AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(2): 2925–2937.
DOI: 10.3934/math.2024144
Received: 24 August 2023
Revised: 26 November 2023
Accepted: 06 December 2023
Published: 02 January 2024

Research article

On Riemannian warped-twisted product submersions

Richa Agarwal[1], Fatemah Mofarreh[2], Sarvesh Kumar Yadav[3], Shahid Ali[1] and Abdul Haseeb[4,*]

[1] Department of Mathematics, Faculty of Sciences, Aligarh Muslim University, Aligarh-202002, India

[2] Mathematical Science Department, Faculty of Sciences, Princess Nourah Bint Abdulrahman University, Riyadh-11546, Kingdom of Saudi Arabia

[3] Department of Mathematics, Faculty of Sciences, Jamia Millia Islamia, New Delhi-110025, India

[4] Department of Mathematics, College of Science, Jazan University, Jazan-45142, Kingdom of Saudi Arabia

* **Correspondence:** Email: malikhaseeb80@gmail.com, haseeb@jazanu.edu.sa.

Abstract: In this paper, we introduce the concepts of Riemannian warped-twisted product submersions and examine their fundamental properties, including total geodesicity, total umbilicity and minimality. Additionally, we investigate the Ricci tensor of Riemannian warped-twisted product submersions, specifically about the horizontal and vertical distributions. Finally, we obtain Einstein condition for base manifold if the horizontal and vertical distributions of the ambient manifold is Einstein.

Keywords: warped product; warped-twisted product; Riemannian submersion
Mathematics Subject Classification: 53C15, 53C40, 53C50

1. Introduction

In the 1960's, B. O'Neill [1] introduced the notion of Riemannian submersion as a tool to study the geometry of a manifold in terms of the simpler components, namely, fibers and base space. A. L. Besse considered warped product Riemannian submersion [2]. Further, I. K. Erken and C. Murathan [3] studied warped product Riemannian submersion and obtained fundamental geometric properties.

J. F. Nash [4] started the study of warped product manifolds and proved that every warped product manifold can be embedded as a Riemannian submanifold in some Euclidean spaces. In 1969, B. O'Neill and R. L. Bishop [5] studied the warped product manifold as a fruitful generalization of the Riemannian product manifold.

Warped product manifolds play key roles in mathematical physics [6]. H. M. Tastan and S. B. Aydin [7, 8] introduced the concept of a warped-twisted product as an extension of the twisted product and consequently, the warped product. The warped-twisted product, denoted as $\mathcal{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$, refers to the product manifold $\mathcal{M}_1 \times \mathcal{M}_2$ endowed with the metric tensor g, which is defined by

$$g = (f_2 \circ \varphi_2)^2 \varphi_1^* (g_1) + f_1^2 \varphi_2^* (g_2), \tag{1.1}$$

where, $\varphi_i : \mathcal{M}_1 \times \mathcal{M}_2 \longrightarrow \mathcal{M}_i$ is the natural projections, for $i \in \{1, 2\}$. The function $f_2 \in C^\infty (\mathcal{M}_2)$ is named a warping function, and the function $f_1 \in C^\infty (\mathcal{M}_1 \times \mathcal{M}_2)$ is named a twisting function of $\mathcal{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$. If the function f_1 solely depends on the points of $\mathcal{M}_2$ in this instance, the resulting warped-twisted product can be classified as a base conformal warped product [8]. A warped-twisted product is considered non-trivial if it does not fall into any categories of a doubly warped product, a warped product, or a base conformal warped product. For more details about the concerned studies, we refer the papers [9–22].

The following is our definition of warped-twisted product submersions:

Definition 1.1. *Suppose that $\mathcal{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ and $\aleph = {}_{\rho_2}\aleph_1 \times_{\rho_1} \aleph_2$ are warped-twisted product manifolds and $\varphi_i : \mathcal{M}_i \to \aleph_i, i \in \{1, 2\}$, are Riemannian submersion between the manifolds $\mathcal{M}_i$ and $\mathcal{N}_i$. Then the map*

$$\varphi = \varphi_1 \times \varphi_2 : \mathcal{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2 \to \aleph = {}_{\rho_2}\aleph_1 \times_{\rho_1} \aleph_2 \tag{1.2}$$

given by $\varphi(x_1, x_2) = (\varphi_1(x_1), \varphi_2(x_2))$ is a Riemannian submersion, which is called warped-twisted product submersion.

Our primary objective of this paper is to investigate the fundamental geometric properties associated with warped-twisted product submersions. The notion of warped product generalizes usual products, which is further generalized by the twisted product and doubly warped product. Non-trivial wraped-twisted product is neither twisted nor base conformal nor direct product. The definition of warped product submersion and its geometrical properties was discussed by Murathan, C. in his article "Riemannian warped product submersions". These results, which are presented in that article, serve as our motivation.

The paper is organized in the following way. In Section 2, we recall definitions and some fundamental results of Riemannian submersions and warped-twisted product manifolds which are useful for this paper. In Section 3, we defined Riemannian warped-twisted product submersion and discuss some geometrical properties for this submersion. In Section 4, we obtain the Ricci tensors for Riemannian warped-twisted product submerion and discuss Einstein's condition on vertical and horizontal distributions of total manifold.

2. Preliminaries

In this section, we recall some definitions, results and notations that are necessary for the paper.

2.1. Riemannian submersion

Let $(\mathcal{M}, g_\mathcal{M})$ and $(\aleph, g_\aleph)$ be two Riemannian manifolds with $dim\mathcal{M} = m$ and $dim\aleph = n$, where $m > n$. A smooth map $\varphi : (\mathcal{M}, g_\mathcal{M}) \longrightarrow (\aleph, g_\aleph)$ is said to be Riemannian submersion if the following axioms are satisfied:

1) φ_*(derivative map of φ) is onto,
2) φ_* preserves the length of horizontal vectors, i.e.,

$$g_{\aleph}(\varphi_* X, \varphi_* Y) = g_{\mathcal{M}}(X, Y).$$

For each $p_2 \in \aleph$, $\phi^{-1}(p_2)$ is a submanifold of dimension $(m-n)$ called fibers. If the fibers are orthogonal then a vector field on $\mathcal{M}$ is referred to as horizontal and it is referred to as vertical if the fibers are tangent. Let $\varphi : (\mathcal{M}, g_{\mathcal{M}}) \longrightarrow (\aleph, g_{\aleph})$ be a smooth map. Then $\Gamma(T\mathcal{M})$ has the following decomposition:

$$T\mathcal{M} = (ker\varphi_*) \oplus (ker\varphi_*)^{\perp}.$$

B. O'Neill [1] first introduced the fundamental tensors of submersions, and are defined by

$$T(E, F) = T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.1}$$

$$A(E, F) = A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \tag{2.2}$$

where E and F are vector fields on $\mathcal{M}$; $\mathcal{H}$ and $\mathcal{V}$ are the projection morphism on the distribution $(ker\varphi*)^{\perp}$ and $(ker\varphi_*)$, respectively. We observe that the tensor fields T and A satisfy

1) $T_U V = T_V U, \qquad U, V \in \Gamma(ker\varphi_*)$,
2) $A_X Y = -A_Y X, \quad X, Y \in \Gamma(ker\varphi_*)^{\perp}$.

