Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(2): 2824–2853.
DOI: 10.3934/math.2024140
Received: 24 October 2023
Revised: 10 December 2023
Accepted: 12 December 2023
Published: 02 January 2024

Research article

Convergence of smooth solutions to parabolic equations with an oblique derivative boundary condition

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Abstract: In this paper, the parabolic equation with oblique derivative boundary condition is considered. The long time behavior of the solution is derived by selecting the appropriate auxiliary functions and making priori estimates. Through blow up analysis, time-dependent gradient estimates are obtained, followed by second-order derivative estimates. Then, the convergence of smooth solution to parabolic equations with the oblique derivative boundary condition is obtained using standard theory.

Keywords: convergence; uniformly parabolic equations; oblique derivative boundary condition; long time behavior; derivative estimates
Mathematics Subject Classification: 35B45, 35G30

1. Introduction

In this paper, we consider the long time behavior of smooth solutions to the following parabolic equations with oblique derivative boundary value problems,

$$\begin{cases} u_t - F(\nabla^2 u) = 0 & \text{in} \quad \Omega \times [0, \infty), \\ u(x, 0) = u_0(x) & \text{on} \quad \overline{\Omega} \times \{0\}, \\ \dfrac{\partial u}{\partial \beta} = \varphi(x) & \text{on} \quad \partial\Omega \times [0, \infty), \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in R^n, F is a smooth real function defined on S^n, S^n means $n \times n$ real symmetric matrix space. φ is a given function defined on $\overline{\Omega}$, β is the inward unit vector along $\partial\Omega$, and satisfies the condition $< \nu, \beta >= \beta_n = \cos\theta \geq c_0 > 0$, where ν is the inner normal vector to $\partial\Omega$, $\frac{\partial u}{\partial \beta} =< \nabla u, \beta >$, where $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots \frac{\partial u}{\partial x_n})$ and $u_0 \in C^\infty(\overline{\Omega})$ satisfies $\frac{\partial u_0}{\partial \beta} = \varphi(x)$.

At present, there are many results on various boundary value problems of partial differential equations [1–7], and the oblique derivative boundary value problems of partial differential equations have been widely studied. The related problems of the oblique derivative boundary value problems of linear and quasilinear elliptic equations can be seen in the book [8–10]. The related results of nonlinear differential equations can be found in the literature [11–16]. In [13], Bao established the global Hölder gradient estimates for the $W^{2,p}$ solution of the nonlinear oblique derivative problems for the second-order fully nonlinear elliptic equations using the perturbation idea of Caffarelli. In [17], they studied the long time behavior of the solution in the classical senses through a blow up skill for the following parabolic equation

$$\begin{cases} u_t - F(\nabla^2 u) = 0 & \text{in } \ \Omega \times [0, \infty), \\ u(x, 0) = u_0(x) & \text{on } \ \overline{\Omega} \times \{0\}, \\ \dfrac{\partial u}{\partial \nu} = \varphi(x) & \text{on } \ \partial\Omega \times [0, \infty), \end{cases}$$

where ν is the inward unit normal vector. In this paper, we will consider the long-time behavior of the solution to the above problem when the boundary condition becomes the oblique cases.

We need to make some structural assumptions about F :

(F_1) $\quad \forall r \in S^n, \quad \lambda I \le F_r(r), \quad |F(r)| \le \mu_0 |r|,$

(F_2) $\quad \forall r, X \in S^n, \quad |F_X(r)| \le \mu_1 |X|,$

(F_3) $\quad \forall r, X \in S^n, \quad F_{XX}(r) \le 0,$

where λ, μ_0, μ_1 are positive constsnts. Besides, we suppose

(F_4) There exists a smooth function F_∞ , such that

$$s^{-1} F(sr) \to F_\infty(r) \quad locally \quad uniformly \quad in \quad C^1(S^n), \quad as \quad s \to +\infty.$$

First, we state our major results of this paper.

Theorem 1.1. *Suppose* $\Omega \subset R^n$ $(n \ge 2)$ *is a bounded domain with smooth boundary. If* F *satisfies* (F_1)–(F_4), $\varphi \in C^\infty(\overline{\Omega})$, *then the smooth solution* $u(x, t)$ *of* (1.1) *converges to* $U + \tau t$, *namely,* $\forall$ $D \subset\subset \Omega$, $\zeta < 1$ *and* $0 < \alpha < 1$,

$$\lim_{t \to +\infty} \|u(\cdot, t) - (U(\cdot) + \tau t)\|_{C^{1+\zeta}(\overline{\Omega})} = 0, \quad \lim_{t \to +\infty} \|u(\cdot, t) - (U(\cdot) + \tau t)\|_{C^{4+\alpha}(\overline{D})} = 0, \tag{1.2}$$

where (U, τ) *is a suitable solution to*

$$\begin{cases} F(\nabla^2 U) = \tau & in \quad \Omega, \\ \dfrac{\partial U}{\partial \beta} = \varphi(x) & on \quad \partial\Omega. \end{cases} \tag{1.3}$$

The constant τ depends only on Ω, φ and F. The solution to (1.3) is unique up to a constant.

