1.$

Assumption 9.

$$2\alpha\exp\{-\nu\Delta\} + 4\Delta[24\Delta((\bar{\mu}-\underline{\mu})^2\|B\|^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\|W\|^2F^2 + 8\bar{\mu}^2\|W\|^2F^2)]\alpha/\nu$$
$$\times\exp\{2\Delta[24\Delta(\bar{\mu}^2\|B\|^2 + \bar{\mu}^2\|K\|^2F^2 + 3\bar{\mu}^2\|W\|^2F^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 8\bar{\mu}^2\|W\|^2F^2)]\} < 1.$$

Lemma 2. *Let A1–A6 hold, and let $l(t)$ be a solution of SPCGNN (5.1). For all $t \in R^+$, the inequality*

$$E\|l(\varrho(t))\|^2 \le \bar{\lambda}E\|l(t)\|^2 \tag{5.3}$$

holds, where $\bar{\lambda} = 3\Big\{1 - [9\theta^2\bar{\mu}^2F^2\|W\|^2 + 9\theta(3\theta\bar{\mu}^2\|B\|^2 + 3\theta\bar{\mu}^2F^2\|K\|^2 + \sigma^2)(1 + 3\theta^2\bar{\mu}^2F^2\|W\|^2)\exp(3\theta(3\theta\bar{\mu}^2\|B\|^2 + 3\theta\bar{\mu}^2F^2\|K\|^2 + \sigma^2))]\Big\}^{-1}$.

Proof. Fix $t \in R^+$, $k \in N$ such that $t \in [\theta_k, \theta_{k+1})$, $\varrho(t) = \vartheta_k$,

$$\begin{aligned}
&E\|l(t)\|^2\\
\le&3E\|l(\vartheta_k)\|^2 + 3\theta E\sum_{i=1}^{n}\int_{\vartheta_k}^{t}\Big|\bar{\mu}[-b_i(l_i(s)) + \sum_{j=1}^{n}k_{ij}f_j(l_j(s)) + \sum_{j=1}^{n}w_{ij}f_j(l_j(\vartheta_k))]\Big|^2\mathrm{d}s\\
&+3E\sum_{i=1}^{n}\Big|\int_{\vartheta_k}^{t}\sigma_i l_i(s)\mathrm{d}\omega(s)\Big|^2\\
\le&3E\|l(\vartheta_k)\|^2 + 9\theta\bar{\mu}^2\|B\|^2\int_{\vartheta_k}^{t}E\|l(s)\|^2\mathrm{d}s + 9\theta\bar{\mu}^2\|K\|^2F^2\int_{\vartheta_k}^{t}E\|l(s)\|^2\mathrm{d}s\\
&+9\theta^2\bar{\mu}\|W\|^2F^2E\|l(\vartheta_k)\|^2 + 3\sigma^2\int_{\vartheta_k}^{t}E\|l(s)\|^2\mathrm{d}s\\
=&3(1 + 3\theta^2\bar{\mu}^2\|W\|^2F^2)E\|l(\vartheta_k)\|^2 + 3(3\theta\bar{\mu}^2\|B\|^2 + 3\theta\bar{\mu}^2\|K\|^2F^2 + \sigma^2)\int_{\vartheta_k}^{t}E\|l(s)\|^2\mathrm{d}s.
\end{aligned}\tag{5.4}$$

From the Gronwall-Bellman lemma, we have

$$E\|l(t)\|^2 \le 3(1 + 3\theta^2\bar{\mu}^2\|W\|^2F^2)E\|l(\vartheta_k)\|^2\exp\{3\theta(3\theta\bar{\mu}^2\|B\|^2 + 3\theta\bar{\mu}^2\|K\|^2F^2 + \sigma^2)\}. \tag{5.5}$$

Similary, for $t \in [\theta_k, \theta_{k+1})$, from (5.4), we have

$$\begin{aligned}
E\|l(\vartheta_k)\|^2 \le&3E\|l(t)\|^2 + 9\theta^2\bar{\mu}^2\|W\|^2F^2E\|l(\vartheta_k)\|^2 + 3(3\theta\bar{\mu}^2\|B\|^2 + 3\theta\bar{\mu}^2\|K\|^2F^2\\
&+\sigma^2)\int_{\vartheta_k}^{t}E\|l(s)\|^2\mathrm{d}s\\
\le&3E\|l(t)\|^2 + 9\theta^2\bar{\mu}^2\|W\|^2F^2E\|l(\vartheta_k)\|^2 + 9\theta(3\theta\bar{\mu}^2\|B\|^2 + 3\theta\bar{\mu}^2\|K\|^2F^2 + \sigma^2)\\
&\times(1 + 3\theta^2\bar{\mu}^2\|W\|^2F^2)\exp\{3\theta(3\theta\bar{\mu}^2\|B\|^2 + 3\theta\bar{\mu}^2\|K\|^2F^2 + \sigma^2)\}E\|l(\vartheta_k)\|^2\\
=&3E\|l(t)\|^2 + [9\theta^2\bar{\mu}^2F^2\|W\|^2 + 9\theta(3\theta\bar{\mu}^2\|B\|^2 + 3\theta\bar{\mu}^2\|K\|^2F^2 + \sigma^2)\\
&\times(1 + 3\theta^2\bar{\mu}^2F^2\|W\|^2)\exp\{3\theta(3\theta\bar{\mu}^2\|B\|^2 + 3\theta\bar{\mu}^2\|K\|^2F^2 + \sigma^2)\}]E\|l(\vartheta_k)\|^2\\
=&3E\|l(t)\|^2 + \varpi E\|l(\vartheta_k)\|^2,
\end{aligned}$$

where $\varpi = 9\theta^2\bar{\mu}^2F^2\|W\|^2 + 9\theta(3\theta\bar{\mu}^2\|B\|^2 + 3\theta\bar{\mu}^2\|K\|^2F^2 + \sigma^2)(1 + 3\theta^2\bar{\mu}^2F^2\|W\|^2)\exp\{3\theta(3\theta\bar{\mu}^2\|B\|^2 + 3\theta\bar{\mu}^2\|K\|^2F^2 + \sigma^2)\}$.

From A6, for $t \in [\theta_k, \theta_{k+1})$,

$$E\|l(\varrho(t))\|^2 \leq \frac{3}{1-\varpi}E\|l(t)\|^2 = \bar{\lambda}E\|l(t)\|^2,$$

where $\bar{\lambda} = \frac{3}{1-\varpi}$.