Equations (2.1) and (2.2) give the following lemma.

Lemma 2.1. *[1]. Let $X, Y \in \Gamma(ker\varphi_*)^{\perp}$ and $U, V \in \Gamma(ker\varphi_*)$; then we have*

$$\nabla_U V = T_U V + \hat{\nabla}_U V, \tag{2.3}$$

$$\nabla_U X = \mathcal{H}\nabla_U X + T_U X, \tag{2.4}$$

$$\nabla_X U = A_X U + \mathcal{V}\nabla_X U, \tag{2.5}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y, \tag{2.6}$$

where ∇ is the Levi-Civita connection of $(\mathcal{M}, g_{\mathcal{M}})$ and $\hat{\nabla}_U V = \mathcal{V}\nabla_U V$.

It is noted that if the tensor field A (respectively T) vanishes, then the horizontal distribution $\mathcal{H}$ (respectively, vertical distribution $\mathcal{V}$ or fiber) is integrable. Also, any fiber of Riemannian submersion ϕ is totally umbilical if and only if

$$T_V W = g(V, W)\mathcal{H},$$

where $\mathcal{H}$ is the mean curvature vector field of the fiber given by

$$N = s\mathcal{H},$$

such that

$$N = \sum_{i=1}^{s} T_{U_i} U_i, \tag{2.7}$$

and $\{U_1, U_2, \cdots, U_s\}$ denotes the orthonormal basis of vertical distribution and s denotes the dimension of any fiber. It is easy to see that any fiber of Riemannian submersion ϕ is minimal if and only if the horizontal vector field N vanishes.

2.2. Warped-twisted product manifolds

Let $(\mathcal{M}_i, g_{\mathcal{M}_i})$ be two Riemannian manifolds of dimensions m_1 and m_2, respectively and f_1 and f_2 be two positive differentiable functions on $\mathcal{M}_1$ and $\mathcal{M}_1 \times \mathcal{M}_2$, respectively. Let $\varphi_i : \mathcal{M}_1 \times \mathcal{M}_2 \longrightarrow \mathcal{M}_i$ be the natural projections from product manifold $\mathcal{M}_1 \times \mathcal{M}_2$ to $\mathcal{M}_i$, $i \in \{1, 2\}$. Then the warped-twisted product manifold $\mathcal{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ is a product manifold $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ endowed with the metric $g_{\mathcal{M}}$ such that

$$g_{\mathcal{M}}(X, Y) = (f_2 \circ \varphi_2)^2 g_{\mathcal{M}_1}(\varphi_{1*}(X), \varphi_{1*}(Y)) + {f_1}^2 g_{\mathcal{M}_2}(\varphi_{2*}(X), \varphi_{2*}(Y))$$

for any $X, Y \in \mathcal{M}$.

For any X on $\mathcal{M}_1$, the lift of X to ${}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ is the vector field $\tilde{X}$ whose value at each (p, q) is the lift X_p to (p, q). Thus the lift of X is the unique vector field on ${}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$, that is, φ_1-related to X and φ_2-related to the zero vector field on $\mathcal{M}_2$.

Let ∇ and ∇^i be the Levi-Civita connections of ${}_{f_2}\mathcal{M}_1 \times {}_{f_1}\mathcal{M}_2$ and $\mathcal{M}_i$, respectively for $i \in \{1, 2\}$. The lifts of vector fields on $\mathcal{M}_i$ is denoted by $\mathfrak{L}(\mathcal{M}_i)$.

Then, the covariant derivative formulae for a warped-twisted product manifold are given as [8]:

$$\nabla_X Y = \nabla^1_X Y - g(X, Y)\nabla\left(\ln\left(f_2 \circ \varphi_2\right)\right), \tag{2.8}$$

$$\nabla_X V = \nabla_V X = V\left(\ln\left(f_2 \circ \varphi_2\right)\right) X + X\left(\ln\left(f_1\right)\right) V, \tag{2.9}$$

$$\nabla_U V = \nabla^2_U V + U(\ln f_1)V + V(\ln f_1)U - g(U, V)\nabla\left(\ln\left(f_1\right)\right) \tag{2.10}$$

for $X, Y \in \mathfrak{L}(\mathcal{M}_1)$ and $U, V \in \mathfrak{L}(\mathcal{M}_2)$. Now, for any smooth function ψ on a warped-twisted product $({}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2, g_{\mathcal{M}})$, we have

$$h^{\psi}(X, U) = X(lnf_1)U(\psi) - X(\psi)U(lnf_2)$$

for any $X \in \mathfrak{L}(\mathcal{M}_1)$ and $U \in \mathfrak{L}(\mathcal{M}_2)$, where the definition of the Hessian tensor being used. Now let $\mathbb{S}$ and $\mathbb{S}^i$ be the Ricci tensors of $(\mathcal{M}, g)$ and $(\mathcal{M}_i, g_i)$, respectively. Then we have the following relations:

Lemma 2.2. *[23] Let $\mathbb{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ be a warped-twisted product manifold. Then we have*

$$\begin{aligned}
\mathbb{S}(X, Y) =& \mathbb{S}^1(X, Y) + h^{\ln f_2}(X, Y) - m_2\left\{h_1^{\ln f_1}(X, Y) + X(\ln f_1)Y(\ln f_1)\right\}\\
&- g(X, Y)\{\Delta \ln f_2 + g(\nabla \ln f_2, \nabla \ln f_2)\},\\
\mathbb{S}(X, U) =& (1 - m_2)\, XU(\ln f_1) + (m_1 + m_2 - 2)\, X(\ln f_1)U(\ln f_2),\\
\mathbb{S}(U, V) =& \mathbb{S}^2(U, V) + h^{\ln f_1}(U, V) + (1 - m_2)\, h_2^{\ln f_1}(U, V) + m_2 U(\ln f_1)V(\ln f_1)\\
&- g(U, V)\{\Delta \ln f_1 + g(\nabla \ln f_1, \nabla \ln f_1)\}\\
&- m_1\left\{h_2^{\ln f_2}(U, V) + U(\ln f_2)V(\ln f_2) - U(\ln f_2)V(\ln f_1) - U(\ln f_1)V(\ln f_2)\right\}
\end{aligned}$$

for $X, Y \in \mathcal{L}(\mathcal{M}_1)$ and $U, V \in \mathcal{L}(\mathcal{M}_2)$, where Δ is Laplacian operator and ∇ is gradiant of the function.

3. Riemannian warped-twisted product submersions

In this section, we define Riemannian warped-twisted product submersion and obtain some fruitful results.

Proposition 3.1. *Let* $\mathcal{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ *and* $\aleph = {}_{\rho_2}\aleph_1 \times_{\rho_1} \aleph_2$ *be two warped-twisted product manifolds and let* $\varphi_i : \mathcal{M}_i \to \aleph_i, i \in \{1,2\}$ *be Riemannian submersion between the manifolds* $\mathcal{M}_i$ *and* $\aleph_i$*. Then the map*

$$\varphi = \varphi_1 \times \varphi_2 : \mathcal{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2 \to \aleph = {}_{\rho_2}\aleph_1 \times_{\rho_1} \aleph_2$$

given by $\varphi(x_1, x_2) = (\varphi_1(x_1), \varphi_2(x_2))$ *is a Riemannian submersion, which is called warped-twisted product submersion.*

Proof. Making use of Proposition 2 [3], it is easy to show that the map φ is a Riemannian submersion. Now it is enough to show that the map φ is a warped-twisted product. Since

$$\begin{aligned} g_{\aleph}(\varphi_*(X_1, X_2), \varphi_*(Y_1, Y_2)) &= \rho_2^2 g_{\aleph_1}(\varphi_{1*}(X_1), \varphi_{1*}(Y_1)) \\ &+ \rho_1^2 g_{\aleph_2}(\varphi_{2*}(X_2), \varphi_{2*}(Y_2)) \\ &= (f_2 \circ \varphi_2)^2 g_{\mathcal{M}_1}(X_1, Y_1) + f_1^2 g_{\mathcal{M}_2}(X_2, Y_2) \\ &= g_{\mathcal{M}}((X_1, Y_1), (X_2, Y_2)). \end{aligned}$$

It shows that φ_* preserve the length of the horizontal vector field. Thus φ is a warped-twisted product submersion.

Next, we obtain fundamental tensors for the Riemannian warped-twisted product submersion in the subsequent lemmas:

Lemma 3.1. *Let* $\mathcal{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ *and* $\aleph = {}_{\rho_2}\aleph_1 \times_{\rho_1} \aleph_2$ *are warped-twisted product manifolds and* $\varphi_i : \mathcal{M}_i \to \aleph_i$ *is Riemannian warped-twisted product submersion between the manifolds* $\mathcal{M}_i$ *and* $\aleph_i$*. Then we have*

1) $T_{U_1}V_1 = T^1_{U_1}V_1 - g_{\mathcal{M}}(U_1, V_1)\mathcal{H}\nabla(\ln(f_2 \circ \varphi_2))$,
2) $T_{U_1}U_2 = 0$,
3) $T_{U_2}V_2 = T^2_{U_2}V_2 - g_{\mathcal{M}}(U_2, V_2)\mathcal{H}\nabla(\ln(f_1))$

for any $U_i, V_i \in \Gamma(\mathcal{V}_i)$*,* $i = \{1,2\}$*.*

Proof. From Eq (2.3), we get

$$\nabla_{U_1}V_1 = \hat{\nabla}_{U_1}V_1 + T_{U_1}V_1. \tag{3.1}$$

By using Eq (2.8), we obtain

$$\nabla_{U_1}V_1 = \nabla^1_{U_1}V_1 - g_{\mathcal{M}}(U_1, V_1)\nabla(\ln(f_2 \circ \varphi_2)). \tag{3.2}$$

Using Eq (2.3) in Eq (3.2) and combining the result with Eq (3.1), we get result 1).
By using Eq (2.3), we obtain

$$\nabla_{U_1}V_1 = \hat{\nabla}_{U_1}V_1 + T_{U_1}V_1. \tag{3.3}$$

Making use of Eq (2.9), we have

$$\nabla_{U_1} U_2 = \nabla_{U_2} U_1 = U_2 (\ln (f_2 \circ \pi_2)) U_1 + U_1 (\ln (f_1)) U_2. \tag{3.4}$$

Combining Eq (3.3) with Eq (3.4), we get

$$T_{U_1} U_2 = 0.$$

From Eq (2.1), we obtain

$$T_{U_2} V_2 = \mathcal{H}(\nabla_{U_2} V_2). \tag{3.5}$$

Using Eq (2.10), we get

$$\begin{aligned} \nabla_{U_2} V_2 &= \nabla^2_{U_2} V_2 + U_2(\ln f_1) V_2 + V_2(\ln f_1) U_2 \\ &- g_{\mathcal{M}}(U_2, V_2) \nabla (\ln (f_1)) . \end{aligned} \tag{3.6}$$

From Eq (2.3), we know that

$$\nabla^2_{U_2} V_2 = T^2_{U_2} V_2 + \mathcal{V} \nabla^2_{U_2} V_2. \tag{3.7}$$

By using Eqs (3.6) and (3.7) in Eq (3.5), we get the desired result 3).