Remark. *Note* (1.3) *that* τ *depends only on* F, φ, Ω.

Proof. Assume there exist two pairs (τ_1, u) and (τ_2, v) solving (1.3).

Namely

$$\begin{cases} F(\nabla^2 u) = \tau_1 & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \beta} = \varphi(x) & \text{on} \quad \partial\Omega. \end{cases}$$

$$\begin{cases} F(\nabla^2 v) = \tau_2 & \text{in} \quad \Omega, \\ \dfrac{\partial v}{\partial \beta} = \varphi(x) & \text{on} \quad \partial\Omega. \end{cases}$$

Without loss of generality, we may assume $\tau_1 < \tau_2$, then,

$$\begin{cases} \int_0^1 \frac{\partial F}{\partial u_{\alpha\beta}}[t\nabla^2 u + (1-t)\nabla^2 v]dt(u-v)_{\alpha\beta} < 0, \\ \frac{\partial(u-v)}{\partial\beta} = 0. \end{cases}$$

By maximal principle, the minimum of $u - v$ can be achieved at the boundary, but $\frac{\partial(u-v)}{\partial\beta} = 0$ and strong maximal principle indicate that the minimum can only be reached internally, which is contradictory, thus $\tau_1 = \tau_2$.

The above proof indicates that τ here only depends on F, φ, Ω. □

In [18], Huang and Ye established a convergence result under assumptions of a priori estimate.

Theorem 1.2. *[18] Suppose $\Omega \subset R^n$ $(n \geq 2)$ is a bounded domain with smooth boundary. If F satisfies (F_1) and (F_3), $\varphi \in C^\infty(\overline{\Omega})$. $\forall$ $T > 0$, suppose $u \in C^{4+\alpha, \frac{4+\alpha}{2}}(\overline{\Omega \times (0,T)})$ is a unique solution of the following nonlinear parabolic equation*

$$\begin{cases} u_t - F(\nabla^2 u) = 0 & in \quad \Omega \times [0,T), \\ u(x,0) = u_0(x) & on \quad \overline{\Omega} \times \{0\}, \\ G(x, \nabla u) = 0 & on \quad \partial\Omega \times [0,T), \end{cases} \tag{1.4}$$

and u satisfies

$$\|u_t(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla u(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla^2 u(\cdot,t)\|_{C(\overline{\Omega})} \leq C_1, \tag{1.5}$$

$$\sum_{k=1}^n G_{p_k}(x, \nabla u)\nu_k \geq \frac{1}{C_2}, \tag{1.6}$$

where C_1 and C_2 are positive constants independent of $t > 1$. Then $u(\cdot,t)$ converges to a function $U + \tau t$ in $C^{1+\xi}(\overline{\Omega}) \bigcap C^{4+\alpha'}(\overline{D})$ as $t \to +\infty$, $\forall$ $D \subset\subset \Omega$, $\xi < 1$ and $\alpha' < \alpha$, that is (1.2) is satisfied.

In the paper, we derive the estimate (1.5) for the problem (1.1).

Theorem 1.3. *Suppose $\Omega \subset R^n$ $(n \geq 2)$ is a bounded domain with smooth boundary. Assume F satisfies (F_1)–(F_4), $\varphi \in C^\infty(\overline{\Omega})$, then we get the uniform (in t) estimate (1.5) for the solution to (1.1).*

Actually, in [19], a good proof of convergence result is provided, under the assumption of uniform (in t) $\|u_t(\cdot,t)\|_{C(\overline{\Omega})}$, $\|\nabla u(\cdot,t)\|_{C(\overline{\Omega})}$ estimate of quasilinear equation. In this note, after we establish the estimate of $\|u_t(\cdot,t)\|_{C(\overline{\Omega})}$, $\|\nabla u(\cdot,t)\|_{C(\overline{\Omega})}$, $\|\nabla^2 u(\cdot,t)\|_{C(\overline{\Omega})}$, we use Schauder method and the process in [17] to obtain the convergence result. We can also find more details in the work [18] of Huang and Ye.

First of all, we give some notations.

