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Research article

On the well posedness of a mathematical model for a singular nonlinear fractional pseudo-hyperbolic system with nonlocal boundary conditions and frictional damping terms

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Abstract: This paper is devoted to the study of the well-posedness of a singular nonlinear fractional pseudo-hyperbolic system with frictional damping terms. The fractional derivative is described in Caputo sense. The equations are supplemented by classical and nonlocal boundary conditions. Upon some a priori estimates and density arguments, we establish the existence and uniqueness of the strongly generalized solution for the associated linear fractional system in some Sobolev fractional spaces. On the basis of the obtained results for the linear fractional system, we apply an iterative process in order to establish the well-posedness of the nonlinear fractional system. This mathematical model of pseudo-hyperbolic systems arises mainly in the theory of longitudinal and lateral vibrations of elastic bars (beams), and in some special case it is propounded in unsteady helical flows between two infinite coaxial circular cylinders for some specific boundary conditions.

Keywords: pseudo-hyperbolic system; energy inequality; existence and uniqueness; iterative method; weak solution; Sobolev fractional space; nonlocal boundary condition
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1. Introduction

In the bounded domain $Q = \Omega \times [0, T] = \{(x, t) : 0 < x < b, 0 \leq t \leq T\}$, we are concerned with the well posedness of a nonlinear singular fractional system with two frictional damping terms. More precisely, the model problem is presented in the form

$$\begin{cases} {}^{C}\partial_{0t}^{\beta}u - \frac{1}{x}(xu_x)_x - \frac{1}{x}\frac{\partial}{\partial t}(xu_x)_x + z_1 v + u_t = f(x,t,u,v,u_x,v_x), \\ {}^{C}\partial_{0t}^{\gamma}v - \frac{1}{x}(xv_x)_x - \frac{1}{x}\frac{\partial}{\partial t}(xv_x)_x + z_2 u + v_t = g(x,t,u,v,u_x,v_x), \\ u(x,0) = \varphi_1(x), \quad u_t(x,0) = \varphi_2(x), \\ v(x,0) = \psi_1(x), \quad v_t(x,0) = \psi_2(x), \\ u_x(b,t) = 0, \quad v_x(b,t) = 0, \quad \int\limits_0^b xu dx = 0, \quad \int\limits_0^b xv dx = 0. \end{cases} \tag{1.1}$$

The functions f and g are $L^2(0,T;L^2_\rho(\Omega)$ given Lipschitzian functions, that is there exist two positive constants δ_1 and δ_2 such that

$$\begin{aligned} |f(x,t,u_1,v_1,w_1,d_1) - f(x,t,u_2,v_2,w_2,d_2)| &\le \delta_1(|u_1-u_2| + |v_1-v_2| + |w_1-w_2| + |d_1-d_2|), \\ |g(x,t,u_1,v_1,w_1,d_1) - f(x,t,u_2,v_2,w_2,d_2)| &\le \delta_2(|u_1-u_2| + |v_1-v_2| + |w_1-w_2| + |d_1-d_2|), \end{aligned}$$

for all $(x,t) \in Q$. The functions φ_1, ψ_1, φ_2 and ψ_2 are in $H^1_\rho(\Omega)$, and z_1, z_2 are positive constants. The operator ${}^{C}\partial_{0t}^{\beta}$ denotes the left Caputo fractional derivative, defined in the second section, where $1 < \beta,\ \gamma < 2$.

In the past few decades, fractional order differential equations have emerged as a fundamental tool in diverse scientific disciplines. Their applications span a wide array of fields, including biology [3, 6–8, 10, 11, 14, 25, 27, 37, 52], chaotic dynamical systems control [5, 9, 12, 15, 17, 40, 48, 53], heat transfer and diffusion processes [2,4,22,23,26,31,44,46,55], financial modeling [35,36,38,47,49,50], thermoelasticity [24, 30, 39, 42, 43, 45, 54] and viscoelasticity [13, 18, 32]. These equations have also significantly contributed to advancements in mechanics, engineering, and seismology. The present mathematical model of fractional pseudo-hyperbolic equations arises mainly in the theory of longitudinal and lateral vibrations of elastic bars (beams). Let us say that the fractional approach, incorporating non-integer derivatives, allows for a more realistic description of memory and hereditary properties in materials and processes. These systems are crucial in modeling complex behaviors in fields such as viscoelasticity, where they capture the time-dependent strain response of materials, and in wave propagation, particularly in heterogeneous or absorptive media. Additionally, they find applications in signal processing and control theory, where the ability to model systems with memory effects is essential. Comparatively, the non-fractional counterparts of these equations, namely hyperbolic and pseudo-hyperbolic systems, have been extensively studied [16, 19–21, 28]. In [51], the authors studied a model which is propounded in the investigation of the unsteady helical flows of a generalized Oldroyd-B fluid with fractional calculus between two infinite coaxial circular cylinders with initial conditions and Dirichlet boundary conditions

$$\begin{cases} \lambda_1^{\alpha}\, {}^{C}\partial_{0t}^{\alpha}u + u_t - \nu\frac{1}{x}(xu_x)_x - \lambda_2^{\eta}\, {}^{C}\partial_{0t}^{\eta}\frac{1}{x}(xu_x)_x = 0, \\ \lambda_1^{\alpha}\, {}^{C}\partial_{0t}^{\gamma}v + v_t - \frac{1}{x}(xv_x)_x - \lambda_2^{\eta}\, {}^{C}\partial_{0t}^{\eta}\frac{1}{x}(xv_x)_{xt} = 0. \end{cases}$$

This model can be considered as a particular case of our model (1.1) with $z_1 = 0$, $z_2 = 0$, $\lambda_1^{\alpha} = \lambda_2^{\eta} = 1$, $\eta = 1$, $f = g = 0$, and the Neumann and integral conditions were replaced by Dirichlet conditions.

We should mention that the primary distinction between linear and nonlinear fractional pseudo-hyperbolic systems lies in their response to superposition. Linear systems adhere to the principle of superposition, where the response to a sum of inputs is the sum of the responses to each input independently. This makes linear systems simpler to analyze and solve. In contrast, nonlinear systems do not follow this principle, leading to more complex behaviors such as bifurcations, chaos and amplitude dependence on the input frequency. Nonlinear systems are more representative of real-world scenarios but pose significant analytical and numerical challenges, requiring sophisticated methods for their investigation and solution. Despite the advancements in understanding fractional pseudo-hyperbolic systems, several research gaps remain. One of the primary areas is the comprehensive analysis of nonlinear systems. The complex dynamics introduced by nonlinearity in fractional systems are not fully understood, especially in multi-dimensional and variable coefficient cases. Furthermore, the development of robust numerical methods for solving these equations efficiently and accurately is still an ongoing area of research. For establishing the well posedness of the nonlinear fractional problem, we used an iterative process. The iterative process presents distinct advantages in solving complex fractional pseudo-hyperbolic systems, especially in the context of nonlinear problems. Unlike direct methods, which may be impractical for nonlinear systems due to their complexity, iterative methods allow for a step by step approximation of the solution, improving accuracy with each iteration. This approach is particularly beneficial in handling the intricacies of nonlinearity, where small changes in input can lead to significant differences in output.

The organization of this paper is as follows: Section 2 lays the foundational groundwork by introducing the requisite function spaces, establishing key inequalities and delineating essential fractional calculus relations, which are pivotal for the subsequent analysis. Section 3 is dedicated to reformulating the fractional linear system, which is intrinsically linked to the nonlinear problem delineated in (1.1), into its operator form. Section 4 is pivotal, as it not only establishes the uniqueness of the solution for the fractional linear system but also discusses the implications of the derived energy estimate (4.1) for the solution. In Section 5, the focus shifts to demonstrating the solvability of the associated linear problem, a critical step in the overall analysis. Finally, in Section 6, on the basis of the results obtained in Sections 4 and 5, and on the use of an iterative process, we prove the existence and uniqueness of the solution of the fractional nonlinear system (1.1).

2. Function spaces and preliminaries

Let $L^2(0,T;L^2_\rho(\Omega))$ be the space consisting of all measurable functions $Q:[0,T]\to L^2_\rho(\Omega)$ with scalar product

$$(Q,Q^*)_{L^2(0,T;L^2_\rho(\Omega))}=\int_0^T (Q,Q^*)_{L^2_\rho(\Omega)}dt, \tag{2.1}$$

and with the associated finite norm

$$\|Q\|^2_{L^2(0,T;L^2_\rho(\Omega))}=\int_0^T \|Q\|^2_{L^2_\rho(\Omega)}dt, \tag{2.2}$$

and we denote by $L^2(0,T;H^1{}_\rho(\Omega))$ the space of functions which are square integrable in the Bochner sense, with the inner product

$$(Q, Q^*)_{L^2(0,T;H^1{}_\rho(\Omega))} = \int_0^T (Q(.,t), Q^*(.,t)_{H^1{}_\rho(\Omega)} dt, \tag{2.3}$$

and the associated norm is

$$||Q||^2_{L^2(0,T;H^1{}_\rho(\Omega))} = \int_0^T ||Q(.,t)||^2_{L^2_\rho(\Omega))} dt + \int_0^T ||Q_x(.,t)||^2_{L^2_\rho(\Omega))} dt. \tag{2.4}$$

We also introduce the fractional functional space $\mathcal{W}^\lambda(Q)$ having the inner product

$$(Q, Q^*)_{\mathcal{W}^\lambda(Q)} = \int_0^T (Q(.t), Q^*(.,t))_{H^1{}_\rho(\Omega)} dt + \int_0^T ({}^C\partial^\lambda_{0t} Q(.,t),\ {}^C\partial^\lambda_{0t} Q^*(.,t))_{H^1{}_\rho(\Omega)} dt, \tag{2.5}$$

and with norm

$$||Q||^2_{\mathcal{W}^\lambda(Q)} = ||Q||^2_{L^2(0,T;H^1{}_\rho(\Omega))} + ||\ {}^C\partial^\lambda_{0t} Q||^2_{L^2(0,T;H^1{}_\rho(\Omega))}. \tag{2.6}$$

We denote by $C(0, T; L^2(\Omega))$ the set of all continuous functions $V^*(.,t) : [0, T] \to L^2(\Omega)$ with the norm

$$||V^*||^2_{C(0,T;L^2(\Omega))} = \sup_{0 \le t \le T} ||V^*(.,t)||^2_{L^2(\Omega)} < \infty. \tag{2.7}$$

We recall some definitions of fractional derivatives and fractional integral [41]. Let $\Gamma(\cdot)$ denotes the Gamma function. For any positive integer n where $n - 1 < \alpha < n$, the Caputo derivative and fractional integral of order α are respectively defined by the left Caputo derivative

$${}^C\partial^\alpha_{0t} v(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{v^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad \forall t \in [0, T], \tag{2.8}$$

the right Caputo derivative

$${}^C\partial^\alpha_{T} v(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^T \frac{v^{(n)}(\tau)}{(\tau - t)^{\alpha-n+1}} d\tau, \quad \forall t \in [0, T], \tag{2.9}$$

and the fractional integral

$$I^\alpha_t v(t) = D^{-\alpha}_{0t} v(t) = \frac{1}{\Gamma(\alpha)} \int_t^T \frac{v(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad \forall t \in [0, T]. \tag{2.10}$$

Lemma 2.1. *[1] Let a nonnegative absolutely continuous function $\mathcal{P}(t)$ satisfy the inequality*

$${}^C\partial^\beta_{0t} \mathcal{P}(t) \le C\mathcal{P}(t) + k(t), \quad 0 < \beta < 1,$$

for almost all $t \in [0, T]$, where C is positive and $k(t)$ is an integrable nonnegative function on $[0, T]$. Then

$$\mathcal{P}(t) \le \mathcal{P}(0) E_\beta(Ct^\beta) + \Gamma(\beta) E_{\beta,\beta}(Ct^\beta) D^{-\beta}_{0t} k(t),$$

where

$$E_\beta(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + 1)} \ \ and \ \ E_{\beta,\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + \alpha)}$$

are the Mittag-Leffler functions.

Lemma 2.2. *[1] For any absolutely continuous function $v(t)$ on $[0, T]$, the following inequality holds:*

$$v(t)\ {}^C\partial_{0t}^\alpha v(t) \geq \frac{1}{2}\ {}^C\partial_{0t}^\alpha v^2(t), \quad 0 < \alpha < 1.$$

We use the following Gronwall-Bellman lemma.

Lemma 2.3. *[29] Let $R(s)$ be nonnegative and absolutely continuous on $[0, T]$, and suppose that for almost all $s \in [0, T]$, the function R satisfies the inequality*

$$\frac{dR}{ds} \leq J(s)R(s) + I(s), \tag{2.11}$$

where the functions $J(s)$ and $I(s)$ are summable and nonnegative on $[0, T]$. Then

$$R(s) \leq \exp\left\{\int_0^s J(t)dt\right\}\left(R(0) + \int_0^s I(t)dt\right). \tag{2.12}$$

We also use the following inequality [1]:

$$D_{0t}^{-\alpha}\|f\|^2_{L^2{}_\rho(\Omega)} \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \|f\|^2_{L^2{}_\rho(\Omega)}d\tau, \tag{2.13}$$

the Cauchy ε-inequality

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \quad \forall\varepsilon > 0, \tag{2.14}$$

where a and b are positive numbers, and the Poincare type inequalities [33]

$$\|\mathcal{J}_x(\xi u)\|^2_{L^2(\Omega)} \leqslant \frac{b^3}{2}\|u(.,t)\|^2_{L^2_\rho(\Omega)}, \tag{2.15}$$

$$\left\|\mathcal{J}_x^2(\xi u)\right\|^2_{L^2(\Omega)} \leqslant \frac{b^2}{2}\,\|\mathcal{J}_x(\xi u)\|^2_{L^2(\Omega)} \leq \frac{b^5}{4}\|u(.,t)\|^2_{L^2_\rho(\Omega)}, \tag{2.16}$$

where

$$\mathcal{J}_x(\xi v) = \int_0^x \xi v(\xi,t)d\xi, \ \ \mathcal{J}_x^2(\xi v) = \int_0^x\int_0^\xi \eta v(\eta,t)d\eta.$$

3. Reformulation of the linear problem

We consider a fractional coupled system of the form

$$\begin{cases} \mathcal{L}_1(u,v) = {}^C\partial_{0t}^{\beta}u - \frac{1}{x}(xu_x)_x - \frac{\partial}{\partial t}\frac{1}{x}(xu_x)_x + z_1 v + u_t = f(x,t), \\ \mathcal{L}_2(u,v) = {}^C\partial_{0t}^{\gamma}v - \frac{1}{x}(xv_x)_x - \frac{\partial}{\partial t}\frac{1}{x}(xv_x)_x + z_2 u + v_t = g(x,t), \end{cases} \tag{3.1}$$

supplemented by the initial conditions

$$\begin{cases} \ell_1 u = u(x,0) = \varphi_1(x), \quad \ell_2 u = u_t(x,0) = \varphi_2(x), \\ \ell_3 v = v(x,0) = \psi_1(x), \quad \ell_4 v = v_t(x,0) = \psi_2(x), \end{cases} \tag{3.2}$$

and the Neumann and integral boundary conditions

$$u_x(b,t) = 0, \ v_x(b,t) = 0, \int_0^b xu dx = 0, \ \int_0^b xv dx = 0. \tag{3.3}$$

We assume that there exists a solution $(u,v) \in (C^{2,2}(\overline{Q}))^2$ consisting of the set of functions together with their partial derivatives of order 2 in x and t, which are continuous on $\overline{Q}$.