Lemma 3.2. *Let $\mathcal{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ and $\aleph = {}_{\rho_2}\aleph_1 \times_{\rho_1} \aleph_2$ are warped-twisted product manifolds and $\varphi_i : \mathcal{M}_i \to \aleph_i$ is Riemannian warped-twisted product submersion between the manifolds $\mathcal{M}_i$ and $\aleph_i$. Then we have*

1) $\mathcal{H}\nabla_{X_1} Y_1 = \mathcal{H}\nabla^1_{X_1} Y_1 - g_{\mathcal{M}}(X_1, Y_1)\mathcal{H}\nabla (\ln (f_2 \circ \varphi_2))$,
   $A_{X_1} Y_1 = A^1_{X_1} Y_1 - g_{\mathcal{M}}(X_1, Y_1)\mathcal{V}\nabla (\ln (f_2 \circ \varphi_2))$,
2) $\mathcal{H}\nabla_{X_1} X_2 = \mathcal{H}\nabla_{X_2} X_1 = X_2(\ln (f_2 \circ \varphi_2)) X_1 + X_1(\ln (f_1)) X_2$,
   $A_{X_2} X_1 = 0 = A_{X_1} X_2$,
3) $A_{X_2} Y_2 = A^2_{X_2} Y_2$ *and* $\mathcal{V}\nabla (\ln (f_1)) = 0$,
   $\mathcal{H}\nabla_{X_2} Y_2 = \mathcal{H}\nabla^2_{X_2} Y_2 + X_2(\ln f_1) Y_2 + Y_2(\ln f_1) X_2 - g_{\mathcal{M}}(X_2, Y_2)\mathcal{H}\nabla (\ln (f_1))$

for any $X_i, Y_i \in \Gamma(\mathcal{H}_i)$, $i = \{1, 2\}$.

Proof. From Eq (2.7), we have

$$\nabla_{X_1} Y_1 = \mathcal{H}(\nabla_{X_1} Y_1) + A_{X_1} Y_1. \tag{3.8}$$

From Eqs (2.8) and (3.8), we get

$$\mathcal{H}\nabla_{X_1} Y_1 + A_{X_1} Y_1 = \mathcal{H}\nabla^1_{X_1} Y_1 + A^1_{X_1} Y_1 - g_{\mathcal{M}}(X_1, Y_1) \nabla (\ln (f_2 \circ \varphi_2)) . \tag{3.9}$$

Separating the horizontal and vertical parts in Eq (3.9), we obtain

$$\begin{aligned} \mathcal{H}\nabla_{X_1} Y_1 &= \mathcal{H}\nabla^1_{X_1} Y_1 - g_{\mathcal{M}}(X_1, Y_1)\mathcal{H}\nabla (\ln (f_2 \circ \varphi_2)) , \\ A_{X_1} Y_1 &= A^1_{X_1} Y_1 - g_{\mathcal{M}}(X_1, Y_1)\mathcal{V}\nabla (\ln (f_2 \circ \varphi_2)) . \end{aligned}$$

From Eq (2.7), we have

$$\begin{aligned} \nabla_{X_1} X_2 &= \mathcal{H}\nabla_{X_1} X_2 + A_{X_1} X_2. && (3.10)\\ \nabla_{X_2} X_1 &= \mathcal{H}\nabla_{X_2} X_1 + A_{X_2} X_1. && (3.11) \end{aligned}$$

From Eq (2.9), we get

$$\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = X_2(\ln(f_2 \circ \varphi_2))\, X_1 + X_1(\ln(f_1))\, X_2. \tag{3.12}$$

Combining Eqs (3.10)–(3.12), we obtain result 2). We know that

$$A_{X_2} Y_2 = \mathcal{V}\nabla_{X_2} Y_2.$$

Using Eq (2.10) in the above equation, we get

$$\begin{aligned} A_{X_2} Y_2 &= \mathcal{V}[\nabla^2_{X_2} Y_2 - g_{\mathcal{M}}(X_2, Y_2)\nabla(\ln(f_1))] \\ &= \mathcal{V}\nabla^2_{X_2} Y_2 - g_{\mathcal{M}}(X_2, Y_2)\mathcal{V}\nabla(\ln(f_1)) \\ &= A^2_{X_2} Y_2 - g_{\mathcal{M}}(X_2, Y_2)\mathcal{V}\nabla(\ln(f_1)), && (3.13) \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}\nabla_{X_2} Y_2 &= \mathcal{H}\nabla^2_{X_2} Y_2 + X_2(\ln f_1) Y_2 + Y_2(\ln f_1) X_2 && (3.14)\\ &\quad - g_{\mathcal{M}}(X_2, Y_2)\mathcal{H}\nabla(\ln(f_1)). \end{aligned}$$

Since A and A^2 are skew-symmetric tensor fields and $g_{\mathcal{M}}$ is a symmetric tensor field, by using Eqs (3.13) and (3.14), we obtain the required result 3).

Lemma 3.3. *Let $\mathcal{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ and $\aleph = {}_{\rho_2}\aleph_1 \times_{\rho_1} \aleph_2$ be warped-twisted product manifolds and let $\varphi_i : \mathcal{M}_i \to \aleph_i$ be Riemannian warped-twisted product submersion between the manifolds $\mathcal{M}_i$ and $\aleph_i$. Then we have*

1) $\mathcal{H}\nabla_{V_1} X_1 = \mathcal{H}\nabla^1_{V_1} X_1$ *and* $T_{V_1} X_1 = T^1_{V_1} X_1$,
2) $T_{V_1} X_2 = X_2(\ln(f_2 \circ \varphi_2))\, V_1 = \mathcal{V}\nabla_{X_2} V_1$ *and*
   $A_{X_2} V_1 = V_1(\ln(f_1))\, X_2 = \mathcal{H}\nabla_{V_2} X_1$,
3) $T_{V_2} X_1 = X_1(\ln(f_1))\, V_2 = \mathcal{V}\nabla_{X_1} V_2$ *and*
   $A_{X_1} V_2 = V_2(\ln(f_2 \circ \varphi_2))\, X_1 = \mathcal{H}\nabla_{V_2} X_1$,
4) $T_{V_2} X_2 = T^2_{V_2} X_2 + X_2(\ln f_1) V_2$ *and*
   $\mathcal{H}\nabla_{V_2} X_2 = \mathcal{H}\nabla^2_{V_2} X_2 + V_2(\ln f_1) X_2$

for any $V_i \in \Gamma(\mathcal{V}_i)$, and $X_i \in \Gamma(\mathcal{H}_i)$ where $i = \{1, 2\}$.