Suppose $\Omega \subset R^n$ $(n \geq 2)$ is a bounded domain, $\partial\Omega \in C^3$. Set

$$d(x) = \text{dist}(x, \partial\Omega),$$

and

$$\Omega_\mu = \{x \in \Omega : d(x) < \mu\}.$$

Then there exists a positive constant $\mu_1 > 0$ such that $\forall\ \mu \leq \mu_1, d(x) \in C^3(\overline{\Omega}_\mu)$. As mentioned in Lieberman [8], we can prolong ν as Dd in Ω_μ which is a C^2 vector field. We also have the following expressions

$$\begin{aligned} &|\nabla \nu| + |\nabla^2 \nu| \leq \tilde{C}(n, \Omega) && \text{in } \Omega_\mu, \\ &\sum_{1\leq i\leq n} \nu^i \nabla_i \nu^j = 0 && \text{in } \Omega_\mu, \\ &|\nu| = 1 && \text{in } \Omega_\mu. \end{aligned} \tag{1.7}$$

Furthermore, in this paper, to simplify the proof of the theorems, we use $O(z)$ to represent an expression that there exists a uniform constant $C > 0$ satisfying $|O(z)| \leq Cz$.

In the following part of the paper, we make the following arrangement. In the second section, we think about the special case of $F(\nabla^2 u) = \triangle u$, and use a blow-up technique to control $\|u(\cdot, t)\|_{C(\bar{\Omega})}$ and then derive the estimate of $\|\nabla u(\cdot, t)\|_{C(\bar{\Omega})}$ and $\|\nabla^2 u(\cdot, t)\|_{C(\bar{\Omega})}$. In the third section , we study the general $F(\nabla^2 u)$ and derive the priori estimates.

2. Long time behavior for the diffusion equation

In this part, we discuss the long time behavior of the following diffusion equation with oblique derivative boundary conditions

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \Omega \times [0, T), \\ u(x, 0) = u_0(x), & \text{on } \overline{\Omega} \times \{0\}, \\ u_\beta = \varphi, & \text{on } \partial\Omega \times [0, T), \end{cases} \tag{2.1}$$

where $\Omega \subset R^n$ is a bounded smooth domain, $\varphi(x)$, $u_0(x) \in C^\infty(\overline{\Omega})$, and $u_{0\beta} = \varphi(x)$ on $\partial\Omega$.

As before, we denote by ν the inner normal vector field along $\partial\Omega$. Set $\{T_l\}_{l=1}^{n-1}$ to be the unit tangent vector fields which joint with ν form a unit normal frame along $\partial\Omega$. Assume $\beta = \beta_n \nu + \Sigma_{l=1}^{n-1} \beta_l T_l$, therefore, $\varphi(x) = \frac{\partial u}{\partial \beta} =< \nabla u, \beta >= \frac{\partial u}{\partial \nu}\beta_n + \Sigma \beta_l u_l$, where $u_l =< \nabla u, T_l >$.

Lemma 2.1. *Suppose $\Omega \subset R^n$ $(n \geq 2)$ is a bounded domain with smooth boundary. If $u(x, t)$ is a smooth solution to* (2.1)*, then*

$$\sup_{\Omega\times[0,T)} |u_t|^2 = \sup_{\Omega} |u_t(x, 0)|^2,$$

so there exists a constant $C = C(u_0) > 0$, such that $\forall (x, t) \in \Omega \times [0, T)$,

$$|u_t|(x, t) \leq C.$$

Proof. Because $(\Delta - \frac{\partial}{\partial t})(u_t^2) = \Delta u_t^2 - \frac{\partial}{\partial t}(u_t^2) = 2u_t \Delta u_t + 2|Du_t|^2 - 2u_t u_{tt} = 2|Du_t|^2 \geq 0$, from the weak maximum principle, we have

$$\sup_{\Omega\times(0,T)} |u_t|^2 = \sup_{\Omega\times\{0\}\cup\partial\Omega\times(0,T)} |u_t|^2.$$

On the other hand, $(u_t^2)_\beta = 2u_t u_{t\beta} = 2u_t \varphi_t = 0$.

Hopf lemma shows that the maximum cannot appear on $\partial\Omega \times (0, T)$, then

$$\sup_{\Omega\times(0,T)} |u_t|^2 = \sup_{\Omega\times\{0\}} |u_t|^2 = \sup_{\Omega} |\Delta u_0|^2.$$

□

Take $x_0 \in \Omega$ and let $v(x,t) = u(x,t) - u(x_0,t)$, in the following, we first give a time independent bound of $|v|$ by using a blow-up method. With the C^0 estimate of v, we then obtain the C^2 estimate of v. Naturally, the estimates for $|\nabla u|$ and $|\nabla^2 u|$ follow. Finally, the convergence results are obtained by using [18].

