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Research article

Exact Jacobi elliptic solutions of some models for the interaction of long and short waves

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Abstract: Some systems were recently put forth by Nguyen et al. as models for studying the interaction of long and short waves in dispersive media. These systems were shown to possess synchronized Jacobi elliptic solutions as well as synchronized solitary wave solutions under certain constraints, i.e., vector solutions, where the two components are proportional to one another. In this paper, the exact periodic traveling wave solutions to these systems in general were found to be given by Jacobi elliptic functions. Moreover, these cnoidal wave solutions are unique. Thus, the explicit synchronized solutions under some conditions obtained by Nguyen et al. are also indeed unique.

Keywords: periodic solutions; cnoidal solutions; NLS-equation; KdV-equation; BBM-equation; NLS-KdV system
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1. Introduction

The following four systems, termed Schrödinger KdV-KdV, Schrödinger BBM-BBM, Schrödinger KdV-BBM and Schrödinger BBM-KdV, respectively,

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mu_0 \dfrac{\partial u}{\partial x} + a_0 \dfrac{\partial^3 u}{\partial x^3} + ib \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial (uv)}{\partial x} - i\mu_1 uv, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial x} + c\dfrac{\partial^3 v}{\partial x^3} = -\dfrac{1}{2}\dfrac{\partial |u|^2}{\partial x}, \end{cases} \tag{1.1}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mu_0 \dfrac{\partial u}{\partial x} - a_1 \dfrac{\partial^3 u}{\partial x^2 \partial t} + ib \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial (uv)}{\partial x} - i\mu_1 uv, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial x} - c\dfrac{\partial^3 v}{\partial x^2 \partial t} = -\dfrac{1}{2}\dfrac{\partial |u|^2}{\partial x}, \end{cases} \tag{1.2}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mu_0 \dfrac{\partial u}{\partial x} + a_0 \dfrac{\partial^3 u}{\partial x^3} + ib\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial (uv)}{\partial x} - i\mu_1 uv, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial x} - c\dfrac{\partial^3 v}{\partial x^2 \partial t} = -\dfrac{1}{2}\dfrac{\partial |u|^2}{\partial x}, \end{cases} \tag{1.3}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mu_0 \dfrac{\partial u}{\partial x} - a_1 \dfrac{\partial^3 u}{\partial x^2 \partial t} + ib\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial (uv)}{\partial x} - i\mu_1 uv, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial x} + c\dfrac{\partial^3 v}{\partial x^3} = -\dfrac{1}{2}\dfrac{\partial |u|^2}{\partial x} \end{cases} \tag{1.4}$$

were recently advocated in [1, 2] (see also [3]) as more suitable models for studying the interaction of long and short waves in dispersive media due to their consistent derivation when compared to the nonlinear Schrödinger-KdV system [4]:

$$\begin{cases} iu_t + u_{xx} + a|u|^2 u = -buv, \\ v_t + cvv_x + v_{xxx} = -\dfrac{b}{2}(|u|^2)_x. \end{cases} \tag{1.5}$$

Here, the function $u(x,t)$ is a complex-valued function, while $v(x,t)$ is a real-valued function and $x, t \in \mathbb{R}$, where $\mu_0, \mu_1, a_0, a_1, b$ and c are real constants with $\mu_0, \mu_1, a_0, a_1, c > 0$. For a detailed discussion on these systems, we refer our readers to the papers [1–3].

A traveling-wave solution to the above four systems is a vector solution $(u(x,t), v(x,t))$ of the form

$$u(x,t) = e^{i\omega t} e^{iB(x-\sigma t)} f(x - \sigma t), \qquad v(x,t) = g(x - \sigma t), \tag{1.6}$$

where f and g are smooth, real-valued functions with speed $\sigma > 0$ and phase shifts $B, \omega \in \mathbb{R}$. Substituting the traveling-wave ansatz (1.6) into the four systems and separating the real and imaginary parts, the following associated systems of ordinary differential equations (ODE) are obtained:

$$\begin{cases} f'g + fg' + a_0 f''' + (\mu_0 - \sigma - 3a_0 B^2 - 2bB)f' = 0, \\ (B + \mu_1)fg + (3a_0 B + b)f'' + (\omega + B\mu_0 - B\sigma - a_0 B^3 - bB^2)f = 0, \\ ff' + gg' + cg''' + (1 - \sigma)g' = 0, \end{cases} \tag{1.7}$$

$$\begin{cases} f'g + fg' + a_1 \sigma f''' + (\mu_0 + 2a_1 B\omega - 3a_1 B^2 \sigma - \sigma - 2bB)f' = 0, \\ (B + \mu_1)fg + (3a_1 B\sigma + b - a_1\omega)f'' + (\omega + B\mu_0 + a_1 B^2\omega - a_1 B^3 \sigma - B\sigma - bB^2)f = 0, \\ ff' + gg' + c\sigma g''' + (1 - \sigma)g' = 0, \end{cases} \tag{1.8}$$

$$\begin{cases} f'g + fg' + a_0 f''' + (\mu_0 - \sigma - 3a_0 B^2 - 2bB)f' = 0, \\ (B + \mu_1)fg + (3a_0 B + b)f'' + (\omega + B\mu_0 - B\sigma - a_0 B^3 - bB^2)f = 0, \\ ff' + gg' + c\sigma g''' + (1 - \sigma)g' = 0, \end{cases} \tag{1.9}$$

and

$$\begin{cases} f'g + fg' + a_1 \sigma f''' + (\mu_0 + 2a_1 B\omega - 3a_1 B^2 \sigma - \sigma - 2bB)f' = 0, \\ (B + \mu_1)fg + (3a_1 B\sigma + b - a_1\omega)f'' + (\omega + B\mu_0 + a_1 B^2\omega - a_1 B^3 \sigma - B\sigma - bB^2)f = 0, \\ ff' + gg' + cg''' + (1 - \sigma)g' = 0. \end{cases} \tag{1.10}$$

We refer to semi-trivial solutions as solutions where at least one component is a constant (possibly zero). Of course, the trivial solution $(0,0)$ is always a solution. In the case when f is a constant multiple of g, the vector solution is termed a synchronized solution. Among the traveling-wave solutions, attention is often given to the solitary-wave and periodic solutions due to the roles they sometimes play in the evolution equations. Solitary waves are smooth traveling-wave solutions that are symmetric around a single maximum and rapidly decay to zero away from the maximum while periodic solutions are self-explanatory. Even though less common, the term solitary waves are also sometimes used to describe traveling-wave solutions that are symmetric around a single maximum, but that approach nonzero constants as $\xi \to \pm\infty$.

The topic of existence of synchronized traveling-wave solutions to these four systems has been addressed previously [5]. Notice that when f is a constant multiple of g, i.e., $u = Av$ for some proportional constant A, the three equations in each of the four associated ODEs (1.7)–(1.10) can be collapsed into four single equations of the form

$$f'^2 = k_3 f^3 + k_2 f^2 + k_1 f + k_0, \tag{1.11}$$

under certain constraints. In [5], it was shown that the systems possess synchronized solitary waves with the usual hyperbolic sech^2-profile typical of dispersive equations. In [6], a novel approach was first employed to establish the existence of periodic traveling-wave solutions for these systems, namely, the topological degree theory for positive operators that was introduced by Krasnosel'skii [7, 8] and used in several different models [9–11]. The explicit synchronized periodic solutions $u = Av$, where v is given by the Jacobi elliptic function

$$v(x - \sigma t) :\equiv v(\xi) = C_0 + C_2 \operatorname{cn}^2(\alpha\xi + \beta, m), \tag{1.12}$$

are then obtained by demanding the coefficients in each of the four cases to satisfy certain constraints. (A brief description of the Jacobi elliptic functions is recalled below.) Neither approach, however, guarantees uniqueness of the periodic solutions obtained due to several factors, such as the form of v as a priori assumption because of (1.11) as well as the nature of the topological degree theory approach.

It is worth it to point out that explicit solitary wave solutions have been found for another system [12, 13], the *abcd*-system

$$\begin{cases} \eta_t + w_x + (w\eta)_x + a w_{xxx} - b\eta_{xxt} = 0, \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = 0, \end{cases} \tag{1.13}$$

where a, b, c and d are real constants satisfying

$$a + b = \frac{1}{2}(\theta^2 - \frac{1}{3}), \quad c + d = \frac{1}{2}(1 - \theta^2) \geq 0, \quad a + b + c + d = \frac{1}{3},$$

and $\theta \in [0, 1]$. This system is used to model small-amplitude, long wavelength, gravity waves on the surface of water [14, 15]. Here, $\eta(x,t)$ and $w(x,t)$ are real valued functions and $x, t \in \mathbb{R}$. However, the existence of periodic traveling-wave solutions for this system are still not well understood. The only result that we are aware of is for the special case when $a = c = 0$ and $b = d = 1/6$, where the solutions are given in term of the Jacobi elliptic cnoidal function [10].

The manuscript is organized as follows. In Section 2, some facts about the Jacobi elliptic functions are reviewed and the results are summarized. In Section 3, the explicit cnoidal solutions to the four systems are established, and how these solutions limit to the solitary-wave solutions are analyzed. Section 4 is devoted to discussion of the obtained results. To preserve the self-completeness without affecting the flow of the paper, some tedious formulae and expressions are delegated to the Appendix.

2. Preliminaries and statement of results

For the readers' convinience, some notions of the Jacobi elliptic functions are briefly recalled here. Let

$$v = \int_0^\phi \frac{1}{\sqrt{1-m^2\sin^2 t}}dt, \qquad \text{for } 0 \le m \le 1,$$

then $v = F(\phi, m)$ or, equivalently, $\phi = F^{-1}(v, m) = \mathrm{am}(v, m)$, which is the Jacobi amplitude. The two basic Jacobi elliptic functions $\mathrm{cn}(v, m)$ and $\mathrm{sn}(v, m)$ are defined as

$$\mathrm{sn}(v, m) = \sin(\phi) = \sin\left(F^{-1}(v, m)\right) \quad \text{and} \quad \mathrm{cn}(v, m) = \cos(\phi) = \cos\left(F^{-1}(v, m)\right),$$

where m is referred to as the Jacobi elliptic modulus. These functions are generalizations of the trigonometric and hyperbolic functions, which satisfy

$$\begin{array}{ll} \mathrm{sn}(v, 0) = \sin(v), & \mathrm{cn}(v, 0) = \cos(v), \\ \mathrm{cn}(v, 1) = \mathrm{sech}(v), & \mathrm{sn}(v, 1) = \tanh(v). \end{array}$$

We recall the following relations between these functions:

$$\begin{cases} \mathrm{sn}^2(\lambda\xi, m) = 1 - \mathrm{cn}^2(\lambda\xi, m), \\ \mathrm{dn}^2(\lambda\xi, m) = 1 - m^2 + m^2\,\mathrm{cn}^2(\lambda\xi, m), \\ \dfrac{d}{d\xi}\,\mathrm{cn}(\lambda\xi, m) = -\lambda\,\mathrm{sn}(\lambda\xi, m)\,\mathrm{dn}(\lambda\xi, m), \\ \dfrac{d}{d\xi}\,\mathrm{sn}(\lambda\xi, m) = \lambda\,\mathrm{cn}(\lambda\xi, m)\,\mathrm{dn}(\lambda\xi, m), \\ \dfrac{d}{d\xi}\,\mathrm{dn}(\lambda\xi, m) = -m^2\lambda\,\mathrm{cn}(\lambda\xi, m)\,\mathrm{sn}(\lambda\xi, m). \end{cases}$$

