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Research article

On the stability of Fractal interpolation functions with variable parameters

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Abstract: Fractal interpolation function (FIF) is a fixed point of the Read–Bajraktarević operator defined on a suitable function space and is constructed via an iterated function system (IFS). In this paper, we considered the generalized affine FIF generated through the IFS defined by the functions $W_n(x, y) = (a_n(x) + e_n, \alpha_n(x)y + \psi_n(x))$, $n = 1, \dots, N$. We studied the shift of the fractal interpolation curve, by computing the error estimate in response to a small perturbation on $\alpha_n(x)$. In addition, we gave a sufficient condition on the perturbed IFS so that it satisfies the continuity condition. As an application, we computed an upper bound of the maximum range of the perturbed FIF.

Keywords: iterated function system (IFS); generalized affine fractal interpolation function; hölder and Lipschitz functions
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1. Introduction

Fractal interpolation provides a framework for interpolating sets of data that may exhibit some self-similarity at different scales. While classical interpolation techniques, such as polynomial interpolation and spline interpolation generate smooth interpolants, fractal interpolation is able to model irregular sets of data. The concept of the fractal interpolation functions (FIFs) was first introduced by Barnsley [1–3]. Since then, this theory has become a powerful and useful tool in applied science and engineering, especially when dealing with real-world signals such as financial series, time series, climate data and bioelectric recordings.

The iterated function systems (IFSs in short) are useful to construct some fractal sets [4–8]).

Specifically, IFS employs contractive functions over a complete metric space $(\mathbb{X}, d)$, where the existence and uniqueness of the fixed point are guaranteed by Banach's theorem. This is done using the Hutchinson operator, which also is a contraction mapping over $\mathcal{H}(\mathbb{X})$, where $\mathcal{H}(\mathbb{X})$ is the space of all compact subsets of $\mathbb{X}$ (see, for instance, [4, 9–14] for some extension of Hutchinson's framework). Recently, the existence of FIFs through different well-known results given by the fixed point theory have been studied by many researchers [15–20]. Moreover, the wide range of FIFs constructed and studied proved many important properties including calculus, dimensionality, stability, smoothness and disturbance error [17, 20–26].

Let $N \geq 2$. In this paper, we consider the generalized affine FIF defined by

$$\begin{cases} L_n(x) = a_n x + e_n \\ F_n(x,y) = \alpha_n(x)y + \psi_n(x), \end{cases} \qquad n \in J := \{1, \dots N\},$$

where, for each n, α_n is a Lipschitz function, ψ_n is a continuous function and a_n and e_n are real numbers. When the functions $\{\alpha_n\}_n$ are constants (they are called vertical scaling factors), this system is extensively studied by many authors [21–23, 27–31]. In this case, the set of vertical scaling factors has a decisive influence on the properties and shape of the corresponding FIF. In particular, the smoothness of FIFs can be described through the vertical scaling factor [30–32] and, therefore by choosing the appropriate vertical scale factor, they can fit the real rough curve precisely. Moreover, all aforementioned works are dealing with different choices of functions ψ_n and highlighted some choice of these functions. In the section 2 we recall some preliminaries concerning IFS and FIF, and in Section 3, we examine the change of the fractal interpolation curve in response to a minor perturbation on $\alpha_n(x)$. We compute a sufficient condition so that the new (perturbed) IFS satisfies the continuity condition. Additionally, we discuss how these changes influence FIFs by computing the error estimate. In Section 4, we consider the case when $F_n(x, y)$ are defined as follows

$$F_n(x,y) = \alpha_n(x)y + f(L_n(x)) - \alpha_n(x)b(x),$$

where the functions $\alpha_n : I \longrightarrow \mathbb{R}$ are Lipschitz functions and b, f are continuous functions on $[x_0, x_N]$ such that $b(x_0) = f(x_0)$ and $b(x_N) = f(x_N)$. The FIF interpolates the function f at the nodes of the partition : $x_0 < x_1 < x_2 \cdots < x_N$ [33–35]. We will compute in this case an appropriate upper bound of the maximum range of the perturbed FIF.

2. Preliminaries

2.1. Iterated function systems

Let $(\mathbb{X}, d)$ be a complete metric space. A mapping $g : \mathbb{X} \longrightarrow \mathbb{X}$ is called a contraction if there exists $c \in [0, 1)$ such that $d(g(x_1), g(x_2)) \leq c\, d(x_1, x_2)$ for all $x_1, x_2 \in \mathbb{X}$. We define on $\mathcal{H}(\mathbb{X})$, the space of compact subsets of $\mathbb{X}$, the Hausdorff metric d_H as

$$d_H(K_1, K_2) = \max\{d(K_1, K_2), d(K_2, K_1)\}, \quad K_1, K_2 \in \mathcal{H}(\mathbb{X}),$$

where $d(K_1, K_2) = \sup_{x \in K_1} \inf_{y \in K_2} d(x, y)$, then $(\mathcal{H}(\mathbb{X}), d_H)$ is complete space [5] and compact whenever $\mathbb{X}$ is compact. Now, we consider $\mathbb{I} = \{\mathbb{X}\,,\ w_n\ ;\ n \in J\}$ to be an IFS, where $w_n : \mathbb{X} \longrightarrow \mathbb{X}$ is a continuous mapping for each $n \in J$. We define also the Hutchinson operator $W : \mathcal{H}(\mathbb{X}) \longrightarrow \mathcal{H}(\mathbb{X})$ as

$$W(A) = \bigcup_{n=1}^{N} w_n(A), \quad \forall\, A \in \mathcal{H}(\mathbb{X}).$$

It is well know that any IFS admits at least one attractor, that is, a set $G \in \mathcal{H}(\mathbb{X})$ such that $W(G) = G$ [2]. In addition [2, 5], the Hutchinson operator is a contraction mapping on $(\mathcal{H}(\mathbb{X}), d_H)$ whenever $\mathbb{I}$ is hyperbolic; that is, for each $n \in J$, w_n is a contraction.