Lemma 2.2. *Let $\Omega \subset R^n$ $(n \geq 2)$ be a bounded domain with smooth boundary. If $u(x,t)$ is a smooth solution to* (2.1)*, $v(x,t)$ as defined above, then there exists a constant $A_0 > 0$, independent of T, so that*

$$||v||_{C^0(\Omega\times[0,T))} \leq A_0. \tag{2.2}$$

Proof. *Let $A = ||v||_{C^0(\Omega\times[0,T))}$. Without loss of generality, we assume $A \geq \delta = \delta(u_0) > 0$, (otherwise we get a constant solution to* (2.1)*). Assume A is unbounded, i.e., $A \to \infty$, as $T \to \infty$. Let*

$$w(x,t) = \frac{v(x,t)}{A}.$$

Then, $w(x_0,t) = 0, t \in [0,T), |w|_{C^0(\overline{\Omega}\times[0,T))} = 1$, *and satisfies*

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w = -\dfrac{u_t(x_0,t)}{A} & in \quad \Omega \times [0,T), \\ w(x,0) = \dfrac{(u_0(x) - u_0(x_0))}{A} & on \quad \overline{\Omega} \times \{0\}, \\ \dfrac{\partial w}{\partial \beta} = \dfrac{1}{A}\varphi(x) & on \quad \partial\Omega \times [0,T). \end{cases} \tag{2.3}$$

To finish the proof, we need the following propositions.

Proposition 2.3. *Let $w \in C^{3,2}(\Omega \times [0,T))$ and satisfy*

$$\frac{\partial w}{\partial t} - \Delta w = f(t), \ |w| \leq 1, in\ \Omega \times [0,T).$$

Then, $\forall\ \Omega' \subset\subset \Omega$,

$$\sup_{\Omega'\times[0,T)} |\nabla w| \leq C(dist(\Omega', \partial\Omega), |f|_{L^\infty([0,T))}).$$

Remark. *We can see the proof process of this proposition in [17], so we skip it here. Note that $f = -\frac{u_t(x_0,t)}{A}$, we have*

$$\sup_{\Omega'\times[0,T)} |\nabla w| \leq C(\mathrm{dist}(\Omega', \partial\Omega), |f|_{L^\infty([0,T))}) = C(\mathrm{dist}(\Omega', \partial\Omega), u_0). \tag{2.4}$$

Proposition 2.4. *Let $\Omega \subset R^n$ $(n \geq 2)$ be a bounded domain with smooth boundary. Assume that $w \in C^{3,2}(\overline{\Omega} \times [0,T)$ is a solution to* (2.3)*, Then there is a constant $C = C(\Omega, n, u_0, ||\varphi||_{C^3(\overline{\Omega})})$ such that for $\sigma \leq \sigma_1$,*

$$\sup_{\Omega_\sigma\times[0,T)} |\nabla w| \leq C. \tag{2.5}$$

Proof. For $0 < T' < T$, We will prove that we can give $|\nabla w|$ a bound independent of T' on $\partial\Omega \times [0,T']$ and then take a limit.

Let $\varphi' = \frac{\varphi(x)}{A} = \frac{\partial w}{\partial \beta} =< \nabla w, \beta >= \frac{\partial w}{\partial \nu}\beta_n + \sum\limits_{l=1}^{n-1} \beta_l w_l$ and $\rho = w - \frac{\varphi' d}{\cos\theta}$, then $w = \rho + \frac{\varphi' d}{\cos\theta}$ and $\varphi'(x) = \frac{\partial(\rho + \frac{\varphi' d}{\cos\theta})}{\partial \nu}\beta_n + \sum\limits_{l=1}^{n-1} \beta_l(\rho + \frac{\varphi' d}{\cos\theta})_l \Rightarrow \frac{\partial \rho}{\partial \nu}\beta_n + \sum\limits_{l=1}^{n-1} \beta_l \rho_l = 0 \Rightarrow \frac{\partial \rho}{\partial \nu} = -\sum\limits_{l=1}^{n-1} \frac{\beta_l}{\beta_n}\rho_l$.

Thus

$$(\frac{\partial \rho}{\partial \nu})^2 \cos^2\theta = (-\sum_{l=1}^{n-1} \beta_l \rho_l)^2 \le \sum_{l=1}^{n-1} \beta_l^2 \sum_{l=1}^{n-1} \rho_l^2 = (|\nabla \rho|^2 - (\frac{\partial \rho}{\partial \nu})^2) \sin^2\theta. \tag{2.6}$$

Therefore,

$$(\frac{\partial \rho}{\partial \nu})^2 \le |\nabla \rho|^2 \sin^2\theta. \tag{2.7}$$

Let

$$\phi = |\nabla \rho|^2 - (\sum_{i=1}^{n} \rho_i d_i)^2 = \sum_{i,j=1}^{n} (\delta_{ij} - d_i d_j)\rho_i \rho_j \triangleq \sum_{i,j=1}^{n} C^{ij} \rho_i \rho_j,$$

and

$$\Phi = \log\phi + \tau d + \mu d^2,$$

where τ, μ are two constants to be determined later.

Suppose that the maximum value of Φ on $\Omega_\sigma \times [0, T']$ $(\sigma \le \sigma_1)$ is obtained at (x_0, t_0). Let us discuss it in several cases:

Case 1. $t_0 = 0$. If this happens, it is easy to get the gradient estimate.

Case 2. $x_0 \in \partial\Omega_\sigma \bigcap \Omega$. In this way, the estimate is transformed into interior gradient estimate.