In this manuscript, the existence of periodic traveling-wave solutions to the above four associated ODE systems (1.7)–(1.10) in general are analyzed. The periodic traveling-wave solutions sought here are given by

$$f(\xi) = \sum_{r=0}^{n} d_r\,\mathrm{cn}^r(\lambda\xi, m) \quad \text{and} \quad g(\xi) = \sum_{r=0}^{n} h_r\,\mathrm{cn}^r(\lambda\xi, m), \tag{2.1}$$

where $d_r, h_r \in \mathbb{R}$, $\lambda > 0$ and $0 \le m \le 1$. Using the above relations, the following is revealed:

$$\begin{cases} \dfrac{d}{d\xi}\,\mathrm{cn}^r = -r\lambda\,\mathrm{cn}^{r-1}\,\mathrm{sn}\,\mathrm{dn}, \\ \dfrac{d^2}{d\xi^2}\,\mathrm{cn}^r = -r\lambda^2[(r+1)m^2\,\mathrm{cn}^{r+2} + r(1-2m^2)\,\mathrm{cn}^r + (r-1)(m^2-1)\,\mathrm{cn}^{r-2}], \\ \dfrac{d^3}{d\xi^3}\,\mathrm{cn}^r = r\lambda^3\,\mathrm{sn}\,\mathrm{dn}\,[(r+1)(r+2)m^2\,\mathrm{cn}^{r+1} + r^2(1-2m^2)\,\mathrm{cn}^{r-1} + (r-1)(r-2)(m^2-1)\,\mathrm{cn}^{r-3}], \end{cases} \tag{2.2}$$

where the argument $(\lambda\xi, m)$ has been dropped for clarity reasons. Notice that each of the above four associated ODE systems (1.7)–(1.10) involves three equations. Plugging (2.2) into these systems, the following generic form is obtained:

$$\begin{cases} \operatorname{sn}(\lambda\xi, m)\operatorname{dn}(\lambda\xi, m)\displaystyle\sum_{q=0}^{2n-1} k_{1,q}\operatorname{cn}^q(\lambda\xi, m) = 0, \\ \displaystyle\sum_{q=0}^{2n} k_{2,q}\operatorname{cn}^q(\lambda\xi, m) = 0, \\ \operatorname{sn}(\lambda\xi, m)\operatorname{dn}(\lambda\xi, m)\displaystyle\sum_{q=0}^{2n-1} k_{3,q}\operatorname{cn}^q(\lambda\xi, m) = 0, \end{cases} \tag{2.3}$$

where the subscripts j and q in the coefficient $k_{j,q}$ indicate the equation and the power on the cnoidal function cn, respectively. Notice that as (2.3) must hold true for all $(\lambda\xi, m)$, it must be the case that $k_{j,q} = 0$ for each j and q. Moreover, from the third equation in all four systems, the sum $(ff' + gg')$ contributes the highest order term of cn^{2n-1}. While the next highest order term is from g''', which is cn^{n+1}, by balancing these highest order terms, it reveals that when $n \geq 3$, the highest order term is

$$k_{3,2n-1}\operatorname{cn}^{2n-1} = -n\lambda(d_n^2 + h_n^2)\operatorname{cn}^{2n-1}.$$

Since $\lambda, n > 0$, requiring $k_{3,2n-1} = 0$ implies that $d_n = h_n = 0$, holding true for all $n \geq 3$. Thus, the periodic traveling-wave ansatz (2.1) reduces to

$$f(\xi) = d_0 + d_1\operatorname{cn}(\lambda\xi, m) + d_2\operatorname{cn}^2(\lambda\xi, m) \quad \text{and} \quad g(\xi) = h_0 + h_1\operatorname{cn}(\lambda\xi, m) + h_2\operatorname{cn}^2(\lambda\xi, m). \tag{2.4}$$

Next, by demanding all the coefficients $k_{j,q} = 0$, a set of 13 equations is obtained for each of the four systems involving 11 unknowns $d_i, h_i, B, \lambda, \omega, \sigma$ and m with $i = 0, 1, 2$ (Eqs (A1)–(A4)). For the Schrödinger KdV-KdV and Schrödinger BBM-BBM, the first and last equations in (1.7) and (1.8), respectively, further yield $d_1 = h_1 = 0$. In particular, the only nontrivial periodic solutions for the systems (1.1) and (1.2) are of the form

$$f(\xi) = d_0 + d_2\operatorname{cn}^2(\lambda\xi, m) \quad \text{and} \quad g(\xi) = h_0 + h_2\operatorname{cn}^2(\lambda\xi, m). \tag{2.5}$$

Under these conditions, the sets of 13 equations involving 11 unknowns (Eqs (A1) and (A2)) reduce to sets of seven equations with nine unknowns. Similarly, for the Schrödinger KdV-BBM system and the Schrödinger BBM-KdV sytem, the first and last equations in (1.9) and (1.10), respectively, reveal that $h_1 = 0$. Additionally, when substituting $h_1 = 0$ into (A3) and (A4), the coefficients $k_{3,2}$ and $k_{3,0}$ in both systems require that either $d_1 = 0$ or $d_0 = d_2 = 0$. When $d_1 = h_1 = 0$, we have solutions of the form (2.5), where the sets of 13 equations involving 11 unkowns (Eqs (A3) and (A4)) reduce to seven equations with nine unknowns. When $d_0 = d_2 = h_1 = 0$, we have that the only nontrivial periodic solutions for the systems (1.3) and (1.4) are of the form

$$f(\xi) = d_1\operatorname{cn}(\lambda\xi, m) \quad \text{and} \quad g(\xi) = h_0 + h_2\operatorname{cn}^2(\lambda\xi, m),$$

in which case the sets of 13 equations involving 11 unknowns (Eqs (A3) and (A4)) reduce to sets of six equations with eight unknowns.

The exact, explicit periodic traveling-wave solutions to the four systems (1.1)–(1.4) could then be established by solving those reduced nonlinear systems with the help of the software Maple. As there are two degrees of freedom, in principle any pair of two unknowns can be chosen as "free parameters" so long as solutions can be found consistently. In most physical situations, though, it is more desirable to think of the wave speed σ and elliptic modulus m as "independent" parameters; that is, the cnoidal solutions are found for fixed elliptic modulus $m \in [0, 1]$ and a certain range of wave speed $\sigma > 0$. Indeed, for some cases, it is necessary to assume this condition to have solutions. For the Schrödinger KdV-KdV system (1.1), these nontrivial periodic traveling-wave solutions are established for each wave speed $\sigma > 0$ with $2c > a_0 > 0$, while for the Schrödinger BBM-BBM system (1.2), $\sigma > 0$ with $2c > a_1 > 0$. For the Schrödinger KdV-BBM (1.3), the range of wave speed is $\sigma > \frac{a_0}{2c} > 0$, while for the Schrödinger BBM-KdV (1.4), $0 < \sigma < \frac{2c}{a_1}$. Moreover, for all four systems, the coefficients d_2 and h_2 are constant multiples of each other with the ratios being controlled by the coefficients of the third derivatives in the KdV-KdV and BBM-BBM cases, as well as the wave speed in the KdV-BBM and BBM-KdV cases. Precisely, their ratio is an expression of only a_0 and c in the KdV-KdV case; a_1 and c in the BBM-BBM case; a_0, c and σ in the KdV-BBM case; a_1, c and σ in the BBM-KdV case.

3. Exact Jacobi elliptic solutions

For conciseness, let

$$R = \pm \sqrt{m^4 - m^2 + 1}, \tag{3.1}$$

then $R \in \mathbb{R}$ as $m \in [0, 1]$.

3.1. Schrödinger KdV-KdV

Setting all $k_{j,q} = 0$ gives us the following set of parameters, whenever $2c > a_0 > 0$:

$$\begin{cases}
B = \frac{a_0\mu_1 - b}{2a_0}, \\
d_1 = h_1 = 0, \\
d_0 = \frac{\left(m^4 - 2m^2R - m^2 + R + 1\right)\sqrt{2c - a_0}\left(3a_0^2\mu_1^2 - 2a_0b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)}{8\sqrt{a_0}R^2(a_0 - c)}, \\
d_2 = \frac{3\sqrt{2c - a_0}\left(3a_0^2\mu_1^2 - 2a_0b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)m^2}{8\sqrt{a_0}R(a_0 - c)}, \\
h_0 = \frac{-1}{8a_0R(a_0 - c)}\Big(6a_0^3m^2\mu_1^2 - 3a_0^3\mu_1^2R + 6a_0^2c\mu_1^2R - 3a_0^3\mu_1^2 - 4a_0^2bm^2\mu_1 \\
\quad + 2a_0^2b\mu_1R - 4a_0bc\mu_1R + 2a_0^2b\mu_1 - 8a_0^2m^2\mu_0 + 4a_0^2\mu_0R - 8a_0^2R\sigma - 2a_0b^2m^2 \\
\quad + a_0b^2R - 8a_0c\mu_0R + 8a_0cR\sigma - 2b^2cR + 8a_0^2m^2 + 4a_0^2\mu_0 + 4a_0^2R + a_0b^2 - 4a_0^2\Big), \\
h_2 = \frac{3\left(3a_0^2\mu_1^2 - 2a_0b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)m^2}{8R(a_0 - c)}, \\
\lambda = \sqrt{\frac{3a_0^2\mu_1^2 - 2a_0b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0}{16a_0R(a_0 - c)}}, \\
\omega = -\left(a_0\mu_1^2 - \mu_1 b - \mu_0 + \sigma\right)\mu_1, \\
\sigma > 0, \\
m \in [0, 1].
\end{cases}$$

Thus, explicit periodic traveling-wave solutions to the Schrödinger KdV-KdV system $(u(x,t), v(x,t)) = (e^{i\omega t}e^{iB(x-\sigma t)}f(x-\sigma t), g(x-\sigma t))$, given in term of the Jacobi cnoidal function

$$f(\xi) = d_0 + d_2\,\mathrm{cn}^2(\lambda\xi, m) \quad \text{and} \quad g(\xi) = h_0 + h_2\,\mathrm{cn}^2(\lambda\xi, m)$$

are established. Notice that $\frac{h_2}{d_2} = \sqrt{\frac{a_0}{2c-a_0}}$ and that as m approaches one, R limits to ± 1. When $m = R = 1$, the above coefficients simplify to $d_0 = 0$ and

$$\begin{cases} \tilde{h}_0 = -\frac{1}{4\,a_0\,(a_0-c)}\left(3a_0^2c\mu_1^2 - 2a_0bc\mu_1 + 4a_0^2 - 4a_0c\mu_0 - b^2c - 4a_0^2\sigma + 4a_0c\sigma\right), \\ \tilde{d}_2 = \frac{3\,\sqrt{2\,c-a_0}\left(3\,a_0{}^2\mu_1{}^2-2\,a_0\,b\mu_1-4\,a_0\,\mu_0-b^2+4\,a_0\right)}{8\,\sqrt{a_0}(a_0-c)}, \\ \tilde{h}_2 = \frac{3\left(3\,a_0{}^2\mu_1{}^2-2\,a_0\,b\mu_1-4\,a_0\,\mu_0-b^2+4\,a_0\right)}{8\,(a_0-c)}, \\ \tilde{\lambda} = \sqrt{\frac{3\,a_0{}^2\mu_1{}^2-2\,a_0\,b\mu_1-4\,a_0\,\mu_0-b^2+4\,a_0}{16\,a_0\,(a_0-c)}}, \\ \omega = -\left(a_0\,\mu_1{}^2 - \mu_1\,b - \mu_0 + \sigma\right)\mu_1, \end{cases}$$

from which one obtains the following solitary-wave solution to system (1.1):

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}\tilde{f}(x-\sigma t) \quad \text{and} \quad v(x,t) = \tilde{h}_0 \pm \sqrt{\frac{a_0}{2c-a_0}}\tilde{f}(x-\sigma t),$$

where $\tilde{f}(\xi) = \tilde{d}_2\,\mathrm{sech}^2(\tilde{\lambda}\xi)$. Furthermore, when $\sigma = \frac{4a_0^2+3a_0^2c\mu_1^2-2a_0bc\mu_1-4a_0c\mu_0-b^2c}{4a_0(a_0-c)}$, one has $\tilde{h}_0 = 0$, and the synchronized solitary-wave solution established in [5] is recovered.