2.2. FIF

Let $\mathbb{N}^*$ be the set of positive integers and $I = [x_0, x_N]$ be a real compact interval. We consider the set of data $\Delta = \{(x_n, y_n) \in I \times \mathbb{R}\ ;\ n \in J_0 := \{0, 1, \ldots, N\}\}$, where $N \in \mathbb{N}^*$, $x_0 < x_1 < \cdots < x_N$, $y_i \in [a, b]$, with $-\infty < a < b < \infty$. Now we consider, for $n \in J$, the set $I_n = [x_{n-1}, x_n]$ and we define the contractive homeomorphism $L_n : I \longrightarrow I_n$ such that

$$\begin{aligned} L_n(x_0) = x_{n-1}, &\qquad L_n(x_N) = x_n \\ |L_n(x) - L_n(x')| \le l|x - x'|, &\qquad \forall\, x, x' \in I, \end{aligned} \tag{2.1}$$

for some $l \in [0, 1)$. We define also N continuous mappings $F_n : K := I \times [a, b] \longrightarrow \mathbb{R}$, such that

$$F_n(x_0, y_0) = y_{n-1}, \qquad F_n(x_N, y_N) = y_n \tag{2.2}$$

$$|F_n(x, y) - F_n(x, y')| \le |\alpha_n||y - y'|, \qquad \forall x \in I,\ y, y' \in [a, b], \tag{2.3}$$

where $\alpha_n \in (-1, 1), n \in J$. Now, we define the mapping $W_n : K \longrightarrow I_n \times \mathbb{R}$, as

$$W_n(x, y) = \big(L_n(x), F_n(x, y)\big), \quad \forall (x, y) \in K, n \in J.$$

Under the conditions (2.1), (2.2) and (2.3), the IFS $\{K, W_n : n \in J\}$ has a unique attractor G. Furthermore, G is the graph of continuous function f that passes through the points $\{(x_n, y_n)\}_{n=1}^{N}$ [2]. On the other hand, we define the complete metric space $(\mathcal{G}, \rho)$, where

$$\mathcal{G} = \Big\{g : I \longrightarrow \mathbb{R}, \text{ such that } g \text{ is continuous, } g(x_0) = x_0 \text{ and } g(x_N) = x_N\Big\}$$

and

$$\rho(g, h) = \|g - h\|_\infty := \max\Big\{|g(x) - h(x)| : x \in I\Big\}, \quad \forall g, h \in \mathcal{G}.$$

Now, we define the Read-Bajraktarevic operator T, defined on $(\mathcal{G}, \rho)$ by

$$T(g(x)) = F_n\Big(L_n^{-1}(x), g(L_n^{-1}(x))\Big), \quad x \in I_n, n \in J,$$

then, using (2.3), we obtain $\|T(f) - T(g)\| \le \alpha\|f - g\|_\infty$, where $\alpha := \max_n |\alpha_n|$. Hence, T is a contraction mapping and possesses a unique fixed point f on $\mathcal{G}$. From this, the FIF is the unique function satisfying the following functional relation

$$f(x) = F_n\Big(L_n^{-1}(x), f(L_n^{-1}(x))\Big), \qquad \forall x \in I_n, n \in J. \tag{2.4}$$

The most widely studied FIFs are defined by the following system

$$\begin{cases} L_n(x) = a_n x + e_n \\ F_n(x,y) = \alpha_n y + \psi_n(x), \end{cases} \quad n \in J, \tag{2.5}$$

where the real constants a_n and e_n are determined by condition (2.1), ψ_n are continuous functions such that conditions (2.2) and (2.3) hold and $-1 < \alpha_n < 1$ are free parameters called vertical scaling factors. Recently, many authors managed to construct more general FIFs that do not have to exhibit the strict self similarity (see, for instance, [15]). In this work, we consider the IFS with variable parameters [32] defined by :

$$W_n(x,y) = \big(a_n x + e_n, \alpha_n(x) y + \psi_n(x), \big), \quad \forall (x,y) \in K, n \in J,$$

where $\alpha_n : I \longrightarrow \mathbb{R}$ are Lipschitz functions such that $\|\alpha_n\|_\infty := \sup\{\alpha_n(x); x \in I\} < 1$. In this case, the FIF will be called generalized affine FIF and denoted by $f^{\alpha}_{\Delta,N}$ where $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_N)$ (or simply by f if there is no ambiguity). Moreover, consider that generalized affine FIF provides a wide variety of systems for different approximations problems. In Figure 1, we plot the FIF with constant parameters : $\alpha_1(x) = 0.4$, $\alpha_2(x) = 0.3$, $\alpha_3(x) = 0.5$ and $\alpha_4(x) = 0.2$. Meanwhile in Figure 2, we plot the FIF with variable parameters : $\alpha_1(x) = 0.4\sin(5x) + 0.2$, $\alpha_2(x) = 0.3\cos(10x)$, $\alpha_3(x) = 0.5\exp(-2x) + 0.3$ and $\alpha_4(x) = 0.2\exp(x)\sin(x) + 0.1$. As we can see, the self-similarity in Figure 2 is weaker than that of the affine fractal interpolation curve shown in Figure 1 [32].

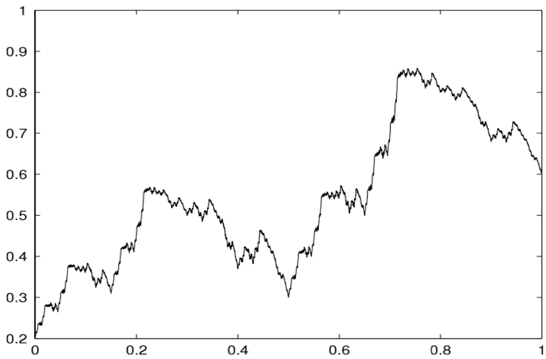

Figure 1. FIF with constant parameters.

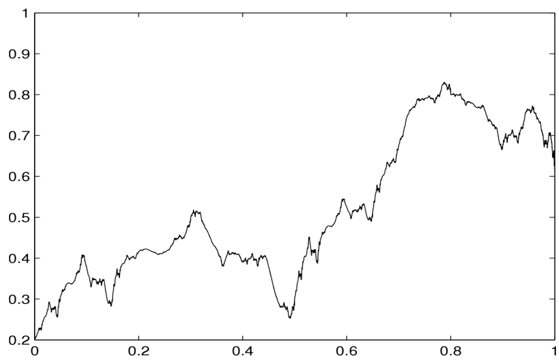

Figure 2. FIF with variable parameters.