Case 3. $x_0 \in \partial\Omega$. Select a suitable coordinate at x_0, so that $\frac{\partial}{\partial x_n} = \nu$, and $\frac{\partial}{\partial x_i}(i = 1, \cdots, n-1)$ are tangent along $\partial\Omega$. Then, we have

$$d_n = 1, d_i = 0, \quad \frac{\partial^2 d}{\partial x_n \partial x_\alpha} = 0, \quad \frac{\partial^2 d}{\partial x_i \partial x_j} = -\kappa_i \delta_{ij},$$

where $1 \le i, j < n,\ 1 \le \alpha \le n-1$, and κ_i is the principal curvatures of $\partial\Omega$ at x_0.

Because x_0 is the maximum point of Φ, then we have,

$$\Phi_i = 0, 1 \le i < n-1, \tag{2.8}$$

and

$$0 \ge \Phi_n = \frac{\phi_n}{\phi} + \tau. \tag{2.9}$$

By (2.8), for $1 \le i < n-1$, we have

$$\begin{aligned} 0 = \Phi_i =& (|\nabla \rho|^2)_i - (\sum_{\alpha=1}^{n} \rho_\alpha d_\alpha)_i^2 \\ =& 2\sum_{j=1}^{n-1} \rho_j \rho_{ji} - 2\sum_{j=1}^{n-1} \rho_n \rho_j d_{ij} \\ =& 2\sum_{j=1}^{n-1} \rho_j \rho_{ji} + 2\rho_n \rho_i \kappa_i. \end{aligned} \tag{2.10}$$

Using (2.10) to calculate ϕ_n , we obtain

$$\begin{aligned}\phi_n =&(|\nabla\rho|^2)_n - (\sum_{\alpha=1}^{n}\rho_\alpha d_\alpha)_n^2 = 2\sum_{i=1}^{n}\rho_i\rho_{in} - 2\rho_n\rho_{nn} = 2\sum_{i=1}^{n-1}\rho_i\rho_{in}\\ =&2\sum_{i=1}^{n-1}\rho_i\rho_{in} + 2\sum_{i,j=1}^{n-1}\rho_i\rho_j\kappa_{ij} = 2\sum_{i,l=1}^{n-1}\rho_i(-\frac{\beta_l}{\beta_n}\rho_l)_i + 2\sum_{i,j=1}^{n-1}\kappa_{ij}\rho_i\rho_j\\ =&-2\sum_{i,l=1}^{n-1}\frac{\rho_i\rho_{li}\beta_l}{\beta_n} - 2\sum_{i,l=1}^{n-1}\rho_i\rho_l(\frac{\beta_l}{\beta_n})_i + 2\sum_{i,j=1}^{n-1}\kappa_{ij}\rho_i\rho_j\\ =&2\frac{\rho_n}{\beta_n}\sum_{l=1}^{n-1}\rho_l\kappa_l\beta_l - 2\sum_{i,l=1}^{n-1}\rho_i\rho_l(\frac{\beta_l}{\beta_n})_i + 2\sum_{i,j=1}^{n-1}\kappa_{ij}\rho_i\rho_j,\end{aligned} \tag{2.11}$$

where we denote by κ_{ij} the Weingarten matrix.

Thus,

$$0 \geq \Phi_n = \frac{2\frac{\rho_n}{\beta_n}\sum\limits_{l=1}^{n-1}\kappa_l\beta_l\rho_l - 2\sum\limits_{i,l=1}^{n-1}\rho_i\rho_l(\frac{\beta_l}{\beta_n})_i + 2\sum\limits_{i,j=1}^{n-1}\kappa_{ij}\rho_i\rho_j}{\phi} + \tau. \tag{2.12}$$

From (2.7), we have

$$c_0^2|\nabla\rho|^2 \leq |\nabla\rho|^2\cos^2\theta \leq \phi \leq |\nabla\rho|^2.$$

If we make τ large enough determined by the geometry of $\partial\Omega, c_0$ and $|\beta|_{C^1(\partial\Omega)}$, this case can not happen.

Case 4. $x_0 \in \Omega_\sigma$, and $t_0 > 0$.

First, we show that $|\nabla w|^2$ gets the maximum value at the boundary.

By simple calculation, we have $\Delta(|\nabla w|^2) - (|\nabla w|^2)_t \geq 0$, then

$$\sup_{\Omega\times[0,T']}|\nabla w|^2 \leq \sup_{\partial\Omega\times[0,T']\bigcup\Omega\times\{0\}}|\nabla w|^2. \tag{2.13}$$

Choose a special coordinate, such that $\rho_1 = |\nabla\rho|,\ \ \rho_l = 0\ \ (l = 2, 3, \cdots, n)$ and $(\rho_{ij})\ \ (2 \leq i, j \leq n)$ is diagonal. We assume that $|\nabla w|$ is large enough such that $|\nabla\rho|,\quad |\nabla w|$ are equivalent at this point.