For all $t \in R^+$, (5.3) holds due to the arbitrariness of t and k. □

Theorem 3. *Let A1–A9 hold, and CGNN (5.2) be globally exponentially stable. SPCGNN (5.1) is MSGES and also ASGES if* $|\sigma| < \frac{\bar{\sigma}}{\sqrt{2}}$, *and* $\theta < \min(\frac{\Delta}{2}, \bar{\theta})$, *where* $\bar{\sigma}$ *is the unique positive solution* $\hat{w}$ *of Eq (5.6):*

$$\begin{aligned}&2\alpha\exp\{-\nu\Delta\} + 4\Delta[24\Delta((\bar{\mu}-\underline{\mu})^2\|B\|^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\|W\|^2F^2\\&+ 8\bar{\mu}^2\|W\|^2F^2) + 4\hat{w}^2]\alpha/\nu \times \exp\{2\Delta[24\Delta(\bar{\mu}^2\|B\|^2 + \bar{\mu}^2\|K\|^2F^2\\&+ 3\bar{\mu}^2\|W\|^2F^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 8\bar{\mu}^2\|W\|^2F^2) + 4\hat{w}^2]\} = 1\end{aligned} \tag{5.6}$$

and $\bar{\theta}$ *is the unique positive solution* $\check{w}$ *of Eq (5.7)*

$$\begin{aligned}&2\alpha\exp\{-\nu(\Delta-\check{w})\} + 4\Delta[24\Delta((\bar{\mu}-\underline{\mu})^2\|B\|^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\|W\|^2F^2\\&+ 2\bar{\mu}^2\|W\|^2F^2\bar{\bar{\lambda}}) + 2\sigma^2]\alpha/\nu \times \exp\{2\Delta[24\Delta(\bar{\mu}^2\|B\|^2 + \bar{\mu}^2\|K\|^2F^2 + 3\bar{\mu}^2\|W\|^2F^2\\&+ (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\|W\|^2F^2\bar{\bar{\lambda}}) + 2\sigma^2]\} = 1,\end{aligned} \tag{5.7}$$

where $\bar{\bar{\lambda}} = 3(1 - 9\check{w}^2\bar{\mu}^2F^2\|W\|^2 - 9\check{w}(3\check{w}\bar{\mu}^2\|B\|^2 + 3\check{w}\bar{\mu}^2\|K\|^2F^2 + \sigma^2)(1 + 3\check{w}^2\bar{\mu}^2\|W\|^2F^2)\exp\{3\check{w}(3\check{w}\bar{\mu}^2\|B\|^2 + 3\check{w}\bar{\mu}^2F^2\|K\|^2 + \sigma^2)\})^{-1}$, *and* $\Delta > \frac{ln(\alpha)}{\nu} > 0$.

Proof. For convenience, we write $h(t; t_0, h_0) \equiv h(t)$ and $l(t; t_0, l_0) \equiv l(t)$.

Combined with (5.1), (5.2), and Lemma 2, $\forall t \geq t_0 > 0$,

$$\begin{aligned}&E\|h(t)-l(t)\|^2\\
\leq& 2E\sum_{i=1}^{n}\Big|\int_{t_0}^{t}\Big\{\big(a_i(l_i(s))b_i(l_i(s)) - a_i(h_i(s))b_i(h_i(s))\big) + \sum_{j=1}^{n}k_{ij}\big(a_i(h_i(s))f_j(h_j(s)) - a_i(l_i(s))f_j(l_j(s))\big)\\
&+ \sum_{j=1}^{n}w_{ij}\big(a_i(h_i(s))f_j(h_j(s)) - a_i(l_i(s))f_j(l_j(\varrho(s)))\big)\Big\}\mathrm{d}s\Big|^2 + 2E\sum_{i=1}^{n}\Big|\int_{t_0}^{t}\sigma_i l_i(s)\mathrm{d}\omega(s)\Big|^2\\
\leq& 2E\sum_{i=1}^{n}\Big\{\int_{t_0}^{t}\Big[\bar{\mu}B_i|l_i(s)-h_i(s)| + (\bar{\mu}-\underline{\mu})B_i|h_i(s)| + \sum_{j=1}^{n}k_{ij}\bar{\mu}F_j|h_j(s)-l_j(s)| + \sum_{j=1}^{n}k_{ij}(\bar{\mu}-\underline{\mu})F_j|l_j(s)|\\
&+ \sum_{j=1}^{n}w_{ij}\bar{\mu}F_j|h_j(s)-l_j(s)| + \sum_{j=1}^{n}w_{ij}(\bar{\mu}F_j|l_j(s)| - \underline{\mu}F_j|l_j(\varrho(s))|)\Big]\mathrm{d}s\Big\}^2 + 2E\sum_{i=1}^{n}\int_{t_0}^{t}\big|\sigma_i l_i(s)\big|^2\mathrm{d}s\\
\leq& 12(t-t_0)\Big[\bar{\mu}^2\|B\|^2\int_{t_0}^{t}E\|h(s)-l(s)\|^2\mathrm{d}s + (\bar{\mu}-\underline{\mu})^2\|B\|^2\int_{t_0}^{t}E\|h(s)\|^2\mathrm{d}s + \bar{\mu}^2\|K\|^2F^2\int_{t_0}^{t}E\|h(s)\\
&- l(s)\|^2\mathrm{d}s + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2\int_{t_0}^{t}E\|l(s)\|^2\mathrm{d}s + \bar{\mu}^2\|W\|^2F^2\int_{t_0}^{t}E\|h(s)-l(s)\|^2\mathrm{d}s + 2\bar{\mu}\|W\|^2F^2\end{aligned}$$

$$\times \int_{t_0}^{t} E\|l(s)\|^2 ds + 2\bar{\mu}^2\bar{\lambda}\|W\|^2F^2 \int_{t_0}^{t} E\|l(s)\|^2 ds\Big] + 4\sigma^2 \int_{t_0}^{t} E\|h(s)-l(s)\|^2 ds + 4\sigma^2 \int_{t_0}^{t} E\|h(s)\|^2 ds$$
$$\leq [12(t-t_0)(\bar{\mu}^2\|B\|^2 + \bar{\mu}^2\|K\|^2F^2 + \bar{\mu}^2\|W\|^2F^2) + 4\sigma^2 + 12(t-t_0)[(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\|W\|^2F^2$$
$$+ 2\bar{\mu}^2\bar{\lambda}\|W\|^2F^2]] \int_{t_0}^{t} E\|h(s)-l(s)\|^2 ds + [12(t-t_0)[(\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\|W\|^2F^2$$
$$+ 2\bar{\mu}^2\bar{\lambda}\|W\|^2F^2] + 12(t-t_0)(\bar{\mu}-\underline{\mu})^2\|B\|^2 + 4\sigma^2] \int_{t_0}^{t} E\|h(s)\|^2 ds$$
$$\leq [12(t-t_0)(\bar{\mu}^2\|B\|^2 + \bar{\mu}^2\|K\|^2F^2 + 3\bar{\mu}^2\|W\|^2F^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\bar{\lambda}\|W\|^2F^2) + 4\sigma^2]$$
$$\times \int_{t_0}^{t} E\|h(s)-l(s)\|^2 ds + [12(t-t_0)((\bar{\mu}-\underline{\mu})^2\|B\|^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\|W\|^2F^2$$
$$+ 2\bar{\mu}^2\bar{\lambda}\|W\|^2F^2) + 4\sigma^2]\alpha/\nu\|l_0\|^2(t-t_0). \tag{5.8}$$

By using the Gronwall-Bellman Lemma, for $t_0 + \theta \leq t \leq t_0 + 2\Delta$, from (5.8) we get

$$E\|h(t)-l(t)\|^2$$
$$\leq 2\Delta[24\Delta((\bar{\mu}-\underline{\mu})^2\|B\|^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\|W\|^2F^2 + 2\bar{\mu}^2\bar{\lambda}\|W\|^2F^2) + 4\sigma^2]\alpha/\nu\|l_0\|^2$$
$$\exp\{2\Delta[24\Delta((\bar{\mu}^2\|B\|^2 + \bar{\mu}^2\|K\|^2F^2 + 3\bar{\mu}^2\|W\|^2F^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\bar{\lambda}\|W\|^2F^2)) + 4\sigma^2]\}.$$