3. Perturbation on FIFs

In this section we consider the FIFs generated through the following IFS

$$\begin{cases} L_n(x) = a_n x + e_n \\ F_n(x,y) = \alpha_n(x) y + \psi_n(x), \end{cases} \quad n \in J, \tag{3.1}$$

where a_n and e_n are determined by condition (2.1) and the functions ψ_n are continuous functions such that conditions (2.2) and (2.3) hold. Assume that $\alpha_n : I \longrightarrow \mathbb{R}$ are Lipschitz functions, with Lipschitz constant C_n, such that $\alpha := \max_n \|\alpha_n\|_\infty < 1$. Now we define, for $x \in I$,

$$\begin{cases} L_{n_1 n_2 \dots n_k}(x) := L_{n_1} \circ L_{n_2} \circ \cdots \circ L_{n_k}(x) \\ L_{n_1 n_2 \dots n_k}(I) := L_{n_1} \circ L_{n_2} \circ \cdots \circ L_{n_k}(I), \end{cases}$$

where $n_j \in J$, $k \geq 1$, $j \in \{1, \dots, k\}$. Finally, let σ^j denote the j-fold composition of σ with itself such that

$$L_{\sigma^j(n_1 n_2 \dots n_k)}(x) = L_{n_{j+1} \dots n_k}(x), \qquad 1 \leq j \leq k-1,$$

while $L_{\sigma^k(n_1 n_2 \dots n_k)}(x) = x$. Using a successive iteration and induction (see [32, 36]), we have, for all $n_j \in J$, $j = 1, \dots, k$,

$$L_{n_1 n_2 \dots n_k}(x) = \Big(\prod_{j=1}^{k} a_{n_j}\Big)x + \sum_{r=1}^{k} \Big(\prod_{j=1}^{r-1} a_{n_j}\Big) e_{n_r}, \tag{3.2}$$

with the convention $\prod_{j=1}^{0} S_j(x) = 1$ for any family of functions $\{S_j\}_j$. In this section we consider the following perturbation of the IFS defined in (3.1),

$$\begin{cases} L_n(x) = a_n x + e_n \\ \widetilde{F}_n(x, y) = \beta_n(x) y + \psi_n(x) + \lambda_n(x), \end{cases} \tag{3.3}$$

where β_n and λ_n are continuous functions such that $\|\beta_n\|_\infty < 1$. We define also

$$\delta_n(x) := \beta_n(x) - \alpha_n(x) \qquad \text{and} \qquad \gamma_n := \lambda_n(x_N) - \lambda_n(x_0).$$

We will say that the IFS satisfies the continuous condition if, for $1 \leq n < N$, we have

$$\widetilde{F}_n(x_N, y_N) = \widetilde{F}_{n+1}(x_0, y_0).$$

We assume, for all $1 \leq n < N$, that

$$\lambda_{n+1}(x_0) = \Big(\sum_{j=1}^{n} \delta_j(x_N)\Big) y_N - \Big(\sum_{j=2}^{n+1} \delta_j(x_N)\Big) y_0 + \Big(\sum_{j=1}^{n} \gamma_j\Big) + \lambda_1(x_0). \tag{3.4}$$

Notice that by using (3.2) and the fact that $|a_n| < 1$ for $n \in J$ that

$$x = \lim_{k \to \infty} L_{n_1 n_2 \dots n_k}(x) = \sum_{r=1}^{\infty} \Big(\prod_{j=1}^{r-1} a_{n_j}\Big) e_{n_r},$$

for all $x \in I$. Our main result in this section is the following.

Theorem 1. *Let f and $\widetilde{f}$ be the FIFs generated by the IFSs* (3.1) *and* (3.3)*, respectively.*

1. *Under* (3.4) *the IFS defined by the system* (3.3) *satisfies the continuous condition.*
2. *For a given $x \in I$, let $\{n_j\}, n_j \in J$ be sequence such that $x = \sum_{r=1}^{\infty} \Big(\prod_{j=1}^{r-1} a_{n_j}\Big) e_{n_r}$, then*

$$\widetilde{f}(x) - f(x) \;\; = \;\; \sum_{r=1}^{\infty} \Big[\prod_{j=1}^{r-1} \beta_{n_j}(\mathsf{L}^j[x]) - \prod_{j=1}^{r-1} \alpha_{n_j}(\mathsf{L}^j[x]) \Big] \psi_{n_r}(\mathsf{L}^r[x])$$

$$+ \sum_{r=1}^{\infty}\Big[\prod_{j=1}^{r-1}\beta_{n_j}(\mathsf{L}^j[x])\Big]\lambda_{n_r}(\mathsf{L}^r[x]),$$

where $\mathsf{L}^r[x] = \sum_{l=1}^{\infty}\Big(\prod_{j=1}^{l-1} a_{n_{r+j}}\Big)e_{n_{r+l}}$.

Proof. 1. The continuous condition implies, for each $n = 1, \dots, N-1$, that $\widetilde{F}_n(x_N, y_N) = \widetilde{F}_{n+1}(x_0, y_0)$, then

$$\beta_{n+1}(x_0)y_0 + \psi_{n+1}(x_0) + \lambda_{n+1}(x_0) = \beta_n(x_N)y_N + \psi_n(x_N) + \lambda_n(x_N).$$

Thus, we obtain $\delta_{n+1}(x_0)y_0 + \lambda_{n+1}(x_0) = \delta_n(x_N)y_N + \lambda_n(x_N)$ and then

$$\begin{aligned}
\lambda_{n+1}(x_0) &= \delta_n(x_N)y_N - \delta_{n+1}(x_0)y_0 + \lambda_n(x_N)\\
&= \delta_n(x_N)y_N - \delta_{n+1}(x_0)y_0 + \gamma_n + \lambda_n(x_0)\\
&= \Big(\delta_n(x_N) + \delta_{n-1}(x_N)\Big)y_N - \Big(\delta_{n+1}(x_0) + \delta_n(x_0)\Big)y_0 + \gamma_n + \gamma_{n-1} + \lambda_{n-1}(x_0)\\
&\vdots\\
&= \Big(\sum_{j=1}^{n}\delta_j(x_N)\Big)y_N - \Big(\sum_{j=2}^{n+1}\delta_j(x_0)\Big)y_0 + \Big(\sum_{j=1}^{n}\gamma_j\Big) + \lambda_1(x_0).
\end{aligned}$$

2. Using (3.2), we have

$$\begin{aligned}
f(L_{n_1n_2\cdots n_k}(x)) =& \Big[\prod_{j=1}^{k}\alpha_{n_j}(L_{\sigma^j(n_1n_2\dots n_k)}(x))\Big]f(x)\\
&+ \sum_{r=1}^{k}\Big[\prod_{j=1}^{r-1}\alpha_{n_j}(L_{\sigma^j(n_1n_2\dots n_k)}(x))\Big]\psi_{n_r}(L_{\sigma^r(n_1n_2\dots n_k)}(x))
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{f}(L_{n_1n_2\cdots n_k}(x)) =& \Big[\prod_{j=1}^{k}\beta_{n_j}(L_{\sigma^j(n_1n_2\dots n_k)}(x))\Big]\widetilde{f}(x) + \sum_{r=1}^{k}\Big[\prod_{j=1}^{r-1}\beta_{n_j}(L_{\sigma^j(n_1n_2\dots n_k)}(x))\Big]\\
&\cdot \Big[\psi_{n_r}(L_{\sigma^r(n_1n_2\dots n_k)}(x)) + \lambda_{n_r}(L_{\sigma^r(n_1n_2\dots n_k)}(x))\Big].
\end{aligned}$$