Proof. For $V_1 \in \Gamma(\mathcal{V}_i)$ and $X_1 \in \Gamma(\mathcal{H}_i)$, by using Eq (2.4), we have

$$\nabla_{V_1} X_1 = \mathcal{H}\nabla_{V_1} X_1 + T_{V_1} X_1. \tag{3.15}$$

Making use of Eqs (2.8) and (2.4), we obtain

$$\nabla_{V_1} X_1 = \nabla^1_{V_1} X_1 - g_{\mathcal{M}}(V_1, X_1)\nabla(\ln(f_2 \circ \varphi_2))$$

$$= \mathcal{H}\nabla^{1}_{V_1}X_1 + T^{1}_{V_1}X_1. \tag{3.16}$$

Combining Eqs (3.15) and (3.16) and comparing the vertical and the horizontal parts in the resulting expression, we get result 1).
From Eq (2.9), we have

$$\nabla_{V_1}X_2 = \nabla_{X_2}V_1 = X_2\left(\ln\left(f_2 \circ \varphi_2\right)\right)V_1 + V_1\left(\ln\left(f_1\right)\right)X_2. \tag{3.17}$$

From Eqs (2.4) and (2.5), we have

$$\nabla_{V_1}X_2 = \mathcal{H}\nabla_{V_1}X_2 + T_{V_1}X_2, \tag{3.18}$$
$$\nabla_{X_2}V_1 = A_{X_2}V_1 + \mathcal{V}\nabla_{X_2}V_1. \tag{3.19}$$

Combining Eqs (3.17)–(3.19) and comparing the vertical and the horizontal parts in the resulting expression, we obtain result 2). On a similar line, we get result 3).
Further, using Eqs (2.4) and (2.10), we obtain result 4).

Further, using the above lemmas, we obtained the fundamental geometric properties of Riemannian warped-twisted product submersion in the consequent theorems.

Theorem 3.1. *Let $\mathcal{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ and $\aleph = {}_{\rho_2}\aleph_1 \times_{\rho_1} \aleph_2$ be warped-twisted product manifolds and let $\varphi_i : \mathcal{M}_i \to \aleph_i$ be Riemannian warped-twisted product submersion between the manifolds $\mathcal{M}_i$ and $\aleph_i$ with $\dim \mathcal{M}_1 = m_1, \dim \mathcal{M}_2 = m_2, \dim \aleph_1 = n_1$ and $\dim \aleph_2 = n_2$. Then*
(i) φ has totally geodesic fibers if and only if φ_1 and φ_2 have totally geodesic fibers and f_1 and f_2 are constants,
(ii) The fundamental metric tensor T satisfies the following inequality

$$\|T\|^2 \geq -\left(n_1 - m_1\right)\|\mathcal{H}(\nabla \ln f_2)\|^2 - \left(n_2 - m_2\right)\|\mathcal{H}(\nabla \ln f_1)\|^2$$

with the equality holding if and only if φ_1 and φ_2 have totally geodesic fibers.

Proof. (i) Let $e_k \in \Gamma\left(\mathcal{V}_1\right)$ and $k = 1, \ldots, m_1 - n_1$ and $e_c \in \Gamma\left(\mathcal{V}_2\right), c = m_1 - n_1 + 1, \ldots, m_1 - n_1 + m_2 - n_2$ be orthonormal vectors of vertical spaces of submersion π. Then using Lemma (3.2), we have

$$\begin{aligned}\|T\|^2 &= \sum_{k,k_1=1}^{m_1-n_1} g_{\mathcal{M}}\left(T\left(e_k, e_{k_1}\right), T\left(e_k, e_{k_1}\right)\right) + \sum_{c,d=m_1-n_1+1}^{m_1-n_1+m_2-n_2} g_{\mathcal{M}}\left(T\left(e_c, e_d\right), T\left(e_c, e_d\right)\right) \\ &= \left\|T^1\right\|^2 + \left\|T^2\right\|^2 + \left(m_1 - n_1\right)\|\mathcal{H}(\nabla \ln f_2)\|^2 + \left(m_2 - n_2\right)\|\mathcal{H}(\nabla \ln f_1)\|^2.\end{aligned}$$

(ii) It follows from the above equation.

Theorem 3.2. *Let $\mathcal{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ and $\aleph = {}_{\rho_2}\aleph_1 \times_{\rho_1} \aleph_2$ be warped-twisted product manifolds and let $\varphi_i : \mathcal{M}_i \to \aleph_i$ be Riemannian warped-twisted product submersion between the manifolds $\mathcal{M}_i$ and $\aleph_i$. Then φ has totally umbilical fibers if and only if φ_1 and φ_2 have totally geodesic fibers and $\vec{\mathcal{H}}^{\varphi} = \mathcal{H}(\nabla \ln f_1) = \mathcal{H}(\nabla \ln f_2)$, where $\vec{\mathcal{H}}^{\varphi}$ denotes the mean curvature of φ.*

Proof. From Lemma (3.1) and the fact that φ has totally umbilical fibers, we have

$$\begin{aligned} T_{U_1}V_1 &= T^1_{U_1}V_1 - g_M(U_1,V_1)\mathcal{H}(\nabla \ln f_2) = g_{\mathcal{M}}(U_1,V_1)\vec{\mathcal{H}}^{\varphi}, \\ T_{U_1}U_2 &= 0 = g_{\mathcal{M}}(U_1,U_2)\vec{\mathcal{H}}^{\varphi} = 0\vec{\mathcal{H}}^{\varphi}, \\ T_{U_2}V_2 &= T^2_{U_2}V_2 - g_M(U_2,V_2)\mathcal{H}(\nabla \ln f_1) = g_{\mathcal{M}}(U_2,V_2)\vec{\mathcal{H}}^{\varphi}, \end{aligned}$$

for any $U_i, V_i \in \Gamma(\mathcal{V}_i)$, $i \in \{1,2\}$ which gives the following relation

$$\vec{\mathcal{H}}^{\varphi} = -\mathcal{H}(\nabla \ln f_1),\ \ \vec{\mathcal{H}}^{\varphi} = -\mathcal{H}(\nabla \ln f_2) \text{ and } T^1_{U_1}V_1 = 0 = T^2_{U_2}V_2. \tag{3.20}$$

Converse follow easily.