Therefore, for $t_0 + \theta \leq t \leq t_0 + 2\Delta$,

$$E\|l(t)\|^2$$
$$\leq 2\alpha\|l_0\|^2 \exp\{-\nu(t-t_0)\} + 4\Delta[24\Delta((\bar{\mu}-\underline{\mu})^2\|B\|^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\|W\|^2F^2$$
$$+ 2\bar{\mu}^2\bar{\lambda}\|W\|^2F^2) + 4\sigma^2]\alpha/\nu\|l_0\|^2 \times \exp\{2\Delta[24\Delta((\bar{\mu}^2\|B\|^2 + \bar{\mu}^2\|K\|^2F^2$$
$$+ 3\bar{\mu}^2\|W\|^2F^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\bar{\lambda}\|W\|^2F^2)) + 4\sigma^2]\}$$

and thus, for $t_0 - \theta + \Delta \leq t \leq t_0 - \theta + 2\Delta$,

$$E\|l(t)\|^2$$
$$\leq 2\alpha\|l_0\|^2 \exp\{-\nu(\Delta-\theta)\} + 4\Delta[24\Delta((\bar{\mu}-\underline{\mu})^2\|B\|^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\|W\|^2F^2$$
$$+ 2\bar{\mu}^2\bar{\lambda}\|W\|^2F^2) + 4\sigma^2]\alpha/\nu\|l_0\|^2 \times \exp\{2\Delta[24\Delta((\bar{\mu}^2\|B\|^2 + \bar{\mu}^2\|K\|^2F^2$$
$$+ 3\bar{\mu}^2\|W\|^2F^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\bar{\lambda}\|W\|^2F^2)) + 4\sigma^2]\}.$$

Let $M(\sigma,\theta) = 2\alpha\exp\{-\nu(\Delta-\theta)\} + 4\Delta[24\Delta((\bar{\mu}-\underline{\mu})^2\|B\|^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\|W\|^2F^2 + 2\bar{\mu}^2\bar{\lambda}\|W\|^2F^2) + 4\sigma^2]\alpha/\nu \times \exp\{2\Delta[24\Delta((\bar{\mu}^2\|B\|^2 + \bar{\mu}^2\|K\|^2F^2 + 3\bar{\mu}^2\|W\|^2F^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\bar{\lambda}\|W\|^2F^2)) + 4\sigma^2]\}$. From A9, we can get $M(0,0) < 1$. Moreover, it is easy to know that $M(\sigma,0)$ is strictly increasing for σ. So, there exists a positive constant $\bar{\sigma}$ such that $M(\bar{\sigma},0) = 1$. $M(\sigma,\theta)$ is obviously strictly increasing for θ, so there exists a positive constant $\bar{\theta}$, and when $\theta < \min\{\frac{\Delta}{2},\bar{\theta}\}$, $|\sigma| < \frac{\bar{\sigma}}{\sqrt{2}}$ such that $M(\sigma,\theta) < 1$.

Letting

$$\delta = -\ln\Big\{2\alpha\|l_0\|^2 \exp\{-\nu(\Delta-\theta)\} + 4\Delta[24\Delta((\bar{\mu}-\underline{\mu})^2\|B\|^2 + (\bar{\mu}-\underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\|W\|^2F^2$$

$$+ 2\bar{\mu}^2\bar{\lambda}\|W\|^2F^2) + 4\sigma^2]\alpha/\nu \times \exp\{2\Delta[24\Delta((\bar{\mu}^2\|B\|^2 + \bar{\mu}^2\|K\|^2F^2 + 3\bar{\mu}^2\|W\|^2F^2$$

$$+ (\bar{\mu} - \underline{\mu})^2\|K\|^2F^2 + 2\bar{\mu}^2\bar{\lambda}\|W\|^2F^2)) + 4\sigma^2]\}\Big\}/\Delta$$

we obtain

$$E\|l(t)\|^2 \leq \exp\{-\delta\Delta\}\|l_0\|^2. \tag{5.9}$$

From the uniqueness of the solution of SPCGNN (5.1), there exists a positive integer ε such that

$$l(t; t_0, l_0) = l(t; t_0 + (\varepsilon - 1)\Delta, l(t_0 + (\varepsilon - 1)\Delta; t_0, l_0)). \tag{5.10}$$

Then, from (5.9) and (5.10), for $t \geq t_0 - \theta + \varepsilon\Delta$,

$$\begin{aligned} &E\|l(t; t_0, l_0)\|^2 \\ &\leq \exp\{-\delta\Delta\}\|l(t_0 + (\varepsilon - 1)\Delta; t_0, l_0)\|^2 \\ &\leq ... \\ &\leq \exp\{-\varepsilon\delta\Delta\}\|l_0\|^2. \end{aligned} \tag{5.11}$$

Thus, for any $t > t_0 - \theta + \Delta$, there is an integer $\varepsilon > 0$ such that $t_0 - \theta + (\varepsilon - 1)\Delta \leq t \leq t_0 - \theta + \varepsilon\Delta$,

$$E\|l(t; t_0, l_0)\|^2 \leq \exp\{-\delta(t - t_0)\}\exp\{-\delta(\Delta - \theta)\}\|l_0\|^2. \tag{5.12}$$

Clearly, (5.12) also holds for $t_0 \leq t \leq t_0 - \theta + \Delta$. So, system (5.1) is MSGES and also is ASGES.

□

Remark 3. *In Section 5, the discussed system is hybrid and the robustness of CGNNs with PCA and SDs is considered. Based on the GES of CGNN (5.2), the perturbed SPCGNN (5.1) can stand MSGES and ASGES when both values are below the obtained upper bounds.*

Remark 4. *We discuss the robustness of CGNNs with PCA and SDs in Theorem 3. The research topics of literature [8, 9] are related to the robustness of RNNs. The system studied is a more general in this paper. When the amplification function $a_i(l_i(t)) = 1$ in a CGNN, the CGNN is converted to an RNN or HNN, so the research results in this paper are generic. Similarly, we also characterize the robustness of CGNNs by the upper bounds of the perturbation factors obtained by solving transcendental equations. Aiming at the difficulties caused by the existence of the amplification function $a_i(l_i(t))$, we solve the influence of the amplification function on a CGNN by a hypothesis. Finally, the desired results are obtained.*

6. Numerical examples

The following section presents three instances to substantiate the obtained findings.

Example 1. Consider a two-dimensional SCGNN

$$\begin{aligned} dl_1(t) =&(1.5 + 0.5\sin(2l_1(t)))[-0.005l_1(t) + 0.004f(l_1(t)) + 0.002f(l_2(t))]dt + \sigma l_1(t)d\omega(t), \\ dl_2(t) =&(1.5 + 0.5\cos(4l_2(t)))[-0.005l_2(t) + 0.006f(l_1(t)) + 0.003f(l_2(t))]dt + \sigma l_2(t)d\omega(t) \end{aligned} \tag{6.1}$$

where $f(l) = \tanh(l)$, σ is the noise intensity, and $\omega(t)$ is a scalar Brownian motion defined in the probability space.