Now remark that $\bigcap_{k=1}^{\infty} L_{n_1n_2\dots n_k}(I)$ consists of a single point in I for any sequence $\{n_k\}$ of integers such that $1 \le n_k \le N$. Moreover, for any fixed $x \in I$, there exists a sequence $\{n_k\}$ satisfying

$$\{x\} = \bigcap_{k=1}^{\infty} L_{n_1n_2\dots n_k}(I) = \lim_{k\to\infty} L_{n_1n_2\dots n_k}(I).$$

This implies by using (3.2) that

$$x = \lim_{k\to\infty} L_{n_1n_2\dots n_k}(x') = \sum_{r=1}^{\infty}\Big(\prod_{j=1}^{r-1} a_{n_j}\Big)e_{n_r},$$

where x' is any point belonging to I . Therefore, there exists $x' \in I$ such that

$$f(x) = \lim_{k\to\infty} \sum_{r=1}^{k} \Big[\prod_{j=1}^{r-1} \alpha_{n_j}(L_{\sigma^j(n_1 n_2 \dots n_k)}(x')) \Big] \psi_{n_r}(L_{\sigma^r(n_1 n_2 \dots n_k)}(x')).$$

$$= \sum_{r=1}^{\infty} \Big[\prod_{j=1}^{r-1} \alpha_{n_j}(\mathsf{L}^j[x]) \Big] \psi_{n_r}(\mathsf{L}^r[x]).$$

Similarly, we have

$$\widetilde{f}(x) = \sum_{r=1}^{\infty} \Big[\prod_{j=1}^{r-1} \beta_{n_j}(\mathsf{L}^j[x]) \Big] \Big(\psi_{n_r}(\mathsf{L}^r[x]) + \lambda_{n_r}(\mathsf{L}^r[x]) \Big),$$

then we obtain the desired result.

□

Example 1. *We consider the Weierstrass function (see Figure 3) that can be seen as a classical fractal function since it is continuous everywhere, yet differentiable nowhere. Therefore, while its graph is connected, it looks jagged when viewed on arbitrarily small scales. There are many works on fractal dimensions of their graphs, including box dimension, Hausdorff dimension, and other types of dimensions [37, 38]. Let $l \geq 2$ be an integer, $1/l < \lambda < 1$ and $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a $\mathbb{Z}$-periodic real analytic function. We define*

$$f^{\phi}_{\lambda,l}(x) = \sum_{k=0}^{\infty} \lambda^k \phi(l^k x), \qquad x \in \mathbb{R}. \tag{3.5}$$

In fact, such a function is real analytic, or the Hausdorff dimension of its graph is equal to $2 + \log_l(\lambda)$ [39].

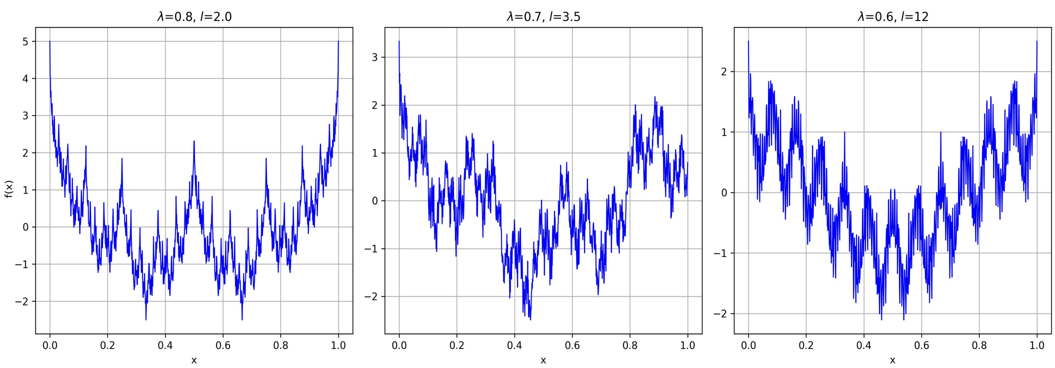

Figure 3. Weierstrass function for different choice of variables λ and l with $\phi(x) = \cos(2\pi x)$.

In this example, we consider the classical Weierstrass function f; that is, when $\phi(x) = \cos(2\pi x)$. Let $I = [0, 1]$, $N = l = 3$ and choose in definition (3.5) the special case $\lambda = 1/2$, then we obtain $f(0) = f(1) = 2$. Now, consider

$$W_n(x, y) = \Big(\frac{x + n - 1}{3}, \alpha_n(x) y + \phi\Big(\frac{x + n - 1}{3}\Big) \Big), \qquad (x, y) \in I \times \mathbb{R},$$

where $\alpha_n(x) = \frac{1}{2} + (-1)^{1+\lfloor n/2 \rfloor}\frac{\sin(2\pi x)}{4}$, *for* $n \in \{1,2,3\}$, *where* $\lfloor n/2 \rfloor$ *means the integer part of x. In this case, we have that the Weierstrass function* f *is the FIF defined by* $\{W_n\}_{n=1}^3$ *[40]. Now, consider the following perturbed system defined by*

$$\begin{cases} L_n(x) = \frac{x+n-1}{3} \\ \widetilde{F}(x,y) = \beta_n(x)y + \phi(\frac{x+n-1}{3}) + \lambda_n(x), \end{cases} \qquad (x,y) \in I \times \mathbb{R}, \tag{3.6}$$

where $\beta_n(x) = (-1)^{1+\lfloor n/2 \rfloor}\frac{\sin(2\pi x)}{4}$ *for* $n \in \{1,2,3\}$. *In this case, we have* $\delta_n(x) := \delta_n = -\frac{1}{2}$. *Choose* $\lambda_1(x) = x$ *and then, by* (3.4), *we have*

$$\lambda_{n+1}(x_0) = 2\Big(\sum_{j=1}^{n}\delta_j\Big) - 2\Big(\sum_{j=2}^{n+1}\delta_j\Big) + \Big(\sum_{j=1}^{n}\gamma_j\Big) = \sum_{j=1}^{n}\gamma_j.$$

Hence, if we take $\lambda_2(x) = \lambda_3(x) = x + 1$, *the system defined in* (3.6) *satisfies the continuous condition.*

In the following, we consider a special case of Theorem 1. For this, we define the following perturbation of the IFS defined in (2.5):

$$\begin{cases} L_n(x) = a_n x + e_n \\ \widetilde{F}_n(x,y) = (\alpha_n + \delta_n)y + \psi_n(x) + \lambda_n, \end{cases} \tag{3.7}$$

where δ_n and λ_n are constants such that $|\alpha_n + \delta_n| < 1$. In this situation, we have $\gamma_n = 0$ and we obtain the following result.