The solution of system (3.1)–(3.3) can be regarded as the solution of the operator equation $\mathcal{X}W = \mathcal{F}$, where W, $\mathcal{X}W$ and $\mathcal{F}$ are respectively the pairs $W = (u,v)$, $\mathcal{X}W = (L_1u, L_2v)$, $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$, with

$$L_1 u = \{\mathcal{L}_1 u, \ell_1 u, \ell_2 u\}, \quad L_2 v = \{\mathcal{L}_2 v, \ell_3 v, \ell_4 v\},$$

and

$$\mathcal{F}_1 = \{f, \varphi_1, \varphi_2\}, \quad \mathcal{F}_2 = \{g, \psi_1, \psi_2\}.$$

The operator $\mathcal{X}$ is considered from a space B into a space H, where B is a Banach space consisting of all functions $(u,v) \in \left(L^2(0,T;L^2_\rho(\Omega))\right)^2$ satisfying conditions (3.3) and having the finite norm

$$\|W\|_B^2 = \|u\|^2_{W^\beta(Q_T)} + \|v\|^2_{W^\gamma(Q_T)} + \|u\|^2_{C(0,t.,H^1{}_\rho(\Omega))} + \|v\|^2_{C(0,t.,H^1{}_\rho(\Omega))},$$

and $H = \left(L^2(Q_T)\right)^2 \times \left(H^1_\rho(\Omega)\right)^4$ is the Hilbert space consisting of vector-valued functions $\mathcal{S} = (\{f, \varphi_1, \psi_1\}, \{g, \varphi_2, \psi_2\})$ with norm

$$\|\mathcal{S}\|_H^2 = \|f\|^2_{L^2(0,T;L^2_\rho(\Omega))} + \|g\|^2_{L^2(0,T;L^2_\rho(\Omega))} + \|\varphi_1\|^2_{H^1_\rho(\Omega)} + \|\varphi_2\|^2_{H^1_\rho(\Omega)} + \|\psi_1\|^2_{H^1_\rho(\Omega)} + \|\psi_2\|^2_{H^1_\rho(\Omega)}.$$

Let $D(\mathcal{X})$ be the domain of definition of the operator $\mathcal{X}$, defined by

$$D(\mathcal{X}) = \begin{cases} (u,v) \in \left(L^2(0,T;L^2_\rho(\Omega))\right)^2 \text{ such that } {}^C\partial_{0t}^{\beta}u, \ {}^C\partial_{0t}^{\gamma}v, u_x, \\ v_x, u_{xx}, v_{xx}, u_{tx}, v_{tx}, u_{txx}, v_{txx} \in L^2(0,T;L^2_\rho(\Omega)), \\ u_x(b,t) = 0, \ v_x(b,t) = 0, \ \int_0^b xu dx = 0, \ \int_0^b xv dx = 0. \end{cases}$$

4. Uniqueness of the solution

In this section, we prove the uniqueness result for the fractional system (3.1)–(3.3), that is we establish an energy inequality for the operator $\mathcal{X}$ and we give some of its consequences.

Theorem 4.1. *For any $(u, v) \in D(\mathcal{X})$, f, $g \in L^2(0, T; L^2_\rho(\Omega))$, and $\varphi_1, \psi_1, \varphi_2, \psi_2 \in H^1_\rho(\Omega)$, the solution of the problem (3.1)–(3.3) verifies the a priori bound*

$$\begin{aligned}
&\|u\|^2_{\mathcal{W}^\beta(Q_T)} + \|u\|^2_{\mathcal{W}^\gamma(Q_T)} + \|u\|^2_{C(0,t.,H^1_\rho(\Omega)} + \|v\|^2_{C(0,t.,H^1_\rho(\Omega)} \\
\leq\ &\mathcal{M}\Big(\|f\|^2_{L^2(0,T;L^2_\rho(\Omega))} + \|g\|^2_{L^2(0,T;L^2_\rho(\Omega))} + \|\varphi_1\|^2_{H^1_\rho(\Omega)} + \|\psi_1\|^2_{H^1_\rho(\Omega)} + \|\varphi_2\|^2_{H^1_\rho(\Omega)} + \|\psi_2\|^2_{H^1_\rho(\Omega)}\Big),
\end{aligned} \tag{4.1}$$

where $\mathcal{M} = \mathcal{Y}^{**}e^{T\mathcal{Y}^{**}}$ is a positive constant with

$$\begin{cases}
\mathcal{Y}^{**} = \max\left(1, \mathcal{Y}^*\right), \ \mathcal{Y}^* = \frac{\mathcal{Y}}{\min\left(1, \frac{b^2}{4}\right)}, \ \mathcal{Y} = \chi^*\chi \max\left(\frac{T^{\beta-1}}{\Gamma(\beta)}, \frac{T^{\gamma-1}}{\Gamma(\gamma)}\right), \\
\chi^* = \Gamma(\beta-1)E_{\beta-1,\beta-1}(\chi t^{\beta-1}) \max\left\{1, \frac{T^{\beta-1}}{(\beta-1)\Gamma(\beta-1)}\right\}, \ \chi = D^{**}\left(1 + D^{**}e^{D^{**}T}\right), \\
D^{**} = D^* \max\left\{1, \frac{b^4}{2}, \frac{T^{2-\beta}}{(2-\beta)\Gamma(2-\beta)}, \frac{T^{2-\gamma}}{(2-\gamma)\Gamma(2-\gamma)}\right\}, \\
D^* = 2\max\left\{3, \frac{b^6}{8} + \frac{1}{2}, \frac{b^4}{8} + \frac{5}{2}\right\}.
\end{cases} \tag{4.2}$$

Proof. The fractional partial differential equations in (3.1), and the following fractional integro-differential operators:

$$\mathcal{M}_1 u = {}^C\partial^\beta_{0t}u + u_t - \mathcal{J}^2_x(\xi u_t) \text{ and } \mathcal{M}_2 v = {}^C\partial^\gamma_{0t}v + v_t - \mathcal{J}^2_x(\xi v_t),$$

lead to

$$\begin{aligned}
&2\left({}^C\partial^\beta_{0t}u, u_t\right)_{L^2_\rho(\Omega)} - \left({}^C\partial^\beta_{0t}u, \mathcal{J}^2_x(\xi u_t)\right)_{L^2_\rho(\Omega)} - \left(\frac{1}{x}(xu_x)_x, u_t\right)_{L^2_\rho(\Omega)} + \left(\frac{1}{x}(xu_x)_x, \mathcal{J}^2_x(\xi u_t)\right)_{L^2_\rho(\Omega)} \\
&+\left({}^C\partial^\beta_{0t}u, {}^C\partial^\beta_{0t}u\right)_{L^2_\rho(\Omega)} - \left(\frac{1}{x}(xu_x)_x, {}^C\partial^\beta_{0t}u\right)_{L^2_\rho(\Omega)} - \left(\frac{1}{x}(xu_x)_{xt}, {}^C\partial^\beta_{0t}u\right)_{L^2_\rho(\Omega)} \\
&+\left({}^C\partial^\beta_{0t}u, z_1 v\right)_{L^2_\rho(\Omega)} + \left({}^C\partial^\gamma_{0t}v, {}^C\partial^\gamma_{0t}v\right)_{L^2_\rho(\Omega)} - \left(\frac{1}{x}(xv_x)_x, {}^C\partial^\gamma_{0t}v\right)_{L^2_\rho(\Omega)} - \left(\frac{1}{x}(xv_x)_{xt}, {}^C\partial^\gamma_{0t}v\right)_{L^2_\rho(\Omega)} \\
&+2\left({}^C\partial^\gamma_{0t}v, v_t\right)_{L^2_\rho(\Omega)} + \left({}^C\partial^\gamma_{0t}v, z_2 u\right)_{L^2_\rho(\Omega)} - \left(\frac{1}{x}(xu_x)_{xt}, u_t\right)_{L^2_\rho(\Omega)} + \left(\frac{1}{x}(xu_x)_{xt}, \mathcal{J}^2_x(\xi u_t)\right)_{L^2_\rho(\Omega)} \\
&+(z_1 v, u_t)_{L^2_\rho(\Omega)} - \left(z_1 v, \mathcal{J}^2_x(\xi u_t)\right)_{L^2_\rho(\Omega)} - \left({}^C\partial^\gamma_{0t}v, \mathcal{J}^2_x(\xi v_t)\right)_{L^2_\rho(\Omega)} \\
&-\left(\frac{1}{x}(xv_x)_x, v_t\right)_{L^2_\rho(\Omega)} + \left(\frac{1}{x}(xv_x)_x, \mathcal{J}^2_x(\xi v_t)\right)_{L^2_\rho(\Omega)} - \left(\frac{1}{x}(xv_x)_{xt}, v_t\right)_{L^2_\rho(\Omega)} \\
&+\left(\frac{1}{x}(xv_x)_{xt}, \mathcal{J}^2_x(\xi v_t)\right)_{L^2_\rho(\Omega)} + (z_2 u, v_t)_{L^2_\rho(\Omega)} - \left(z_2 u, \mathcal{J}^2_x(\xi v_t)\right)_{L^2_\rho(\Omega)} \\
&+\|u_t\|^2_{L^2_\rho(\Omega)} + \|v_t\|^2_{L^2_\rho(\Omega)} - \left(u_t, \mathcal{J}^2_x(\xi u_t)\right)_{L^2_\rho(\Omega)} - \left(v_t, \mathcal{J}^2_x(\xi v_t)\right)_{L^2_\rho(\Omega)} \\
=\ &(f, u_t)_{L^2_\rho(\Omega)} - \left(f, \mathcal{J}^2_x(\xi u_t)\right)_{L^2_\rho(\Omega)} + (g, v_t)_{L^2_\rho(\Omega)} - (g, \mathcal{J}^2_x(\xi v_t))_{L^2_\rho(\Omega)} + \left(f, {}^C\partial^\beta_{0t}u\right)_{L^2_\rho(\Omega)} + \left(g, {}^C\partial^\gamma_{0t}v\right)_{L^2_\rho(\Omega)}.
\end{aligned} \tag{4.3}$$

Using boundary conditions (3.3), we evaluate the following terms on the LHS of (4.3) as follows:

$$-\left({}^{C}\partial_{0t}^{\beta}u, \mathcal{J}_x^2(\xi u_t)\right)_{L^2_\rho(\Omega)} = \left({}^{C}\partial_{0t}^{\beta}\left(\mathcal{J}_x(\xi u)\right), \mathcal{J}_x(\xi u_t)\right)_{L^2(\Omega)}, \tag{4.4}$$

$$-\left(\frac{1}{x}(xu_x)_x, u_t\right)_{L^2_\rho(\Omega)} = \frac{1}{2}\frac{\partial}{\partial t}\|u_x\|^2_{L^2_\rho(\Omega)}, \tag{4.5}$$

$$\left(\frac{1}{x}(xu_x)_x, \mathcal{J}_x^2(\xi u_t)\right)_{L^2_\rho(\Omega)} = -\left(u_x, \mathcal{J}_x(\xi u_t)\right)_{L^2_\rho(\Omega)}, \tag{4.6}$$

$$-\left(\frac{1}{x}(xu_x)_{xt}, u_t\right)_{L^2_\rho(\Omega)} = \|u_{xt}\|^2_{L^2_\rho(\Omega)}, \tag{4.7}$$

$$\left(\frac{1}{x}(xu_x)_{xt}, \mathcal{J}_x^2(\xi u_t)\right)_{L^2_\rho(\Omega)} = -\left(u_{xt}, \mathcal{J}_x(\xi u_t)\right)_{L^2_\rho(\Omega)}, \tag{4.8}$$

$$-\left(z_1 v, \mathcal{J}_x^2(\xi u_t)\right)_{L^2_\rho(\Omega)} = -z_1\left(\mathcal{J}_x^2(\xi v), u_t\right)_{L^2_\rho(\Omega)}, \tag{4.9}$$

$$\left({}^{C}\partial_{0t}^{\beta}u, {}^{C}\partial_{0t}^{\beta}u\right)_{L^2_\rho(\Omega)} = \|{}^{C}\partial_{0t}^{\beta-1}u_t\|^2_{L^2_\rho(\Omega)}, \tag{4.10}$$

$$-\left(u_t, \mathcal{J}_x^2(\xi u_t)\right)_{L^2_\rho(\Omega)} = \|\mathcal{J}_x(\xi u_t)\|^2_{L^2(\Omega)}, \tag{4.11}$$

$$-\left(v_t, \mathcal{J}_x^2(\xi v_t)\right)_{L^2_\rho(\Omega)} = \|\mathcal{J}_x(\xi v_t)\|^2_{L^2(\Omega)}, \tag{4.12}$$

$$-\left(\frac{1}{x}(xu_x)_x, {}^{C}\partial_{0t}^{\beta}u\right)_{L^2_\rho(\Omega)} = \left({}^{C}\partial_{0t}^{\beta-1}u_t, u_x\right)_{L^2_\rho(\Omega)}, \tag{4.13}$$

$$-\left(\frac{1}{x}(xu_x)_{xt}, {}^{C}\partial_{0t}^{\beta}u\right)_{L^2_\rho(\Omega)} = \left({}^{C}\partial_{0t}^{\beta-1}u_{xt}, u_{xt}\right)_{L^2_\rho(\Omega)}, \tag{4.14}$$

$$\left({}^{C}\partial_{0t}^{\beta}u, z_1 v\right)_{L^2_\rho(\Omega)} = \left({}^{C}\partial_{0t}^{\beta-1}u_t, z_1 v\right)_{L^2_\rho(\Omega)}. \tag{4.15}$$

In the same fashion, and by symmetry, we have the Eqs (4.4)–(4.15) with β replaced by γ, and u replaced by v. Since $0 < \beta - 1 < 1$, then by using Lemma 2.2, we have

$$2\left({}^{C}\partial_{0t}^{\beta}u, u_t\right)_{L^2_\rho(\Omega)} = 2\left({}^{C}\partial_{0t}^{\beta-1}u_t, u_t\right)_{L^2_\rho(\Omega)} \geq {}^{C}\partial_{0t}^{\beta-1}\|u_t\|^2_{L^2_\rho(\Omega)}, \tag{4.16}$$

$$\left({}^{C}\partial_{0t}^{\beta}\left(\mathcal{J}_x(\xi u)\right), \mathcal{J}_x(\xi u_t)\right)_{L^2(\Omega)} = \left({}^{C}\partial_{0t}^{\beta-1}\left(\mathcal{J}_x(\xi u_t)\right), \mathcal{J}_x(\xi u_t)\right)_{L^2(\Omega)} \geq \frac{1}{2}\ {}^{C}\partial_{0t}^{\beta-1}\|\mathcal{J}_x(\xi u_t)\|^2_{L^2_\rho(\Omega)}, \tag{4.17}$$

$$\left({}^{C}\partial_{0t}^{\beta-1}u_{xt}, u_{xt}\right)_{L^2_\rho(\Omega)} \geq \frac{1}{2}\,{}^{C}\partial_{0t}^{\beta-1}\|u_{xt}\|^2_{L^2_\rho(\Omega)}. \tag{4.18}$$

Combination of (4.4)–(4.18) yields

$$\begin{aligned}
&\left\|{}^{C}\partial_{0t}^{\beta-1}u_t\right\|^2_{L^2_\rho(\Omega)} + \left\|{}^{C}\partial_{0t}^{\gamma-1}v_t\right\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\ {}^{C}\partial_{0t}^{\beta-1}\|u_t\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\ {}^{C}\partial_{0t}^{\gamma-1}\|v_t\|^2_{L^2_\rho(\Omega)} \\
&+ \frac{1}{2}\,{}^{C}\partial_{0t}^{\beta-1}\|\mathcal{J}_x(\xi u_t)\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\ {}^{C}\partial_{0t}^{\gamma-1}\|\mathcal{J}_x(\xi v_t)\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\|u_x\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\|v_x\|^2_{L^2_\rho(\Omega)} \\
&+\frac{1}{2}\,{}^{C}\partial_{0t}^{\beta-1}\|u_{xt}\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\ {}^{C}\partial_{0t}^{\gamma-1}\|v_{xt}\|^2_{L^2_\rho(\Omega)} + \|u_t\|^2_{L^2_\rho(\Omega)} + \|v_t\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x(\xi u_t)\|^2_{L^2(\Omega)} + \|\mathcal{J}_x(\xi v_t)\|^2_{L^2(\Omega)}
\end{aligned}$$

$$
\begin{aligned}
\leq\;& (f,u_t)_{L^2_\rho(\Omega)} - \left(f,\mathcal{J}_x^2(\xi u_t)\right)_{L^2_\rho(\Omega)} + (g,v_t)_{L^2_\rho(\Omega)} - (g,\mathcal{J}_x^2(\xi v_t))_{L^2_\rho(\Omega)} + \left(f,\ {}^C\partial_{0t}^{\beta-1}u_t\right)_{L^2_\rho(\Omega)} \\
&+\left(g,\ {}^C\partial_{0t}^{\gamma-1}v_t\right)_{L^2_\rho(\Omega)} - \left({}^C\partial_{0t}^{\beta-1}u_t, z_1 v\right)_{L^2{}_\rho(\Omega)} - \left({}^C\partial_{0t}^{\gamma-1}v_t, z_2 u\right)_{L^2{}_\rho(\Omega)} - \left({}^C\partial_{0t}^{\beta-1}u_t, u_x\right)_{L^2{}_\rho(\Omega)} \\
&-\left({}^C\partial_{0t}^{\gamma-1}v_t, v_x\right)_{L^2{}_\rho(\Omega)} - (z_1 v, u_t)_{L^2{}_\rho(\Omega)} + (u_x,\mathcal{J}_x(\xi u_t))_{L^2{}_\rho(\Omega)} + (u_{xt},\mathcal{J}_x(\xi u_t))_{L^2{}_\rho(\Omega)} + z_1\left(\mathcal{J}_x^2(\xi v),u_t\right)_{L^2{}_\rho(\Omega)} \\
&-(z_2 u, v_t)_{L^2{}_\rho(\Omega)} + (v_x,\mathcal{J}_x(\xi v_t))_{L^2{}_\rho(\Omega)} + (v_{xt},\mathcal{J}_x(\xi v_t))_{L^2{}_\rho(\Omega)} + z_2\left(\mathcal{J}_x^2(\xi u),v_t\right)_{L^2{}_\rho(\Omega)}. \qquad (4.19)
\end{aligned}
$$