Consider the model of a CGNN without SDs:

$$\begin{aligned}\frac{\mathrm{d}l_1(t)}{\mathrm{d}t} =&(1.5 + 0.5\sin(2l_1(t)))[-0.005l_1(t) + 0.004f(l_1(t)) + 0.002f(l_2(t))],\\ \frac{\mathrm{d}l_2(t)}{\mathrm{d}t} =&(1.5 + 0.5\cos(4l_2(t)))[-0.005l_2(t) + 0.006f(l_1(t)) + 0.003f(l_2(t))].\end{aligned} \tag{6.2}$$

It is easy to find that CGNN (7.2) is GES with $\alpha = 0.8$ and $\nu = 0.5$. Let $\triangle = 0.3 > \frac{\ln(2\alpha^2)}{2\nu} = 0.2468$, $\bar{\mu} = 2$, $\underline{\mu} = 1$, and $F = 1$. Substituting them into (3.2), we get

$$1.28[0.00144 + 4\sigma^2] \times \exp\{0.00288 + 2.4\sigma^2\} + 1.28\exp\{-0.3\} = 1.$$

By solving the transcendental equation, we obtain the solution as $\bar{\sigma} = 0.0974$. From Theorem 1, when $|\sigma| < \bar{\sigma}$, SCGNN (7.1) is MSGES and also ASGES. The changing behavior of SCGNN (7.1) is shown in Figure 1.

Example 2. Consider a two-dimensional CGNN with PCA

$$\begin{aligned}\dot{l}_1(t) =&(1.5 + 0.5\sin(2l_1(t)))[-0.005l_1(t) + 0.005f(l_1(t)) + 0.003f(l_2(t))\\ &+ 0.004f(l_1(\varrho(t))) + 0.005f(l_2(\varrho(t)))],\\ \dot{l}_2(t) =&(1.5 + 0.5\cos(4l_2(t)))[-0.005l_2(t) + 0.005f(l_1(t)) + 0.005f(l_2(t))\\ &+ 0.016f(l_1(\varrho(t))) + 0.009f(l_2(\varrho(t)))]\end{aligned} \tag{6.3}$$

where $\theta_k = \frac{k}{16}$, $\vartheta_k = \frac{2k+1}{32}$, and $\varrho(t) = \vartheta_k$, if $t \in [\theta_k, \theta_{k+1})$, $k \in N$, $f(l) = \tanh(l)$.

Consider the CGNN without PCA as follows:

$$\begin{aligned}\dot{l}_1(t) =&(1.5 + 0.5\sin(2l_1(t)))[-0.005l_1(t) + 0.005f(l_1(t)) + 0.003f(l_2(t))\\ &+ 0.004f(l_1(t)) + 0.005f(l_2(t))],\\ \dot{l}_2(t) =&(1.5 + 0.5\cos(4l_2(t)))[-0.005l_2(t) + 0.005f(l_1(t)) + 0.005f(l_2(t))\\ &+ 0.0016f(l_1(t)) + 0.009f(l_2(t))].\end{aligned} \tag{6.4}$$

It is easy to know that CGNN (7.4) is GES with $\alpha = 1.2$ and $\nu = 0.9$. Let $\triangle = 0.3 > \frac{\ln(\alpha)}{\nu} = 0.2025$, $\bar{\mu} = 2$, $\underline{\mu} = 1$, and $F = 1$. Substituting them into (4.4) and (4.5), we get

$$\begin{aligned}&1.2\exp\{-0.9(0.3 - \hat{x})\} + [0.06 + 0.04(1 - \hat{x}(0.04 + 0.04(1 + 0.04\hat{x}) \times \exp(0.04\hat{x})))]\\ &\times 1.2/0.9 \times \exp\{0.078 + 0.024[1 - \hat{x}(0.04 + 0.04(1 + 0.04\hat{x}) \times \exp(0.04\hat{x}))]\} = 1,\end{aligned}$$

$$\check{x}[0.04 + 0.04(1 + 0.04\check{x}) \times \exp(0.04\check{x})] = 1.$$

By solving the equation above, we obtain $\bar{\theta} = 0.0806$, $\bar{\bar{\theta}} = 8.6238$. Thus, in line with Theorem 2, when $\theta < \min\{\frac{\triangle}{2}, \bar{\theta}, \bar{\bar{\theta}}\}$, CGNN (7.3) is GES. Figure 2 depicts the transient states of (7.3).

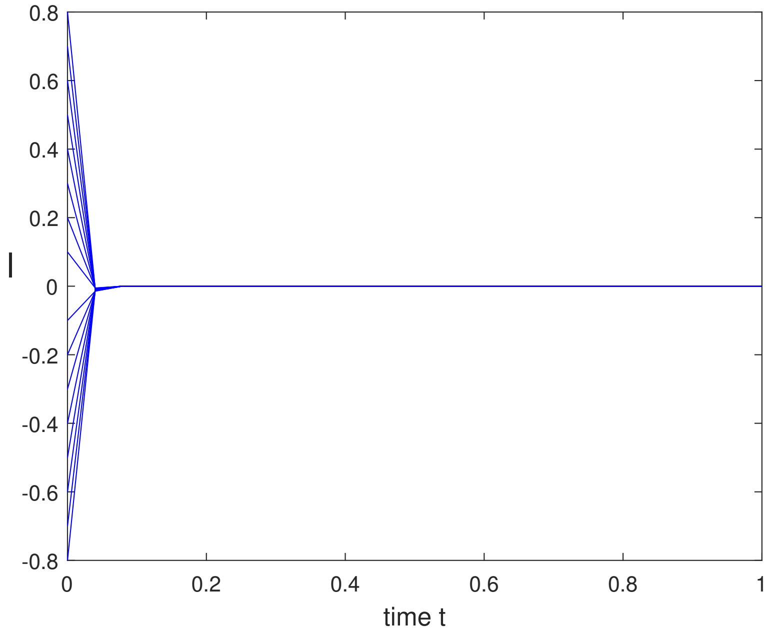

Figure 1. The states of (7.1) with $\sigma = 0.08$.

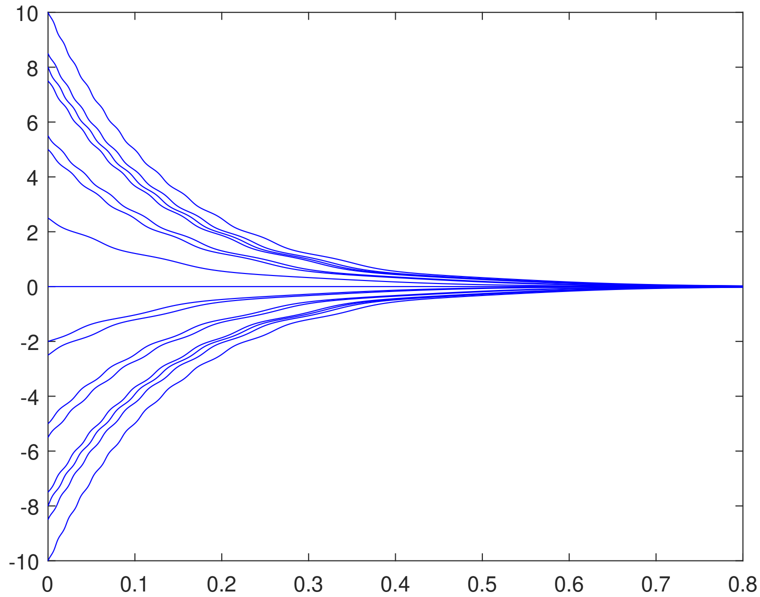

Figure 2. The states of (7.3) with $\theta_k = \{\frac{k}{16}\}$.