When $m = -R = 1$, the above coefficients simplify to

$$\begin{cases} \bar{d}_0 = \frac{\sqrt{2\,c-a_0}\left(3\,a_0{}^2\mu_1{}^2-2\,a_0\,b\mu_1-4\,a_0\,\mu_0-b^2+4\,a_0\right)}{4\,\sqrt{a_0}(a_0-c)}, \\ \bar{h}_0 = \frac{3a_0^3\mu_1^2-2a_0^2b\mu_1-4a_0^2\mu_0-a_0b^2-3a_0^2c\mu_1^2+2a_0bc\mu_1+4a_0c\mu_0+b^2c+4a_0\sigma(a_0-c)}{4\,a_0\,(a_0-c)}, \\ \bar{d}_2 = \frac{3\,\sqrt{2\,c-a_0}\left(3\,a_0{}^2\mu_1{}^2-2\,a_0\,b\mu_1-4\,a_0\,\mu_0-b^2+4\,a_0\right)}{8\,\sqrt{a_0}(a_0-c)}, \\ \bar{h}_2 = -\frac{3\left(3\,a_0{}^2\mu_1{}^2-2\,a_0\,b\mu_1-4\,a_0\,\mu_0-b^2+4\,a_0\right)}{8\,(a_0-c)}, \\ \bar{\lambda} = \sqrt{-\frac{3\,a_0{}^2\mu_1{}^2-2\,a_0\,b\mu_1-4\,a_0\,\mu_0-b^2+4\,a_0}{16\,a_0\,(a_0-c)}}, \\ \omega = -\left(a_0\,\mu_1{}^2 - \mu_1\,b - \mu_0 + \sigma\right)\mu_1, \end{cases}$$

and one arrives at the solitary-wave solution

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}[\bar{d}_0 + \bar{f}(x-\sigma t)] \quad \text{and} \quad v(x,t) = \bar{h}_0 \pm \sqrt{\frac{a_0}{2c-a_0}}\bar{f}(x-\sigma t),$$

where $\bar{f}(\xi) = \bar{d}_2\,\mathrm{sech}^2(\bar{\lambda}\xi)$.

Aside from the above nontrivial solutions, system (1.1) also possessess the following trivial and semi-trivial solutions:

(1)
$$u(x,t) = 0 \quad \text{and} \quad v(x,t) = h_0,$$

for any $h_0 \in \mathbb{R}$.

(2)

$$u(x,t) = e^{i\omega t} e^{iB(x-\sigma t)} d_0 \quad \text{and} \quad v(x,t) = h_0,$$

where $\sigma = \frac{\omega - a_0 B^3 - bB^2 + Bh_0 + B\mu_0 + h_0\mu_1}{B}$, for any $B, d_0, h_0, \omega \in \mathbb{R}$.

(3)

$$u(x,t) = 0 \quad \text{and} \quad v(x,t) = -\frac{2}{3}h_2 + \frac{1}{3}\frac{h_2}{m^2} + \sigma - 1 + h_2 \operatorname{cn}^2(\lambda(x-\sigma t), m),$$

where $\lambda = \sqrt{\frac{h_2}{12cm^2}}$, for any $h_2, \sigma > 0$ and $m \in [\,0, 1\,]$.

3.2. Schrödinger BBM-BBM

Setting all $k_{j,q} = 0$ gives us the following, whenever $2c > a_1 > 0$ and R is as defined in (3.1):

$$\begin{cases}
B = \frac{a_1 \mu_0 \mu_1 - b}{2a_1\sigma\left(a_1 \mu_1{}^2+1\right)}, \\
d_1 = h_1 = 0, \\
d_0 = \frac{\sqrt{a_1 (2c-a_1)}\left(m^4+2m^2R-m^2-R+1\right)}{8 a_1 R^2 \sigma \left(a_1 \mu_1{}^2+1\right)^2 (a_1-c)}\Big(4 a_1{}^3\mu_1{}^4\sigma - 4a_1{}^2 b\mu_1{}^3\sigma - a_1{}^2\mu_0{}^2\mu_1{}^2 - 4a_1{}^2\mu_0\,\mu 1^2\sigma + 8a_1{}^2\mu_1{}^2\sigma \\
\qquad +2a_1 b\mu_0\,\mu 1 - 4a_1 b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\Big), \\
d_2 = \frac{-3m^2\sqrt{a_1(2c-a_1)}}{8a_1R\sigma\left(a_1\mu_1{}^2+1\right)^2(a_1-c)}\Big(4a_1{}^3\mu_1{}^4\sigma - 4a_1{}^2b\mu_1{}^3\sigma - a_1{}^2\mu_0{}^2\mu_1{}^2 - 4a_1{}^2\mu_0\mu_1{}^2\sigma + 8a_1{}^2\mu_1{}^2\sigma \\
\qquad +2a_1 b\mu_0\mu_1 - 4a_1 b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\Big), \\
h_0 = \frac{1}{8a_1R\sigma\left(a_1\mu_1{}^2+1\right)^2(a_1-c)}\Big(8a_1{}^4\mu_1{}^4 r\sigma^2 - 8a_1{}^3c\mu_1{}^4R\sigma^2 + 8a_1{}^4m^2\mu_1{}^4\sigma - 4a_1{}^4\mu_1{}^4R\sigma - 4a_1{}^4\mu_1{}^4\sigma \\
\qquad -8a_1{}^3bm^2\mu_1{}^3\sigma - 4a_1{}^3b\mu_1{}^3R\sigma + 8a_1{}^2bc\mu_1{}^3R\sigma + 4a_1{}^3b\mu_1{}^3\sigma - 2a_1{}^3m^2\mu_0{}^2\mu_1{}^2 \\
\qquad -8a_1{}^3m^2\mu_0\mu_1{}^2\sigma - a_1{}^3\mu_0{}^2\mu_1{}^2R - 4a_1{}^3\mu_0\mu_1{}^2R\sigma + 16a_1{}^3\mu_1{}^2R\sigma^2 + 2a_1{}^2c\mu_0{}^2\mu_1{}^2R \\
\qquad +8a_1{}^2c\mu_0\,\mu 1^2R\sigma - 16a_1{}^2c\mu_1{}^2R\sigma^2 + 16a_1{}^3m^2\mu_1{}^2\sigma + a_1{}^3\mu_0{}^2\mu_1{}^2 + 4a_1{}^3\mu_0\mu_1{}^2\sigma - 8a_1{}^3\mu_1{}^2R\sigma \\
\qquad -8a_1{}^3\mu_1{}^2\sigma + 4a_1{}^2bm^2\mu_0\mu_1 - 8a_1{}^2bm^2\mu_1\sigma + 2a_1{}^2b\mu_0\mu_1R - 4a_1{}^2b\mu_1R\sigma - 4a_1bc\mu_0\mu_1R \\
\qquad +8a_1bc\mu_1R\sigma - 2a_1{}^2b\mu_0\mu_1 + 4a_1{}^2b\mu_1\sigma - 8a_1{}^2m^2\mu_0\sigma - 4a_1{}^2\mu_0R\sigma + 8a_1{}^2R\sigma^2 + 8a_1c\mu_0R\sigma \\
\qquad -8a_1cR\sigma^2 + 8a_1{}^2m^2\sigma + 4a_1{}^2\mu_0\sigma - 4a_1{}^2R\sigma - 2a_1b^2m^2 - a_1b^2R + 2b^2cR - 4a_1{}^2\sigma + a_1b^2\Big), \\
h_2 = \frac{-3\left(4a_1{}^3\mu_1{}^4\sigma - 4a_1{}^2b\mu_1{}^3\sigma - a_1{}^2\mu_0{}^2\mu_1{}^2 - 4a_1{}^2\mu_0\mu_1{}^2\sigma + 8a_1{}^2\mu_1{}^2\sigma + 2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\right)m^2}{8R\sigma\left(a_1\mu_1{}^2+1\right)^2(a_1-c)}, \\
\lambda = \sqrt{\frac{4a_1{}^3\mu_1{}^4\sigma - 4a_1{}^2b\mu_1{}^3\sigma - a_1{}^2\mu_0{}^2\mu_1{}^2 - 4a_1{}^2\mu_0\mu_1{}^2\sigma + 8a_1{}^2\mu_1{}^2\sigma + 2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2}{-16a_1R\sigma^2(a_1-c)\left(a_1\mu_1{}^2+1\right)^2}}, \\
\omega = -\frac{\left(a_1\mu_1{}^2\sigma - b\mu_1 - \mu_0 + \sigma\right)\mu_1}{a_1\mu_1{}^2+1}, \\
\sigma > 0, \\
m \in [\,0, 1\,].
\end{cases}$$

Thus, explicit periodic traveling-wave solutions to the Schrödinger BBM-BBM system $(u(x,t), v(x,t)) = (e^{i\omega t}e^{iB(x-\sigma t)} f(x-\sigma t), g(x-\sigma t))$, given in term of the Jacobi cnoidal function

$$f(\xi) = d_0 + d_2 \operatorname{cn}^2(\lambda\xi, m) \quad \text{and} \quad g(\xi) = h_0 + h_2 \operatorname{cn}^2(\lambda\xi, m)$$

are established. Notice that $\frac{h_2}{d_2} = \sqrt{\frac{a_1}{2c-a_1}}$. When $m = R = 1$, the above coefficients simplify to

$$\begin{cases} \tilde{d}_0 = \frac{\sqrt{a_1(2c-a_1)}}{4a_1\sigma\left(a_1\mu_1^2+1\right)^2(a_1-c)}\Big(4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma \\ \quad +2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\Big), \\ \tilde{d}_2 = \frac{-3\sqrt{a_1(2c-a_1)}}{8a_1\sigma\left(a_1\mu_1^2+1\right)^2(a_1-c)}\Big(4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma \\ \quad +2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\Big), \\ \tilde{h}_0 = \frac{1}{4a_1\sigma\left(a_1\mu_1^2+1\right)^2(a_1-c)}\Big(4a_1^4\mu_1^4\sigma^2 - 4a_1^3c\mu_1^4\sigma^2 - 4a_1^3b\mu_1^3\sigma + 4a_1^2bc\mu_1^3\sigma - a_1^3\mu_0^2\mu_1^2 \\ \quad -4a_1^3\mu_0\mu_1^2\sigma + 8a_1^3\mu_1^2\sigma^2 + a_1^2c\mu_0^2\mu_1^2 + 4a_1^2c\mu_0\mu_1^2\sigma - 8a_1^2c\mu_1^2\sigma^2 + 2a_1^2b\mu_0\mu_1 \\ \quad -4a_1^2b\mu_1\sigma - 2a_1bc\mu_0\mu_1 + 4a_1bc\mu_1\sigma - 4a_1^2\mu_0\sigma + 4a_1^2\sigma^2 + 4a_1c\mu_0\sigma - 4a_1c\sigma^2 \\ \quad -a_1b^2 + b^2c\Big), \\ \tilde{h}_2 = \frac{-3\left(4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma + 2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\right)}{8\sigma\left(a_1\mu_1^2+1\right)^2(a_1-c)}, \\ \tilde{\lambda} = \sqrt{\frac{4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma + 2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2}{-16a_1\sigma^2(a_1-c)\left(a_1\mu_1^2+1\right)^2}}, \\ \omega = -\frac{\left(a_1\mu_1^2\sigma - b\mu_1 - \mu_0 + \sigma\right)\mu_1}{a_1\mu_1^2+1}, \\ \sigma > 0, \end{cases}$$

from which one obtains the following solitary-wave solution to system (1.2):

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}[\tilde{d}_0 + \tilde{f}(x-\sigma t)] \quad \text{and} \quad v(x,t) = \tilde{h}_0 \pm \sqrt{\frac{a_1}{2c-a_1}}\tilde{f}(x-\sigma t),$$

where $\tilde{f}(\xi) = \tilde{d}_2\,\mathrm{sech}^2(\tilde{\lambda}\xi)$.