Under this coordinate and by the assumption that $|\nabla w|$ at (x_0, t_0) is large enough, we first give a basic fact

$$C^{11} \geq \widetilde{C}(\sigma_1, c_0, |\varphi|_{C^1(\Omega)}, |u_0|_{C^1(\Omega)}) > 0. \tag{2.14}$$

In fact, the maximum point of $|\nabla w|$ on $\partial\Omega \times [0, T']$ is denoted by (x_1, t_1), without loss of generality, we suppose that $|\nabla w|(x_1, t_1) \geq 4\sup\limits_{\partial\Omega}|\frac{\varphi'}{\cos\theta}|$.

We propose a precondition that

$$\mu\sigma \leq 1. \tag{2.15}$$

Because of $\Phi(x_1, t_1) \leq \Phi(x_0, t_0)$, (2.7) and (2.13), then we obtain

$$
\begin{aligned}
\phi(x_0,t_0) &\geq e^{-(\tau+1)\sigma_1}\phi(x_1,t_1) = C[|\nabla\rho|^2 - (\frac{\partial\rho}{\partial\nu})^2](x_1,t_1)\\
&\geq C[|\nabla\rho|^2\cos^2\theta](x_1,t_1) \geq C|\nabla\rho|^2(x_1,t_1)\\
&= C|\nabla w - \frac{\varphi'}{\cos\theta}\nu|^2(x_1,t_1) \geq C|\nabla w|^2(x_1,t_1)\\
&\geq C\sup_{\Omega\times[0,T']}|\nabla w|^2\\
&\geq C|\nabla w|^2(x_0,t_0) \geq C|\nabla\rho|^2(x_0,t_0).
\end{aligned}
\tag{2.16}
$$

Note that C may be different in each line of the above processes.

Through an easy observation, it can be seen that

$$C^{11} \geq \widetilde{C} > 0. \tag{2.17}$$

Since (x_0,t_0) is the maximum point, we have

$$0 = \Phi_i = \frac{(C^{kl}\rho_k\rho_l)_i}{\phi} + \tau d_i + 2\mu d d_i = \frac{\phi_i}{\phi} + \tau d_i + 2\mu d d_i. \tag{2.18}$$

Hence one can see that

$$
\begin{aligned}
\frac{\phi_i}{\phi} &= -\tau d_i - 2\mu d d_i,\\
C^{kl}\rho_{ki}\rho_l &= -\frac{\phi}{2}(\tau + 2\mu d)d_i - \frac{C^{kl}{}_{,i}}{2}\rho_k\rho_l.
\end{aligned}
\tag{2.19}
$$

For $i = 1$, we get

$$C^{11}\rho_{11} + \sum_{\delta=2}^{n} C^{\delta 1}\rho_{\delta 1} = -\frac{1}{2}C^{11}{}_{,1}\rho_1 - \frac{\phi}{2\rho_1}(\tau + 2\mu d)d_1. \tag{2.20}$$

For $\delta > 1$, we have

$$C^{11}\rho_{1\delta} + C^{1\delta}\rho_{\delta\delta} = -\frac{1}{2}C^{11}{}_{,\delta}\rho_1 - \frac{\phi}{2\rho_1}(\tau + 2\mu d)d_\delta. \tag{2.21}$$

Then

$$\rho_{1\delta} = -\frac{C^{1\delta}}{C^{11}}\rho_{\delta\delta} - \frac{C^{11}{}_{,\delta}}{2C^{11}}\rho_1 - \frac{(\tau + 2\mu d)d_\delta}{2}\rho_1 = -\frac{C^{1\delta}}{C^{11}}\rho_{\delta\delta} + O(|\nabla\rho|). \tag{2.22}$$

Replace (2.22) back to (2.20), we have

$$
\begin{aligned}
\rho_{11} =& (\frac{C^{1\delta}}{C^{11}})^2\rho_{\delta\delta} + \frac{C^{\delta 1}C^{11}{}_{,\delta}}{2(C^{11})^2}\rho_1 + \frac{C^{\delta 1}\rho_1(\tau + 2\mu d)d_\delta}{2C^{11}} - \frac{C^{11}{}_{,1}}{2C^{11}}\rho_1 - \frac{\rho_1(\tau + 2\mu d)d_1}{2}\\
=& (\frac{C^{1\delta}}{C^{11}})^2\rho_{\delta\delta} + O(|\nabla\rho|).
\end{aligned}
\tag{2.23}
$$

At this point we still have

$$0 \leq \Phi_t = \frac{\phi_t}{\phi} = \frac{2C^{kl}\rho_k\rho_{lt}}{\phi}, \tag{2.24}$$

and

$$0 \geq \triangle\Phi = \frac{\triangle\phi}{\phi} - (\frac{\nabla\phi}{\phi})^2 + (\tau + 2\mu d)\triangle d + 2\mu|\nabla d|^2. \tag{2.25}$$

Combining (2.19), (2.24) and (2.25), we gain

$$
\begin{aligned}
0 \geq \triangle\Phi - \Phi_t =& \frac{\triangle\phi - \phi_t}{\phi} + (\tau + 2\mu d)\triangle d + 2\mu|\nabla d|^2 - (\tau + 2\mu d)^2|\nabla d|^2 \\
\geq& \frac{\triangle\phi - \phi_t}{\phi} + [2\mu - (\tau + 2\mu d)^2]|\nabla d|^2 - (\tau + 2\mu d)k_0.
\end{aligned} \tag{2.26}
$$

Where $\triangle d \geq -k_0$ on Ω_{σ_1}.