Example 3. Consider a one-dimensional CGNN

$$\dot{h}(t) = (1.5 + 0.5\sin(2h(t)))[-0.01h(t) + 0.03f(h(t))] \tag{6.5}$$

where $f(l) = \tanh(l)$. We can readily determine that CGNN (7.5) is GES with $\alpha = 1.2$ and $\nu = 3$ by many of the current criteria.

The corresponding disturbed SPCGNN is given by

$$dl(t) = (1.5 + 0.5\sin(2l(t)))[-0.01l(t) + 0.01f(l(t)) + 0.02fl(\varrho(t))]dt + \sigma l(t)d\omega(t) \tag{6.6}$$

where $f(l) = \tanh(l)$. When the mathematical parameters are put into (5.6), the result is

$$2.4\exp\{-1.5\} + 0.8(0.1944 + 4\hat{w})\exp\{0.222 + 4\hat{w}^2\} = 1.$$

Thus, we have $\bar{\sigma} = 0.218$. Note that $|\sigma| < \frac{\bar{\sigma}}{\sqrt{2}}$, which means that $|\sigma| < 0.1541$.

Combined with $\bar{\sigma} = 0.218$, substituting the other computing parameters into (5.7), we get

$$\begin{aligned}&2.4\exp\{-3(0.5 - \check{w})\} + [0.10937 + 0.09216[1 - 0.0144\check{w}^2 - 9\check{w}(0.0024\check{w} + 0.0757)(1 + 0.0048\check{w}^2)\\ &\times \exp\{3\check{w}(0.0024\check{w} + 0.0757)\}]] \times \exp\{0.1643 + 0.1152(1 - 0.0144\check{w}^2 - 9\check{w}(0.024\check{w}\\ &+ 0.0757)(1 + 0.0048\check{w}^2 \exp\{3\check{w}(0.0024\check{w} + 0.0757)\}))\} = 1.\end{aligned}$$

Hence, it can be derived that $\bar{\theta} = 0.1166$. Note that $\theta < \min(\frac{\triangle}{2}, \bar{\theta})$, and therefore $\theta < 0.1166$. Figure 3 depicts the transient states of (7.6) with $\bar{\sigma} = 0.1$ and $\{\theta_k\} = \frac{k}{20}$. This shows that the state of (7.6) is MSGES and also ASGES.

Figure 4 displays the change in behavior of SPCGNN (7.6) with $\sigma = 0.8 > \frac{\bar{\sigma}}{\sqrt{2}}$ and $\{\theta_k\} = \frac{k}{2}$. The unstable states of SPCGNN (7.6) are depicted in Figure 5 with $\sigma = 0.8$ and $\{\theta_k\} = \frac{k}{20}$. Figure 6 illustrates that system (7.6) is unstable with $\sigma = 0.1$ and $\{\theta_k\} = \frac{k}{2}$. Consequently, the CGNN can turn unstable if the conditions in the theorems cannot be fulfilled.

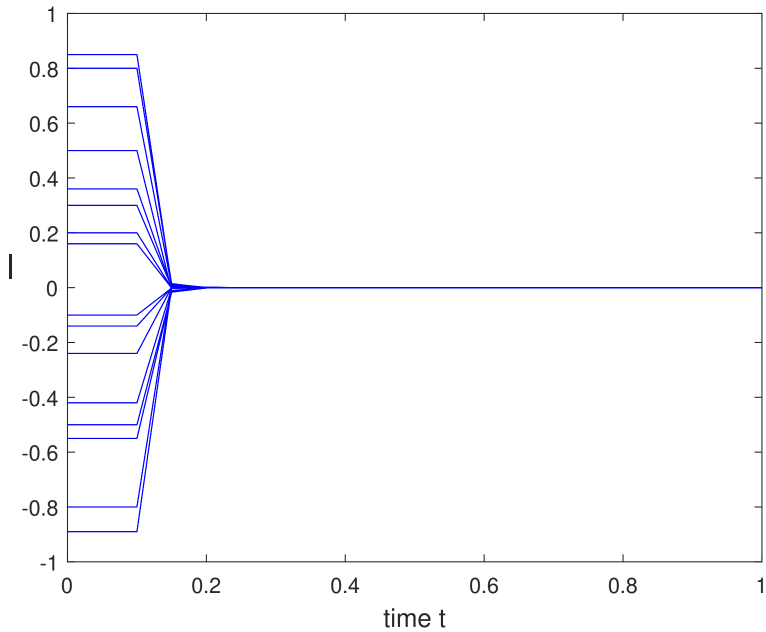

Figure 3. The states of (7.6) with $\sigma = 0.1$ and $\theta_k = \{\frac{k}{20}\}$.

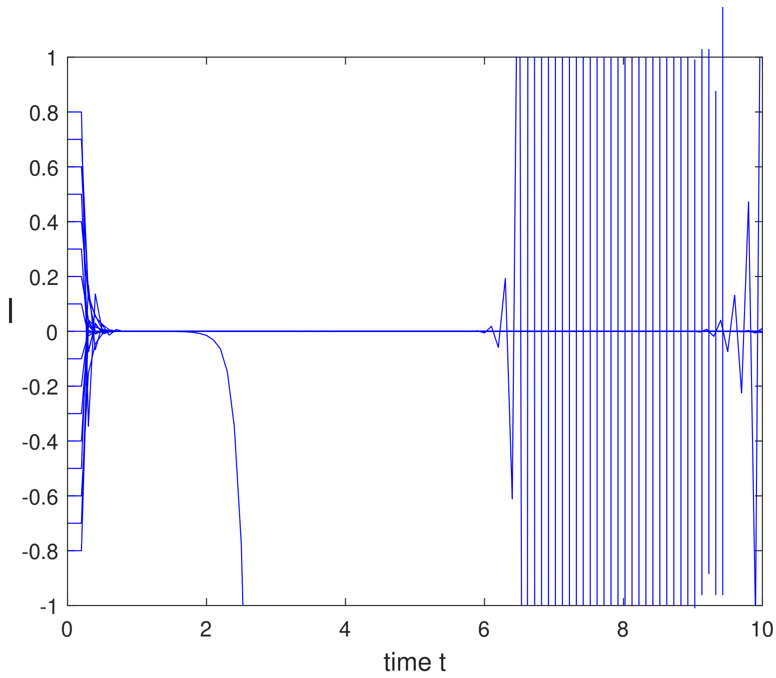

Figure 4. The states of (7.6) with $\sigma = 0.8$ and $\theta_k = \{\frac{k}{20}\}$.

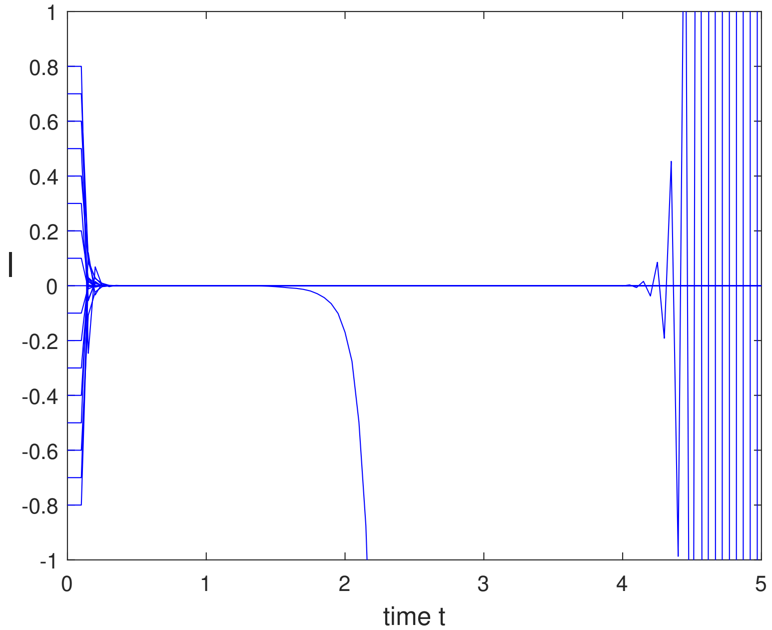

Figure 5. The states of (7.6) with $\sigma = 0.8$ and $\theta_k = \{\frac{k}{20}\}$.