When $m = -R = 1$, the above coefficients simplify to $d_0 = 0$ and

$$\begin{cases} \bar{d}_2 = \frac{3\sqrt{a_1(2c-a_1)}}{8a_1\sigma\left(a_1\mu_1^2+1\right)^2(a_1-c)}\Big(4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma \\ \quad +2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\Big), \\ \bar{h}_0 = \frac{1}{-8a_1\sigma\left(a_1\mu_1^2+1\right)^2(a_1-c)}\Big(-8a_1^4\mu_1^4\sigma^2 + 8a_1^3c\mu_1^4\sigma^2 + 8a_1^4\mu_1^4\sigma - 8a_1^2bc\mu_1^3\sigma - 16a_1^3\mu_1^2\sigma^2 \\ \quad -2a_1^2c\mu_0^2\mu_1^2 - 8a_1^2c\mu_0\mu_1^2\sigma + 16a_1^2c\mu_1^2\sigma^2 + 16a_1^3\mu_1^2\sigma + 4a_1bc\mu_0\mu_1 - 8a_1bc\mu_1\sigma \\ \quad -8a_1^2\sigma^2 - 8a_1c\mu_0\sigma + 8a_1c\sigma^2 + 8a_1^2\sigma - 2b^2c\Big), \\ \bar{h}_2 = \frac{3\left(4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma + 2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2\right)}{8\sigma\left(a_1\mu_1^2+1\right)^2(a_1-c)}, \\ \bar{\lambda} = \sqrt{\frac{4a_1^3\mu_1^4\sigma - 4a_1^2b\mu_1^3\sigma - a_1^2\mu_0^2\mu_1^2 - 4a_1^2\mu_0\mu_1^2\sigma + 8a_1^2\mu_1^2\sigma + 2a_1b\mu_0\mu_1 - 4a_1b\mu_1\sigma - 4a_1\mu_0\sigma + 4a_1\sigma - b^2}{16a_1\sigma^2(a_1-c)\left(a_1\mu_1^2+1\right)^2}}, \\ \omega = -\frac{\left(a_1\mu_1^2\sigma - b\mu_1 - \mu_0 + \sigma\right)\mu_1}{a_1\mu_1^2+1}, \end{cases}$$

and one arrives at the solitary-wave solution

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}\bar{f}(x-\sigma t) \quad \text{and} \quad v(x,t) = \bar{h}_0 \pm \sqrt{\frac{a_1}{2c-a_1}}\bar{f}(x-\sigma t),$$

where $\bar{f}(\xi) = \bar{d}_2 \operatorname{sech}^2(\bar{\lambda}\xi)$. Furthermore, when $B = \frac{a_1 \mu_0 \mu_1 - b}{2\sigma a_1 (a_1 \mu_1^2 + 1)}$ satisfies the following equation:

$$(a_1^2 c\mu_0\mu_1 - a_1 bc)B^2 + (2a_1 bc\mu_1 + 2a_1 c\mu_0 - 2a_1^3\mu_1^2 - 2a_1^2)B + (a_1^2\mu_0\mu_1 + bc - a_1 b - a_1 c\mu_0\mu_1) = 0,$$

one has $\bar{h}_0 = 0$, and the synchronized solitary-wave solution established in [5] is recovered.

Aside from the above nontrivial solutions, system (1.2) also possessess the following trivial and semi-trivial solutions:

(1)
$$u(x,t) = 0 \quad \text{and} \quad v(x,t) = h_0,$$
for any $h_0 \in \mathbb{R}$.

(2)
$$u(x,t) = e^{i\omega t} e^{iB(x-\sigma t)} d_0 \quad \text{and} \quad v(x,t) = h_0,$$
where $\sigma = \frac{a_1 B^2\omega - bB^2 + Bh_0 + B\mu_0 + h_0\mu_1 + \omega}{B(a_1 B^2 + 1)}$, for any $B, d_0, h_0, \omega \in \mathbb{R}$.

(3)
$$u(x,t) = 0 \quad \text{and} \quad v(x,t) = -\frac{2}{3}h_2 + \frac{1}{3}\frac{h_2}{m^2} + \sigma - 1 + h_2 \operatorname{cn}^2(\lambda(x - \sigma t), m),$$
where $\lambda = \sqrt{\frac{h_2}{12cm^2\sigma}}$, for any $h_2, \sigma > 0$ and $m \in [0, 1]$.

3.3. *Schrödinger KdV-BBM*

Setting all $k_{j,q} = 0$ gives us the following set of parameters, with R as defined in (3.1):

$$\begin{cases}
B = \frac{a_0 \mu_1 - b}{2 a_0}, \\
d_1 = h_1 = 0, \\
d_0 = \frac{\left(m^4 - 2m^2 R - m^2 + R + 1\right)\sqrt{2c\sigma - a_0}\left(3a_0^2\mu_1^2 - 2a_0 b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)}{8\sqrt{a_0}R^2(a_0 - c\sigma)}, \\
d_2 = \frac{3\sqrt{2c\sigma - a_0}\left(3a_0^2\mu_1^2 - 2a_0 b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)m^2}{8\sqrt{a_0}R(a_0 - c\sigma)}, \\
h_0 = \frac{1}{8a_0 R(c\sigma - a_0)}\Big(6a_0^2 c\mu_1^2 R\sigma + 6a_0^3 m^2\mu_1^2 - 3a_0^3\mu_1^2 R - 4a_0 bc\mu_1 R\sigma - 3a_0^3\mu_1^2 - 4a_0^2 bm^2\mu_1 \\
\quad +2a_0^2 b\mu_1 R - 8a_0 c\mu_0 R\sigma + 8a_0 cR\sigma^2 - 2b^2 cR\sigma + 2a_0^2 b\mu_1 - 8a_0^2 m^2\mu_0 + 4a_0^2\mu_0 R \\
\quad -8a_0^2 R\sigma - 2a_0 b^2 m^2 + a_0 b^2 R + 8a_0^2 m^2 + 4a_0^2\mu_0 + 4a_0^2 R + a_0 b^2 - 4a_0^2\Big), \\
h_2 = \frac{3\left(3a_0^2\mu_1^2 - 2a_0 b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)m^2}{8R(a_0 - c\sigma)}, \\
\lambda = \sqrt{\frac{3a_0^2\mu_1^2 - 2a_0 b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0}{16a_0 R(a_0 - c\sigma)}}, \\
\omega = -\left(a_0\mu_1^2 - \mu_1 b - \mu_0 + \sigma\right)\mu_1, \\
\sigma > \frac{a_0}{2c}, \\
m \in [0, 1].
\end{cases}$$

Thus, explicit periodic traveling-wave solutions to the Schrödinger KdV-BBM system $(u(x,t), v(x,t)) = (e^{i\omega t}e^{iB(x-\sigma t)}f(x - \sigma t), g(x - \sigma t))$, given in term of the Jacobi cnoidal function

$$f(\xi) = d_0 + d_2 \operatorname{cn}^2(\lambda\xi, m) \quad \text{and} \quad g(\xi) = h_0 + h_2 \operatorname{cn}^2(\lambda\xi, m)$$

are established. Notice that $\frac{h_2}{d_2} = \sqrt{\frac{a_0}{2c\sigma - a_0}}$. When $m = R = 1$, the above coefficients simplify to $d_0 = 0$ and

$$\begin{cases} \tilde{d}_2 = \frac{3\sqrt{2c\sigma - a_0}\left(3a_0^2\mu_1^2 - 2a_0 b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)}{8\sqrt{a_0}(a_0 - c\sigma)}, \\ \tilde{h}_0 = \frac{1}{8a_0(c\sigma - a_0)}\left(6a_0^2 c\mu_1^2\sigma - 4a_0 bc\mu_1\sigma - 8a_0 c\mu_0\sigma + 8a_0 c\sigma^2 - 2b^2 c\sigma - 8a_0^2\sigma + 8a_0^2\right), \\ \tilde{h}_2 = \frac{3\left(3a_0^2\mu_1^2 - 2a_0 b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)}{8(a_0 - c\sigma)}, \\ \tilde{\lambda} = \sqrt{\frac{3a_0^2\mu_1^2 - 2a_0 b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0}{16a_0(a_0 - c\sigma)}}, \\ \omega = -\left(a_0\mu_1^2 - \mu_1 b - \mu_0 + \sigma\right)\mu_1, \\ \sigma > \frac{a_0}{2c}, \end{cases}$$

from which one obtains the following solitary-wave solution to system (1.3):

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}\tilde{f}(x - \sigma t) \quad \text{and} \quad v(x,t) = \tilde{h}_0 \pm \sqrt{\frac{a_0}{2c\sigma - a_0}}\tilde{f}(x - \sigma t),$$

where $\tilde{f}(\xi) = \tilde{d}_2 \operatorname{sech}^2(\tilde{\lambda}\xi)$. Furthermore, when σ satisfies the condition $\frac{\sigma + 3a_0B^2 + 2bB - \mu_0}{a_0} = \frac{\sigma - 1}{c\sigma}$, one has $\tilde{h}_0 = 0$, and the synchronized solitary-wave solution established in [5] is recovered.

When $m = -R = 1$, the above coefficients simplify to

$$\begin{cases} \bar{d}_0 = \frac{\sqrt{2c\sigma - a_0}\left(3a_0^2\mu_1^2 - 2a_0 b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)}{4\sqrt{a_0}(a_0 - c\sigma)}, \\ \bar{d}_2 = \frac{3\sqrt{2c\sigma - a_0}\left(3a_0^2\mu_1^2 - 2a_0 b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)}{8\sqrt{a_0}(c\sigma - a_0)}, \\ \bar{h}_0 = \frac{1}{4a_0(c\sigma - a_0)}\left(3a_0^2 c\mu_1^2\sigma - 3a_0^3\mu_1^2 - 2a_0 bc\mu_1\sigma + 2a_0^2 b\mu_1 - 4a_0 c\mu_0\sigma + 4a_0 c\sigma^2 \right. \\ \quad \left. - b^2 c\sigma + 4a_0^2\mu_0 - 4a_0^2\sigma + a_0 b^2\right), \\ \bar{h}_2 = \frac{3\left(3a_0^2\mu_1^2 - 2a_0 b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0\right)}{8(c\sigma - a_0)}, \\ \bar{\lambda} = \sqrt{\frac{3a_0^2\mu_1^2 - 2a_0 b\mu_1 - 4a_0\mu_0 - b^2 + 4a_0}{16a_0(c\sigma - a_0)}}, \\ \omega = -\left(a_0\mu_1^2 - \mu_1 b - \mu_0 + \sigma\right)\mu_1, \\ \sigma > \frac{a_0}{2c}, \end{cases}$$

and one arrives at the solitary-wave solution

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}[\bar{d}_0 + \bar{f}(x - \sigma t)] \quad \text{and} \quad v(x,t) = \bar{h}_0 \pm \sqrt{\frac{a_0}{2c\sigma - a_0}}\bar{f}(x - \sigma t),$$

where $\bar{f}(\xi) = \bar{d}_2 \operatorname{sech}^2(\bar{\lambda}\xi)$.

Aside from the above nontrivial solutions, system (1.3) also possessess the following trivial and semi-trivial solutions:

(1)

$$u(x,t) = 0 \quad \text{and} \quad v(x,t) = h_0,$$

for any $h_0 \in \mathbb{R}$.

(2)

$$u(x,t) = e^{i\omega t} e^{iB(x-\sigma t)} d_0 \quad \text{and} \quad v(x,t) = h_0,$$

where $\sigma = \frac{\omega - a_0 B^3 - bB^2 + Bh_0 + B\mu_0 + h_0\mu_1}{B}$, for any $B, d_0, h_0, \omega \in \mathbb{R}$.

(3)

$$u(x,t) = e^{i\omega t} e^{iB(x-\sigma t)} d_1 \operatorname{cn}(\lambda(x-\sigma t), m) \quad \text{and} \quad v(x,t) = h_0 + h_2 \operatorname{cn}^2(\lambda(x-\sigma t), m),$$

for any $m \in [0,1]$, where $h_0 = \frac{9\,a_0{}^2cm^2\mu_1{}^2 - 6\,a_0\,bcm^2\mu_1 - 12\,a_0\,ch_2\,m^2 - 12\,a_0\,cm^2\mu_0 - 3\,b^2cm^2 + 2\,a_0{}^2m^2 + 6\,a_0\,ch_2}{12\,a_0\,cm^2}$; $h_2 > 0$ such that $9a_0{}^2m^2\mu_1{}^2 - 6a_0bm^2\mu_1 - 4a_0h_2m^2 - 12a_0m^2\mu_0 - 3b^2m^2 + 2a_0h_2 + 12a_0m^2 < 0$; $d_1 = \pm\frac{\sqrt{-6\,a_0h_2m^2\left(9\,a_0{}^2m^2\mu_1{}^2 - 6\,a_0\,bm^2\mu_1 - 4\,a_0\,h_2\,m^2 - 12\,a_0\,m^2\mu_0 - 3\,b^2m^2 + 2\,a_0\,h_2 + 12\,a_0\,m^2\right)}}{6\,a_0\,m^2}$, $B = \frac{a_0\,\mu_1 - b}{2\,a_0}$, $\omega = \frac{-\mu_1\left(6\,a_0\,c\mu_1{}^2 - 6\,bc\mu_1 - 6\,c\mu_0 + a_0\right)}{6\,c}$, $\lambda = \sqrt{\frac{h_2}{2\,a_0m^2}}$ and $\sigma = \frac{a_0}{6\,c}$.