By applying Cauchy-ε-inequality (2.14) and Poincare type inequalities (2.15) and (2.16) to the right-hand side of (4.19), we obtain the inequalities

$$
\begin{aligned}
(f,u_t)_{L^2_\rho(\Omega)} &\leq \frac{\eta_1}{2}\|f\|^2_{L^2{}_\rho(\Omega)} + \frac{1}{2\eta_1}\|u_t\|^2_{L^2{}_\rho(\Omega)}, && (4.20)\\
-\left(f,\mathcal{J}_x^2(\xi u_t)\right)_{L^2_\rho(\Omega)} &\leq \frac{1}{2\eta_2}\|f\|^2_{L^2{}_\rho(\Omega)} + \frac{\eta_2 b^6}{8}\|u_t\|^2_{L^2{}_\rho(\Omega)}, && (4.21)\\
-z_1(v,u_t)_{L^2{}_\rho(\Omega)} &\leq \frac{z_1^2}{2\eta_3}\|v\|^2_{L^2{}_\rho(\Omega)} + \frac{\eta_3}{2}\|u_t\|^2_{L^2{}_\rho(\Omega)}, && (4.22)\\
(u_x,\mathcal{J}_x(\xi u_t))_{L^2{}_\rho(\Omega)} &\leq \frac{1}{2}\|u_x\|^2_{L^2{}_\rho(\Omega)} + \frac{1}{2}\|\mathcal{J}_x(\xi u_t)\|^2_{L^2{}_\rho(\Omega)}, && (4.23)\\
(u_{xt},\mathcal{J}_x(\xi u_t))_{L^2{}_\rho(\Omega)} &\leq \frac{\eta_4}{2}\|u_{xt}\|^2_{L^2{}_\rho(\Omega)} + \frac{1}{2\eta_4}\|\mathcal{J}_x(\xi u_t)\|^2_{L^2{}_\rho(\Omega)}, && (4.24)\\
z_1\left(\mathcal{J}_x^2(\xi v),u_t\right)_{L^2{}_\rho(\Omega)} &\leq \frac{z_1^2 b^4}{8\eta_5}\|v\|^2_{L^2{}_\rho(\Omega)} + \frac{\eta_5}{2}\|u_t\|^2_{L^2{}_\rho(\Omega)}, && (4.25)\\
\left(f,\ {}^C\partial_{0t}^{\beta-1}u_t\right)_{L^2_\rho(\Omega)} &\leq \frac{\eta_{11}}{2}\|\,{}^C\partial_{0t}^{\beta-1}u_t\|^2_{L^2{}_\rho(\Omega)} + \frac{1}{2\eta_{11}}\|f\|^2_{L^2{}_\rho(\Omega)}, && (4.26)\\
\left(g,\ {}^C\partial_{0t}^{\gamma-1}v_t\right)_{L^2_\rho(\Omega)} &\leq \frac{\eta_{12}}{2}\|\,{}^C\partial_{0t}^{\gamma-1}v_t\|^2_{L^2{}_\rho(\Omega)} + \frac{1}{2\eta_{12}}\|g\|^2_{L^2{}_\rho(\Omega)}, && (4.27)\\
-z_1\left({}^C\partial_{0t}^{\beta-1}u_t, v\right)_{L^2{}_\rho(\Omega)} &\leq \frac{\eta_{13}}{2}\|\,{}^C\partial_{0t}^{\beta-1}u_t\|^2_{L^2{}_\rho(\Omega)} + \frac{z_1^2}{2\eta_{13}}\|v\|^2_{L^2{}_\rho(\Omega)}, && (4.28)\\
-\left({}^C\partial_{0t}^{\gamma-1}v_t, u\right)_{L^2{}_\rho(\Omega)} &\leq \frac{\eta_{14}}{2}\|\,{}^C\partial_{0t}^{\gamma-1}v_t\|^2_{L^2{}_\rho(\Omega)} + \frac{1}{2\eta_{14}}\|u\|^2_{L^2{}_\rho(\Omega)}, && (4.29)\\
-\left({}^C\partial_{0t}^{\beta-1}u_t, u_x\right)_{L^2{}_\rho(\Omega)} &\leq \frac{\eta_{15}}{2}\|\,{}^C\partial_{0t}^{\beta-1}u_t\|^2_{L^2{}_\rho(\Omega)} + \frac{1}{2\eta_{15}}\|u_x\|^2_{L^2{}_\rho(\Omega)}, && (4.30)\\
-z_2\left({}^C\partial_{0t}^{\gamma-1}v_t, v_x\right)_{L^2{}_\rho(\Omega)} &\leq \frac{\eta_{16}}{2}\|\,{}^C\partial_{0t}^{\gamma-1}v_t\|^2_{L^2{}_\rho(\Omega)} + \frac{z_2^2}{2\eta_{15}}\|v_x\|^2_{L^2{}_\rho(\Omega)}, && (4.31)\\
(g,v_t)_{L^2_\rho(\Omega)} &\leq \frac{\eta_6}{2}\|g\|^2_{L^2{}_\rho(\Omega)} + \frac{1}{2\eta_6}\|v_t\|^2_{L^2{}_\rho(\Omega)}, && (4.32)\\
-\left(g,\mathcal{J}_x^2(\xi v_t)\right)_{L^2_\rho(\Omega)} &\leq \frac{1}{2\eta_7}\|g\|^2_{L^2{}_\rho(\Omega)} + \frac{\eta_7 b^6}{8}\|v_t\|^2_{L^2{}_\rho(\Omega)}, && (4.33)\\
-z_2(u,v_t)_{L^2{}_\rho(\Omega)} &\leq \frac{z_2^2}{2\eta_8}\|u\|^2_{L^2{}_\rho(\Omega)} + \frac{\eta_8}{2}\|v_t\|^2_{L^2{}_\rho(\Omega)}, && (4.34)
\end{aligned}
$$

$$
(v_x,\mathcal{J}_x(\xi v_t))_{L^2{}_\rho(\Omega)} \leq \frac{1}{2}\|v_x\|^2_{L^2{}_\rho(\Omega)} + \frac{1}{2}\|\mathcal{J}_x(\xi v_t)\|^2_{L^2{}_\rho(\Omega)}, \qquad (4.35)
$$

$$(v_{xt},\mathcal{J}_x(\xi v_t))_{L^2{}_\rho(\Omega)} \le \frac{\eta_9}{2}\|v_{xt}\|^2_{L^2{}_\rho(\Omega)} + \frac{1}{2\eta_9}\|\mathcal{J}_x(\xi v_t)\|^2_{L^2{}_\rho(\Omega)}, \tag{4.36}$$

$$z_2\left(\mathcal{J}_x^2(\xi u), v_t\right)_{L^2{}_\rho(\Omega)} \le \frac{z_2^2 b^4}{8\eta_{10}}\|u\|^2_{L^2{}_\rho(\Omega)} + \frac{\eta_{10}}{2}\|v_t\|^2_{L^2{}_\rho(\Omega)}. \tag{4.37}$$

By inserting (4.20)–(4.37) into (4.19), and taking $\eta_1 = \eta_2 = \eta_3 = \eta_5 = \eta_6 = \eta_7 = \eta_8 = \eta_{10} = 1$, $\eta_4 = \eta_9 = 1$, $\eta_{11} = \eta_{12} = \eta_{13} = \eta_{14} = \eta_{15} = \eta_{16} = 1/4$, gives

$$\begin{aligned}
&\| {}^C\partial_{0t}^{\beta-1} u_t\|^2_{L^2{}_\rho(\Omega)} + \| {}^C\partial_{0t}^{\gamma-1} v_t\|^2_{L^2{}_\rho(\Omega)} + {}^C\partial_{0t}^{\beta-1}\|u_t\|^2_{L^2{}_\rho(\Omega)} + {}^C\partial_{0t}^{\gamma-1}\|v_t\|^2_{L^2{}_\rho(\Omega)} + \frac{\partial}{\partial t}\|u_x\|^2_{L^2{}_\rho(\Omega)} \\
&+\frac{\partial}{\partial t}\|v_x\|^2_{L^2{}_\rho(\Omega)} + {}^C\partial_{0t}^{\beta-1}\|\mathcal{J}_x(\xi u_t)\|^2_{L^2{}_\rho(\Omega)} + {}^C\partial_{0t}^{\gamma-1}\|\mathcal{J}_x(\xi v_t)\|^2_{L^2{}_\rho(\Omega)} \\
&+ {}^C\partial_{0t}^{\beta-1}\|u_{tx})\|^2_{L^2{}_\rho(\Omega)} + {}^C\partial_{0t}^{\gamma-1}\|v_{tx})\|^2_{L^2{}_\rho(\Omega)} \\
\le\ & D^*\Big(\|u_t\|^2_{L^2{}_\rho(\Omega)} + \|v_t\|^2_{L^2{}_\rho(\Omega)} + \|u\|^2_{L^2{}_\rho(\Omega)} + \|v\|^2_{L^2{}_\rho(\Omega)} + \|u_x\|^2_{L^2{}_\rho(\Omega)} + \|v_x\|^2_{L^2{}_\rho(\Omega)} + \|u_{xt}\|^2_{L^2{}_\rho(\Omega)} \\
&+\|v_{xt}\|^2_{L^2{}_\rho(\Omega)} + \|\mathcal{J}_x(\xi u_t)\|^2_{L^2{}_\rho(\Omega)} + \|\mathcal{J}_x(\xi v_t)\|^2_{L^2{}_\rho(\Omega)} + \|f\|^2_{L^2{}_\rho(\Omega)} + \|g\|^2_{L^2{}_\rho(\Omega)}\Big),
\end{aligned} \tag{4.38}$$

where

$$D^* = 2\max\left\{3, \frac{b^6}{8} + \frac{3}{2}, \frac{(z_1^2 + z_2^2)b^4}{8} + \frac{5}{2}\right\}. \tag{4.39}$$

Replacing t by τ and integrating both sides of (4.38) with respect to τ over $[0, t]$, we obtain

$$\begin{aligned}
&\| {}^C\partial_{0t}^{\beta-1} u_t\|^2_{L^2(0,t;L^2{}_\rho(\Omega))} + \| {}^C\partial_{0t}^{\gamma-1} v_t\|^2_{L^2(0,t;L^2{}_\rho(\Omega))} + D_{0t}^{\beta-2}\left(\|u_t\|^2_{L^2{}_\rho(\Omega)} + \|\mathcal{J}_x(\xi u_t)\|^2_{L^2{}_\rho(\Omega)}\right) \\
&+D_{0t}^{\gamma-2}\left(\|v_t\|^2_{L^2{}_\rho(\Omega)} + \|\mathcal{J}_x(\xi v_t)\|^2_{L^2{}_\rho(\Omega)}\right) + \|u_x\|^2_{L^2{}_\rho(\Omega)} + \|v_x\|^2_{L^2{}_\rho(\Omega)} + D_{0t}^{\beta-2}\|u_{tx}\|^2_{L^2{}_\rho(\Omega)} + D_{0t}^{\gamma-2}\|u_{tx}\|^2_{L^2{}_\rho(\Omega)} \\
\le\ & D^*\left(\int_0^t \left(\|u_s\|^2_{L^2{}_\rho(\Omega)} + \|\mathcal{J}_x(\xi u_s)\|^2_{L^2{}_\rho(\Omega)}\right)ds + \int_0^t \left(\|v_s\|^2_{L^2{}_\rho(\Omega)} + \|\mathcal{J}_x(\xi v_s)\|^2_{L^2{}_\rho(\Omega)}\right)ds\right. \\
&\left.+\int_0^t \left(\|u_x\|^2_{L^2{}_\rho(\Omega)} + \|v_x\|^2_{L^2{}_\rho(\Omega)}\right)ds + \int_0^t \left(\|u\|^2_{L^2{}_\rho(\Omega)} + \|v\|^2_{L^2{}_\rho(\Omega)}\right)ds\right) \\
&+\frac{t^{2-\beta}D^*}{(2-\beta)\Gamma(2-\beta)}\left(\|\varphi_2\|^2_{L^2{}_\rho(\Omega)} + \|\mathcal{J}_x(\xi\varphi_2)\|^2_{L^2{}_\rho(\Omega)}\right) + D^*\left(\int_0^t \|f\|^2_{L^2{}_\rho(\Omega)}ds + \int_0^t \|g\|^2_{L^2{}_\rho(\Omega)}ds\right) \\
&+\frac{t^{2-\gamma}D^*}{(2-\gamma)\Gamma(2-\gamma)}\left(\|\psi_2\|^2_{L^2{}_\rho(\Omega)} + \|\mathcal{J}_x(\xi\psi_2)\|^2_{L^2{}_\rho(\Omega)}\right) + \frac{t^{2-\beta}D^*}{(2-\beta)\Gamma(2-\beta)}\|\frac{\partial\varphi_2}{\partial x}\|^2_{L^2{}_\rho(\Omega)} \\
&+\frac{t^{2-\gamma}D^*}{(2-\gamma)\Gamma(2-\gamma)}\|\frac{\partial\psi_2}{\partial x}\|^2_{L^2{}_\rho(\Omega)} + D^*\left(\|\frac{\partial\varphi_1}{\partial x}\|^2_{L^2{}_\rho(\Omega)} + \|\frac{\partial\psi_1}{\partial x}\|^2_{L^2{}_\rho(\Omega)}\right).
\end{aligned} \tag{4.40}$$