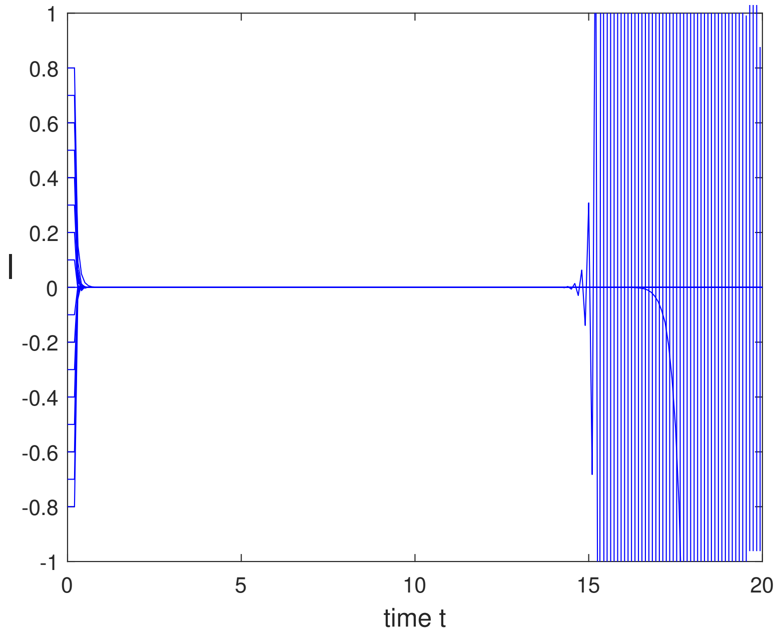

Figure 6. The states of (7.6) with $\sigma = 0.1$ and $\theta_k = \{\frac{k}{2}\}$.

Example 4. When the amplification function $a_i(l_i(t)) = 1$, the CGNN is converted to an RNN or HNN, and then SCGNN (3.1) is converted to

$$\begin{aligned} dl_1(t) =& [-l_1(t) - 2f(l_1(t)) + 2f(l_2(t))]dt + \sigma l_1(t)d\omega(t), \\ dl_2(t) =& [-l_2(t) + 2f(l_1(t)) - 2f(l_2(t))]dt + \sigma l_2(t)d\omega(t) \end{aligned} \tag{6.7}$$

where $f(l) = \tanh(l)$, σ is the noise intensity, and $\omega(t)$ is a scalar Brownian motion.

Consider model (7.7) without SDs:

$$\begin{aligned} \frac{dl_1(t)}{dt} =& -l_1(t) - 2f(l_1(t)) + 2f(l_2(t)), \\ \frac{dl_2(t)}{dt} =& -l_2(t) + 2f(l_1(t)) - 2f(l_2(t)). \end{aligned} \tag{6.8}$$

It is easy to find that system (7.8) is GES with $\alpha = 0.8$ and $\nu = 0.5$. Let $\triangle = 0.3$, $\bar{\mu} = \underline{\mu} = 1$, and $F = 1$. Substituting them into (3.2), we get

$$5.12\sigma^2 \times \exp\{48.96 + 2.4\sigma^2\} + 1.28\exp\{-0.3\} = 1.$$

By solving the transcendental equation, we obtain the solution as $\bar{\sigma} = 2.3488 \times 10^{-12}$. According to Theorem 1, when $|\sigma| < \bar{\sigma}$, SCGNN (7.7) is MSGES and also ASGES.

Remark 5. *In Theorem 1 of reference [8], the author studied the robustness of RNNs with SDs. In Example 4, when the amplification function $a_i(l_i(t)) = 1$, the SCGNN is converted to an RNN or HNN, and then SCGNN (3.1) is similar to the system of reference [8]. This shows that the results of this paper are more general.*

Examples 5 and 6 are variations of Examples 2 and 3 when the amplification function $a_i(l_i(t)) = 1$ in the CGNN.

Example 5. Consider the following model with PCA

$$\begin{aligned}\dot{l}_1(t) =& -0.005l_1(t) + 0.004f(l_1(t)) + 0.006f(l_2(t)) + 0.009f(l_1(\varrho(t))) + 0.008f(l_2(\varrho(t))),\\ \dot{l}_2(t) =& -0.005l_2(t) + 0.002f(l_1(t)) + 0.012f(l_2(t)) + 0.005f(l_1(\varrho(t))) + 0.004f(l_2(\varrho(t)))\end{aligned} \tag{6.9}$$

where $\varrho(t) = \vartheta_k$, if $t \in [\theta_k, \theta_{k+1})$, $k \in N$, and $\theta_k = \frac{k}{16}$, $\vartheta_k = \frac{2k+1}{32}$, $f(l) = \tanh(l)$.

Consider model (7.9) without PCA as follows:

$$\begin{aligned}\dot{l}_1(t) =& -0.005l_1(t) + 0.004f(l_1(t)) + 0.006f(l_2(t)) + 0.009f(l_1(t)) + 0.008f(l_2(t)),\\ \dot{l}_2(t) =& -0.005l_2(t) + 0.002f(l_1(t)) + 0.012f(l_2(t)) + 0.005f(l_1(t)) + 0.004f(l_2(t))].\end{aligned} \tag{6.10}$$

It is known that (7.10) is GES with $\alpha = 1.2$ and $\nu = 0.9$. Let $\triangle = 0.3$, $\bar{\mu} = \underline{\mu} = 1$, and $F = 1$. Substituting them into (4.4) and (4.5), we get

$$\begin{aligned}&1.2\exp\{-0.9(0.3 - \hat{x})\} + [0.02 + 0.02(1 - \hat{x}(0.04 + 0.04(1 + 0.04\hat{x}) \times \exp(0.04\hat{x})))]\\ &\times 1.2/0.9 \times \exp\{0.036 + 0.012[1 - \hat{x}(0.04 + 0.04(1 + 0.04\hat{x}) \times \exp(0.04\hat{x}))]\} = 1,\end{aligned}$$

$$\check{x}[0.04 + 0.04(1 + 0.04\check{x}) \times \exp(0.04\check{x})] = 1.$$

By solving the equations above, we obtain $\bar{\theta} = 0.0335$, $\bar{\bar{\theta}} = 8.6238$. From Theorem 2, when $\theta < \min\{\frac{\triangle}{2}, \bar{\theta}, \bar{\bar{\theta}}\}$, system (7.9) is GES.

Example 6. Consider the system

$$\dot{h}(t) = -0.01h(t) + 0.03f(h(t)) \tag{6.11}$$

where $f(l) = \tanh(l)$. We can readily determine that (7.11) is GES with $\alpha = 1.2$ and $\nu = 3$ by many of the current criteria.