(4)

$$u(x,t) = 0 \quad \text{and} \quad v(x,t) = -\frac{2}{3}\,h_2 + \frac{1}{3}\,\frac{h_2}{m^2} + \sigma - 1 + h_2 \operatorname{cn}^2(\lambda(x-\sigma t), m),$$

where $\lambda = \sqrt{\frac{h_2}{12cm^2\sigma}}$, for any $h_2, \sigma > 0$ and $m \in [\,0,1\,]$.

3.4. Schrödinger BBM-KdV

Setting all $k_{j,q} = 0$ gives us the following set of parameters, with R as defined in (3.1):

$$\begin{cases}
B = \frac{a_1\,\mu_0\,\mu_1 - b}{2a_1\sigma\left(a_1\,\mu_1{}^2+1\right)},\\
d_1 = h_1 = 0,\\
d_0 = \frac{\sqrt{2\,c - a_1\,\sigma}\left(m^4 - 2\,m^2R - m^2 + R + 1\right)}{8\,R^2\,\sqrt{a_1\,\sigma}\left(a_1\,\mu_1{}^2+1\right)^2(a_1\,\sigma - c)}\Big(4\,a_1{}^3\mu_1{}^4\sigma - 4\,a_1{}^2b\mu_1{}^3\sigma - a_1{}^2\mu_0{}^2\mu_1{}^2 - 4\,a_1{}^2\mu_0\,\mu_1{}^2\sigma + 8\,a_1{}^2\mu_1{}^2\sigma\\
\quad +2\,a_1\,b\mu_0\,\mu_1 - 4\,a_1b\mu_1\,\sigma - 4\,a_1\mu_0\,\sigma + 4\,a_1\,\sigma - b^2\Big),\\
d_2 = \frac{3\,m^2\,\sqrt{2\,c - a_1\,\sigma}}{8R\,\sqrt{a_1\,\sigma}\left(a_1\,\mu_1{}^2+1\right)^2(a_1\,\sigma - c)}\Big(4\,a_1{}^3\mu_1{}^4\sigma - 4\,a_1{}^2b\mu_1{}^3\sigma - a_1{}^2\mu_0{}^2\mu_1{}^2 - 4\,a_1{}^2\mu_0\,\mu_1{}^2\sigma + 8\,a_1{}^2\mu_1{}^2\sigma\\
\quad +2\,a_1\,b\mu_0\,\mu_1 - 4\,a_1b\mu_1\,\sigma - 4\,a_1\mu_0\,\sigma + 4\,a_1\,\sigma - b^2\Big),\\
h_0 = \frac{-1}{8\,a_1\,R\sigma\left(a_1\,\mu_1{}^2+1\right)^2(a_1\,\sigma - c)}\Big(-8\,a_1{}^4\mu_1{}^4R\sigma^3 + 8\,a_1{}^4m^2\mu_1{}^4\sigma^2 + 4\,a_1{}^4\mu_1{}^4R\sigma^2 + 8\,a_1{}^3c\mu_1{}^4R\sigma^2\\
\quad -4\,a_1{}^4\mu_1{}^4\sigma^2 - 8\,a_1{}^3bm^2\mu_1{}^3\sigma^2 + 4\,a_1{}^3b\mu_1{}^3R\sigma^2 + 4\,a_1{}^3b\mu_1{}^3\sigma^2 - 2\,a_1{}^3m^2\mu_0{}^2\mu_1{}^2\sigma\\
\quad -8\,a_1{}^3m^2\mu_0\,\mu_1{}^2\sigma^2 + a_1{}^3\mu_0{}^2\mu_1{}^2\sigma R + 4\,a_1{}^3\mu_0\,\mu_1{}^2R\sigma^2 - 16\,a_1{}^3\mu_1{}^2R\sigma^3 - 8\,a_1^2bc\mu_1{}^3R\sigma\\
\quad +16\,a_1{}^3m^2\mu_1{}^2\sigma^2 + a_1{}^3\mu_0{}^2\mu_1{}^2\sigma + 4\,a_1{}^3\mu_0\,\mu_1{}^2\sigma^2 + 8\,a_1{}^3\mu_1{}^2R\sigma^2 - 2\,a_1{}^2c\mu_0{}^2\mu_1{}^2R\\
\quad -8\,a_1{}^2c\mu_0\,\mu_1{}^2R\sigma + 16\,a_1{}^2c\mu_1{}^2R\sigma^2 - 8\,a_1{}^3\mu_1{}^2\sigma^2 + 4\,a_1{}^2bm^2\mu_0\,\mu 1\,\sigma - 8\,a_1{}^2bm^2\mu_1\,\sigma^2\\
\quad -2\,a_1{}^2b\mu_0\,\mu_1R\sigma + 4\,a_1{}^2b\mu_1\,R\sigma^2 - 8\,a_1{}^2m^2\mu_0\,\sigma^2 + 4\,a_1{}^2\mu_0\,R\sigma^2 - 8\,a_1{}^2R\sigma^3 + 4\,a_1\,bc\mu_0\,\mu_1\,R\\
\quad -8\,a_1bc\mu_1\,R\sigma + 8\,a_1{}^2m^2\sigma^2 + 4\,a_1{}^2\mu_0\,\sigma^2 + 4\,a_1{}^2R\sigma^2 - 2\,a_1b^2m^2\sigma + a_1b^2R\sigma - 8\,a_1c\mu_0\,R\sigma\\
\quad +8\,a_1\,cR\sigma^2 - 4\,a_1{}^2\sigma^2 + a_1b^2\sigma - 2\,b^2cR\Big),
\end{cases}$$

$$\begin{cases} h_2 = \frac{3\left(4a_1^3\mu_1^4\sigma-4a_1^2b\mu_1^3\sigma-a_1^2\mu_0^2\mu_1^2-4a_1^2\mu_0\mu_1^2\sigma+8a_1^2\mu_1^2\sigma+2a_1b\mu_0\mu_1-4a_1b\mu_1\sigma-4a_1\mu_0\sigma+4a_1\sigma-b^2\right)m^2}{8R\left(a_1\mu_1^2+1\right)^2(a_1\sigma-c)}, \\ \lambda = \sqrt{\frac{4a_1^3\mu_1^4\sigma-4a_1^2b\mu_1^3\sigma-a_1^2\mu_0^2\mu_1^2-4a_1^2\mu_0\mu_1^2\sigma+8a_1^2\mu_1^2\sigma+2a_1b\mu_0\mu_1-4a_1b\mu_1\sigma-4a_1\mu_0\sigma+4a_1\sigma-b^2}{16a_1R\sigma\left(a_1\mu_1^2+1\right)^2(a_1\sigma-c)}}, \\ \omega = -\frac{\left(a_1\mu_1^2\sigma-b\mu_1-\mu_0+\sigma\right)\mu_1}{a_1\mu_1^2+1}, \\ \sigma < \frac{2c}{a_1}, \\ m \in [\,0,1\,]. \end{cases}$$

Thus, explicit periodic traveling-wave solutions to the Schrödinger BBM-KdV system $(u(x,t), v(x,t)) = (e^{i\omega t}e^{iB(x-\sigma t)}f(x-\sigma t), g(x-\sigma t))$, given in term of the Jacobi cnoidal function

$$f(\xi) = d_0 + d_2\,\mathrm{cn}^2(\lambda\xi, m) \quad \text{and} \quad g(\xi) = h_0 + h_2\,\mathrm{cn}^2(\lambda\xi, m)$$

are established. Notice that $\frac{h_2}{d_2} = \sqrt{\frac{a_1\sigma}{2c-a_1\sigma}}$. When $m = R = 1$, the above coefficients simplify to $d_0 = 0$ and

$$\begin{cases} \tilde{d}_2 = \frac{3\sqrt{2c-a_1\sigma}}{8\sqrt{a_1\sigma}\left(a_1\mu_1^2+1\right)^2(a_1\sigma-c)}\Big(4a_1^3\mu_1^4\sigma-4a_1^2b\mu_1^3\sigma-a_1^2\mu_0^2\mu_1^2-4a_1^2\mu_0\mu_1^2\sigma+8a_1^2\mu_1^2\sigma \\ \qquad +2a_1b\mu_0\mu_1-4a_1b\mu_1\sigma-4a_1\mu_0\sigma+4a_1\sigma-b^2\Big), \\ \tilde{h}_0 = \frac{-1}{8a_1\sigma\left(a_1\mu_1^2+1\right)^2(a_1\sigma-c)}\Big(-8a_1^4\mu_1^4\sigma^3+8a_1^4\mu_1^4\sigma^2+8a_1^3c\mu_1^4\sigma^2-16a_1^3\mu_1^2\sigma^3-8a_1^2bc\mu_1^3\sigma \\ \qquad +16a_1^3\mu_1^2\sigma^2-2a_1^2c\mu_0^2\mu_1^2-8a_1^2c\mu_0\mu_1^2\sigma+16a_1^2c\mu_1^2\sigma^2-8a_1^2\sigma^3+4a_1bc\mu_0\mu_1 \\ \qquad -8a_1bc\mu_1\sigma+8a_1^2\sigma^2-8a_1c\mu_0\sigma+8a_1c\sigma^2-2b^2c\Big), \\ \tilde{h}_2 = \frac{3\left(4a_1^3\mu_1^4\sigma-4a_1^2b\mu_1^3\sigma-a_1^2\mu_0^2\mu_1^2-4a_1^2\mu_0\mu_1^2\sigma+8a_1^2\mu_1^2\sigma+2a_1b\mu_0\mu_1-4a_1b\mu_1\sigma-4a_1\mu_0\sigma+4a_1\sigma-b^2\right)}{8\left(a_1\mu_1^2+1\right)^2(a_1\sigma-c)}, \\ \tilde{\lambda} = \sqrt{\frac{4a_1^3\mu_1^4\sigma-4a_1^2b\mu_1^3\sigma-a_1^2\mu_0^2\mu_1^2-4a_1^2\mu_0\mu_1^2\sigma+8a_1^2\mu_1^2\sigma+2a_1b\mu_0\mu_1-4a_1b\mu_1\sigma-4a_1\mu_0\sigma+4a_1\sigma-b^2}{16a_1\sigma\left(a_1\mu_1^2+1\right)^2(a_1\sigma-c)}}, \\ \omega = -\frac{\left(a_1\mu_1^2\sigma-b\mu_1-\mu_0+\sigma\right)\mu_1}{a_1\mu_1^2+1}, \\ \sigma < \frac{2c}{a_1}, \end{cases}$$

from which one obtains the following solitary-wave solution to system (1.4):

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}\tilde{f}(x-\sigma t) \quad \text{and} \quad v(x,t) = \tilde{h}_0 \pm \sqrt{\frac{a_1\sigma}{2c-a_1\sigma}}\tilde{f}(x-\sigma t),$$

where $\tilde{f}(\xi) = \tilde{d}_2\,\mathrm{sech}^2(\tilde{\lambda}\xi)$. Furthermore, when $B = \frac{a_1\mu_0\mu_1-b}{2\sigma a_1\left(a_1\mu_1^2+1\right)}$ satisfies the condition

$$\begin{aligned} &(2a_1^2bc\mu_1^2 + 2a_1bc - 2a_1^3c\mu_0\mu_1^3 - 2a_1^2c\mu_0\mu_1)B^3 + (-4a_1^2bc\mu_1^3 - 4a_1^2c\mu_0\mu_1^2 - 4a_1bc\mu_1 - 4a_1c\mu_0)B^2 \\ &+ (2a_1^3\mu_0\mu_1^3 + 2a_1^2c\mu_0\mu_1^3 + 2a_1^2\mu_0\mu_1 + 2a_1c\mu_0\mu_1 - 2a_1^2b\mu_1^2 - 2a_1b - 2a_1bc\mu_1^2 - 2bc)B \\ &+ (2a_1b\mu_0\mu_1 - a_1^2\mu_0^2\mu_1^2 - b^2) = 0, \end{aligned}$$

one has $\tilde{h}_0 = 0$, and the synchronized solitary-wave solution established in [5] is recovered.