Next, we calculate the term $\frac{\triangle\phi-\phi_t}{\phi}$. Note that

$$
\begin{aligned}
\overline{I} &= \triangle\phi - \phi_t = \triangle(C^{ij}\rho_i\rho_j) - \phi_t \\
&= 2[C^{ij}(\triangle\rho)_i\rho_j - C^{ij}\rho_i\rho_{tj}] + 2C^{ij}\rho_{ik}\rho_{jk} + 4C^{ij}{}_{,k}\rho_{ik}\rho_j + \triangle C^{ij}\rho_i\rho_j \\
&= I + II + III + IV.
\end{aligned} \tag{2.27}
$$

For the term I,

$$
\begin{aligned}
I =& 2[C^{ij}(\triangle\rho)_i\rho_j - C^{ij}\rho_i\rho_{tj}] = 2C^{ij}\{[(\triangle w)_i - (\triangle\frac{\varphi' d}{\cos\theta})_i]\rho_j - \rho_i w_{tj}\} \\
=& 2C^{ij}\{[w_{ti} - (\triangle\frac{\varphi' d}{\cos\theta})_i]\rho_j - \rho_i w_{tj}\} = -2C^{ij}(\triangle\frac{\varphi' d}{\cos\theta})_i\rho_j = O(|\nabla\rho|).
\end{aligned} \tag{2.28}
$$

For the term IV,

$$
IV = C^{ij}{}_{,kk}\rho_i\rho_j = O(|\nabla\rho|^2). \tag{2.29}
$$

For the term III,

$$
\begin{aligned}
III =& 4C^{ij}{}_{,k}\rho_{ik}\rho_j = 4\rho_1 C^{i1}{}_{,k}\rho_{ik} \\
=& 4\rho_1 C^{11}{}_{,1}\rho_{11} + 4\rho_1 C^{\delta 1}{}_{,1}\rho_{1\delta} + 4\rho_1 C^{11}{}_{,\delta}\rho_{1\delta} + 4\rho_1 C^{\delta 1}{}_{,\delta}\rho_{\delta\delta} \\
=& III_1 + III_2 + III_3 + III_4,
\end{aligned} \tag{2.30}
$$

where

$$
\begin{aligned}
III_1 =& 4\rho_1 C^{11}{}_{,1}\rho_{11} = 4\rho_1 C^{11}{}_{,1}[(\frac{C^{1\delta}}{C^{11}})^2\rho_{\delta\delta} + O(\rho_1)] \\
=& O(|\nabla w|)\rho_{\delta\delta} + O(|\nabla w|^2),
\end{aligned}
$$

$$
\begin{aligned}
III_2 + III_3 =& 4\rho_1(C^{\delta 1}{}_{,1} + C^{11}{}_{,\delta})\rho_{1\delta} = 4\rho_1(C^{\delta 1}{}_{,1} + C^{11}{}_{,\delta})(-\frac{C^{1\delta}}{C^{11}}\rho_{\delta\delta} + O(|\nabla\rho|)) \\
=& O(|\nabla w|)\rho_{\delta\delta} + O(|\nabla w|^2),
\end{aligned}
$$

then

$$
III = O(|\nabla w|)\rho_{\delta\delta} + O(|\nabla w|^2). \tag{2.31}
$$

For the term II,

$$
II = 2C^{ij}\rho_{ik}\rho_{jk} = 2C^{1i}\rho_{ik}\rho_{1k} + 2C^{i\delta}\rho_{ik}\rho_{\delta k} = II_1 + II_2, \tag{2.32}
$$

where

$$
\begin{aligned}
II_1 =& 2C^{1i}\rho_{ik}\rho_{1k} = (-C^{i1}{}_{,k}\rho_i - \frac{\phi}{\rho_1}(\tau + 2\mu d)d_k)\rho_{1k} \\
=& (-C^{11}{}_{,1}\rho_1 - \rho_1(\tau + 2\mu d)d_1)\rho_{11} + (-C^{11}{}_{,\delta}\rho_1 - \rho_1(\tau + 2\mu d)d_\delta)\rho_{1\delta} \\
=& II_{11} + II_{12},
\end{aligned}
$$