The corresponding disturbed system is given by

$$dl(t) = [-0.01l(t) + 0.01f(l(t)) + 0.02fl(\varrho(t))]dt + \sigma l(t)d\omega(t) \tag{6.12}$$

where $f(l) = \tanh(l)$. When the mathematical parameters are put into (5.6), we have

$$2.4\exp\{-1.5\} + 0.8(0.048 + 4\hat{w})\exp\{0.0552 + 4\hat{w}^2\} = 1.$$

Thus, we have $\bar{\sigma} = 0.2925$. Note that $|\sigma| < \frac{\bar{\sigma}}{\sqrt{2}}$, which means that $|\sigma| < 0.2068$.

Combined with $\bar{\sigma} = 0.2925$, substituting the other computing parameters into (5.7), we get

$$\begin{aligned}&2.4\exp\{-3(0.5 - \check{w})\} + [0.14457 + 0.002304[1 - 0.0144\check{w}^2 - 9\check{w}(0.0024\check{w} + 0.0757)(1 + 0.0048\check{w}^2)\\ &\times \exp\{3\check{w}(0.0024\check{w} + 0.0757)\}]] \times \exp\{0.1879 + 0.0288(1 - 0.0144\check{w}^2 - 9\check{w}(0.024\check{w}\\ &+ 0.0757)(1 + 0.0048\check{w}^2\exp\{3\check{w}(0.0024\check{w} + 0.0757)\}))\} = 1.\end{aligned}$$

Hence, it can be derived that $\bar{\theta} = 0.1318$. Note that $\theta < \min(\frac{\triangle}{2}, \bar{\theta})$, and therefore $\theta < 0.1318$. From Theorem 3, (7.12) is MSGES and also ASGES.

Examples 7 and 8 are four-dimensional numerical examples of Theorems 1 and 2.

Example 7. Consider the four-dimensional SCGNN

$$\begin{aligned} \mathrm{d}l_1(t) =&(1.5 + 0.5\sin(4l_1(t)))[-0.02l_1(t) + 0.01f(l_1(t)) + 0.003f(l_2(t)) \\ &+ 0.001f(l_3(t)) + 0.005f(l_4(t))]\mathrm{d}t + \sigma l_1(t)\mathrm{d}\omega(t), \\ \mathrm{d}l_2(t) =&(1.5 + 0.5\cos(6l_2(t)))[-0.01l_2(t) + 0.012f(l_1(t)) + 0.02f(l_2(t)) \\ &+ 0.018f(l_3(t)) + 0.024f(l_4(t))]\mathrm{d}t + \sigma l_2(t)\mathrm{d}\omega(t), \\ \mathrm{d}l_3(t) =&(1.5 + 0.5\sin(5l_3(t)))[-0.01l_3(t) + 0.003f(l_1(t)) + 0.012f(l_2(t)) \\ &+ 0.007f(l_3(t)) + 0.002f(l_4(t))]\mathrm{d}t + \sigma l_3(t)\mathrm{d}\omega(t), \\ \mathrm{d}l_4(t) =&(1.5 + 0.5\cos(2l_4(t)))[-0.02l_4(t) + 0.015f(l_1(t)) + 0.014f(l_2(t)) \\ &+ 0.004f(l_3(t)) + 0.001f(l_4(t))]\mathrm{d}t + \sigma l_4(t)\mathrm{d}\omega(t) \end{aligned} \tag{6.13}$$

where $f(l) = \tanh(l)$, σ is the noise intensity, and $\omega(t)$ is a scalar Brownian motion.

Consider the model of a CGNN without SDs:

$$\begin{aligned} \frac{\mathrm{d}l_1(t)}{\mathrm{d}t} =&(1.5 + 0.5\sin(4l_1(t)))[-0.02l_1(t) + 0.01f(l_1(t)) + 0.003f(l_2(t)) \\ &+ 0.001f(l_3(t)) + 0.005f(l_4(t))], \\ \frac{\mathrm{d}l_2(t)}{\mathrm{d}t} =&(1.5 + 0.5\cos(6l_2(t)))[-0.01l_2(t) + 0.012f(l_1(t)) + 0.02f(l_2(t)) \\ &+ 0.018f(l_3(t)) + 0.024f(l_4(t))], \\ \frac{\mathrm{d}l_3(t)}{\mathrm{d}t} =&(1.5 + 0.5\sin(5l_3(t)))[-0.01l_3(t) + 0.003f(l_1(t)) + 0.012f(l_2(t)) \\ &+ 0.007f(l_3(t)) + 0.002f(l_4(t))], \\ \frac{\mathrm{d}l_4(t)}{\mathrm{d}t} =&(1.5 + 0.5\cos(2l_4(t)))[-0.02l_4(t) + 0.015f(l_1(t)) + 0.014f(l_2(t)) \\ &+ 0.004f(l_3(t)) + 0.001f(l_4(t))]. \end{aligned} \tag{6.14}$$

It is easily find that CGNN (7.14) is GES with $\alpha = 0.8$ and $\nu = 0.5$. Let $\triangle = 0.3$, $\bar{\mu} = 2$, $\underline{\mu} = 1$, and $F = 1$. Substituting them into (3.2), we get

$$1.28\exp\{-0.3\} + 1.28[0.03264 + 4\sigma^2] \times \exp\{0.06912 + 2.4\sigma^2\} = 1.$$

By solving the transcendental equation, we obtain the solution as $\bar{\sigma} = 0.0353$. From Theorem 1, when $|\sigma| < \bar{\sigma}$, SCGNN (7.13) is MSGES and also ASGES. The changing behavior of SCGNN (7.13) is shown in Figure 7.

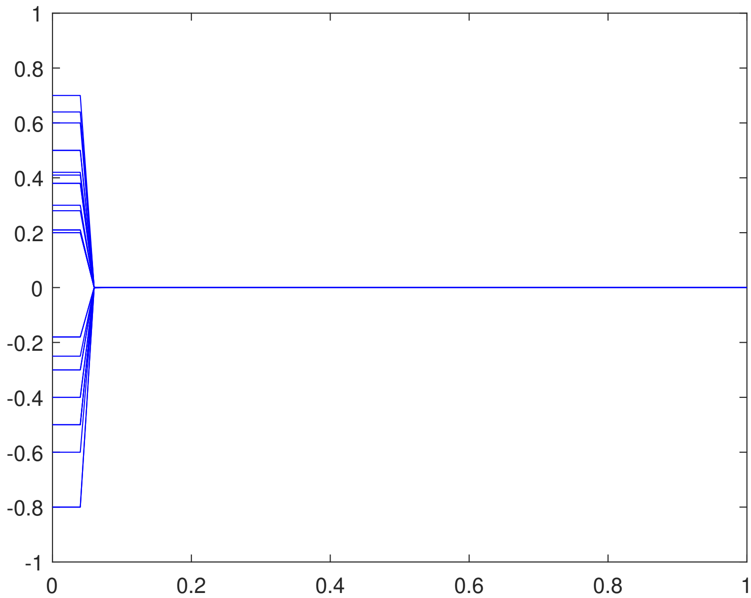

Figure 7. The states of SCGNN (7.13) with $\sigma = 0.02$.