Boundary integral conditions allow us to use the Poincare inequalities

$$\|u\|^2_{L^2_\rho(\Omega)} \le \frac{b^2}{4}\|u_x\|^2_{L^2_\rho(\Omega)}, \quad \|v\|^2_{L^2_\rho(\Omega)} \le \frac{b^2}{4}\|v_x\|^2_{L^2_\rho(\Omega)}, \tag{4.41}$$

to get rid of the fourth integral term on the right-hand side of (4.40), and in the mean time, we use Poincare type inequality (2.15), we then have

$$\begin{aligned}
& \| {}^{C}\partial_{0t}^{\beta} u\|^2_{L^2(0,t;L^2_{\rho}(\Omega))} + \| {}^{C}\partial_{0t}^{\gamma} v\|^2_{L^2(0,t;L^2_{\rho}(\Omega))} + \| {}^{C}\partial_{0t}^{\beta} u_x\|^2_{L^2(0,t;L^2_{\rho}(\Omega))} + \| {}^{C}\partial_{0t}^{\gamma} v_x\|^2_{L^2(0,t;L^2_{\rho}(\Omega))} \\
& + D_{0t}^{\beta-2}\left(\|u_t\|^2_{L^2_{\rho}(\Omega)} + \|\mathcal{J}_x(\xi u_t)\|^2_{L^2_{\rho}(\Omega)}\right) + D_{0t}^{\gamma-2}\left(\|v_t\|^2_{L^2_{\rho}(\Omega)} + \|\mathcal{J}_x(\xi v_t)\|^2_{L^2_{\rho}(\Omega)}\right) + D_{0t}^{\beta-2}\|u_{tx}\|^2_{L^2_{\rho}(\Omega)} \\
& + D_{0t}^{\gamma-2}\|u_{tx}\|^2_{L^2_{\rho}(\Omega)} + \|u_x\|^2_{L^2_{\rho}(\Omega)} + \|v_x\|^2_{L^2_{\rho}(\Omega)} \\
\leq\ & D^{**}\left(\int_0^t \left(\|u_s\|^2_{L^2_{\rho}(\Omega)} + \|\mathcal{J}_x(\xi u_s)\|^2_{L^2_{\rho}(\Omega)}\right) ds + \int_0^t \left(\|v_s\|^2_{L^2_{\rho}(\Omega)} + \|\mathcal{J}_x(\xi v_s)\|^2_{L^2_{\rho}(\Omega)}\right) ds \right. \\
& \left. + \int_0^t \left(\|u_x\|^2_{L^2_{\rho}(\Omega)} + \|v_x\|^2_{L^2_{\rho}(\Omega)}\right) ds \right) + D^{**}\left(\|f\|^2_{L^2(0,t;L^2_{\rho}(\Omega))} + \|g\|^2_{L^2(0,t;L^2_{\rho}(\Omega))} + \|\varphi_1\|^2_{H^1_{\rho}(\Omega)} \right. \\
& \left. + \|\psi_1\|^2_{H^1_{\rho}(\Omega)} + \|\varphi_2\|^2_{H^1_{\rho}(\Omega)} + \|\psi_2\|^2_{H^1_{\rho}(\Omega)}\right), \qquad (4.42)
\end{aligned}$$

where

$$D^{**} = D^{*} \max\left\{1, \frac{b^4}{2}, \frac{T^{2-\beta}}{(2-\beta)\Gamma(2-\beta)}, \frac{T^{2-\gamma}}{(2-\gamma)\Gamma(2-\gamma)}\right\}. \qquad (4.43)$$

If we leave only the last two terms on the left-hand side in inequality (4.42), and use the Gronwall-Bellman Lemma 2.3, with

$$\begin{aligned}
R(t) &= \int_0^t \left(\|u_x\|^2_{L^2_{\rho}(\Omega)} + \|v_x\|^2_{L^2_{\rho}(\Omega)}\right) ds, \quad R(0) = 0, \\
\frac{\partial R(t)}{\partial t} &= \|u_x\|^2_{L^2_{\rho}(\Omega)} + \|v_x\|^2_{L^2_{\rho}(\Omega)},
\end{aligned}$$

we obtain

$$\begin{aligned}
R(t) \leq\ & D^{**} e^{D^{**}T}\left(\int_0^t \left(\|u_s\|^2_{L^2_{\rho}(\Omega)} + \|\mathcal{J}_x(\xi u_s)\|^2_{L^2_{\rho}(\Omega)}\right) ds + \int_0^t \left(\|v_s\|^2_{L^2_{\rho}(\Omega)} + \|\mathcal{J}_x(\xi v_s)\|^2_{L^2_{\rho}(\Omega)}\right) ds \right. \\
& \left. + \|f\|^2_{L^2(0,t;L^2_{\rho}(\Omega))} + \|g\|^2_{L^2(0,t;L^2_{\rho}(\Omega))} + \|\varphi_1\|^2_{H^1_{\rho}(\Omega)} + \|\psi_1\|^2_{H^1_{\rho}(\Omega)} + \|\varphi_2\|^2_{H^1_{\rho}(\Omega)} + \|\psi_2\|^2_{H^1_{\rho}(\Omega)}\right). \qquad (4.44)
\end{aligned}$$

Now by keeping only the fifth and sixth terms on the left-hand side of (4.42), and by taking into account the inequality (4.44), we have

$$\begin{aligned}
& D_{0t}^{\beta-2}\left(\|u_t\|^2_{L^2_{\rho}(\Omega)} + \|\mathcal{J}_x(\xi u_t)\|^2_{L^2_{\rho}(\Omega)}\right) + D_{0t}^{\gamma-2}\left(\|v_t\|^2_{L^2_{\rho}(\Omega)} + \|\mathcal{J}_x(\xi v_t)\|^2_{L^2_{\rho}(\Omega)}\right) \\
\leq\ & \chi\left(\int_0^t \left(\|u_s\|^2_{L^2_{\rho}(\Omega)} + \|\mathcal{J}_x(\xi u_s)\|^2_{L^2_{\rho}(\Omega)}\right) ds + \int_0^t \left(\|v_s\|^2_{L^2_{\rho}(\Omega)} + \|\mathcal{J}_x(\xi v_s)\|^2_{L^2_{\rho}(\Omega)}\right) ds \right. \\
& \left. + \|f\|^2_{L^2(0,t;L^2_{\rho}(\Omega))} + \|g\|^2_{L^2(0,t;L^2_{\rho}(\Omega))} + \|\varphi_1\|^2_{H^1_{\rho}(\Omega)} + \|\psi_1\|^2_{H^1_{\rho}(\Omega)} + \|\varphi_2\|^2_{H^1_{\rho}(\Omega)} + \|\psi_2\|^2_{H^1_{\rho}(\Omega)}\right), \qquad (4.45)
\end{aligned}$$

where

$$\chi = D^{**}\left(1 + D^{**} e^{D^{**}T}\right). \qquad (4.46)$$

By Lemma 2.1, with

$$\begin{cases} \mathcal{P}_1(t) = \int\limits_0^t \left(\|u_s\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x(\xi u_s)\|^2_{L^2_\rho(\Omega)} \right) ds, \ \ \mathcal{P}_1(0) = 0, \\ {}^C\partial_{0t}^{\beta-1}\mathcal{P}_1 = D_{0t}^{\beta-2} \left(\|u_t\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x(\xi u_t)\|^2_{L^2_\rho(\Omega)} \right), \\ \mathcal{P}_2(t) = \int\limits_0^t \left(\|v_s\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x(\xi v_s)\|^2_{L^2_\rho(\Omega)} \right) ds, \ \ \mathcal{P}_2(0) = 0, \\ {}^C\partial_{0t}^{\gamma-1}\mathcal{P}_2 = D_{0t}^{\gamma-2} \left(\|v_t\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x(\xi v_t)\|^2_{L^2_\rho(\Omega)} \right), \end{cases} \tag{4.47}$$

we see from (4.45), that

$$\begin{aligned} & \mathcal{P}_1(t) + \mathcal{P}_2(t) \\ \leq \ & \chi^* \left(D^{-\beta}\|f\|^2_{L^2_\rho(\Omega)} + D^{-\gamma}\|g\|^2_{L^2_\rho(\Omega))} + \|\varphi_1\|^2_{H^1_\rho(\Omega)} + \|\psi_1\|^2_{H^1_\rho(\Omega)} + \|\varphi_2\|^2_{H^1_\rho(\Omega)} + \|\psi_2\|^2_{H^1_\rho(\Omega)} \right), \end{aligned} \tag{4.48}$$

where

$$\begin{aligned} \chi^* \ = \ & \Gamma(\beta-1)E_{\beta-1,\,\beta-1}(\chi t^{\beta-1}) \max\left\{1, \frac{T^{\beta-1}}{(\beta-1)\Gamma(\beta-1)}\right\} \\ & +\Gamma(\gamma-1)E_{\gamma-1,\gamma-1}(\chi t^{\gamma-1}) \max\left\{1, \frac{T^{\gamma-1}}{(\gamma-1)\Gamma(\gamma-1)}\right\}. \end{aligned} \tag{4.49}$$

Owing to the inequalities

$$D^{-\beta}\|f\|^2_{L^2_\rho(\Omega)} \leq \frac{t^{\beta-1}}{\Gamma(\beta)} \int\limits_0^t \|f\|^2_{L^2_\rho(\Omega)} ds, \ \ D^{-\gamma}\|g\|^2_{L^2_\rho(\Omega)} \leq \frac{t^{\gamma-1}}{\Gamma(\gamma)} \int\limits_0^t \|g\|^2_{L^2_\rho(\Omega)} ds, \tag{4.50}$$

we deduce from inequalities (4.42), (4.44) and (4.48) that

$$\begin{aligned} & \| {}^C\partial_{0t}^{\beta} u\|^2_{L^2(0,t;L^2_\rho(\Omega))} + \| {}^C\partial_{0t}^{\beta} u_x\|^2_{L^2(0,t;L^2_\rho(\Omega))} + \| {}^C\partial_{0t}^{\gamma} v\|^2_{L^2(0,t;L^2_\rho(\Omega))} \\ & +\| {}^C\partial_{0t}^{\gamma} v_x\|^2_{L^2(0,t;L^2_\rho(\Omega))} + \|u_x\|^2_{L^2_\rho(\Omega)} + \|v_x\|^2_{L^2_\rho(\Omega)} \\ \leq \ & \mathcal{Y} \left(\|f\|^2_{L^2(0,t;L^2_\rho(\Omega))} + \|g\|^2_{L^2(0,t;L^2_\rho(\Omega))} + \|\varphi_1\|^2_{H^1_\rho(\Omega)} + \|\psi_1\|^2_{H^1_\rho(\Omega)} + \|\varphi_2\|^2_{H^1_\rho(\Omega)} + \|\psi_2\|^2_{H^1_\rho(\Omega)} \right), \end{aligned} \tag{4.51}$$

where

$$\mathcal{Y} = \chi^* \chi \max\left(\frac{T^{\beta-1}}{\Gamma(\beta)}, \frac{T^{\gamma-1}}{\Gamma(\gamma)} \right). \tag{4.52}$$

By virtue of Poincare inequalities (4.41), and equivalence of norms, the inequality (4.51) takes the form

$$\begin{aligned} & \| {}^C\partial_{0t}^{\beta} u\|^2_{L^2(0,t;H^1_\rho(\Omega))} + \| {}^C\partial_{0t}^{\gamma} v\|^2_{L^2(0,t;H^1_\rho(\Omega))} + \|u\|^2_{H^1_\rho(\Omega)} + \|v\|^2_{H^1_\rho(\Omega)} \\ \leq \ & \mathcal{Y}^* \left(\|f\|^2_{L^2(0,t;L^2_\rho(\Omega))} + \|g\|^2_{L^2(0,t;L^2_\rho(\Omega))} + \|\varphi_1\|^2_{H^1_\rho(\Omega)} + \|\psi_1\|^2_{H^1_\rho(\Omega)} + \|\varphi_2\|^2_{H^1_\rho(\Omega)} + \|\psi_2\|^2_{H^1_\rho(\Omega)} \right), \end{aligned} \tag{4.53}$$

where

$$\mathcal{Y}^* = \frac{\mathcal{Y}}{\min\left(1, \frac{b^2}{4}\right)}. \tag{4.54}$$

Now by adding the quantity $\|u\|^2_{L^2(0,t;H^1{}_\rho(\Omega))} + \|v\|^2_{L^2(0,t;H^1{}_\rho(\Omega))}$ to both sides of (4.53), we have

$$\begin{aligned}
& \| {}^C\partial^\beta_{0t} u\|^2_{L^2(0,t;H^1{}_\rho(\Omega))} + \|u\|^2_{L^2(0,t;H^1{}_\rho(\Omega))} + \|u\|^2_{H^1{}_\rho(\Omega)} + \| {}^C\partial^\gamma_{0t} v\|^2_{L^2(0,t;H^1{}_\rho(\Omega))} \\
& + \|v\|^2_{L^2(0,t;H^1{}_\rho(\Omega))} + \|v\|^2_{H^1{}_\rho(\Omega)} \\
\leq\ & \mathcal{Y}^{**}\Big(\|f\|^2_{L^2(0,t;L^2{}_\rho(\Omega))} + \|g\|^2_{L^2(0,t;L^2{}_\rho(\Omega))} + \|\varphi_1\|^2_{H^1{}_\rho(\Omega)} + \|\psi_1\|^2_{H^1{}_\rho(\Omega)} + \|\varphi_2\|^2_{H^1{}_\rho(\Omega)} \\
& + \|\psi_2\|^2_{H^1{}_\rho(\Omega)} + \|u\|^2_{L^2(0,t;H^1{}_\rho(\Omega))} + \|v\|^2_{L^2(0,t;H^1{}_\rho(\Omega))}\Big), \qquad (4.55)
\end{aligned}$$

where

$$\mathcal{Y}^{**} = \max\left(1, \mathcal{Y}^{*}\right). \qquad (4.56)$$

Application of Gronwall's lemma to (4.55) gives the inequality

$$\begin{aligned}
& \|u\|^2_{\mathcal{W}^\beta(Q_t)} + \|v\|^2_{\mathcal{W}^\gamma(Q_t)} + \|u\|^2_{H^1{}_\rho(\Omega)} + \|v\|^2_{H^1{}_\rho(\Omega)} \\
\leq\ & \mathcal{Y}^{**} e^{T\mathcal{Y}^{**}}\Big(\|f\|^2_{L^2(0,T;L^2{}_\rho(\Omega))} + \|g\|^2_{L^2(0,T;L^2{}_\rho(\Omega))} + \|\varphi_1\|^2_{H^1{}_\rho(\Omega)} + \|\psi_1\|^2_{H^1{}_\rho(\Omega)} \\
& + \|\varphi_2\|^2_{H^1{}_\rho(\Omega)} + \|\psi_2\|^2_{H^1{}_\rho(\Omega)}\Big). \qquad (4.57)
\end{aligned}$$

The independence of the right-hand side on t in (4.57), gives

$$\begin{aligned}
& \|u\|^2_{\mathcal{W}^\beta(Q_T)} + \|u\|^2_{\mathcal{W}^\gamma(Q_T)} + \|u\|^2_{C(0,T.,H^1{}_\rho(\Omega)} + \|v\|^2_{C(0,T.,H^1{}_\rho(\Omega)} \\
\leq\ & \mathcal{M}\Big(\|f\|^2_{L^2(0,T;L^2{}_\rho(\Omega))} + \|g\|^2_{L^2(0,T;L^2{}_\rho(\Omega))} + \|\varphi_1\|^2_{H^1{}_\rho(\Omega)} + \|\psi_1\|^2_{H^1{}_\rho(\Omega)} \\
& + \|\varphi_2\|^2_{H^1{}_\rho(\Omega)} + \|\psi_2\|^2_{H^1{}_\rho(\Omega)}\Big), \qquad (4.58)
\end{aligned}$$

where $\mathcal{M} = \mathcal{Y}^{**} e^{T\mathcal{Y}^{**}}$.

It can be proved in a standard way that the operator $\mathcal{X} : B \to H$ is closable. Let $\overline{\mathcal{X}}$ be its closure.

Proposition 4.1. *The operator $\mathcal{X} : B \to H$ has a closure.*

Proof. The proof can be established in a similar way as in [34].

These are some consequences of Theorem 4.1.

Corollary 4.1. *There exists a positive constant C such that*

$$\|W\|_B \leq C\|\overline{\mathcal{X}}W\|_H, \quad \forall W \in D(\overline{\mathcal{X}}), \qquad (4.59)$$

where $C = \sqrt{C_7}$.

The inequality (4.59) means that inequality (4.1) can be extended to strong solutions after passing to limit.

We can deduce from inequality (4.59) that a strong solution of the system (3.1)–(3.3) is unique and depends continuously on $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) \in H$, where $\mathcal{F}_1 = \{f, \varphi_1, \varphi_2\}$ and $\mathcal{F}_2 = \{g, \psi_1, \psi_2\}$, and that the image $R(\overline{\mathcal{X}})$ of $\overline{\mathcal{X}}$ is closed in H and $R(\overline{\mathcal{X}}) = \overline{R(\mathcal{X})}$. So in order to prove that the system (3.1)–(3.3) has a strong solution for arbitrary $(\mathcal{F}_1, \mathcal{F}_2) \in H$, it is sufficient to prove that the range of $\mathcal{X}$ is dense in H, that is $\overline{R(\mathcal{X})} = H$.