Theorem 2. *Let* f *and* $\widetilde{f}$ *be the functions generated by the IFSs* (2.5) *and* (3.7), *respectively. For a given* $x \in I$, *let* $\{n_j\}, n_j \in J$ *be sequence such that* $x = \sum_{r=1}^{\infty}\Big(\prod_{j=1}^{r-1} a_{n_j}\Big)e_{n_r}$.

1. *Assume for all* $1 \le n < N$ *that*

$$\lambda_{n+1} = \Big(\sum_{j=1}^{n}\delta_j\Big)y_N - \Big(\sum_{j=2}^{n+1}\delta_j\Big)y_0 + \lambda_1,$$

then the IFS defined by the system (3.7) *satisfies the continuous condition.*

2. *For a given* $x \in I$, *let* $\{n_j\}, n_j \in J$ *be sequence such that* $x = \sum_{r=1}^{\infty}\Big(\prod_{j=1}^{r-1} a_{n_j}\Big)e_{n_r}$, *then*

$$\Big|\widetilde{f}(x) - f(x)\Big| \le \sum_{r=1}^{\infty}\Big|(s+t)^{r-1} - t^{r-1}\Big|\psi_{n_r}\Big(\sum_{l=1}^{\infty}\Big(\prod_{j=1}^{l-1} a_{n_{r+j}}\Big)e_{n_{r+l}}\Big) + \sum_{r=1}^{\infty}\Big(\prod_{j=1}^{r-1}\beta_{n_j}\Big)\lambda_{n_r},$$

where $t := \max_n |\alpha_n|$ *and* $s = \max_n |\beta_n - \alpha_n|$.

Proof. The first assertion is a direct application of Theorem 1, and we will only prove the second one. Using Theorem 1, we obtain

$$\widetilde{f}(x) - f(x) = \sum_{r=1}^{\infty}\Big[\prod_{j=1}^{r-1}\beta_{n_j} - \prod_{j=1}^{r-1}\alpha_{n_j}\Big]\psi_{n_r}\Big(\sum_{l=1}^{\infty}\Big(\prod_{j=1}^{l-1} a_{n_{r+j}}\Big)e_{n_{r+l}}\Big) + \sum_{r=1}^{\infty}\Big(\prod_{j=1}^{r-1}\beta_{n_j}\Big)\lambda_{n_r}. \tag{3.8}$$

Note that (3.8) can be found in [41, Theorem 3]. Moreover, using [32, Lemma 3.2], we have

$$\left|\widetilde{f}(x) - f(x)\right| \leq \sum_{r=1}^{\infty} \left|(s+t)^{r-1} - t^{r-1}\right| \psi_{n_r}\Big(\sum_{l=1}^{\infty} \Big(\prod_{j=1}^{l-1} a_{n_{r+j}} \Big) e_{n_{r+l}} \Big) + \sum_{r=1}^{\infty} \Big(\prod_{j=1}^{r-1} \beta_{n_j} \Big) \lambda_{n_r},$$

where $t := \max_n |\alpha_n|$ and $s = \max_n |\beta_n - \alpha_n|$. □

Remark 1. *Consider the IFS defined by* (3.7) *such that* $\alpha := \alpha_1 = \cdots = \alpha_N$. *Choose* $\delta = \frac{1-\alpha}{2}$ *so that we have* $\alpha + \delta < 1$. *Therefore, by a direct computation using* (2.4)*, we obtain for all* $x \in I$

$$\begin{aligned} |\widetilde{f}(x) - f(x)| &\leq \max_{n \in J} \left| \widetilde{F}_n(L_n^{-1}(x), \widetilde{f} \circ L_n^{-1}(x)) - F_n(L_n^{-1}(x), f \circ L_n^{-1}(x)) \right| \\ &= \max_{n \in J} \left| (\alpha + \delta) \widetilde{f}(L_n^{-1}(x)) + \lambda_n - \alpha f(L_n^{-1}(x)) \right| \\ &\leq \alpha \|\widetilde{f} - f\|_\infty + \delta \|\widetilde{f} - f\|_\infty + \delta \|f\|_\infty + \lambda_\infty, \end{aligned}$$

where $\lambda_\infty := \max_{n \in J} \lambda_n$. *It follows that*

$$\|\widetilde{f} - f\|_\infty \leq \frac{\delta \|f\|_\infty + \lambda_\infty}{1 - \alpha - \delta} \leq \frac{2}{1-\alpha} \Big(\|f\|_\infty + \lambda_\infty \Big).$$

Now, applying Theorem 2, we get

$$\begin{aligned} \left|\widetilde{f}(x) - f(x)\right| &\leq \sum_{r=1}^{\infty} \Big(\frac{1+\alpha}{2} \Big)^{r-1} \max_n \|\psi_n\|_\infty + \sum_{r=1}^{\infty} \Big(\frac{1+\alpha}{2} \Big)^{r-1} \lambda_\infty \\ &\leq \frac{2}{1-\alpha} \Big(\max_n \|\psi_n\|_\infty + \lambda_\infty \Big). \end{aligned}$$

Example 2. *In this example, we consider a special case of Theorem 2 by choosing all the parameters* λ_n *as equal for* $n \in J$. *This is done by choosing* $y_0 = y_N = 0$. *For this, let us consider the sets of data points:*

$$\Delta := \Big\{ (0,0), (1/3, 1), (2/3, -1), (1, 0) \Big\},$$

and we consider the IFS defined by (2.5) *where*

$$L_n(x) = \frac{x}{3} + \frac{n-1}{N}, \quad \psi_1(x) = x, \quad \psi_2(x) = 1 - 2x, \quad \psi_3(x) = x - 1 \quad \textit{and} \quad \alpha_1 = \alpha_2 = \alpha_3 = -0.5.$$

Now, we consider the following perturbed system

$$\begin{cases} L_n(x) = \frac{x}{3} + \frac{n-1}{N} \\ \widetilde{F}_n(x,y) = (\alpha_n + \delta_n) y + \psi_n(x) + \lambda_n. \end{cases} \tag{3.9}$$

We choose $\lambda_1 = 0$ *and we collect the different values of* δ_n *in the Table 1. Moreover, different perturbed FIF are plotted in Figure 4. As we may see, we obtain smooth or non-smooth FIF depending on the choice of* δ_n. *One can describe the self-similar structures of the graph of FIF by computing the box dimension D (also known as the Minkowski–Bouligand dimension or the box-counting dimension),* *which is a mathematical object used to describe the complexity of certain figures and proved to be an appropriate and effective method for fractal dimension estimate. More precisely,*

$$D = \lim_{\epsilon \to 0} \frac{\log \mathsf{N}_\epsilon}{\log(1/\epsilon)}, \tag{3.10}$$

where N_ϵ *is the minimum number of* $\epsilon \times \epsilon$ *squares needed to cover the graph of* f.