Example 8. Consider the four-dimensional CGNN with PCA

$$\begin{aligned}
\dot{l}_1(t) =&(1.5 + 0.5\sin(3l_1(t)))[-0.03l_1(t) + 0.022f(l_1(t)) + 0.012f(l_2(t)) + 0.008f(l_3(t)) + 0.02f(l_4(t)) \\
&+ 0.003f(l_1(\varrho(t))) + 0.002f(l_2(\varrho(t))) + 0.01f(l_3(\varrho(t))) + 0.003f(l_4(\varrho(t)))], \\
\dot{l}_2(t) =&(1.5 + 0.5\cos(2l_2(t)))[-0.02l_2(t) + 0.01f(l_1(t)) + 0.006f(l_2(t)) + 0.003f(l_3(t)) + 0.016f(l_4(t)) \\
&+ 0.005f(l_1(\varrho(t))) + 0.018f(l_2(\varrho(t))) + 0.007f(l_3(\varrho(t))) + 0.014f(l_4(\varrho(t)))], \\
\dot{l}_3(t) =&(1.5 + 0.5\sin(4l_3(t)))[-0.01l_3(t) + 0.081f(l_1(t)) + 0.012f(l_2(t)) + 0.015f(l_3(t)) + 0.004f(l_4(t)) \\
&+ 0.014f(l_1(\varrho(t))) + 0.015f(l_2(\varrho(t))) + 0.015f(l_3(\varrho(t))) + 0.02f(l_4(\varrho(t)))], \\
\dot{l}_4(t) =&(1.5 + 0.5\sin(3l_4(t)))[-0.02l_4(t) + 0.02f(l_1(t)) + 0.01f(l_2(t)) + 0.015f(l_3(t)) + 0.006f(l_4(t)) \\
&+ 0.016f(l_1(\varrho(t))) + 0.012f(l_2(\varrho(t))) + 0.018f(l_3(\varrho(t))) + 0.003f(l_4(\varrho(t)))] \qquad (6.15)
\end{aligned}$$

where $\theta_k = \frac{k}{16}$, $\vartheta_k = \frac{2k+1}{32}$, and $\varrho(t) = \vartheta_k$, if $t \in [\theta_k, \theta_{k+1})$, $k \in N$, $f(l) = \tanh(l)$.

Consider the CGNN without PCA as follows:

$$\begin{aligned}
\dot{l}_1(t) =&(1.5 + 0.5\sin(3l_1(t)))[-0.03l_1(t) + 0.022f(l_1(t)) + 0.012f(l_2(t)) + 0.008f(l_3(t)) + 0.02f(l_4(t))] \\
&+ 0.003f(l_1(t)) + 0.002f(l_2(t)) + 0.01f(l_3(t)) + 0.003f(l_4(t))], \\
\dot{l}_2(t) =&(1.5 + 0.5\cos(2l_2(t)))[-0.02l_2(t) + 0.01f(l_1(t)) + 0.006f(l_2(t)) + 0.003f(l_3(t)) + 0.016f(l_4(t))] \\
&+ 0.005f(l_1(t)) + 0.018f(l_2(t)) + 0.007f(l_3(t)) + 0.014f(l_4(t))], \\
\dot{l}_3(t) =&(1.5 + 0.5\sin(4l_3(t)))[-0.01l_3(t) + 0.008f(l_1(t)) + 0.012f(l_2(t)) + 0.015f(l_3(t)) + 0.004f(l_4(t))] \\
&+ 0.014f(l_1(t)) + 0.015f(l_2(t)) + 0.015f(l_3(t)) + 0.02f(l_4(t))], \\
\dot{l}_4(t) =&(1.5 + 0.5\sin(3l_4(t)))[-0.02l_4(t) + 0.02f(l_1(t)) + 0.01f(l_2(t)) + 0.015f(l_3(t)) + 0.006f(l_4(t))] \\
&+ 0.016f(l_1(t)) + 0.012f(l_2(t)) + 0.018f(l_3(t)) + 0.003f(l_4(t))]. \qquad (6.16)
\end{aligned}$$

It is easy to know that CGNN (7.16) is GES with $\alpha = 1.2$ and $\nu = 0.9$. Let $\triangle = 0.3$, $\bar{\mu} = 2$, $\underline{\mu} = 1$, and $F = 1$. Substituting them into (4.4) and (4.5), we get

$$\begin{aligned}&1.2\exp\{-0.9(0.3-\hat{x})\} + [0.24 + 0.1(1-\hat{x}(0.04 + 0.04(1+0.04\hat{x})\times\exp(0.04\hat{x})))]\\&\times 1.2/0.9\times\exp\{0.324 + 0.06[1-\hat{x}(0.04+0.04(1+0.04\hat{x})\times\exp(0.04\hat{x}))]\} = 1,\end{aligned}$$

$$\check{x}[0.1 + 0.28(1+0.1\check{x})\times\exp(0.28\check{x})] = 1.$$

By figuring out the equations above, we obtain $\bar{\theta} = 1.1937$, $\bar{\bar{\theta}} = 1.6291$. Thus, according to Theorem 2, when $\theta < \min\{\frac{\triangle}{2}, \bar{\theta}, \bar{\bar{\theta}}\}$, CGNN (7.15) is GES. Figure 8 depicts the transient states of (7.15).

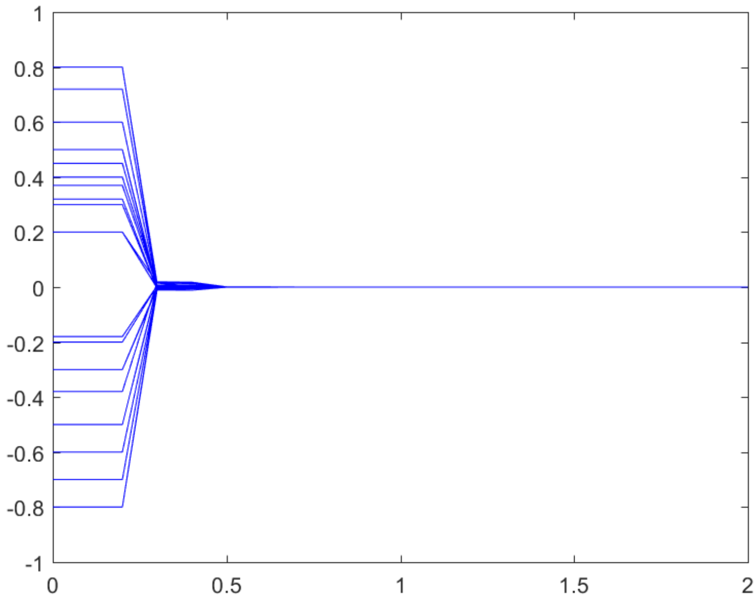

Figure 8. The states of (7.15) with $\theta_k = \{\frac{k}{10}\}$.

7. Conclusions

The main content in this article is about the robustness analysis of CGNNs with PCA and SDs. For an originally stable CGNN, we discuss the problem that how much the PCA and noise intensity the CGNN can withstand to be globally exponentially stable in the presence of PCA and SDs. We apply inequality techniques and stochastic analysis theory to obtain the upper bounds of PCA and SDs that the CGNN can withstand without losing stability by solving transcendental equations. It provides a theoretical basis for the designs and applications of CGNNs. Future work can explore the influence of delay and parameter uncertainty or other factors on the robustness of CGNNs, and the finite-time stability of CGNNs with PCA and SDs can also be considered to obtain richer results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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