Theorem 3.3. *Let $\mathcal{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ and $\aleph = {}_{\rho_2}\aleph_1 \times_{\rho_1} \aleph_2$ be warped-twisted product manifolds and let $\varphi_i : \mathcal{M}_i \to \aleph_i$ be Riemannian warped-twisted product submersion between the manifolds $\mathcal{M}_i$ and $\aleph_i$. Then φ has minimal fibers if and only if the mean curvature of φ_1 and φ_2 is given by $\vec{\mathcal{H}}_1 = \frac{m_2-n_2}{m_1-n_1}\mathcal{H}(\nabla^1 \ln f_1)$ and $\vec{\mathcal{H}}_2 = \frac{m_1-n_1}{m_2-n_2}\mathcal{H}(\nabla \ln f_2) + \mathcal{H}(\nabla^2 \ln f_1)$.*

Proof. We suppose that φ has minimal fibers for $\mathcal{M}$. Let $e_k \in \Gamma(\mathcal{V}_1)$ and $k = 1,\ldots,m_1-n_1$ and $e_c \in \Gamma(\mathcal{V}_2), c = m_1-n_1+1\ldots,m_1-n_1+m_2-n_2$ be orthonormal frames of vertical spaces of submersion φ. Then using Eq (2.7) and Lemma (3.1), we have

$$\begin{aligned} \vec{\mathcal{H}} &= \frac{1}{m_1-n_1+m_2-n_2}\left(\sum_{k=1}^{m_1-n_1} T(e_k,e_k) + \sum_{c=m_1-n_1+1}^{m_1-n_1+m_2-n_2} T(e_c,e_c)\right) \\ &= \frac{1}{m_1-n_1+m_2-n_2}\left(\begin{array}{c} \sum_{k=1}^{m_1-n_1} T^1(e_k,e_k) - g_M(e_k,e_k)\mathcal{H}(\nabla \ln f_2) \\ + \sum_{c=m_1-n_1+1}^{m_1-n_1+m_2-n_2}\left(T^2(e_c,e_c) - g_M(e_c,e_c)\mathcal{H}(\nabla \ln f_1)\right) \end{array}\right) \\ &= \frac{1}{m_1-n_1+m_2-n_2}\left(\begin{array}{c} (m_1-n_1)\left(\vec{\mathcal{H}}_1 - \mathcal{H}(\nabla \ln f_2)\right) \\ +(m_2-n_2)\left(\vec{\mathcal{H}}_2 - \mathcal{H}(\nabla \ln f_1)\right) \end{array}\right). \end{aligned}$$

Since, $\mathcal{H}(\nabla \ln f_2) \in \Gamma(\mathcal{H}_2)$ and $\mathcal{H}(\nabla \ln f_1) \in \Gamma(\mathcal{H}_1 \times \mathcal{H}_2)$. So, we can write $\mathcal{H}(\nabla \ln f_1) = \mathcal{H}(\nabla^1 \ln f_1) + \mathcal{H}(\nabla^2 \ln f_1)$, where $\mathcal{H}(\nabla^1 \ln f_1) \in \Gamma(\mathcal{H}_1)$ and $\mathcal{H}(\nabla^2 \ln f_1) \in \Gamma(\mathcal{H}_2)$. Then, we obtain $\vec{\mathcal{H}}_1 = \frac{m_2-n_2}{m_1-n_1}\mathcal{H}(\nabla^1 \ln f_1)$ and $\vec{\mathcal{H}}_2 = \frac{m_1-n_1}{m_2-n_2}\mathcal{H}(\nabla \ln f_2) + \mathcal{H}(\nabla^2 \ln f_1)$. Converse follows quickly from the above relation.

4. Riemannian warped-twisted product submersion as an Einstein manifold

In this section, we obtain the Ricci tensor for Riemannian warped-twisted product submersion. Further, we discuss Einstein's condition on vertical and horizontal spaces of $\mathcal{M}_i$ for Riemannian warped-twisted product submersion.

Definition 4.1. *[2]. A Riemannian manifold $(\mathcal{M}, g_{\mathcal{M}})$ is called an Einstein manifold if*

$$\mathbb{S} = \lambda g_{\mathcal{M}}, \tag{4.1}$$

where λ is a real constant, and $\mathbb{S}$ is the Ricci tensor on $\mathcal{M}$.

Making use of Lemma 2.2, we have

Lemma 4.1. *Let $\mathcal{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ and $\aleph = {}_{\rho_2}\aleph_1 \times_{\rho_1} \aleph_2$ be warped-twisted product manifolds and $\varphi_i : \mathcal{M}_i \to \aleph_i$ be Riemannian warped-twisted product submersion between the manifolds $\mathcal{M}_i$ and $\aleph_i$ with $dim\mathcal{M}_i = m_i$ and $dim\aleph_i = n_i$; $i \in \{1, 2\}$. Let $\mathbb{S}^1$ and $\mathbb{S}^2$ be the lifts of the Ricci curvatures on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Then for any $X_i, Y_i \in \Gamma(\mathcal{H}_i)$ and $U_i, V_i \in \Gamma(\mathcal{V}_i)$, we have following relations:*

1) $\mathbb{S}(X_1, U_1) = (1 - m_2)X_1U_1(\ln f_1) + (m_1 + m_2 - 2)X_1(\ln f_1)U_1(\ln f_2)$,

2) $\mathbb{S}(X_1, X_2) = (1 - m_2)X_1X_2(\ln f_1) + (m_1 + m_2 - 2)X_1(\ln f_1)X_2(\ln f_2)$,

3) $\mathbb{S}(X_1, Y_1) = \mathbb{S}^1(X_1, Y_1) + h^{\ln f_2}(X_1, Y_1) - m_2\{h_1^{\ln f_1}(X_1, Y_1) + X_1(\ln f_1)Y_1(\ln f_1)\}$
$- g_{\mathcal{M}}(X_1, Y_1)\{\Delta \ln f_2 + g_{\mathcal{M}}(\nabla \ln f_2, \nabla \ln f_2)\}$,

4) $\mathbb{S}(X_2, Y_2) = \mathbb{S}^2(X_2, Y_2) + h^{\ln f_1}(X_2, Y_2) + (1 - m_2)h_2^{\ln f_1}(X_2, Y_2) + m_2X_2(\ln f_1)Y_2(\ln f_1)$
$-g_{\mathcal{M}}(X_2, Y_2)\{\Delta \ln f_1 + g_{\mathcal{M}}(\nabla \ln f_1, \nabla \ln f_1)\} - m_1\{h_2^{\ln f_2}(X_2, Y_2) + X_2(\ln f_2)Y_2(\ln f_2) - X_2(\ln f_2)Y_2(\ln f_1) -$
$X_2(\ln f_1)Y_2(\ln f_2)\}$,