5. Existence of the solution of the linear system

Proposition 5.1. *If for some function: $Y^*(x,t) = (y_1^*(x,t), y_2^*(x,t)) \in (L^2(0,T;L^2_\rho(\Omega)))^2$, and for all $W(x,t) = (u(x,t), v(x,t)) \in D_0(\mathcal{X}) = \{W/W \in D(\mathcal{X}) : \ell_1 u = 0, \ell_2 u = 0, \ell_3 v = 0, \ell_4 v = 0\}$, we have*

$$(\mathcal{L}W, Y^*)_{L^2(0,T;L^2_\rho(\Omega))} = (\mathcal{L}_1 u, y_1^*)_{L^2(0,T;L^2_\rho(\Omega))} + (\mathcal{L}_2 v, y_2^*)_{L^2(0,T;L^2_\rho(\Omega))} = 0, \tag{5.1}$$

then Y^* vanishes a.e in the domain Q.

Proof. We first set

$$W = (u,v) = (\mathcal{J}_t^2(p_1), \mathcal{J}_t^2(p_2)), \tag{5.2}$$

$$Y^*(x,t) = (y_1^*(x,t), y_2^*(x,t)) = (\mathcal{J}_t(p_1) - \mathcal{J}_x^2(\xi\mathcal{J}_t(p_1)), \mathcal{J}_t(p_2) - \mathcal{J}_x^2(\xi\mathcal{J}_t(p_2))), \tag{5.3}$$

where

$$\mathcal{J}_t(p_i) = \int_0^t p_i(x,s)ds, \quad \mathcal{J}_t^2(p_i) = \int_0^t\int_0^s p_i(x,z)dzds,$$

$$\mathcal{J}_x^2(\xi\mathcal{J}_t(p_i)) = \int_0^x\int_0^\xi\int_0^t \eta\, p_i(\eta,s)dsd\eta d\xi, \quad i = 1,2.$$

We suppose that the functions $p_i(x,t)$ satisfy conditions (3.3) and such that

$$p_i,\ p_{ix},\ \mathcal{J}_t(p_i),\ \mathcal{J}_t^2(p_i),\ x\mathcal{J}_t^2(p_{ix}),\ \mathcal{J}_x^2(\xi\mathcal{J}_t(p_i)),\ {}^C\partial_{0t}^\beta p_i,\ {}^C\partial_{0t}^\gamma p_i \in L^2(Q),\ \ i = 1,2.$$

Now by replacing (5.2) and (5.3) in the relation (5.1), we obtain

$$\begin{aligned}
&\left({}^C\partial_{0t}^\beta\left(\mathcal{J}_t^2(p_1)\right), \mathcal{J}_t(p_1)\right)_{L^2{}_\rho(\Omega)} - \left(\left(x\left(\mathcal{J}_t^2(p_{1x})\right)\right)_x, \mathcal{J}_t(p_1)\right)_{L^2(\Omega)} - \left(\left(x\left(\mathcal{J}_t^2(p_{1x})\right)\right)_{xt}, \mathcal{J}_t(p_1)\right)_{L^2(\Omega)}\\
&+\left(\mathcal{J}_t^2(p_2), \mathcal{J}_t(p_1)\right)_{L^2{}_\rho(\Omega)} - \left({}^C\partial_{0t}^\beta\left(\mathcal{J}_t^2(p_1)\right), \mathcal{J}_x^2(\xi\mathcal{J}_t(p_1))\right)_{L^2{}_\rho(\Omega)}\\
&+\left(\left(x\left(\mathcal{J}_t^2(p_{1x})\right)\right)_x, \mathcal{J}_x^2(\xi\mathcal{J}_t(p_1))\right)_{L^2(\Omega)} + \left(\left(x\left(\mathcal{J}_t^2(p_{1x})\right)\right)_{xt}, \mathcal{J}_x^2(\xi\mathcal{J}_t(p_1))\right)_{L^2(\Omega)}\\
&-\left(\mathcal{J}_t^2(p_2), \mathcal{J}_x^2(\xi\mathcal{J}_t(p_1))\right)_{L^2{}_\rho(\Omega)} + \left({}^C\partial_{0t}^\gamma\left(\mathcal{J}_t^2(p_2)\right), \mathcal{J}_t(p_2)\right)_{L^2{}_\rho(\Omega)} - \left(\left(x\left(\mathcal{J}_t^2(p_{2x})\right)\right)_x, \mathcal{J}_t(p_2)\right)_{L^2(\Omega)}\\
&-\left(\left(x\left(\mathcal{J}_t^2(p_{2x})\right)\right)_{xt}, \mathcal{J}_t(p_2)\right)_{L^2(\Omega)} + \left(\mathcal{J}_t^2(p_1), \mathcal{J}_t(p_2)\right)_{L^2{}_\rho(\Omega)} - \left({}^C\partial_{0t}^\gamma\left(\mathcal{J}_t^2(p_2)\right), \mathcal{J}_x^2(\xi\mathcal{J}_t(p_2))\right)_{L^2{}_\rho(\Omega)}\\
&+\left(\left(x\left(\mathcal{J}_t^2(p_{2x})\right)\right)_x, \mathcal{J}_x^2(\xi\mathcal{J}_t(p_2))\right)_{L^2(\Omega)} + \left(\left(x\left(\mathcal{J}_t^2(p_{2x})\right)\right)_{xt}, \mathcal{J}_x^2(\xi\mathcal{J}_t(p_2))\right)_{L^2(\Omega)}\\
&-\left(\mathcal{J}_t^2(p_1), \mathcal{J}_x^2(\xi\mathcal{J}_t(p_2))\right)_{L^2{}_\rho(\Omega)} = 0.
\end{aligned} \tag{5.4}$$

Since

$$\|\mathcal{J}_t^2(p_i)\|^2_{L^2{}_\rho(\Omega)} \le \frac{T^2}{2}\|\mathcal{J}_t(p_i)\|^2_{L^2{}_\rho(\Omega)},\ i = 1,2,$$

then, using conditions (3.3), and computation of each term of (5.4), gives

$$\left({}^C\partial_{0t}^\beta\left(\mathcal{J}_t^2(p_1)\right), \mathcal{J}_t(p_1)\right)_{L^2{}_\rho(\Omega)} = \left({}^C\partial_{0t}^{\beta-1}\left(\mathcal{J}_t(p_1)\right), \mathcal{J}_t(p_1)\right)_{L^2{}_\rho(\Omega)} \ge \frac{1}{2}\ {}^C\partial_{0t}^{\beta-1}\|\mathcal{J}_t(p_1)\|^2_{L^2{}_\rho(\Omega)}, \tag{5.5}$$

$$-\left(\left(x\left(\mathcal{J}_t^2(p_{1x})\right)\right)_x, \mathcal{J}_t(p_1)\right)_{L^2(\Omega)} = \frac{1}{2}\frac{\partial}{\partial t}\|\mathcal{J}_t^2(p_{1x})\|^2_{L^2_\rho(\Omega)}, \tag{5.6}$$

$$-\left(\left(x\left(\mathcal{J}_t^2(p_{1x})\right)\right)_{xt}, \mathcal{J}_t(p_1)\right)_{L^2(\Omega)} = \|\mathcal{J}_t(p_{1x})\|^2_{L^2_\rho(\Omega)}, \tag{5.7}$$

$$\begin{aligned}-\left(\mathcal{J}_t^2(p_2), \mathcal{J}_t(p_1)\right)_{L^2_\rho(\Omega)} &\le \frac{1}{2}\|\mathcal{J}_t^2(p_2)\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\|\mathcal{J}_t(p_1)\|^2_{L^2_\rho(\Omega)} \\ &\le \frac{T^2}{4}\|\mathcal{J}_t(p_2)\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\|\mathcal{J}_t(p_1)\|^2_{L^2_\rho(\Omega)},\end{aligned} \tag{5.8}$$

$$\begin{aligned}-\left({}^C\partial_{0t}^{\beta}\left(\mathcal{J}_t^2(p_1)\right), \mathcal{J}_x^2\left(\xi\mathcal{J}_t(p_1)\right)\right)_{L^2_\rho(\Omega)} &= -\left({}^C\partial_{0t}^{\beta-1}\left(\mathcal{J}_t(p_1)\right), \mathcal{J}_x^2\left(\xi\mathcal{J}_t(p_1)\right)\right)_{L^2_\rho(\Omega)} \\ &= \left({}^C\partial_{0t}^{\beta-1}\left(\mathcal{J}_x\left(\xi\mathcal{J}_t(p_1)\right)\right), \mathcal{J}_x\left(\xi\mathcal{J}_t(p_1)\right)\right)_{L^2(\Omega)} \\ &\ge \frac{1}{2b}\ {}^C\partial_{0t}^{\beta-1}\|\mathcal{J}_x\left(\xi\mathcal{J}_t(p_1)\right)\|^2_{L^2_\rho(\Omega)},\end{aligned} \tag{5.9}$$

$$\begin{aligned}\left(\left(x\left(\mathcal{J}_t^2(p_{1x})\right)\right)_x, \mathcal{J}_x^2\left(\xi\mathcal{J}_t(p_1)\right)\right)_{L^2(\Omega)} &= -\left(\mathcal{J}_t^2(p_{1x}), \mathcal{J}_x\left(\xi\mathcal{J}_t(p_1)\right)\right)_{L^2_\rho(\Omega)} \\ &\le \frac{1}{T^2}\|\mathcal{J}_t^2(p_{1x})\|^2_{L^2_\rho(\Omega)} + \frac{T^2}{4}\|\mathcal{J}_x\left(\xi\mathcal{J}_t(p_1)\right)\|^2_{L^2_\rho(\Omega)} \\ &\le \frac{1}{2}\|\mathcal{J}_t(p_{1x})\|^2_{L^2_\rho(\Omega)} + \frac{T^2}{4}\|\mathcal{J}_x\left(\xi\mathcal{J}_t(p_1)\right)\|^2_{L^2_\rho(\Omega)},\end{aligned} \tag{5.10}$$

$$\begin{aligned}\left(\left(x\left(\mathcal{J}_t^2(p_{1x})\right)\right)_{xt}, \mathcal{J}_x^2\left(\xi\mathcal{J}_t(p_1)\right)\right)_{L^2(\Omega)} &= -\left(\mathcal{J}_t(p_{1x}), \mathcal{J}_x\left(\xi\mathcal{J}_t(p_1)\right)\right)_{L^2_\rho(\Omega)} \\ &\le \frac{1}{2}\|\mathcal{J}_t(p_{1x})\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\|\mathcal{J}_x\left(\xi\mathcal{J}_t(p_1)\right)\|^2_{L^2_\rho(\Omega)},\end{aligned} \tag{5.11}$$

$$\begin{aligned}\left(\mathcal{J}_t^2(p_2), \mathcal{J}_x^2\left(\xi\mathcal{J}_t(p_1)\right)\right)_{L^2_\rho(\Omega)} &\le \frac{1}{2}\|\mathcal{J}_t^2(p_2)\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\|\mathcal{J}_x^2\left(\xi\mathcal{J}_t(p_1)\right)\|^2_{L^2_\rho(\Omega)} \\ &\le \frac{T^2}{4}\|\mathcal{J}_t(p_2)\|^2_{L^2_\rho(\Omega)} + \frac{b^6}{8}\|\mathcal{J}_t(p_1)\|^2_{L^2_\rho(\Omega)}.\end{aligned} \tag{5.12}$$

Combination of (5.5)–(5.12) and (5.4) yields

$$\begin{aligned}&{}^C\partial_{0t}^{\beta-1}\left(\|\mathcal{J}_t(p_1)\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x\left(\xi\mathcal{J}_t(p_1)\right)\|^2_{L^2_\rho(\Omega)}\right) + \frac{\partial}{\partial t}\|\mathcal{J}_t^2(p_{1x})\|^2_{L^2_\rho(\Omega)} \\ &+ {}^C\partial_{0t}^{\gamma-1}\left(\|\mathcal{J}_t(p_2)\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x\left(\xi\mathcal{J}_t(p_2)\right)\|^2_{L^2_\rho(\Omega)}\right) + \frac{\partial}{\partial t}\|\mathcal{J}_t^2(p_{2x})\|^2_{L^2_\rho(\Omega)} \\ \le\ & M_1\left(\|\mathcal{J}_t(p_1)\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x\left(\xi\mathcal{J}_t(p_1)\right)\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_t(p_2)\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x\left(\xi\mathcal{J}_t(p_2)\right)\|^2_{L^2_\rho(\Omega)}\right),\end{aligned} \tag{5.13}$$

where

$$M_1 = \frac{\max\left\{T^2, 1+\frac{b^6}{4}, 1+\frac{T^2}{2}\right\}}{\min\left\{1, \frac{1}{b}\right\}}.$$

After integration, we entail from (5.13) that

$$\begin{aligned}
& D_{0t}^{\beta-2}\|\mathcal{J}_t(p_1)\|^2_{L^2_\rho(\Omega)} + D_{0t}^{\beta-2}\|\mathcal{J}_x(\xi\mathcal{J}_t(p_1))\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_t^2(p_{1x})\|^2_{L^2_\rho(\Omega)} \\
& + D_{0t}^{\gamma-2}\|\mathcal{J}_t(p_2)\|^2_{L^2_\rho(\Omega)} + D_{0t}^{\gamma-2}\|\mathcal{J}_x(\xi\mathcal{J}_t(p_2))\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_t^2(p_{2x})\|^2_{L^2_\rho(\Omega)} \\
\leq\ & M_1\left[\int_0^t\left(\|\mathcal{J}_\tau(p_1)\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x(\xi\mathcal{J}_\tau(p_1))\|^2_{L^2_\rho(\Omega)}\right)d\tau\right. \\
& \left. + \int_0^t\left(\|\mathcal{J}_\tau(p_2)\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x(\xi\mathcal{J}_\tau(p_2))\|^2_{L^2_\rho(\Omega)}\right)d\tau\right]. \qquad (5.14)
\end{aligned}$$

If we drop the last four terms on the left-hand side of (5.14), apply Lemma 2.1, and use inequality (2.13), we have

$$\begin{aligned}
& \int_0^t\left(\|\mathcal{J}_\tau(p_1)\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x(\xi\mathcal{J}_\tau(p_1))\|^2_{L^2_\rho(\Omega)}\right)d\tau \\
\leq\ & M_1\Gamma(\beta-1)E_{\beta-1,\beta-1}(M_1T^{\beta-1})D_{0t}^{-\beta}\left(\|\mathcal{J}_\tau(p_2)\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x(\xi\mathcal{J}_\tau(p_2))\|^2_{L^2_\rho(\Omega)}\right). \qquad (5.15)
\end{aligned}$$

Application of inequality (2.13), reduces (5.15) to

$$\begin{aligned}
& \int_0^t\left(\|\mathcal{J}_\tau(p_1)\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x(\xi\mathcal{J}_\tau(p_1))\|^2_{L^2_\rho(\Omega)}\right)d\tau \\
\leq\ & M_2\left(\int_0^t\left(\|\mathcal{J}_\tau(p_2)\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x(\xi\mathcal{J}_\tau(p_2))\|^2_{L^2_\rho(\Omega)}\right)d\tau\right), \qquad (5.16)
\end{aligned}$$

where

$$M_2 = M_1\Gamma(\beta-1)E_{\beta-1,\beta-1}(M_1T^{\beta-1})\frac{T^{\beta-1}}{\Gamma(\beta)}. \qquad (5.17)$$

We infer from inequalities (5.16) and (5.14) that

$$\begin{aligned}
& D_{0t}^{\gamma-2}\|\mathcal{J}_t(p_2)\|^2_{L^2_\rho(\Omega)} + D_{0t}^{\gamma-2}\|\mathcal{J}_x(\xi\mathcal{J}_t(p_2))\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_t^2(p_{2x})\|^2_{L^2_\rho(\Omega)} \\
& + D_{0t}^{\beta-2}\|\mathcal{J}_t(p_1)\|^2_{L^2_\rho(\Omega)} + D_{0t}^{\beta-2}\|\mathcal{J}_x(\xi\mathcal{J}_t(p_1))\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_t^2(p_{1x})\|^2_{L^2_\rho(\Omega)} \\
\leq\ & M_3\left[\int_0^t\left(\|\mathcal{J}_\tau(p_2)\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x(\xi\mathcal{J}_\tau(p_2))\|^2_{L^2_\rho(\Omega)}\right)d\tau\right], \qquad (5.18)
\end{aligned}$$

where

$$M_3 = M_1(1 + M_2).$$

If we now discard the last four terms in the left-hand side of (5.18), and apply Lemma 2.1, we get

$$\int_0^t\left(\|\mathcal{J}_\tau(p_2)\|^2_{L^2_\rho(\Omega)} + \|\mathcal{J}_x(\xi\mathcal{J}_\tau(p_2))\|^2_{L^2_\rho(\Omega)}\right)d\tau \leq M_4\left(D_{0t}^{-\gamma}(0)\right) = 0,$$

with $M_4 = \Gamma(\gamma-1)E_{\gamma-1,\gamma-1}(M_3T^{\gamma-1})$.