When $m = -R = 1$, the above coefficients simplify to

$$\begin{cases} \bar{d}_0 = \frac{\sqrt{2c-a_1\sigma}}{4\sqrt{a_1\sigma}\left(a_1\mu_1^2+1\right)^2(a_1\sigma-c)}\Big(4a_1^3\mu_1^4\sigma-4a_1^2b\mu_1^3\sigma-a_1^2\mu_0^2\mu_1^2-4a_1^2\mu_0\mu_1^2\sigma+8a_1^2\mu_1^2\sigma \\ \quad +2a_1b\mu_0\mu_1-4a_1b\mu_1\sigma-4a_1\mu_0\sigma+4a_1\sigma-b^2\Big), \\ \bar{d}_2 = \frac{-3\sqrt{2c-a_1\sigma}}{8\sqrt{a_1\sigma}\left(a_1\mu_1^2+1\right)^2(a_1\sigma-c)}\Big(4a_1^3\mu_1^4\sigma-4a_1^2b\mu_1^3\sigma-a_1^2\mu_0^2\mu_1^2-4a_1^2\mu_0\mu_1^2\sigma+8a_1^2\mu_1^2\sigma \\ \quad +2a_1b\mu_0\mu_1-4a_1b\mu_1\sigma-4a_1\mu_0\sigma+4a_1\sigma-b^2\Big), \\ \bar{h}_0 = \frac{1}{4a_1\sigma\left(a_1\mu_1^2+1\right)^2(a_1\sigma-c)}\Big(4a_1^4\mu_1^4\sigma^3-4a_1^3c\mu_1^4\sigma^2-4a_1^3b\mu_1^3\sigma^2-a_1^3\mu_0^2\mu_1^2\sigma \\ \quad -4a_1^3\mu_0\mu_1^2\sigma^2+8a_1^3\mu_1^2\sigma^3+4a_1^2bc\mu_1^3\sigma+a_1^2c\mu_0^2\mu_1^2+4a_1^2c\mu_0\mu_1^2\sigma \\ \quad -8a_1^2c\mu_1^2\sigma^2+2a_1^2b\mu_0\mu_1\sigma-4a_1^2b\mu_1\sigma^2-4a_1^2\mu_0\sigma^2+4a_1^2\sigma^3-2a_1bc\mu_0\mu_1 \\ \quad +4a_1bc\mu_1\sigma-a_1b^2\sigma+4a_1c\mu_0\sigma-4a_1c\sigma^2+b^2c\Big), \\ \bar{h}_2 = \frac{-3\left(4a_1^3\mu_1^4\sigma-4a_1^2b\mu_1^3\sigma-a_1^2\mu_0^2\mu_1^2-4a_1^2\mu_0\mu_1^2\sigma+8a_1^2\mu_1^2\sigma+2a_1b\mu_0\mu_1-4a_1b\mu_1\sigma-4a_1\mu_0\sigma+4a_1\sigma-b^2\right)}{8\left(a_1\mu_1^2+1\right)^2(a_1\sigma-c)}, \\ \bar{\lambda} = \sqrt{\frac{4a_1^3\mu_1^4\sigma-4a_1^2b\mu_1^3\sigma-a_1^2\mu_0^2\mu_1^2-4a_1^2\mu_0\mu_1^2\sigma+8a_1^2\mu_1^2\sigma+2a_1b\mu_0\mu_1-4a_1b\mu_1\sigma-4a_1\mu_0\sigma+4a_1\sigma-b^2}{-16a_1\sigma\left(a_1\mu_1^2+1\right)^2(a_1\sigma-c)}}, \\ \omega = -\frac{\left(a_1\mu_1^2\sigma-b\mu_1-\mu_0+\sigma\right)\mu_1}{a_1\mu_1^2+1}, \\ \sigma < \frac{2c}{a_1}, \end{cases}$$

and one arrives at the solitary-wave solution

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}[\bar{d}_0 + \bar{f}(x-\sigma t)] \quad \text{and} \quad v(x,t) = \bar{h}_0 \pm \sqrt{\frac{a_1\sigma}{2c-a_1\sigma}}\bar{f}(x-\sigma t),$$

where $\bar{f}(\xi) = \bar{d}_2\,\mathrm{sech}^2(\bar{\lambda}\xi)$.

Aside from the above nontrivial solutions, system (1.4) also possessess the following trivial and semi-trivial solutions:

(1)

$$u(x,t) = 0 \quad \text{and} \quad v(x,t) = h_0,$$

for any $h_0 \in \mathbb{R}$.

(2)

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}d_0 \quad \text{and} \quad v(x,t) = h_0,$$

where $\sigma = \frac{a_1B^2\omega - bB^2 + Bh_0 + B\mu_0 + h_0\mu_1 + \omega}{B(a_1B^2+1)}$, for any $B, d_0, h_0, \omega \in \mathbb{R}$.

(3)

$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}d_1\,\mathrm{cn}\,(\lambda(x-\sigma t), m) \quad \text{and} \quad v(x,t) = h_0 + h_2\,\mathrm{cn}^2\,(\lambda(x-\sigma t), m),$$

where

$$\begin{aligned} d_1 = \pm\frac{1}{6cm^2(a_1\mu_1^2+1)}\Big(&3ch_2m^2\,(8a_1^2ch_2m^2\mu_1^4 - 4a_1^2ch_2\mu_1^4 - 24a_1^2cm^2\mu_1^4 + a_1^2m^2\mu_0^2\mu_1^2 \\ &+ 24a_1bcm^2\mu_1^3 + 16a_1ch_2m^2\mu_1^2 + 24a_1cm^2\mu_0\mu_1^2 - 2a_1bm^2\mu_0\mu_1 - 8a_1ch_2\mu_1^2 \\ &- 48a_1cm^2\mu_1^2 + 24bcm^2\mu_1 + b^2m^2 + 8ch_2m^2 + 24cm^2\mu_0 - 4ch_2 - 24cm^2)\Big)^{1/2}; \end{aligned}$$

$$h_0 = \frac{-1}{24\,(a_1{}^2\mu_1{}^4 + 2\,a_1\mu_1{}^2 + 1)\,a_1\,cm^2}\Big(24\,a_1{}^3ch_2\,m^2\mu_1{}^4 - 12\,a_1{}^3ch_2\,\mu_1{}^4 - 144\,a_1{}^2c^2m^2\mu_1{}^4 + a_1{}^3m^2\mu_0{}^2\mu_1{}^2 + 24\,a_1{}^2bcm^2\mu_1{}^3 + 48\,a_1{}^2ch_2\,m^2\mu_1{}^2 + 24\,a_1{}^2cm^2\mu_0\,\mu_1{}^2 - 2\,a_1{}^2bm^2\mu_0\,\mu_1 - 24\,a_1{}^2ch_2\,\mu_1{}^2 - 288\,a_1\,c^2m^2\mu_1{}^2 + 24\,a_1\,bcm^2\mu_1 + a_1\,b^2m^2 + 24\,a_1\,ch_2\,m^2 + 24\,a_1\,cm^2\mu_0 - 12\,a_1\,ch_2 - 144\,c^2m^2\Big);$$

any $h_2 > 0$ such that $8\,a_1{}^2ch_2\,m^2\mu_1{}^4 - 4\,a_1{}^2ch_2\mu_1{}^4 - 24\,a_1{}^2cm^2\mu_1{}^4 + a_1{}^2m^2\mu_0{}^2\mu_1{}^2 + 24\,a_1\,bcm^2\mu_1{}^3 + 16\,a_1\,ch_2\,m^2\mu_1{}^2 + 24\,a_1\,cm^2\mu_0\,\mu_1{}^2 - 2\,a_1\,bm^2\mu_0\,\mu_1 - 8\,a_1\,ch_2\mu_1{}^2 - 48\,a_1\,cm^2\mu_1{}^2 + 24\,bcm^2\mu_1 + b^2m^2 + 8\,ch_2\,m^2 + 24\,cm^2\mu_0 - 4\,ch_2 - 24\,cm^2 > 0$; $B = \frac{a_1\,\mu_0\,\mu_1 - b}{12\,c\left(a_1\mu_1{}^2+1\right)}$; $\lambda = \sqrt{\frac{h_2}{12\,cm^2}}$; $\omega = \frac{\mu_1\left(-6\,a_1\,c\mu_1{}^2 + a_1 b\mu_1 + a_1\,\mu_0 - 6\,c\right)}{\left(a_1\mu_1{}^2+1\right)a_1}$; $\sigma = \frac{6\,c}{a_1}$; and any $m \in [\,0, 1\,]$.

(4)

$$u(x,t) = 0 \quad \text{and} \quad v(x,t) = -\frac{2}{3}\,h_2 + \frac{1}{3}\,\frac{h_2}{m^2} + \sigma - 1 + h_2\,\mathrm{cn}^2\left(\lambda(x-\sigma t), m\right),$$

where $\lambda = \sqrt{\frac{h_2}{12cm^2}}$, for any $h_2, \sigma > 0$ and $m \in [\,0, 1\,]$.

4. Conclusions

The periodic traveling wave solutions for the four systems (1.1)–(1.4) were found. Our results showed that all periodic solutions to the four systems were given by (1.6) and (2.4). These cnoidal solutions were limited to the solitary-wave solutions when $m \to 1$. This was expected since it is well known that the ODE equation

$$f'^2 = k_3 f^3 + k_2 f^2 + k_1 f + k_0$$

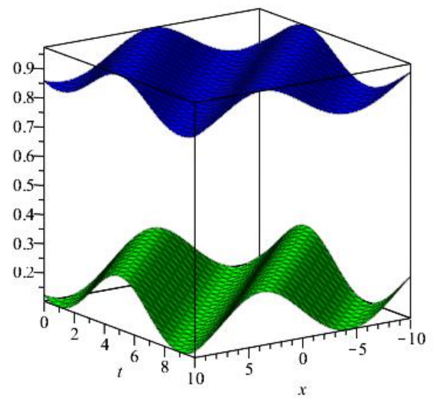
has a unique solitary-wave solution as well as a periodic cnoidal solution, and that the periodic cnoidal solution limits to the solitary-wave solution when the Jacobi elliptic modulus m approaches one. Consequently, we obtained solitary-wave solutions for all four systems as the byproducts. All of the synchronized solitary-wave solutions established in [5] are special cases of those obtained here. Another direct consequence was that the synchronized periodic solutions previously obtained in [6] are indeed unique, a fact that wasn't established therein. Since those synchronized periodic solutions approach the synchronized solitary-wave solutions obtained in [5], it would be interesting to know whether these synchronized solitary-wave solutions are also indeed unique. This question was not pursued here.