	δ	λ	box dimension
Initial system	$\delta_n = 0$	$\lambda_n = 0$	1.369
First perturbation	$\delta_n = 1.25$	$\lambda_n = 0$	1.738
Second perturbation	$\delta_1 = 0.65$ $\delta_2 = 0.6$ $\delta_3 = 0.85$	$\lambda_n = 0$	1
Third perturbation	$\delta_1 = 1$ $\delta_2 = 0.65$ $\delta_3 = -0.3$	$\lambda_n = 0$	1.338

Table 1. The Box dimension, computed from (3.10), of the different perturbation systems.

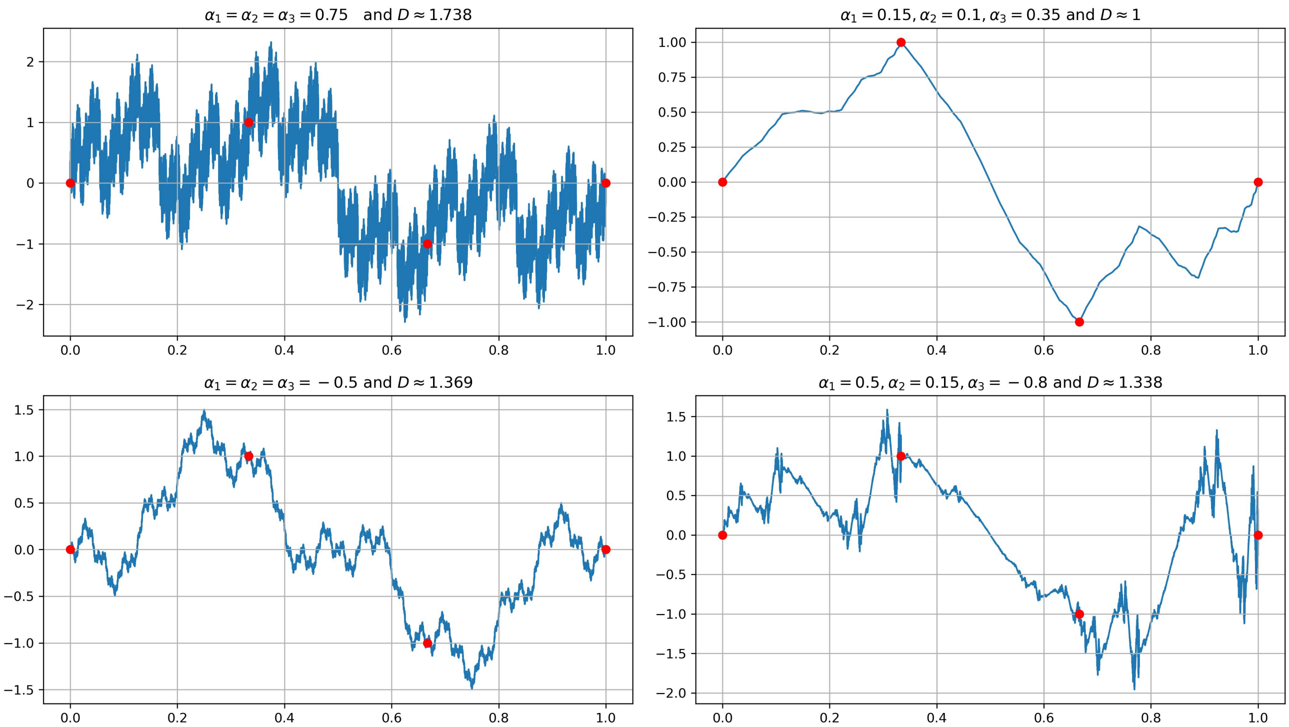

Figure 4. The graphs of FIFs obtained from (3.9) with different δ_n and their corresponding box dimension D.

As an application of Theorem 2, the following result concerning the Weierstrass function is considered in Example 1.

Corollary 1. *Let f be the Weierstrass function defined in* (3.5) *with* $\phi(0) = 0$, *then there exists an FIF* $\widetilde{f}$ *satisfying*

$$\|\widetilde{f} - f\|_\infty \leq \frac{2\|\phi\|_\infty}{1-\lambda}.$$

Proof. Let $I = [0, 1]$ and the interpolating points $x_0 = 0 < x_1 < \cdots < x_N = 1$ such that $x_n - x_{n-1} = 1/N$ ($N = l$). We consider the following system defined as

$$\begin{cases} L_n(x) = \frac{x}{N} + \frac{n-1}{N} \\ F_n(x, y) = \alpha y + \phi(\frac{x+n-1}{N}), \end{cases}$$

where $\alpha = \lambda$. It is well known that the function f is an FIF [40]. Indeed, consider for $n \in J$ the function

$$W_n(x, y) = \Big(\frac{x+n-1}{N}, \alpha y + \phi(\frac{x+n-1}{N})\Big), \qquad (x, y) \in [0, 1] \times \mathbb{R}.$$

It follows that

$$f(L_n(x)) = f\Big(\frac{x+n-1}{N}\Big) = \phi(\frac{x+n-1}{N}) + \alpha \sum_{k=0}^{\infty} \alpha^k \phi(N^k x) = \phi(\frac{x+n-1}{N}) + \alpha f(x)$$

and, thus,

$$C_f = \bigcup_{n=1}^{N} W_n(C_f).$$

Now, we consider the following perturbed system

$$\begin{cases} L_n(x) = \frac{x}{N} + \frac{n-1}{N} \\ F_n(x, y) = (\alpha + \delta)y + \phi(\frac{x+n-1}{N}) + \lambda_n, \end{cases}$$

where $\delta = (1-\alpha)/2$ and $\lambda_1 = 0$, and we obtain

$$\lambda_{n+1} = \frac{n}{2}(1-\alpha)y_N - \frac{n}{2}(1-\alpha)y_0 = \frac{n}{2}(1-\alpha)[y_N - y_0].$$

Furthermore, using Remark 1, we get

$$\begin{aligned} \|\widetilde{f} - f\|_\infty &\leq \frac{2}{1-\alpha}\Big(\|\phi\|_\infty + \lambda_\infty\Big) \\ &\leq \frac{2}{1-\alpha}\Big(\|\phi\|_\infty + \frac{N}{2}(1-\alpha)(y_N - y_0)\Big). \end{aligned}$$

Finally, we get the desired result since ϕ is a $\mathbb{Z}$-periodic function with $\phi(0) = 0$, which implies that $f(0) = f(1)$ and $y_N = y_0$.