5) $\mathbb{S}(X_1, U_2) = (1 - m_2)X_1U_2(\ln f_1) + (m_1 + m_2 - 2)X_1(\ln f_1)U_2(\ln f_2)$,

6) $\mathbb{S}(X_2, U_1) = (1 - m_2)U_1X_2(\ln f_1) + (m_1 + m_2 - 2)U_1(\ln f_1)X_2(\ln f_2)$,

7) $\mathbb{S}(X_2, U_2) = (1 - m_2)X_2U_2(\ln f_1) + (m_1 + m_2 - 2)X_2(\ln f_1)U_2(\ln f_2)$,

8) $\mathbb{S}(U_1, V_1) = \mathbb{S}^1(U_1, V_1) + h^{\ln f_2}(U_1, V_1) - m_2\{h_1^{\ln f_1}(U_1, V_1) + U_1(\ln f_1)V_1(\ln f_1)\}$
$- g_{\mathcal{M}}(U_1, V_1)\{\Delta \ln f_2 + g_{\mathcal{M}}(\nabla \ln f_2, \nabla \ln f_2)\}$,

9) $\mathbb{S}(U_1, U_2) = (1 - m_2)U_1U_2(\ln f_1) + (m_1 + m_2 - 2)U_1(\ln f_1)U_2(\ln f_2)$,

10) $\mathbb{S}(U_2, V_2) = \mathbb{S}^2(U_2, V_2) + h^{\ln f_1}(U_2, V_2) + (1 - m_2)h_2^{\ln f_1}(U_2, V_2) + m_2U_2(\ln f_1)V_2(\ln f_1)$
$- g_{\mathcal{M}}(U_2, V_2)\{\Delta \ln f_1 + g_{\mathcal{M}}(\nabla\ln f_1, \nabla \ln f_1)\} - m_1\{h_2^{\ln f_2}(U_2, V_2) + U_2(\ln f_2)V_2(\ln f_2) -$
$U_2(\ln f_2)V_2(\ln f_1) - U_2(\ln f_1)V_2(\ln f_2)\}$.

Now, we study Einstein conditions for the horizontal and vertical distributions of Riemannian warped-twisted product submersion.

Theorem 4.1. *Let $\mathcal{M} = {}_{f_2}\mathcal{M}_1 \times_{f_1} \mathcal{M}_2$ and $\aleph = {}_{\rho_2}\aleph_1 \times_{\rho_1} \aleph_2$ be warped-twisted product manifolds and $\varphi_i : \mathcal{M}_i \to \aleph_i$ be Riemannian warped-twisted product submersion between the manifolds $\mathcal{M}_i$ and $\aleph_i$ with $dim\mathcal{M}_i = m_i$ and $dim\aleph_i = n_i$; $i \in \{1, 2\}$. Then*
(*i*) *If the vertical space $\mathcal{V}_1$ (or horizontal space $\mathcal{H}_1$) of $\mathcal{M}$ is Einstein, then vertical space of $\mathcal{M}_1$ (resp. horizontal space $\mathcal{H}_1$) is Einstein assuming that $h_1^{\ln f_1}, h^{\ln f_2}$ and, $U_1(\ln f_1)V_1(\ln f_1)$ is proportional to constant times the metric $g_{\mathcal{M}}$ and $\Delta \ln f_2 + g_{\mathcal{M}}(\nabla \ln f_2, \nabla \ln f_2)$ is constant,*
(*ii*) *If the vertical space $\mathcal{V}_2$ (or horizontal space $\mathcal{H}_2$) of $\mathcal{M}$ is Einstein, then vertical space of $\mathcal{M}_2$ (resp. horizontal space $\mathcal{H}_2$) is Einstein assuming that $h_2^{\ln f_2}, h_2^{\ln f_1}, h^{\ln f_1}$ $U_2(\ln f_1)V_2(\ln f_1)$ and $U_2(\ln f_2)V_2(\ln f_2) - U_2(\ln f_2)V_2(\ln f_1) - U_2(\ln f_1)V_2(\ln f_2)\}$ are proportional to the metric $g_{\mathcal{M}}$ and, $\{\Delta \ln f_1 + g_{\mathcal{M}}(\nabla \ln f_1, \nabla \ln f_1)\}$ is constant.*

Proof. (i) Suppose, $\mathcal{V}_1$ of $\mathcal{M}$ is Einstein's manifold. Then, from Eq (4.1) and Lemma (4.1) we have

$$\begin{aligned}\mathbb{S}^1(U_1, V_1) &= -h^{\ln f_2}(U_1, V_1) + m_2\{h_1^{\ln f_1}(U_1, V_1) + U_1(\ln f_1)V_1(\ln f_1)\}\\ &+ g_{\mathcal{M}}(U_1, V_1)\{\lambda + \Delta \ln f_2 + g_{\mathcal{M}}(\nabla \ln f_2, \nabla \ln f_2)\}.\end{aligned}$$

Now if $h_1^{\ln f_1}, h^{\ln f_2}$ and $U_1(\ln f_1)V_1(\ln f_1)$ is proportional to constant times the metric $g_{\mathcal{M}}$ and $\Delta \ln f_2 + g_{\mathcal{M}}(\nabla \ln f_2, \nabla \ln f_2)$ are constants, then $\mathcal{M}_1$ is Einstein's manifold.

(ii) For vertical space $\mathcal{V}_2$ of $\mathcal{M}$, using definition of Einstein's manifold and relation 10 of Lemma (4.1), we obtain the required result.

5. Conclusions

The Einstein equations are of significant importance within the framework of the general theory of relativity, as they form the foundation for the gravitational and cosmological models. The Einstein equation can be expressed as $\mathbb{S} = \lambda g$, which is a non-linear second-order system of differential equations. In the context of this system, the symbol λ is referred to as the Einstein constant, whereas physicists commonly refer to it as the cosmological constant [2]. The geometric properties of warped-twisted products encompass a broader class than both warped products and twisted products. This class exhibits a multitude of applications, not only within the realm of geometry but also in the field of theoretical physics. The exact solution of Einstein's equations exhibits a warped product structure. The technique of Riemannian submersion is commonly employed in the construction of Riemannian manifolds showing positive sectional curvature. Additionally, it is employed in the construction of widely recognized instances of Einstein manifolds. Riemannian submersion finds extensive application within the domains of Kaluza-Klein theory, Yang-Mills theory, superstring and supergravity theory in physics [24–26]. The objective of this study is to decompose the Einstein's equation for warped-twisted product into their constituent elements, specifically the base and fiber components.