Figure 1 below shows some graphs for the cnoidal wave solutions for the four systems (1.1)–(1.4). Recall that a traveling-wave solution to the above four systems is a vector solution $(u(x,t), v(x,t))$ of the form

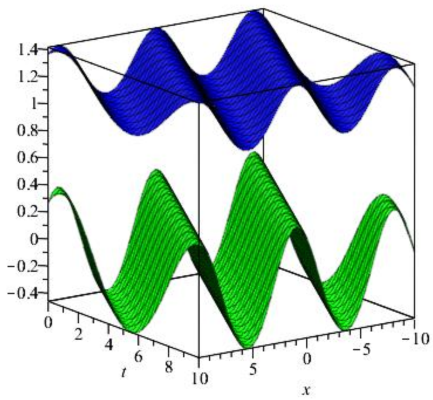
$$u(x,t) = e^{i\omega t}e^{iB(x-\sigma t)}f(x-\sigma t), \qquad v(x,t) = g(x-\sigma t),$$

where f and g are smooth, real-valued functions with speed $\sigma > 0$ and phase shifts $B, \omega \in \mathbb{R}$. For ease of graphing, the imaginary terms in $u(x,t)$ were supressed, as they define a phase shift and, thus, a rotation of the real function f, which is graphed below. For all four vector solutions, $m = \frac{1}{2}$ and $R = \frac{\sqrt{13}}{4}$ were chosen, while the remaining parameters were then fixed to ensure real solutions and are listed here; KdV-KdV: $\sigma = 2,\ a_0 = 1,\ b = -1,\ c = \frac{3}{2},\ \mu_0 = 1,\ \mu_1 = \frac{1}{4}$; BBM-BBM: $\sigma = 1,\ a_1 =$

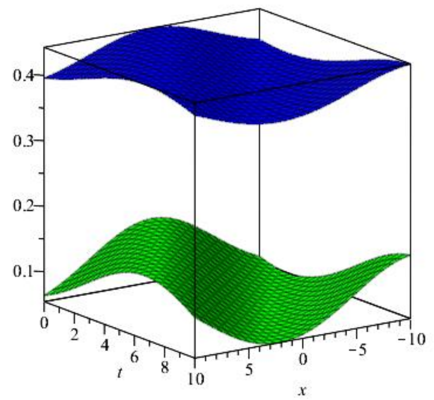
1, $b = -1$, $c = \frac{5}{2}$, $\mu_0 = 1$, $\mu_1 = 1$; KdV-BBM: $\sigma = \frac{3}{2}$, $a_0 = 1$, $b = -1$, $c = \frac{3}{2}$, $\mu_0 = 1$, $\mu_1 = \frac{1}{4}$; BBM-KdV: $\sigma = \frac{1}{2}$, $a_1 = 1$, $b = -1$, $c = \frac{3}{2}$, $\mu_0 = 1$, $\mu_1 = \frac{1}{4}$. The graphs are now listed below, with $u(x,t)$ in blue and $v(x,t)$ in green.



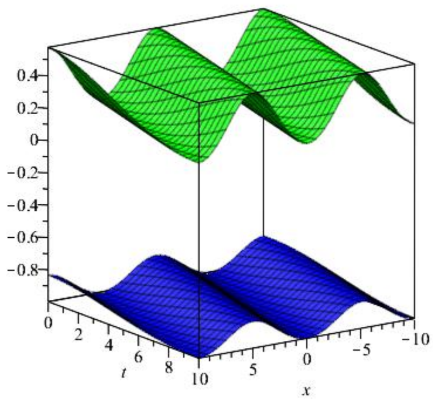
(a) Cnoidal solution for the KdV-KdV system.



(b) Cnoidal solution for the BBM-BBM system.



(c) Cnoidal solution for the KdV-BBM system.



(d) Cnoidal solution for the BBM-KdV system.

Figure 1. Graphs of some cnoidal solutions for the four systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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Appendix

For the Schrödinger KdV-KdV system (1.1), the $k_{j,q}$ in (2.3) are:

$$\begin{cases} k_{1,3} = 4\,\lambda^2\,a_0\,d_2\,m^2 - \frac{2}{3}\,d_2\,h_2, \\ k_{1,2} = \lambda^2\,a_0\,d_1\,m^2 - \frac{1}{2}\,d_1\,h_2 - \frac{1}{2}\,d_2\,h_1, \\ k_{1,1} = d_2\,a_0\,B^2 - \frac{8}{3}\,\lambda^2\,a_0\,d_2\,m^2 + \frac{2}{3}\,d_2\,bB + \frac{4}{3}\,d_2\,a_0\,\lambda^2 - \frac{1}{3}\,d_0\,h_2 - \frac{1}{3}\,d_1\,h_1 - \frac{1}{3}\,d_2\,h_0 \\ \qquad - \frac{1}{3}\,d_2\,\mu_0 + \frac{1}{3}\,d_2\,\sigma, \\ k_{1,0} = \frac{1}{2}\,B^2 a_0\,d_1 - \frac{1}{3}\,\lambda^2\,a_0\,d_1\,m^2 + \frac{1}{3}\,Bbd_1 + \frac{1}{6}\,\lambda^2\,a_0\,d_1 - \frac{1}{6}\,d_0\,h_1 - \frac{1}{6}\,d_1\,h_0 - \frac{1}{6}\,d_1\,\mu_0 + \frac{1}{6}\,d_1\,\sigma, \\ k_{2,4} = -18\,Ba_0\,d_2\,\lambda^2\,m^2 - 6\,bd_2\,\lambda^2\,m^2 + Bd_2\,h_2 + d_2\,h_2\,\mu_1, \\ k_{2,3} = -6\,Ba_0\,d_1\,\lambda^2\,m^2 - 2\,bd_1\,\lambda^2\,m^2 + Bd_1\,h_2 + Bd_2\,h_1 + d_1\,h_2\,\mu_1 + d_2\,h_1\,\mu_1, \\ k_{2,2} = -B^3 a_0\,d_2 + 24\,Ba_0\,d_2\,\lambda^2\,m^2 - B^2 bd_2 - 12\,Ba_0\,d_2\,\lambda^2 + 8\,bd_2\,\lambda^2\,m^2 + Bd_0\,h_2 + Bd_1\,h_1 \\ \qquad + Bd_2\,h_0 + Bd_2\,\mu_0 - Bd_2\,\sigma - 4\,bd_2\,\lambda^2 + d_0\,h_2\,\mu_1 + d_1\,h_1\,\mu_1 + d_2\,h_0\,\mu_1 + d_2\,\omega, \\ k_{2,1} = -B^3 a_0\,d_1 + 6\,Ba_0\,d_1\,\lambda^2\,m^2 - B^2 bd_1 - 3\,Ba_0\,d_1\,\lambda^2 + 2\,bd_1\,\lambda^2\,m^2 + Bd_0\,h_1 \\ \qquad + Bd_1\,h_0 + Bd_1\,\mu_0 - Bd_1\,\sigma - bd_1\,\lambda^2 + d_0\,h_1\,\mu_1 + d_1\,h_0\,\mu_1 + d_1\,\omega, \\ k_{2,0} = -B^3 a_0\,d_0 - 6\,Ba_0\,d_2\,\lambda^2\,m^2 - B^2 bd_0 + 6\,Ba_0\,d_2\,\lambda^2 - 2\,bd_2\,\lambda^2\,m^2 + Bd_0\,h_0 \\ \qquad + Bd_0\,\mu_0 - Bd_0\,\sigma + 2\,bd_2\,\lambda^2 + d_0\,h_0\,\mu_1 + d_0\,\omega, \\ k_{3,3} = \frac{1}{12}\,{d_2}^2 + \frac{1}{12}\,{h_2}^2 - \lambda^2\,ch_2\,m^2, \\ k_{3,2} = -\frac{1}{4}\,\lambda^2\,ch_1\,m^2 + \frac{1}{8}\,d_1\,d_2 + \frac{1}{8}\,h_1\,h_2, \\ k_{3,1} = -\frac{1}{3}\,\lambda^2\,ch_2 + \frac{2}{3}\,\lambda^2\,ch_2\,m^2 + \frac{1}{12}\,d_0\,d_2 + \frac{1}{12}\,h_0\,h_2 - \frac{1}{12}\,h_2\,\sigma + \frac{1}{24}\,{d_1}^2 + \frac{1}{24}\,{h_1}^2 + \frac{1}{12} h_2, \\ k_{3,0} = \frac{1}{12}\,\lambda^2\,ch_1\,m^2 + \frac{1}{24}\,d_0\,d_1 + \frac{1}{24}\,h_0\,h_1 - \frac{1}{24}\,h_1\,\sigma - \frac{1}{24}\,\lambda^2\,ch_1 + \frac{1}{24} h_1. \end{cases} \tag{A1}$$

For the Schrödinger BBM-BBM system (1.2), the $k_{j,q}$ in (2.3) are:

$$\begin{cases} k_{1,3} = -\frac{2}{3}\,d_2\,h_2 + 4\,\lambda^2\,a_1\,d_2\,m^2\sigma, \\ k_{1,2} = -\frac{1}{2}\,d_1\,h_2 - \frac{1}{2}\,d_2\,h_1 + \lambda^2\,a_1\,d_1\,m^2\sigma, \\ k_{1,1} = \frac{2}{3}\,Bbd_2 - \frac{1}{3}\,d_0\,h_2 - \frac{1}{3}\,d_1\,h_1 - \frac{1}{3}\,d_2\,h_0 - \frac{1}{3}\,d_2\,\mu_0 + \frac{1}{3}\,d_2\,\sigma + \frac{4}{3}\,\lambda^2\,a_1\,d_2\,\sigma + B^2 a_1\,d_2\,\sigma \\ \qquad - \frac{2}{3}\,Ba_1\,d_2\,\omega - \frac{8}{3}\,\lambda^2\,a_1\,d_2\,m^2\sigma, \\ k_{1,0} = \frac{1}{3}\,Bbd_1 - \frac{1}{6}\,d_0\,h_1 - \frac{1}{6}\,d_1\,h_0 - \frac{1}{6}\,d_1\,\mu_0 + \frac{1}{6}\,d_1\,\sigma + \frac{1}{2}\,B^2 a_1\,d_1\,\sigma - \frac{1}{3}\,Ba_1\,d_1\,\omega \\ \qquad - \frac{1}{3}\,\lambda^2\,a_1\,d_1\,m^2\sigma + \frac{1}{6}\,\lambda^2\,a_1\,d_1\,\sigma, \\ k_{2,4} = -18\,Ba_1\,d_2\,\lambda^2\,m^2\sigma + 6\,a_1\,d_2\,\lambda^2\,m^2\omega - 6\,bd_2\,\lambda^2\,m^2 + Bd_2\,h_2 + d_2\,h_2\,\mu_1, \\ k_{2,3} = -6\,Ba_1\,d_1\,\lambda^2\,m^2\sigma + 2\,a_1\,d_1\,\lambda^2\,m^2\omega - 2\,bd_1\,\lambda^2\,m^2 + Bd_1\,h_2 + Bd_2\,h_1 \\ \qquad + d_1\,h_2\,\mu_1 + d_2\,h_1\,\mu_1, \end{cases} \tag{A2}$$

$$\begin{cases} k_{2,2} = 6\,Ba_1\,d_2\,\lambda^2\,m^2\sigma + 6\,\left(-B\left(-m^2+1\right)+Bm^2\right)a_1\,d_2\,\lambda^2\,\sigma - 6\,B\left(-m^2+1\right)a_1\,d_2\,\lambda^2\,\sigma \\ \qquad - 2\,a_1\,d_2\,\lambda^2\,m^2\omega - B^3a_1\,d_2\,\sigma + B^2a_1\,d_2\,\omega - 2\,\left(2\,m^2-1\right)a_1\,d_2\,\lambda^2\,\omega + 2\,\left(-m^2+1\right)a_1\,d_2\,\lambda^2\,\omega \\ \qquad + 2\,bd_2\,\lambda^2\,m^2 + Bd_2\,\mu_0 + Bd_0\,h_2 + d_1\,h_1\,\mu_1 + d_2\,h_0\,\mu_1 - B^2bd_2 + d_2\,\omega - Bd_2\,\sigma \\ \qquad + d_0\,h_2\,\mu_1 + Bd_1\,h_1 + Bd_2\,h_0 + 2\,\left(2\,m^2-1\right)bd_2\,\lambda^2 - 2\,\left(-m^2+1\right)bd_2\,\lambda^2, \\ k_{2,1} = 3\,Ba_1\,d_1\,\lambda^2\,m^2\sigma - 3\,B\left(-m^2+1\right)a_1\,d_1\,\lambda^2\,\sigma - a_1\,d_1\,\lambda^2\,m^2\omega + B^2a_1\,d_1\,\omega \\ \qquad - B^3a_1\,d_1\,\sigma + \left(-m^2+1\right)a_1\,d_1\,\lambda^2\,\omega + bd_1\,\lambda^2\,m^2 - B^2bd_1 + Bd_0\,h_1 \\ \qquad + Bd_1\,h_0 + Bd_1\,\mu_0 - Bd_1\,\sigma + d_0\,h_1\,\mu_1 + d_1\,h_0\,\mu_1 + d_1\,\omega - \left(-m^2+1\right)bd_1\,\lambda^2, \\ k_{2,0} = -B^3a_1\,d_0\,\sigma + B^2a_1\,d_0\,\omega + d_0\,\omega - B^2bd_0 + Bd_0\,h_0 + Bd_0\,\mu_0 - Bd_0\,\sigma \\ \qquad + d_0\,h_0\,\mu_1 + 6\,B\left(-m^2+1\right)a_1\,d_2\,\lambda^2\,\sigma - 2\,\left(-m^2+1\right)a_1\,d_2\,\lambda^2\,\omega + 2\,\left(-m^2+1\right)bd_2\,\lambda^2, \\ k_{3,3} = 24\,ch_2\,\lambda^2\,m^2\sigma - 2\,{d_2}^2 - 2\,{h_2}^2, \\ k_{3,2} = 6\,\lambda^2\,ch_1\,m^2\sigma - 3\,d_1\,d_2 - 3\,h_1\,h_2, \\ k_{3,1} = -16\,ch_2\,\lambda^2\,m^2\sigma + 8\,ch_2\,\lambda^2\,\sigma - 2\,d_0\,d_2 - {d_1}^2 - 2\,h_0\,h_2 - {h_1}^2 + 2\,h_2\,\sigma - 2\,h_2, \\ k_{3,0} = -2\,\lambda^2\,ch_1\,m^2\sigma + \lambda^2\,ch_1\,\sigma - d_0\,d_1 - h_0\,h_1 + h_1\,\sigma - h_1. \end{cases}$$