□

4. Upper bound of the maximum range of the perturbed FIF

Let $f \in \mathcal{C}(I)$, the normed space of real valued endowed with the uniform norm continuous function on I. We will say that the function f is Hölder continuous with exponent β and Hölder constant c if

$$|f(x) - f(y)| \leq c|x - y|^{\theta}, \quad x, y \in I.$$

We denote by $\mathsf{H}^{\theta}(I)$ the set of all Hölder continuous functions on I with exponent θ. We consider the interpolation points $P = \{(\frac{n}{N}, y_n) \in \mathbb{R}^2, n \in J\}$ and we denote $D = \{\frac{n}{N} \in I, n \in J_0\}$. We set

$$L^0(D) = D, \quad L(D) = \bigcup_{n=1}^{N} L_n(D), \qquad \text{and} \qquad L^k(D) = L \circ \cdots \circ L(D),$$

k times composition. In this section, we consider

$$\begin{cases} L_n(x) = a_n x + e_n \\ F_n(x, y) = \alpha_n(x) y + f(L_n(x)) - \alpha_n(x) b(x), \end{cases} \tag{4.1}$$

where the real constants a_n and e_n are determined by condition (2.1), the functions $\alpha_n : I \longrightarrow \mathbb{R}$ are Lipschitz functions, with Lipschitz constant C_n such that $\alpha := \max_n \|\alpha_n\|_\infty < 1$ and $b \in \mathcal{C}(I)$ such that $b(x_0) = f(x_0)$ and $b(x_N) = f(x_N)$. The FIF generated by (4.1) will be denoted by f^α, which interpolates f at the nodes of the partition. Moreover, using (2.4), the function f^α satisfies

$$f^\alpha(x) = f(x) + \alpha_n(L_n^{-1}(x))(f^\alpha - b)(L_n^{-1}(x)), \qquad \text{for all } x \in I_n, n \in J. \tag{4.2}$$

We will denote by $\widetilde{f^\alpha}$ the FIF generated by the perturbed system; that is

$$\widetilde{F}_n(x, y) = \beta_n(x) y + f(L_n(x)) - \alpha_n(x) b(x) + \lambda_n(x). \tag{4.3}$$

Lemma 1. *Assume that* $\beta := \max_n \|\beta_n\|_\infty < 1$. *For all* $x \in I_n$ *and* $n \in J$, *we have*

$$\|\widetilde{f^\alpha} - f\|_\infty \leq \frac{\alpha\|f - b\|_\infty + \delta\|f\|_\infty + \|\lambda\|_\infty}{1 - \beta}$$

and

$$\|\widetilde{f^\alpha} - f^\alpha\|_\infty \leq \frac{\beta\|f^\alpha\|_\infty + \|\lambda\|_\infty}{1 - \beta},$$

where $\|\lambda\|_\infty = \max_n \|\lambda_n\|_\infty$, $\alpha = \max_n \|\alpha_n\|_\infty$ *and* $\delta = \max_n \|\delta_n\|_\infty$.

Proof. Using (2.4), we obtain

$$\widetilde{f^\alpha}(x) \quad = \quad f(x) + \beta_n(L_n^{-1}(x))\widetilde{f^\alpha}(L_n^{-1}(x)) - \alpha_n(L_n^{-1}(x)) b(L_n^{-1}(x)) + \lambda_n(L_n^{-1}(x)) \tag{4.4}$$

and

$$(1 - \beta)\|\widetilde{f^\alpha}(x) - f(x)\|_\infty \leq \alpha\|f - b\|_\infty + \delta\|f\|_\infty + \|\lambda\|_\infty$$

as required. Moreover, from (4.4) and (4.2), $\widetilde{f^\alpha}(x) = f^\alpha(x) + \beta_n(L_n^{-1}(x))\widetilde{f^\alpha}(L_n^{-1}(x)) + \lambda_n(L_n^{-1}(x))$ and

$$\|\widetilde{f^\alpha} - f^\alpha\|_\infty \quad \leq \quad \beta\|\widetilde{f^\alpha} - f^\alpha\|_\infty + \beta\|f^\alpha\|_\infty + \|\lambda\|_\infty.$$

□

Now, assume that $f \in \mathsf{H}^{\theta_1}(I)$ and $b \in \mathsf{H}^{\theta_2}(I)$ with Hölder constant H_f, H_b respectively.

Lemma 2. *Let $\widetilde{f}^\alpha$ be the FIF generated by the system* (4.3) *and we assume that* $\beta = \max_n \|\beta_n\|_\infty < 1$. *We denote* $\alpha = \max_n \|\alpha_n\|_\infty$, $\delta = \max_n \|\delta_n\|_\infty$ *and* $A_1 = \dfrac{\beta\Gamma_1 + H_f + \alpha H_b + y_0\delta}{1-\beta}$, *then*

$$\left|\widetilde{f}^\alpha(x) - y_{n-1}\right| \le A_1, \qquad x \in I_n, \forall n \in J.$$

Proof. We define, for $k = 1, 2, \ldots,$

$$\Gamma_k = \max\{|\widetilde{f}^\alpha(x) - y_0|, x \in L^{k-1}(D)\} \quad \text{and} \quad \gamma_k = \max_n\{|\widetilde{f}^\alpha(x) - y_{n-1}|, x \in L^{k-1}(D) \cap I_n\}.$$

First, observe that

$$\begin{aligned} \Gamma_k &\le \max_n\{|\widetilde{f}^\alpha(x) - y_{n-1}|, x \in L^{k-1}(D) \cap I_n\} + \max_n\{|y_{n-1} - y_0|\} \\ &\le \Gamma_1 + \gamma_k. \end{aligned} \tag{4.5}$$

For $x \in L^k(D) \cap I_n$, we have

$$\widetilde{f}^\alpha(x) = f(x) + \beta_n(L_n^{-1}(x))\widetilde{f}^\alpha(L_n^{-1}(x)) - \alpha_n(L_n^{-1}(x))b(L_n^{-1}(x)) + \lambda_n(L_n^{-1}(x))$$

and then

$$\begin{aligned} \widetilde{f}^\alpha(x) - y_{n-1} =& f(x) - y_{n-1} + \beta_n(L_n^{-1}(x))\Big[\widetilde{f}^\alpha(L_n^{-1}(x)) - y_0\Big] - \\ & \alpha_n(L_n^{-1}(x))\Big[b(L_n^{-1}(x)) - y_0\Big] + \lambda_n(L_n^{-1}(x)) + y_0\delta_n(L_n^{-1}(x)) \\ \le& H_f + \beta\Gamma_{k-1} + \alpha H_b + y_0\delta. \end{aligned}$$

We denote by $A = H_f + \alpha H_b + y_0\delta$, which does not depend on k. It follows by using (4.5) that

$$\begin{aligned} \gamma_{k+1} &\le \beta\Gamma_k + A \le \beta\gamma_k + \beta\Gamma_1 + A \\ &\le \beta\Big(\beta\Gamma_{k-1} + A\Big) + \beta\Gamma_1 + A \\ &\le \beta^2\gamma_{k-1} + \beta^2\Gamma_1 + \beta\Gamma_1 + \beta A + A \\ &\vdots \\ &\le \sum_{j=1}^{k}\beta^j\Gamma_1 + \sum_{j=0}^{k-1}\beta^j A \le \frac{\beta\Gamma_1 + A}{1-\beta} = A_1. \end{aligned}$$