6. Remark

In [8, 23], H. M. Tastan and S. G. Aydin considered Einstein like warped-twisted product manifolds and warped-twisted product semi-slant submanifolds. On the other hand, I. K. Erken and C. Murathan [3] studied Riemannian warped product submersion. In this paper, we investigated Riemannian warped-twisted product submersions. In light of these studies, it could be interesting to work on warped product semi-slant submersions and warped-twisted product semi-slant submersions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are really thankful to the learned reviewers for their careful reading of our manuscript and their many insightful comments and suggestions that have improved the quality of our manuscript. The author Fatemah Mofarreh, expresses her gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2024R27), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare no conflicts of interest.

References

1. B. O'Neill, The fundamental equation of submersion, *Michigan Math. J.*, **13** (1966), 458–469. https://doi.org/10.1307/mmj/1028999604
2. A. L. Besse, *Einstein manifolds*, Berlin: Springer-Verlag, 1987. https://doi.org/10.1007/978-3-540-74311-8
3. I. K. Erken, C. Murathan, Riemannian warped product submersions, *Results Math.*, **76** (2021), 1. https://doi.org/10.1007/s00025-020-01310-4
4. J. Nash, The imbedding problem for Riemannian manifolds, *Ann. Math.*, **63** (1956), 20–63. https://doi.org/10.2307/1969989
5. R. L. Bishop, B. O'Neill, Manifolds of negative curvature, *T. Am. Math. Soc.*, **145** (1969), 1–49.
6. B. Y. Chen, *Differential geometry of warped product manifolds and submanifolds*, World Scientific Publishing Co. Pvt. Ltd, 2017. https://doi.org/10.1142/10419
7. S. G. Aydin, H. M. Tastan, On a certain type of warped-twisted product submanifolds, *Turk. J. Math.*, **46** (2022), 2645–2662. https://doi.org/10.55730/1300-0098.3292
8. H. M. Tastan, S. G. Aydin, Warped-twisted product semi-slant submanifolds, *Filomat*, **36** (2022), 1587–1602. https://doi.org/10.2298/FIL2205587T
9. M. A. Khan, S. Uddin, R. Sachdeva, Semi-invariant warped product submanifolds of cosymplectic manifolds, *J. Inequal. Appl.*, **2012** (2012), 19. https://doi.org/10.1186/1029-242X-2012-19
10. K. S. Park, H-semi-invariant submersions, *Taiwanese J. Math.*, **16** (2012), 1865–1878. https://doi.org/10.11650/twjm/1500406802
11. Y. Gunduzalp, Slant submersions from almost product Riemannian manifolds, *Turk. J. Math.*, **37** (2013), 13 https://doi.org/10.3906/mat-1205-64
12. Y. Gunduzalp, Semi slant submersions from almost product Riemannian manifolds, *Demonstr. Math.*, **49** (2016), 345–356. https://doi.org/10.1515/dema-2016-0029
13. B. Sahin, Hemi-slant Riemannian maps, *Mediterr. J. Math.*, **14** (2017), 10. https://doi.org/10.1007/s00009-016-0817-2
14. B. Sahin, *Riemannian submersions, Riemannian maps in Hermitian geometry, and their applications*, Academic Press, 2017.
15. M. A. Khan, C. Ozel, Ricci curvature of contact CR-warped product submanifolds in generalized Sasakian space forms admitting a trans-Sasakian structure, *Filomat*, **35** (2021), 125–146. https://doi.org/10.2298/FIL2101125K

16. Y. Gunduzalp , Warped product pointwise hemi-slant submanifolds of a Para-Kaehler manifold, *Filomat*, **36** (2022), 275–288. https://doi.org/10.2298/FIL2201275G
17. Y. Li, R. Prasad, A. Haseeb, S. Kumar, S. Kumar, A study of Clairaut semi invariant Riemannian maps from cosymplectic manifolds, *Axioms*, **11** (2022), 503. https://doi.org/10.3390/axioms11100503
18. S. Rahman, A. Haseeb, N. Jamal, Geometry of warped product *CR* and semi-slant submanifolds in quasi-para-Sasakian manifolds, *Int. J. Anal. Appl.*, **20** (2022), 59. https://doi.org/10.28924/2291-8639-20-2022-59
19. N. B. Turki, S. Shenawy, H. K. EL-Sayied, N. Syied, C. A. Mantica, ρ-Einstein solitons on warped product manifolds and applications, *J. Math.*, **2022** (2022), 1028339. https://doi.org/10.1155/2022/1028339
20. I. Al-Dayel, F. Aloui, S. Deshmukh, Poisson doubly warped product manifolds, *Mathematics*, **11** (2023), 519. https://doi.org/10.3390/math11030519
21. L. S. Alqahtani, A. Ali, P. Laurian-Ioan, A. H. Alkhaldi, The homology of warped product submanifolds of spheres and their applications, *Mathematics*, **11** (2023), 3405. https://doi.org/10.3390/math11153405
22. S. Kumar, R. Prasad, A. Haseeb, Conformal semi-slant submersions from Sasakian manifolds, *J. Anal.*, **31** (2023), 1855–1872. https://doi.org/10.1007/s41478-022-00540-9
23. S. G. Aydin, H. M. Tastan, Warped-twisted products and Einstein-like manifolds, *Novi Sad J. Math.*, 2022, 1–19. https://doi.org/10.30755/NSJOM.12381
24. J. P. Bourguignon, H. B. Lawson Jr., Stability and isolation phenomena for Yang-Mills fields, *Commun. Math. Phys.*, **79** (1981), 189–230. https://doi.org/10.1007/BF01942061
25. S. Ianus, M. Visinescu, Space-time compactification and Riemannian submersion, In: *The mathematical heritage of C.F. Gauss*, 1991, 358–371. https://doi.org/10.1142/9789814503457_0026
26. M. T. Mustafa, Applications of harmonic morphisms to gravity, *J. Math. Phys.*, **41** (2000), 6918–6929. https://doi.org/10.1063/1.1290381