Hence, we deduce that $Y^*(x,t) = (y_1^*, y_2^*) = (0,0)$ almost everywhere in the domain Q.

Theorem 5.1. *For any $(f,g) \in \left(L^2_\rho(Q)\right)^2$ and any $(\varphi_1,\psi_1),(\varphi_2,\psi_2) \in \left(H^1_\rho(\Omega)\right)^2$, there exists a unique strong solution $W = \overline{\mathcal{X}}^{-1}\mathcal{F} = \overline{\mathcal{X}^{-1}}\mathcal{F}$ of the system (3.1)–(3.3), where $\mathcal{F} = (\mathcal{F}_1,\mathcal{F}_2) \in H$, $\mathcal{F}_1 = \{f,\varphi_1,\varphi_2\}$, $\mathcal{F}_2 = \{g,\psi_1,\psi_2\}$, $W = (u,v)$ and*

$$\|W\|_B \leq C\|\mathcal{X}W\|_H,$$

for a positive constant C, independent of W.

Proof. We show the validity of $\overline{R(\mathcal{X})} = H$. Since H is a Hilbert space, the equality $\overline{R(\mathcal{X})} = H$ holds if

$$\begin{aligned}
(LW,Y)_H &= (\{L_1(u,v),L_2(u,v)\},\{\mathcal{Y}_1,\mathcal{Y}_2\})_H \\
&= (\{(\mathcal{L}_1(u,v),\ell_1u,\ell_2u),(\mathcal{L}_2(u,v),\ell_3v,\ell_4v)\},\{(y_1,y_2,y_3),(y_4,y_5,y_6)\})_H \\
&= (\mathcal{L}_1(u,v),y_1)_{L^2(0,T;L^2_\rho(\Omega))} + (\ell_1u,y_2)_{H^1_\rho(\Omega)} + (\ell_2u,y_3)_{H^1_\rho(\Omega)} + (\mathcal{L}_2(u,v),y_4)_{L^2(0,T;L^2_\rho(\Omega))} \\
&\quad + (\ell_3v,y_5)_{H^1_\rho(\Omega)} + (\ell_4v,y_6)_{H^1_\rho(\Omega)} \\
&= 0 \qquad (5.19)
\end{aligned}$$

implies that $y_1 = y_2 = y_3 = y_4 = y_5 = y_6 = 0$ almost everywhere in the domain Q, where $(\{y_1,y_2,y_3\},\{y_4,y_5,y_6\}) \in R(\mathcal{X})^\perp$.

By putting $W \in D_0(\mathcal{X})$ in (5.19), we have

$$(\mathcal{L}_1(u,v),y_1)_{L^2(0,T;L^2_\rho(\Omega))} + (\mathcal{L}_2(u,v),y_4)_{L^2(0,T;L^2_\rho(\Omega))} = 0, \qquad (5.20)$$

hence Proposition 5.1 implies that $y_1 = y_4 = 0$. Thus (5.19) implies

$$(\ell_1u,y_2)_{H^1_\rho(\Omega)} + (\ell_2u,y_3)_{H^1_\rho(\Omega)} + (\ell_3v,y_5)_{H^1_\rho(\Omega)} + (\ell_4v,y_6)_{H^1_\rho(\Omega)} = 0, \quad \forall W \in D_0(\mathcal{X}). \qquad (5.21)$$

The four sets ℓ_1u, ℓ_2u, ℓ_3v and ℓ_4v are independent, and the images of the trace operator ℓ_1, ℓ_2, ℓ_3 and ℓ_4 are respectively everywhere dense in the Hilbert spaces $H^1_\rho(\Omega)$, then it follows from (5.21), that $y_2 = y_3 = y_5 = y_6 = 0$ almost everywhere in Q.

6. The nonlinear system

We are now in a position to solve the nonlinear system (1.1). Relying on the results obtained previously, we apply an iterative process to establish the existence and uniqueness of the weak solution of the nonlinear system (1.1). If (u,v) is a solution of system (1.1) and (ψ,ϕ) is a solution of the homogeneous system

$$\begin{cases}
{}^C\partial^\beta_{0t}\psi - \frac{1}{x}(x\psi_x)_x - \frac{1}{x}(x\psi_x)_{xt} + z_1\phi + \psi_t = 0, \\
{}^C\partial^\gamma_{0t}\phi - \frac{1}{x}(x\phi_x)_x - \frac{1}{x}(x\phi_x)_{xt} + z_2\psi + \phi_t = 0, \\
\psi(x,0) = \varphi_1(x), \quad \psi_t(x,0) = \varphi_2(x), \\
\phi(x,0) = \psi_1(x), \quad \phi_t(x,0) = \psi_2(x), \\
\psi_x(b,t) = 0, \quad \phi_x(b,t) = 0, \quad \int\limits_0^b x\psi dx = 0, \quad \int\limits_0^b x\phi dx = 0,
\end{cases} \qquad (6.1)$$

then $(U,V)=(u-\psi, v-\phi)$ is a solution of the system

$$\begin{cases} {}^{C}\partial_{0t}^{\beta}U-\frac{1}{x}(xU_x)_x-\frac{1}{x}(xU_x)_{xt}+z_1V+U_t=F(x,t,U,V,U_x,V_x), \\ {}^{C}\partial_{0t}^{\gamma}V-\frac{1}{x}(xV_x)_x-\frac{1}{x}(xV_x)_{xt}+z_2U+V_t=G(x,t,U,V,U_x,V_x), \\ U(x,0)=0, \quad U_t(x,0)=0, \quad V(x,0)=0, \quad V_t(x,0)=0, \\ \int\limits_0^b xUdx=0, \quad \int\limits_0^b xVdx=0, \ U_x(b,t)=0, \quad V_x(b,t)=0, \end{cases} \tag{6.2}$$

where

$$F(x,t,U,V,U_x,V_x)=f(x,t,U+\psi,V+\phi,U_x+\psi_x,V_x+\phi_x),$$

and

$$G(x,t,U,V,U_x,V_x)=g(x,t,U+\psi,V+\phi,U_x+\psi_x,V_x+\phi_x).$$

The functions F and G are Lipschitzian functions

$$\begin{aligned} & |F(x,t,u_1,v_1,w_1,d_1)-F(x,t,u_2,v_2,w_2,d_2)| \\ \leq \ & \delta_1(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|+|d_1-d_2|), \end{aligned} \tag{6.3}$$

$$\begin{aligned} & |G(x,t,u_1,v_1,w_1,d_1)-G(x,t,u_2,v_2,w_2,d_2)| \\ \leq \ & \delta_2(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|+|d_1-d_2|), \end{aligned} \tag{6.4}$$

for all $(x,t)\in Q$.

According to Theorem 5.1, system (6.1) has a unique solution that depends continuously on $(\varphi_1,\varphi_2,\psi_1,\psi_2)\in\left(H_\rho^1(\Omega)\right)^4$.

We must prove that the system (6.2) admits a unique solution.

Suppose that $w,U,V\in C^2(Q)$, such that

$$w(x,T)=0, \quad w_t(x,T)=0, \quad \int\limits_0^b xw(x,t)dx=0. \tag{6.5}$$

Consider the identity

$$\begin{aligned} & (\mathcal{L}_1(U,V),\mathcal{J}_x(\xi w))_{L^2(0,T;L_\rho^2(\Omega))}+(\mathcal{L}_2(U,V),\mathcal{J}_x(\xi w))_{L^2(0,T;L_\rho^2(\Omega))} \\ = \ & (F,\mathcal{J}_x(\xi w))_{L^2(0,T;L_\rho^2(\Omega))}+(G,\mathcal{J}_x(\xi w))_{L^2(0,T;L_\rho^2(\Omega))}. \end{aligned} \tag{6.6}$$

In light of the above assumptions, we obtain

$$\left({}^{C}\partial_{0t}^{\beta}U,\mathcal{J}_x(\xi w)\right)_{L^2(0,T;L_\rho^2(\Omega))}=\left(U,\partial_{tT}^{\beta}\left(\mathcal{J}_x(\xi w)\right)\right)_{L^2(0,T;L_\rho^2(\Omega))}, \tag{6.7}$$

$$-\left(\frac{1}{x}(xU_x)_x,\mathcal{J}_x(\xi w)\right)_{L^2(0,T;L_\rho^2(\Omega))}=(U_x,xw)_{L^2(0,T;L_\rho^2(\Omega))}, \tag{6.8}$$

$$-\left(\frac{1}{x}(xU_x)_{xt},\mathcal{J}_x(\xi w)\right)_{L^2(0,T;L_\rho^2(\Omega))}=-(U_x,xw_t)_{L^2(0,T;L_\rho^2(\Omega))}, \tag{6.9}$$

$$(z_1V,\mathcal{J}_x(\xi w))_{L^2(0,T;L_\rho^2(\Omega))}=-z_1(\mathcal{J}_x(\xi V),w)_{L^2(0,T;L_\rho^2(\Omega))}, \tag{6.10}$$

$$(U_t,\mathcal{J}_x(\xi w))_{L^2(0,T;L_\rho^2(\Omega))}=-(U,\mathcal{J}_x(\xi w_t))_{L^2(0,T;L_\rho^2(\Omega))} \tag{6.11}$$

$$(F,\mathcal{J}_x(\xi w))_{L^2(0,T;L_\rho^2(\Omega))}=-(\mathcal{J}_x(\xi F),w)_{L^2(0,T;L_\rho^2(\Omega))}. \tag{6.12}$$

Using the symmetry in the system, and inserting equations (6.7)–(6.12) into (6.6), yields

$$\begin{aligned}
&\left(U, \partial_{tT}^{\beta}\left(\mathcal{J}_x(\xi w)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} + \left(V, \partial_{tT}^{\gamma}\left(\mathcal{J}_x(\xi w)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} + (U_x, xw)_{L^2(0,T;L^2_\rho(\Omega))} \\
&+ (V_x, xw)_{L^2(0,T;L^2_\rho(\Omega))} - (U_x, xw_t)_{L^2(0,T;L^2_\rho(\Omega))} - (V_x, xw_t)_{L^2(0,T;L^2_\rho(\Omega))} - z_1\left(\mathcal{J}_x(\xi V), w\right)_{L^2(0,T;L^2_\rho(\Omega))} \\
&- z_2\left(\mathcal{J}_x(\xi U), w\right)_{L^2(0,T;L^2_\rho(\Omega))} - (U, \mathcal{J}_x(\xi w_t))_{L^2(0,T;L^2_\rho(\Omega))} - (V, \mathcal{J}_x(\xi w_t))_{L^2(0,T;L^2_\rho(\Omega))} \\
= \;& (F, \mathcal{J}_x(\xi w))_{L^2(0,T;L^2_\rho(\Omega))} + (G, \mathcal{J}_x(\xi w))_{L^2(0,T;L^2_\rho(\Omega))}. \qquad (6.13)
\end{aligned}$$

We write (6.13) in the form

$$A(w, U, V) = (w, \mathcal{J}_x(\xi F))_{L^2(0,T;L^2_\rho(\Omega))} + (w, \mathcal{J}_x(\xi G))_{L^2(0,T;L^2_\rho(\Omega))}, \qquad (6.14)$$

where $A(w, U, V)$ denotes the left-hand side of (6.13).

Definition 6.1. *A function $(U, V) \in (L^2(0, T; H^1_\rho(\Omega)))^2$ is called a weak solution of problem (6.2) if (6.14) and conditions $U_x(b, t) = 0, V_x(b, t) = 0$ hold.*

We now consider the iterated system

$$\begin{cases}
{}^C\partial_{0t}^{\beta} U^{(n)} + -\frac{1}{x}\left(xU_x^{(n)}\right)_x - \frac{1}{x}\left(xU_x^{(n)}\right)_{xt} + z_1 V^{(n)} + U_t^{(n)} = F\left(x, t, U^{(n-1)}, V^{(n-1)}, U_x^{(n-1)}, V_x^{(n-1)}\right), \\
{}^C\partial_{0t}^{\gamma} V^{(n)} + -\frac{1}{x}\left(xV_x^{(n)}\right)_x - \frac{1}{x}\left(xV_x^{(n)}\right)_{xt} + z_2 U^{(n)} + V_t^{(n)} = G\left(x, t, U^{(n-1)}, V^{(n-1)}, U_x^{(n-1)}, V_x^{(n-1)}\right), \\
U^{(n)}(x, 0) = 0, \quad U_t^{(n)}(x, 0) = 0, \quad V^{(n)}(x, 0) = 0, \quad V_t^{(n)}(x, 0) = 0, \\
\int_0^b xU^{(n)} dx = 0, \quad \int_0^b xV^{(n)} dx = 0, \quad U_x^{(n)}(b, t) = 0, \quad V_x^{(n)}(b, t) = 0,
\end{cases} \qquad (6.15)$$

where the iterated sequence $\left\{U^{(n)}, V^{(n)}\right\}_{n\geq 0}$ is constructed in the following way: Given $\left(U^{(0)}, V^{(0)}\right) = (0, 0)$ and the element $\left(U^{(n-1)}, V^{(n-1)}\right)$, then for $n = 1, 2, \ldots$, we solve the problem (6.15). According to Theorem 5.1, for fixed n, each problem (6.15) has a unique solution $\left(U^{(n)}, V^{(n)}\right)$.

If we set $\left(\mathcal{U}^{(n)}(x, t), \mathcal{V}^{(n)}(x, t)\right) = \left(U^{(n+1)}(x, t) - U^{(n)}(x, t), V^{(n+1)}(x, t) - V^{(n)}(x, t)\right)$, then we have the new problem

$$\begin{cases}
{}^C\partial_{0t}^{\beta} \mathcal{U}^{(n)} + -\frac{1}{x}\left(x\mathcal{U}_x^{(n)}\right)_x - \frac{1}{x}\left(x\mathcal{U}_x^{(n)}\right)_{xt} + z_1 \mathcal{V}^{(n)} + \mathcal{U}_t^{(n)} = H_1^{(n-1)}(x, t), \\
{}^C\partial_{0t}^{\gamma} \mathcal{V}^{(n)} + -\frac{1}{x}\left(x\mathcal{V}_x^{(n)}\right)_x - \frac{1}{x}\left(x\mathcal{V}_x^{(n)}\right)_{xt} + z_2 \mathcal{U}^{(n)} + \mathcal{V}_t^{(n)} = H_2^{(n-1)}(x, t), \\
\mathcal{U}^{(n)}(x, 0) = 0, \quad \mathcal{U}_t^{(n)}(x, 0) = 0, \quad \mathcal{V}^{(n)}(x, 0) = 0, \quad \mathcal{V}_t^{(n)}(x, 0) = 0, \\
\int_0^b x\mathcal{U}^{(n)} dx = 0, \quad \int_0^b x\mathcal{V}^{(n)} dx = 0, \quad \mathcal{U}_x^{(n)}(b, t) = 0, \quad \mathcal{V}_x^{(n)}(b, t) = 0,
\end{cases} \qquad (6.16)$$

where

$$H_1^{(n-1)}(x, t) = F\left(x, t, U^{(n)}, U_x^{(n)}, V^{(n)}, V_x^{(n)}\right) - F\left(x, t, U^{(n-1)}, U_x^{(n-1)}, V^{(n-1)}, V_x^{(n-1)}\right), \qquad (6.17)$$

$$H_2^{(n-1)}(x, t) = G\left(x, t, U^{(n)}, U_x^{(n)}, V^{(n)}, V_x^{(n)}\right) - G\left(x, t, U^{(n-1)}, U_x^{(n-1)}, V^{(n-1)}, V_x^{(n-1)}\right). \qquad (6.18)$$

Lemma 6.1. *Assume that conditions (6.3) and (6.4) hold, then for the fractional linearized system (6.16), we have the a priori estimate*

$$\|\mathcal{U}^{(n)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} + \|\mathcal{V}^{(n)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} \le K^*\left(\|\mathcal{U}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} + \|\mathcal{V}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))}\right), \tag{6.19}$$

where K^* is a positive constant given by

$$K^* = 4\mathcal{Y}^{**}e^{T\mathcal{Y}^{**}}T\left(\delta_1^2 + \delta_2^2\right). \tag{6.20}$$