For any $x \in I_n$, there exits a sequence $\{x_j\}_j \in I_n \cap \Big(\bigcup_k L^k(D)\Big)$ such that $x_j \longrightarrow x$ and $\lim_{j\to\infty}|\widetilde{f}^\alpha(x_j) - y_{n-1}| = |\widetilde{f}^\alpha(x) - y_{n-1}|$, by continuity of the function $\widetilde{f}^\alpha$. Therefore, we get

$$\left|\widetilde{f}^\alpha(x) - y_{n-1}\right| \le A_1, \qquad x \in I_n.$$

□

Given a function S defined on I, we define the maximum range R_S of S as

$$R_S(I) = \sup_{s_1, s_2 \in I}\left|S(s_1) - S(s_2)\right|.$$

Theorem 3. *Let f^α be the α-FIF the IFS* (4.1) *with interpolation points P and $\widetilde{f^\alpha}$ as the perturbed FIF defined by* (4.3). *Assume that $\beta = \max_n \|\beta_n\|_\infty < 1$, then*

$$R_{\widetilde{f^\alpha}}(I) \leq \min\left\{N A_1, \frac{2}{1-\beta}(\alpha\|b\|_\infty + \|\lambda\|_\infty + \|f\|_\infty(1+\alpha+\delta-\beta))\right\}$$

where $A_1 = \dfrac{\beta\Gamma_1 + H_f + \alpha H_b + y_0\delta}{1-\beta}$, $\alpha = \max_n \|\alpha_n\|_\infty$ *and* $\delta = \max_n \|\delta_n\|_\infty$.

Proof. From Lemma 2, we have

$$\sup_{I_n}\left|\widetilde{f^\alpha}(x) - y_{n-1}\right| \leq A_1,$$

Now, let $s_1, s_2 \in I$, then there exists $n_1 \leq n_2 \in J$ such that $s_1 \in I_{n_1}$ and $s_2 \in I_{n_2}$. It follows that

$$\begin{aligned}\left|\widetilde{f^\alpha}(s_1) - \widetilde{f^\alpha}(s_2)\right| &\leq \left|\widetilde{f^\alpha}(s_1) - y_{n_1-1}\right| + \left|y_{n_1-1} - y_{n_1}\right| + \cdots + \left|y_{n_2-1} - \widetilde{f^\alpha}(s_2)\right|. \\ &\leq N A_1.\end{aligned}$$

On the other hand, we may estimate the upper bound of the maximum range $R_{\widetilde{f^\alpha}}$ not depending on N. Indeed, using Lemma 1 we get

$$\begin{aligned}R_{\widetilde{f^\alpha}} &\leq 2\|\widetilde{f^\alpha}\|_\infty \leq 2\|\widetilde{f^\alpha} - f\|_\infty + 2\|f\|_\infty \\ &\leq 2\frac{\alpha\|f-b\|_\infty + \delta\|f\|_\infty + \|\lambda\|_\infty}{1-\beta} + 2\|f\|_\infty \\ &\leq \frac{2}{1-\beta}(\alpha\|b\|_\infty + \|\lambda\|_\infty + \|f\|_\infty(1+\alpha+\delta-\beta))\end{aligned}$$

as required. □

Remark 2. *Let f^α be the α-FIF the IFS* (4.1) *with interpolation points P such that $\alpha < 1$, then*

$$R_{f^\alpha}(I) \leq \min\left\{N A_2, \frac{2}{1-\alpha}(\alpha\|b\|_\infty + \|f\|_\infty)\right\}$$

where $A_2 = \dfrac{\alpha\Gamma_1 + H_f + \alpha H_b}{1-\alpha}$.

Example 3. *Let $I = [0,1]$ and $f(x) = x - x^2$. Observe that for any $x, y \in I$, we have*

$$|f(x) - f(y)| \leq |x-y| + |x^2 - y^2| \leq 3|x-y|,$$

then the function f is Hölder continuous with exponent one and Hölder constant $H_f = 3$. In this example, we consider the following perturbed system

$$\begin{cases} L_n(x) = \frac{x}{N} + \frac{n-1}{N} \\ F_n(x,y) = (\alpha+\delta)y + f(L_n(x)) - \alpha b(x) + \lambda_n, \end{cases}$$

where $\delta = (1-\alpha)/2$, $\lambda_1 = 0$ and $b(x) = f(x)/3$. Since $f(0) = f(1) = 0$, we obtain $\lambda_n = 0$ for all $n \in J$. Therefore, using Lemma 1, we have

$$\|\widetilde{f^\alpha} - f\|_\infty \leq \frac{\alpha\|f-b\|_\infty + \delta\|f\|_\infty}{1-\beta} \leq \frac{\alpha/3 + (1-\alpha)/4}{1-\alpha} = \frac{\alpha+3}{12(1-\alpha)}$$

and

$$\|\widetilde{f^{\alpha}} - f^{\alpha}\|_{\infty} \leq \frac{1+\alpha}{1-\alpha}\|f^{\alpha}\|_{\infty}.$$

In particular, if $\alpha = 1/4$, *we obtain*

$$\|\widetilde{f^{\alpha}} - f\|_{\infty} \leq \frac{13}{36} \qquad \textit{and} \qquad \|\widetilde{f^{\alpha}} - f^{\alpha}\|_{\infty} \leq \frac{5}{3}\|f^{\alpha}\|_{\infty}.$$

Therefore, we have

$$\begin{aligned} R_{\widetilde{f^{\alpha}}}(I) &\leq \frac{2}{1-\beta}(\alpha\|b\|_{\infty} + \|\lambda\|_{\infty} + \|f\|_{\infty}(1+\alpha+\delta-\beta)) \\ &\leq \frac{1}{1-\alpha}(\alpha/12 + 1/4), \end{aligned}$$

then $R_{\widetilde{f^{\alpha}}}(I) \leq \frac{13}{36}$ *for* $\alpha = 1/4$.

5. Conclusions

In this paper, a class of generalized affine FIFs with variable parameters, where ordinate scaling is substituted by real-valued control function, is investigated. More precisely, we considered the FIF generated through the IFS defined by the functions $W_n(x, y) = (a_n(x)+e_n, \alpha_n(x)y+\psi_n(x))$, $n = 1, \ldots, N$. We computed the error estimate in response to a small perturbation on $\alpha_n(x)$ and we gave a sufficient condition on the perturbed IFS so that it satisfies the continuity condition.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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