For the Schrödinger KdV-BBM system (1.3), the $k_{j,q}$ in (2.3) are:

$$\begin{cases} k_{1,3} = 4\,\lambda^2\,a_0\,d_2\,m^2 - \frac{2}{3}\,d_2\,h_2, \\ k_{1,2} = \lambda^2\,a_0\,d_1\,m^2 - \frac{1}{2}\,d_1\,h_2 - \frac{1}{2}\,d_2\,h_1, \\ k_{1,1} = d_2\,a_0\,B^2 - \frac{8}{3}\,\lambda^2\,a_0\,d_2\,m^2 + \frac{2}{3}\,d_2\,bB + \frac{4}{3}\,d_2\,a_0\,\lambda^2 - \frac{1}{3}\,d_0\,h_2 - \frac{1}{3}\,d_1\,h_1 - \frac{1}{3}\,d_2\,h_0 \\ \qquad - \frac{1}{3}\,d_2\,\mu_0 + \frac{1}{3}\,d_2\,\sigma, \\ k_{1,0} = \frac{1}{2}\,B^2a_0\,d_1 - \frac{1}{3}\,\lambda^2\,a_0\,d_1\,m^2 + \frac{1}{3}\,Bbd_1 + \frac{1}{6}\,\lambda^2\,a_0\,d_1 - \frac{1}{6}\,d_0\,h_1 - \frac{1}{6}\,d_1\,h_0 - \frac{1}{6}\,d_1\,\mu_0 + \frac{1}{6}\,d_1\,\sigma, \\ k_{2,4} = -18\,Ba_0\,d_2\,\lambda^2\,m^2 - 6\,bd_2\,\lambda^2\,m^2 + Bd_2\,h_2 + d_2\,h_2\,\mu_1, \\ k_{2,3} = -6\,Ba_0\,d_1\,\lambda^2\,m^2 - 2\,bd_1\,\lambda^2\,m^2 + Bd_1\,h_2 + Bd_2\,h_1 + d_1\,h_2\,\mu_1 + d_2\,h_1\,\mu_1, \\ k_{2,2} = -B^3a_0\,d_2 + 24\,Ba_0\,d_2\,\lambda^2\,m^2 - B^2bd_2 - 12\,Ba_0\,d_2\,\lambda^2 + 8\,bd_2\,\lambda^2\,m^2 + Bd_0\,h_2 + Bd_1\,h_1 \\ \qquad + Bd_2\,h_0 + Bd_2\,\mu_0 - Bd_2\,\sigma - 4\,bd_2\,\lambda^2 + d_0\,h_2\,\mu_1 + d_1\,h_1\,\mu_1 + d_2\,h_0\,\mu_1 + d_2\,\omega, \\ k_{2,1} = -B^3a_0\,d_1 + 6\,Ba_0\,d_1\,\lambda^2\,m^2 - B^2bd_1 - 3\,Ba_0\,d_1\,\lambda^2 + 2\,bd_1\,\lambda^2\,m^2 + Bd_0\,h_1 \\ \qquad +Bd_1\,h_0 + Bd_1\,\mu_0 - Bd_1\,\sigma - bd_1\,\lambda^2 + d_0\,h_1\,\mu_1 + d_1\,h_0\,\mu_1 + d_1\,\omega, \\ k_{2,0} = -B^3a_0\,d_0 - 6\,Ba_0\,d_2\,\lambda^2\,m^2 - B^2bd_0 + 6\,Ba_0\,d_2\,\lambda^2 - 2\,bd_2\,\lambda^2\,m^2 + Bd_0\,h_0 \\ \qquad +Bd_0\,\mu_0 - Bd_0\,\sigma + 2\,bd_2\,\lambda^2 + d_0\,h_0\,\mu_1 + d_0\,\omega, \\ k_{3,3} = \frac{1}{3}\,{d_2}^2 + \frac{1}{3}\,{h_2}^2 - 4\,\lambda^2\,ch_2\,m^2\sigma, \\ k_{3,2} = \frac{1}{2}\,d_1\,d_2 + \frac{1}{2}\,h_1\,h_2 - \lambda^2\,ch_1\,m^2\sigma, \\ k_{3,1} = \frac{1}{3}\,d_0\,d_2 + \frac{1}{3}\,h_0\,h_2 - \frac{1}{3}\,h_2\,\sigma - \frac{4}{3}\,\lambda^2\,ch_2\,\sigma + \frac{1}{6}\,{d_1}^2 + \frac{1}{6}\,{h_1}^2 + \frac{1}{3}h_2 + \frac{8}{3}\,\lambda^2\,ch_2\,m^2\sigma, \\ k_{3,0} = \frac{1}{6}h_1 - \frac{1}{6}\,\lambda^2\,ch_1\,\sigma + \frac{1}{6}\,d_0\,d_1 + \frac{1}{6}\,h_0\,h_1 - \frac{1}{6}\,h_1\,\sigma + \frac{1}{3}\,\lambda^2\,ch_1\,m^2\sigma. \end{cases} \tag{A3}$$

For the Schrödinger BBM-KdV system (1.4), the $k_{j,q}$ in (2.3) are:

$$\begin{cases}
k_{1,3} = -\frac{2}{3} d_2 h_2 + 4\lambda^2 a_1 d_2 m^2\sigma, \\
k_{1,2} = -\frac{1}{2} d_1 h_2 - \frac{1}{2} d_2 h_1 + \lambda^2 a_1 d_1 m^2\sigma, \\
k_{1,1} = \frac{2}{3} Bbd_2 - \frac{1}{3} d_0 h_2 - \frac{1}{3} d_1 h_1 - \frac{1}{3} d_2 h_0 - \frac{1}{3} d_2 \mu_0 + \frac{1}{3} d_2 \sigma \\
\qquad + \frac{4}{3}\lambda^2 a_1 d_2 \sigma + B^2 a_1 d_2 \sigma - \frac{2}{3} Ba_1 d_2 \omega - \frac{8}{3}\lambda^2 a_1 d_2 m^2\sigma, \\
k_{1,0} = \frac{1}{3} Bbd_1 - \frac{1}{6} d_0 h_1 - \frac{1}{6} d_1 h_0 - \frac{1}{6} d_1 \mu_0 + \frac{1}{6} d_1 \sigma + \frac{1}{2} B^2 a_1 d_1 \sigma \\
\qquad - \frac{1}{3} Ba_1 d_1 \omega - \frac{1}{3}\lambda^2 a_1 d_1 m^2\sigma + \frac{1}{6}\lambda^2 a_1 d_1 \sigma, \\
k_{2,4} = -18\, Ba_1 d_2 \lambda^2 m^2\sigma + 6\, a_1 d_2 \lambda^2 m^2\omega - 6\, bd_2 \lambda^2 m^2 + Bd_2 h_2 + d_2 h_2 \mu_1, \\
k_{2,3} = -6\, Ba_1 d_1 \lambda^2 m^2\sigma + 2\, a_1 d_1 \lambda^2 m^2\omega - 2\, bd_1 \lambda^2 m^2 + Bd_1 h_2 + Bd_2 h_1 \\
\qquad + d_1 h_2 \mu_1 + d_2 h_1 \mu_1, \\
k_{2,2} = 6\, Ba_1 d_2 \lambda^2 m^2\sigma + 6\left(-B\left(-m^2+1\right) + Bm^2\right) a_1 d_2 \lambda^2 \sigma - 6\, B\left(-m^2+1\right) a_1 d_2 \lambda^2 \sigma \\
\qquad - 2\, a_1 d_2 \lambda^2 m^2\omega - B^3 a_1 d_2 \sigma + B^2 a_1 d_2 \omega - 2\left(2m^2-1\right) a_1 d_2 \lambda^2 \omega \\
\qquad + 2\left(-m^2+1\right) a_1 d_2 \lambda^2 \omega + 2\, bd_2 \lambda^2 m^2 + Bd_2 \mu_0 + Bd_0 h_2 + d_1 h_1 \mu_1 + d_2 h_0 \mu_1 \\
\qquad - B^2 bd_2 + d_2 \omega - Bd_2 \sigma + d_0 h_2 \mu_1 + Bd_1 h_1 + Bd_2 h_0 + 2\left(2m^2-1\right) bd_2 \lambda^2 \\
\qquad - 2\left(-m^2+1\right) bd_2 \lambda^2, \\
k_{2,1} = 3\, Ba_1 d_1 \lambda^2 m^2\sigma - 3\, B\left(-m^2+1\right) a_1 d_1 \lambda^2 \sigma - a_1 d_1 \lambda^2 m^2\omega + B^2 a_1 d_1 \omega \\
\qquad - B^3 a_1 d_1 \sigma + \left(-m^2+1\right) a_1 d_1 \lambda^2 \omega + bd_1 \lambda^2 m^2 - B^2 bd_1 + Bd_0 h_1 \\
\qquad + Bd_1 h_0 + Bd_1 \mu_0 - Bd_1 \sigma + d_0 h_1 \mu_1 + d_1 h_0 \mu_1 + d_1 \omega - \left(-m^2+1\right) bd_1 \lambda^2, \\
k_{2,0} = -B^3 a_1 d_0 \sigma + B^2 a_1 d_0 \omega + d_0 \omega - B^2 bd_0 + Bd_0 h_0 + Bd_0 \mu_0 - Bd_0 \sigma \\
\qquad + d_0 h_0 \mu_1 + 6\, B\left(-m^2+1\right) a_1 d_2 \lambda^2 \sigma - 2\left(-m^2+1\right) a_1 d_2 \lambda^2 \omega + 2\left(-m^2+1\right) bd_2 \lambda^2, \\
k_{3,3} = \frac{1}{12} {d_2}^2 + \frac{1}{12} {h_2}^2 - \lambda^2 ch_2 m^2, \\
k_{3,2} = -\frac{1}{4}\lambda^2 ch_1 m^2 + \frac{1}{8} d_1 d_2 + \frac{1}{8} h_1 h_2, \\
k_{3,1} = -\frac{1}{3}\lambda^2 ch_2 + \frac{2}{3}\lambda^2 ch_2 m^2 + \frac{1}{12} d_0 d_2 + \frac{1}{12} h_0 h_2 - \frac{1}{12} h_2 \sigma + \frac{1}{24} {d_1}^2 + \frac{1}{24} {h_1}^2 + \frac{1}{12} h_2, \\
k_{3,0} = \frac{1}{12}\lambda^2 ch_1 m^2 + \frac{1}{24} d_0 d_1 + \frac{1}{24} h_0 h_1 - \frac{1}{24} h_1 \sigma - \frac{1}{24}\lambda^2 ch_1 + \frac{1}{24} h_1.
\end{cases} \tag{A4}$$

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