Proof. The consideration of the inner products in $L^2_\rho(\Omega)$ of the PDEs in (6.16) and the fractional integro-differential operators

$$\begin{aligned}\mathcal{M}_1\mathcal{U}^{(n)} &= {}^C\partial^\beta_{0t}\mathcal{U}^{(n)} + \mathcal{U}^{(n)}_t - \mathcal{J}^2_x(\xi\mathcal{U}^{(n)}_t),\\ \mathcal{M}_2\mathcal{V}^{(n)} &= {}^C\partial^\gamma_{0t}\mathcal{V}^{(n)} + \mathcal{V}^{(n)}_t - \mathcal{J}^2_x(\xi\mathcal{V}^{(n)}_t),\end{aligned}$$

respectively, gives the equation

$$\begin{aligned}
&\left({}^C\partial^\beta_{0t}\mathcal{U}^{(n)},\ {}^C\partial^\beta_{0t}\mathcal{U}^{(n)} + \mathcal{U}^{(n)}_t - \mathcal{J}^2_x(\xi\mathcal{U}^{(n)}_t)\right)_{L^2{}_\rho(\Omega)} - \left(\frac{1}{x}\left(x\mathcal{U}^{(n)}_x\right)_x,\ {}^C\partial^\beta_{0t}\mathcal{U}^{(n)} + \mathcal{U}^{(n)}_t - \mathcal{J}^2_x(\xi\mathcal{U}^{(n)}_t)\right)_{L^2{}_\rho(\Omega)}\\
&-\left(\frac{1}{x}\left(x\mathcal{U}^{(n)}_x\right)_{xt},\ {}^C\partial^\beta_{0t}\mathcal{U}^{(n)} + \mathcal{U}^{(n)}_t - \mathcal{J}^2_x(\xi\mathcal{U}^{(n)}_t)\right)_{L^2{}_\rho(\Omega)} + \left(z_1\mathcal{V}^{(n)},\ {}^C\partial^\beta_{0t}\mathcal{U}^{(n)} + \mathcal{U}^{(n)}_t - \mathcal{J}^2_x(\xi\mathcal{U}^{(n)}_t)\right)_{L^2{}_\rho(\Omega)}\\
&+\left(\mathcal{U}^{(n)}_t,\ {}^C\partial^\beta_{0t}\mathcal{U}^{(n)} + \mathcal{U}^{(n)}_t - \mathcal{J}^2_x(\xi\mathcal{U}^{(n)}_t)\right)_{L^2{}_\rho(\Omega)} + \left({}^C\partial^\gamma_{0t}\mathcal{V}^{(n)},\ {}^C\partial^\gamma_{0t}\mathcal{V}^{(n)} + \mathcal{V}^{(n)}_t - \mathcal{J}^2_x(\xi\mathcal{V}^{(n)}_t)\right)_{L^2{}_\rho(\Omega)}\\
&-\left(\frac{1}{x}\left(x\mathcal{V}^{(n)}_x\right)_x,\ {}^C\partial^\gamma_{0t}\mathcal{V}^{(n)} + \mathcal{V}^{(n)}_t - \mathcal{J}^2_x(\xi\mathcal{V}^{(n)}_t)\right)_{L^2{}_\rho(\Omega)}\\
&-\left(\frac{1}{x}\left(x\mathcal{V}^{(n)}_x\right)_{xt},\ {}^C\partial^\gamma_{0t}\mathcal{V}^{(n)} + \mathcal{V}^{(n)}_t - \mathcal{J}^2_x(\xi\mathcal{V}^{(n)}_t)\right)_{L^2{}_\rho(\Omega)} + \left(z_2\mathcal{U}^{(n)},\ {}^C\partial^\gamma_{0t}\mathcal{V}^{(n)} + \mathcal{V}^{(n)}_t - \mathcal{J}^2_x(\xi\mathcal{V}^{(n)}_t)\right)_{L^2{}_\rho(\Omega)}\\
&+\left(\mathcal{V}^{(n)}_t,\ {}^C\partial^\gamma_{0t}\mathcal{V}^{(n)} + \mathcal{V}^{(n)}_t - \mathcal{J}^2_x(\xi\mathcal{V}^{(n)}_t)\right)_{L^2{}_\rho(\Omega)}\\
=\ &\left(H^{(n-1)}_1,\ {}^C\partial^\beta_{0t}\mathcal{U}^{(n)} + \mathcal{U}^{(n)}_t - \mathcal{J}^2_x(\xi\mathcal{U}^{(n)}_t)\right)_{L^2_\rho(\Omega)} + \left(H^{(n-1)}_2,\ {}^C\partial^\gamma_{0t}\mathcal{V}^{(n)} + \mathcal{V}^{(n)}_t - \mathcal{J}^2_x(\xi\mathcal{V}^{(n)}_t)\right)_{L^2_\rho(\Omega)}.
\end{aligned} \tag{6.21}$$

As in the proof of Theorem 4.1, we obtain

$$\begin{aligned}&\|\mathcal{U}^{(n)}\|^2_{\mathcal{W}^\beta(Q_t)} + \|\mathcal{V}^{(n)}\|^2_{\mathcal{W}^\gamma(Q_t)} + \|\mathcal{U}^{(n)}\|^2_{H^1{}_\rho(\Omega)} + \|\mathcal{V}^{(n)}\|^2_{H^1{}_\rho(\Omega)}\\ \le\ &\mathcal{Y}^{**}e^{T\mathcal{Y}^{**}}\left(\int_0^T \|H^{(n-1)}_1\|^2_{L^2{}_\rho(\Omega)}d\tau + \int_0^T \|H^{(n-1)}_2\|^2_{L^2{}_\rho(\Omega)}d\tau\right).\end{aligned} \tag{6.22}$$

By dropping the first two terms on the left hand side of (6.22), to get

$$\|\mathcal{U}^{(n)}\|^2_{H^1{}_\rho(\Omega)} + \|\mathcal{V}^{(n)}\|^2_{H^1{}_\rho(\Omega)} \le \mathcal{Y}^{**}e^{T\mathcal{Y}^{**}}\left(\int_0^T \|H^{(n-1)}_1\|^2_{L^2{}_\rho(\Omega)}d\tau + \int_0^T \|H^{(n-1)}_2\|^2_{L^2{}_\rho(\Omega)}d\tau\right). \tag{6.23}$$

According to conditions (6.3) and (6.4), we estimate the right-hand side of (6.23) to obtain

$$\int_0^T \|H_i^{(n-1)}\|^2_{L^2{}_\rho(\Omega)} d\tau \le 4\delta_i^2 \left(\|\mathcal{U}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} + \|\mathcal{V}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} \right), \quad i = 1,2. \tag{6.24}$$

Hence, inequality (6.23) becomes

$$\|\mathcal{U}^{(n)}\|^2_{H^1{}_\rho(\Omega)} + \|\mathcal{V}^{(n)}\|^2_{H^1{}_\rho(\Omega)} \le 4\mathcal{Y}^{**} e^{T\mathcal{Y}^{**}} \left(\delta_1^2 + \delta_2^2\right) \left(\|\mathcal{U}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} + \|\mathcal{V}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} \right). \tag{6.25}$$

By integrating both sides of (6.25) with respect to t over the interval $[0, T]$, we obtain

$$\|\mathcal{U}^{(n)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} + \|\mathcal{V}^{(n)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} \le K^* \left(\|\mathcal{U}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} + \|\mathcal{V}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} \right), \tag{6.26}$$

where K^* is given by (6.20). This achieves the proof of Lemma 6.1.

Theorem 6.1. *Suppose that conditions (6.3) and (6.4) hold, and $K^* < 1/4$, then the nonlinear fractional system (6.2) admits a weak solution in $L^2(0, T; H^1_\rho(\Omega))$.*

Proof. From (6.26), we conclude that the series $\sum_{n=1}^{\infty} \mathcal{U}^{(n)}$ and $\sum_{n=1}^{\infty} \mathcal{V}^{(n)}$ converge if $K^* < 1/4$.

Indeed, inequality (6.26) implies

$$\|\mathcal{U}^{(n)}\|_{L^2(0,T;H^1{}_\rho(\Omega))} \le \sqrt{K^*} \left(\|\mathcal{U}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} + \|\mathcal{V}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} \right)^{1/2}, \tag{6.27}$$

$$\|\mathcal{V}^{(n)}\|_{L^2(0,T;H^1{}_\rho(\Omega))} \le \sqrt{K^*} \left(\|\mathcal{U}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} + \|\mathcal{V}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} \right)^{1/2}. \tag{6.28}$$

It follows from (6.27) and (6.28) that

$$\|\mathcal{U}^{(n)}\|_{L^2(0,T;H^1{}_\rho(\Omega))} + \|\mathcal{V}^{(n)}\|_{L^2(0,T;H^1{}_\rho(\Omega))} \le 2\sqrt{K^*} \left(\|\mathcal{U}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} + \|\mathcal{V}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} \right)^{1/2}. \tag{6.29}$$

Now since

$$\|\mathcal{U}^{(n)} + \mathcal{V}^{(n)}\|_{L^2(0,T;H^1{}_\rho(\Omega))} \le \|\mathcal{U}^{(n)}\|_{L^2(0,T;H^1{}_\rho(\Omega))} + \|\mathcal{V}^{(n)}\|_{L^2(0,T;H^1{}_\rho(\Omega))}, \tag{6.30}$$

then, we infer from (6.29) and (6.30) that

$$\begin{aligned} \|\mathcal{U}^{(n)} + \mathcal{V}^{(n)}\|_{L^2(0,T;H^1{}_\rho(\Omega))} &\le 2\sqrt{K^*} \left(\|\mathcal{U}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} + \|\mathcal{V}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} \right)^{1/2} \\ &\le 2\sqrt{K^*} \left(\|\mathcal{U}^{(n-1)} + \mathcal{V}^{(n-1)}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} \right)^{1/2} \\ &= 2\sqrt{K^*} \|\mathcal{U}^{(n-1)} + \mathcal{V}^{(n-1)}\|_{L^2(0,T;H^1{}_\rho(\Omega))}. \end{aligned} \tag{6.31}$$

Inequality (6.31) shows that the series $\sum_{n=1}^{\infty} \left(\mathcal{U}^{(n)} + \mathcal{V}^{(n)} \right) = \sum_{n=1}^{\infty} \mathcal{U}^{(n)} + \sum_{n=1}^{\infty} \mathcal{V}^{(n)}$ converges if $K^* < 1/4$. Since $\left(\mathcal{U}^{(n)}, \mathcal{V}^{(n)} \right) = (U^{(n+1)} - U^{(n)}, V^{(n+1)} - V^{(n)})$, then it follows that the sequence $\left(U^{(n)}, V^{(n)} \right)_{n \in N}$ with $U^{(n)}$ and $V^{(n)}$ defined by

$$U^{(n)}(x,t) = \sum_{k=0}^{n-1} \mathcal{U}^{(k)}(x,t) + U^{(0)}(x,t) = \sum_{k=0}^{n-1} \left(U^{(k+1)} - U^{(k)} \right) + U^{(0)}(x,t), \quad n = 1, 2, ..., \tag{6.32}$$

and

$$V^{(n)}(x,t) = \sum_{k=0}^{n-1} \mathcal{V}^{(k)}(x,t) + V^{(0)}(x,t) = \sum_{k=0}^{n-1} \left(V^{(k+1)} - V^{(k)}\right) + V^{(0)}(x,t), \quad n = 1,2,..., \tag{6.33}$$

converge to an element $(U,V) \in \left(L^2(0,T;H^1_\rho(\Omega))\right)^2$, which must be proved that it is a solution of problem (6.2). In other words, (U,V) must satisfy (6.14) and the Neumann boundary conditions.

From the iterated system (6.15), we have

$$\begin{aligned} A\left(w, U^{(n)}, V^{(n)}\right) &= \left(w, \mathcal{J}_x\left(\xi F\left(\xi, t, U^{(n-1)}, U^{(n-1)}_\xi, V^{(n-1)}, V^{(n-1)}_\xi\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ &\quad + \left(w, \mathcal{J}_x\left(\xi G\left(\xi, t, U^{(n-1)}, U^{(n-1)}_\xi, V^{(n-1)}, V^{(n-1)}_\xi\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))}. \end{aligned} \tag{6.34}$$

We infer from (6.34) that

$$\begin{aligned} & A\left(w, U^{(n)} - U, V^{(n)} - V\right) + A\left(w, U, V\right) \\ = & \left(w, \mathcal{J}_x\left(\xi F\left(\xi, t, U^{(n-1)}, U^{(n-1)}_\xi, V^{(n-1)}, V^{(n-1)}_\xi\right)\right) - \mathcal{J}_x\left(\xi F\left(\xi, t, U, U_\xi, V, V_\xi\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ & + \left(w, \mathcal{J}_x\left(\xi G\left(\xi, t, U^{(n-1)}, U^{(n-1)}_\xi, V^{(n-1)}, V^{(n-1)}_\xi\right)\right) - \mathcal{J}_x\left(\xi G\left(\xi, t, U, U_\xi, V, V_\xi\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ & + \left(w, \mathcal{J}_x\left(\xi F\left(\xi, t, U, U_\xi, V, V_\xi\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} + \left(w, \mathcal{J}_x\left(\xi G\left(\xi, t, U, U_\xi, V, V_\xi\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))}. \end{aligned} \tag{6.35}$$

Now from the FPDEs in (6.15), we obtain

$$\begin{aligned} & A\left(w, U^{(n)} - U, V^{(n)} - V\right) \\ = & \left(w, \ {}^C\partial^\beta_{0t} \mathcal{J}_x\left(\xi\left(U^{(n)} - U\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} - \left(w, \mathcal{J}_x\left(\frac{\partial}{\partial \xi}\left(\xi \frac{\partial}{\partial \xi}\left(U^{(n)} - U\right)\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ & - \left(w, \frac{\partial}{\partial t}\mathcal{J}_x\left(\frac{\partial}{\partial \xi}\left(\xi \frac{\partial}{\partial \xi}\left(U^{(n)} - U\right)\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} + z_1\left(w, \mathcal{J}_x\left(\xi\left(V^{(n)} - V\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ & + \left(w, \frac{\partial}{\partial t}\mathcal{J}_x\left(\xi\left(U^{(n)} - U\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} + \left(w, \ {}^C\partial^\gamma_{0t} \mathcal{J}_x\left(\xi\left(V^{(n)} - V\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ & - \left(w, \mathcal{J}_x\left(\frac{\partial}{\partial \xi}\left(\xi \frac{\partial}{\partial \xi}\left(V^{(n)} - V\right)\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ & - \left(w, \frac{\partial}{\partial t}\mathcal{J}_x\left(\frac{\partial}{\partial \xi}\left(\xi \frac{\partial}{\partial \xi}\left(V^{(n)} - V\right)\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} + z_2\left(w, \mathcal{J}_x\left(\xi\left(U^{(n)} - U\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} \\ & + \left(w, \frac{\partial}{\partial t}\mathcal{J}_x\left(\xi\left(V^{(n)} - V\right)\right)\right)_{L^2(0,T;L^2_\rho(\Omega))}. \end{aligned} \tag{6.36}$$

Conditions on functions w, U, V, and integration of each term on the right-hand side of (6.36), yield

$$\begin{aligned} & A\left(w, U^{(n)} - U, V^{(n)} - V\right) \\ = & -\left(U^{(n)} - U, \ {}^C\partial^\beta_{tT} \mathcal{J}_x\left(\xi w\right)\right)_{L^2(0,T;L^2_\rho(\Omega))} - \left(\frac{\partial}{\partial x}\left(U^{(n)} - U\right), xw\right)_{L^2(0,T;L^2_\rho(\Omega))} \end{aligned}$$

$$+\left(\frac{\partial}{\partial x}\left(U^{(n)}-U\right),xw_t\right)_{L^2(0,T;L^2_\rho(\Omega))}-z_1\left(V^{(n)}-V,\mathcal{J}_x(\xi w)\right)_{L^2(0,T;L^2_\rho(\Omega))}$$
$$+\left(U^{(n)}-U,\mathcal{J}_x(\xi w_t)\right)_{L^2(0,T;L^2_\rho(\Omega))}+\left(V^{(n)}-V,\mathcal{J}_x(\xi w_t)\right)_{L^2(0,T;L^2_\rho(\Omega))}$$
$$-\left(V^{(n)}-V,\quad {}^C\partial^\gamma_{tT}\mathcal{J}_x(\xi w)\right)_{L^2(0,T;L^2_\rho(\Omega))}-\left(\frac{\partial}{\partial x}\left(V^{(n)}-V\right),xw\right)_{L^2(0,T;L^2_\rho(\Omega))}$$
$$+\left(\frac{\partial}{\partial x}\left(V^{(n)}-V\right),xw_t\right)_{L^2(0,T;L^2_\rho(\Omega))}-z_2\left(U^{(n)}-U,\mathcal{J}_x(\xi w)\right)_{L^2(0,T;L^2_\rho(\Omega))}. \tag{6.37}$$

Application of the Cauchy-Schwarz inequality, to the terms on the right-hand side of (6.37) gives

$$-\left(U^{(n)}-U,\quad {}^C\partial^\beta_{tT}\mathcal{J}_x(\xi w)\right)_{L^2(0,T;L^2_\rho(\Omega))}\leq\left\|U^{(n)}-U\right\|_{L^2(0,T;L^2_\rho(\Omega))}\left\|{}^C\partial^\beta_{tT}\mathcal{J}_x(\xi w)\right\|_{L^2(0,T;L^2_\rho(\Omega))}, \tag{6.38}$$

$$-\left(\frac{\partial}{\partial x}\left(U^{(n)}-U\right),xw\right)_{L^2(0,T;L^2_\rho(\Omega))}\leq b\left\|\frac{\partial}{\partial x}\left(U^{(n)}-U\right)\right\|_{L^2(0,T;L^2_\rho(\Omega))}\|w\|_{L^2(0,T;L^2_\rho(\Omega))}, \tag{6.39}$$

$$+\left(\frac{\partial}{\partial x}\left(U^{(n)}-U\right),xw_t\right)_{L^2(0,T;L^2_\rho(\Omega))}\leq b\left\|\frac{\partial}{\partial x}\left(U^{(n)}-U\right)\right\|_{L^2(0,T;L^2_\rho(\Omega))}\|w_t\|_{L^2(0,T;L^2_\rho(\Omega))}, \tag{6.40}$$

$$-z_1\left(V^{(n)}-V,\mathcal{J}_x(\xi w)\right)_{L^2(0,T;L^2_\rho(\Omega))}\leq z_1\left\|V^{(n)}-V\right\|_{L^2(0,T;L^2_\rho(\Omega))}\|\mathcal{J}_x(\xi w)\|_{L^2(0,T;L^2_\rho(\Omega))}, \tag{6.41}$$

$$-\left(V^{(n)}-V,\quad {}^C\partial^\gamma_{tT}\mathcal{J}_x(\xi w)\right)_{L^2(0,T;L^2(\Omega))}\leq\left\|V^{(n)}-V\right\|_{L^2(0,T;L^2(\Omega))}\left\|{}^C\partial^\gamma_{tT}\mathcal{J}_x(\xi w)\right\|_{L^2(0,T;L^2(\Omega))}, \tag{6.42}$$

$$-\left(\frac{\partial}{\partial x}\left(V^{(n)}-V\right),xw\right)_{L^2(0,T;L^2_\rho(\Omega))}\leq b\left\|\frac{\partial}{\partial x}\left(V^{(n)}-V\right)\right\|_{L^2(0,T;L^2_\rho(\Omega))}\|w\|_{L^2(0,T;L^2_\rho(\Omega))}, \tag{6.43}$$

$$+\left(\frac{\partial}{\partial x}\left(V^{(n)}-V\right),xw_t\right)_{L^2(0,T;L^2_\rho(\Omega))}\leq b\left\|\frac{\partial}{\partial x}\left(V^{(n)}-V\right)\right\|_{L^2(0,T;L^2_\rho(\Omega))}\|w_t\|_{L^2(0,T;L^2_\rho(\Omega))}, \tag{6.44}$$

$$-z_2\left(U^{(n)}-U,\mathcal{J}_x(\xi w)\right)_{L^2(0,T;L^2_\rho(\Omega))}\leq z_2\left\|U^{(n)}-U\right\|_{L^2(0,T;L^2_\rho(\Omega))}\|\mathcal{J}_x(\xi w)\|_{L^2(0,T;L^2_\rho(\Omega))}, \tag{6.45}$$

$$\left(U^{(n)}-U,\mathcal{J}_x(\xi w_t)\right)_{L^2(0,T;L^2_\rho(\Omega))}\leq\left\|U^{(n)}-U\right\|_{L^2(0,T;L^2_\rho(\Omega))}\|\mathcal{J}_x(\xi w_t)\|_{L^2(0,T;L^2_\rho(\Omega)),} \tag{6.46}$$

$$\left(V^{(n)}-V,\mathcal{J}_x(\xi w_t)\right)_{L^2(0,T;L^2_\rho(\Omega))}\leq\left\|V^{(n)}-V\right\|_{L^2(0,T;L^2_\rho(\Omega))}\|\mathcal{J}_x(\xi w_t)\|_{L^2(0,T;L^2_\rho(\Omega))}. \tag{6.47}$$

Combination of equality (6.37) and inequalities (6.38)–(6.47), leads to

$$\begin{aligned}
&A\left(w,U^{(n)}-U,V^{(n)}-V\right)\\
\leq\ & l_1\left(\left\|U^{(n)}-U\right\|_{L^2(0,T;H^1_\rho(\Omega))}\right)\\
&\times\begin{pmatrix}\left\|\partial^\beta_{tT}\mathcal{J}_x(\xi w)\right\|_{L^2(0,T;L^2(\Omega))}+\|\mathcal{J}_x(\xi w)\|_{L^2(0,T;L^2(\Omega))}+\|w\|_{L^2(0,T;L^2(\Omega))}\\ +\|\mathcal{J}_x(\xi w_t)\|_{L^2(0,T;L^2_\rho(\Omega)),}+\|w_t\|_{L^2(0,T;L^2(\Omega))}\end{pmatrix}\\
&+l_2\left(\left\|V^{(n)}-V\right\|_{L^2(0,T;H^1_\rho(\Omega))}\right)\\
&\times\begin{pmatrix}\left\|\partial^\gamma_{tT}\mathcal{J}_x(\xi w)\right\|_{L^2(0,T;L^2(\Omega))}+\|\mathcal{J}_x(\xi w)\|_{L^2(0,T;L^2(\Omega))}+\|w\|_{L^2(0,T;L^2(\Omega))}\\ +\|\mathcal{J}_x(\xi w_t)\|_{L^2(0,T;L^2_\rho(\Omega)),}+\|w_t\|_{L^2(0,T;L^2(\Omega))}\end{pmatrix},
\end{aligned} \tag{6.48}$$

with

$$l_1 = l_2 = \max(1, b, z_1, z_2).$$

On the other side, we have

$$\begin{aligned}&\left(w, \mathcal{J}_x\left(\xi F\left(\xi, t, U^{(n-1)}, U_\xi^{(n-1)}, V^{(n-1)}, V_\xi^{(n-1)}\right)\right) - \mathcal{J}_x\left(\xi F\left(\xi, t, U, U_\xi, V, V_\xi\right)\right)\right)_{L^2(0,T;L^2(\Omega))}\\ \leq\ & \frac{\delta_1 b}{\sqrt{2}} \|w\|_{L^2(0,T;L^2(\Omega))}\left(\left\|U^{(n)} - U\right\|_{L^2(0,T;H_\rho^1(\Omega))} + \left\|V^{(n)} - V\right\|_{L^2(0,T;H_\rho^1(\Omega))}\right),\end{aligned} \tag{6.49}$$

$$\begin{aligned}&\left(w, \mathcal{J}_x\left(\xi G\left(\xi, t, U^{(n-1)}, U_\xi^{(n-1)}, V^{(n-1)}, V_\xi^{(n-1)}\right)\right) - \mathcal{J}_x\left(\xi G\left(\xi, t, U, U_\xi, V, V_\xi\right)\right)\right)_{L^2(0,T;L^2(\Omega))}\\ \leq\ & \frac{\delta_2 b}{\sqrt{2}} \|w\|_{L^2(0,T;L^2(\Omega))}\left(\left\|U^{(n)} - U\right\|_{L^2(0,T;H_\rho^1(\Omega))} + \left\|V^{(n)} - V\right\|_{L^2(0,T;H_\rho^1(\Omega))}\right).\end{aligned} \tag{6.50}$$

As $n \longrightarrow \infty$, it follows from (6.48)–(6.50) and (6.35) that

$$A(w, U, V) = (w, \mathcal{J}_x(\xi F))_{L^2(0,T;L^2(\Omega))} + (w, \mathcal{J}_x(\xi G))_{L^2(0,T;L^2(\Omega))}.$$

To conclude that problem (6.2) admits a weak solution, we must show that conditions $U_x(b,t) = 0$, $V_x(b,t) = 0$ in (6.2) hold. Since $(U, V) \in \left(L^2(0, T; H^1_\rho(\Omega))\right)^2$, then

$$\int_0^t U_x(x, s)ds, \int_0^t V_x(x, s)ds \in C(\overline{Q}),$$

from which we conclude that: $U_x(b,t) = 0$, $V_x(b,t) = 0$, a.e.

It remains now to prove the uniqueness of solution of system (6.2).

Theorem 6.2. *If hypotheses (6.3) and (6.4) are satisfied, then the system (6.2) has only one solution.*

Proof. Suppose that $(U_1, V_1), (U_2, V_2) \in \left(L^2(0, T; H^1_\rho(\Omega))\right)^2$ are two different solutions of the system (6.2), then $(\mathcal{U}, \mathcal{V}) = (U_1 - U_2, V_1 - V_2) \in \left(L^2(0, T; H^1_\rho(\Omega))\right)^2$ verifies

$$\begin{cases} {}^C\partial_{0t}^{\beta}\mathcal{U} - \frac{1}{x}(x\mathcal{U}_x)_x - \frac{1}{x}(x\mathcal{U}_x)_{xt} + \mathcal{V} + \mathcal{U}_n = H_1(x,t), \\ {}^C\partial_{0t}^{\gamma}\mathcal{V} - \frac{1}{x}(x\mathcal{V}_x)_x - \frac{1}{x}(x\mathcal{V}_x)_{xt} + \mathcal{U} + \mathcal{V}_n = H_2(x,t), \\ \mathcal{U}(x,0) = 0, \quad \mathcal{U}_t(x,0) = 0, \quad \mathcal{V}(x,0) = 0, \quad \mathcal{V}_t(x,0) = 0, \\ \int_0^b x\mathcal{U}dx = 0, \quad \int_0^b x\mathcal{V}dx = 0. \quad \mathcal{U}_x(b,t) = 0, \quad \mathcal{V}_x(b,t) = 0, \end{cases} \tag{6.51}$$

where

$$H_1(x,t) = F(x, t, U_1, (U_1)_x, V_1, (V_1)_x) - F(x, t, U_2, (U_2)_x, V_2, (V_2)_x), \tag{6.52}$$

$$H_2(x,t) = G(x, t, U_1, (U_1)_x, V_1, (V_1)_x) - G(x, t, U_2, (U_2)_x, V_2, (V_2)_x). \tag{6.53}$$

We now consider the scalar product in the space $L^2(0, T; L^2(\Omega))$ of the PDEs in (6.51) and the differential operators $\mathcal{M}_1\mathcal{U} = {}^C\partial_{0t}^{\beta}\mathcal{U} + \mathcal{U}_t - \mathcal{J}_x^2(\xi\mathcal{U}_t)$, $\mathcal{M}_2\mathcal{V} = {}^C\partial_{0t}^{\gamma}\mathcal{V} + \mathcal{V}_t - \mathcal{J}_x^2(\xi\mathcal{V}_t)$, and follow the same computations as in Lemma 6.1, we obtain

$$\|\mathcal{U}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} + \|\mathcal{V}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} \leq K^*\left(\|\mathcal{U}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} + \|\mathcal{V}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))}\right), \tag{6.54}$$

where K^* is the same constant as in Lemma 6.1. Since $K^* < 1/4$, we deduce from (6.54) that

$$(1 - K^*)\left(\|\mathcal{U}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))} + \|\mathcal{V}\|^2_{L^2(0,T;H^1{}_\rho(\Omega))}\right) = 0, \tag{6.55}$$

which implies that $(\mathcal{U}, \mathcal{V}) = (U_1 - U_2, V_1 - V_2) = (0, 0)$, and hence

$$U_1 = U_2 \in L^2(0, T; H^1{}_\rho(\Omega)) \text{ and } V_1 = V_2 \in L^2(0, T; H^1{}_\rho(\Omega)).$$

This achieves the proof of Theorem 6.2.

Example 6.1. We consider the following example with $b = 1, \beta = \frac{3}{2}, \gamma = \frac{4}{3}, z_1 = 1, z_2 = 1, f = u^2 - v^2$ and $g = u - v$:

$$\begin{cases} {}^C\partial^{\frac{3}{2}}_{0t}u - \frac{1}{x}(xu_x)_x - \frac{1}{x}\frac{\partial}{\partial t}(xu_x)_x + v + u_t = u^2 - v^2, \\ {}^C\partial^{\frac{4}{3}}_{0t}v - \frac{1}{x}(xv_x)_x - \frac{1}{x}\frac{\partial}{\partial t}(xv_x)_x + u + v_t = u - v, \\ u(x,0) = \varphi_1(x) = \frac{6}{5}x^2 - \frac{12}{5}x + 1, \\ u_t(x,0) = \varphi_2(x) = \frac{5}{8}x^3 - \frac{18}{5}x + 1, \\ v(x,0) = \psi_1(x) = \cos\pi x + \frac{4}{\pi^2}, \\ v_t(x,0) = \psi_2(x) = -\frac{\sin\pi x}{\pi} - (\frac{1}{2} + \frac{\pi^2}{6})\cos\pi x - x, \\ u_x(b,t) = 0, \quad v_x(b,t) = 0, \quad \int\limits_0^b xudx = 0, \quad \int\limits_0^b xvdx = 0. \end{cases} \tag{6.56}$$

We can easily verify that the functions φ_1, φ_2, ψ_1 and ψ_2 satisfy the compatibility conditions

$$\begin{aligned} \left.\frac{\partial\varphi_1}{\partial x}\right|_{x=1} &= 0, \quad \int_0^b x\varphi_1 dx = 0, \quad \left.\frac{\partial\varphi_2}{\partial x}\right|_{x=1} = 0, \quad \int_0^b x\varphi_2 dx = 0, \\ \left.\frac{\partial\psi_1}{\partial x}\right|_{x=1} &= 0, \quad \int_0^b x\psi_1 dx = 0, \quad \left.\frac{\partial\psi_2}{\partial x}\right|_{x=1} = 0, \quad \int_0^b x\psi_2 dx = 0, \end{aligned}$$

and belong to $H^1_\rho(\Omega)$. We also see that $f = u^2 - v^2$ and $g = u - v$ are Lipschitz functions for all $(x, t) \in (0, 1) \times [0, T]$. All conditons are satisfied, problem (6.56) admits a unique solution.

7. Conclusions

A Caputo fractional nonlinear pseudohyperbolic system supplemented by a classical and a nonlocal boundary condition of integral type is investigated. More precisely, in this research work, we search for a function $u(x, t)$ verifying (1.1). The associated fractional linear problem is reformulated, and the uniqueness and existence of the strong solutions are proved in a fractional Sobolev space. A priori bound for the solution is obtained from which the uniqueness of the solution follows. By using some density arguments, the solvability of the linear problem is established. To tackle the well posedness of the fractional nonlinear problem, we relied on the obtained results for the linear fractional system, by applying a certain iterative process. This process is particularly beneficial in handling the intricacies of nonlinearity, where small changes in input can lead to significant differences in output. Our study improves and develops some few existence results for the fractional initial boundary value problems when using the method of functional analysis, the so called energy inequality method. We would like to mention that the application of the used method is a little complicated while dealing with the posed problem in the presence of the nonlinear source terms, the fractional terms, the appearance of the singularity and the nonlocal integral conditions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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