$$
\begin{aligned}
II_{11} =&(-C^{11}{}_{,1}\rho_1 - \rho_1(\tau + 2\mu d)d_1)\rho_{11}\\
=&(-C^{11}{}_{,1}\rho_1 - \rho_1(\tau + 2\mu d)d_1)((\frac{C^{1\delta}}{C^{11}})^2\rho_{\delta\delta} + O(\rho_1))\\
=&\sum_{\delta=2}^{n} O(|\nabla w|)\rho_{\delta\delta} + O(|\nabla w|^2),
\end{aligned}
$$

$$
\begin{aligned}
II_{12} =&(-C^{11}{}_{,\delta}\rho_1 - \rho_1(\tau + 2\mu d)d_\delta)\rho_{1\delta}\\
=&(-C^{11}{}_{,\delta}\rho_1 - \rho_1(\tau + 2\mu d)d_\delta)(-\frac{C^{1\delta}}{C^{11}}\rho_{\delta\delta} + O(|\nabla\rho|))\\
=&\sum_{\delta=2}^{n} O(|\nabla w|)\rho_{\delta\delta} + O(|\nabla w|^2),
\end{aligned}
$$

then

$$
II_1 = \sum_{\delta=2}^{n} O(|\nabla w|)\rho_{\delta\delta} + O(|\nabla w|^2).
$$

Where

$$
\begin{aligned}
II_2 =&2C^{i\delta}\rho_{ik}\rho_{\delta k} = 2C^{1\delta}\rho_{1k}\rho_{\delta k} + 2C^{\alpha\delta}\rho_{\alpha k}\rho_{\delta k}\\
=&2C^{1\delta}\rho_{11}\rho_{\delta 1} + 2C^{1\delta}\rho_{1\delta}\rho_{\delta\delta} + 2C^{\alpha\delta}\rho_{\alpha 1}\rho_{\delta 1} + 2C^{\delta\delta}\rho_{\delta\delta}^2\\
=&II_{21} + II_{22} + II_{23} + II_{24},
\end{aligned}
$$

$$
\begin{aligned}
II_{21} =&2C^{1\delta}\rho_{11}\rho_{\delta 1} = 2[-\sum_{\delta=2}^{n}\frac{(C^{1\delta})^2}{C^{11}}\rho_{\delta\delta} + O(|\nabla\rho|)] \times [\sum_{\delta=2}^{n}(\frac{C^{1\delta}}{C^{11}})^2\rho_{\delta\delta} + O(|\nabla\rho|)]\\
=&-\frac{2}{(C^{11})^3}\sum_{\alpha,\beta=2}^{n} C^{1\alpha}C^{1\beta}(C^{1\alpha}\rho_{\alpha\alpha})(C^{1\beta}\rho_{\beta\beta}) + \sum_{\delta=2}^{n} O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2),
\end{aligned}
$$

$$
II_{22} = 2\sum_{\delta=2}^{n} C^{1\delta}\rho_{1\delta}\rho_{\delta\delta} = -\sum_{\delta=2}^{n}\frac{2(C^{1\delta})^2}{C^{11}}\rho_{\delta\delta}^2 + \sum_{\delta=2}^{n} O(|\nabla\rho|)\rho_{\delta\delta},
$$

$$
\begin{aligned}
II_{23} =&2C^{\alpha\delta}\rho_{\alpha 1}\rho_{\delta 1} = 2\sum_{\alpha,\beta=2}^{n} C^{\alpha\beta}[-\frac{C^{1\alpha}}{C^{11}}\rho_{\alpha\alpha} + O(|\nabla\rho|)] \times [-\frac{C^{1\beta}}{C^{11}}\rho_{\beta\beta} + O(|\nabla\rho|)]\\
=&\frac{2}{(C^{11})^2}\sum_{\alpha,\beta=2}^{n} C^{\alpha\beta}(C^{1\alpha}\rho_{\alpha\alpha})(C^{1\beta}\rho_{\beta\beta}) + \sum_{\delta=2}^{n} O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2),
\end{aligned}
$$

hence

$$
\begin{aligned}
II_2 =&II_{21} + II_{22} + II_{23} + II_{24}\\
=&\frac{2}{(C^{11})^2}\sum_{\alpha,\beta=2}^{n} C^{\alpha\beta}(C^{1\alpha}\rho_{\alpha\alpha})(C^{1\beta}\rho_{\beta\beta}) - \frac{2}{(C^{11})^3}\sum_{\alpha,\beta=2}^{n} C^{1\alpha}C^{1\beta}(C^{1\alpha}\rho_{\alpha\alpha})(C^{1\beta}\rho_{\beta\beta})\\
&+ 2\sum_{\delta=2}^{n} C^{\delta\delta}\rho_{\delta\delta}^2 - \sum_{\delta=2}^{n}\frac{2(C^{1\delta})^2}{C^{11}}\rho_{\delta\delta}^2 + \sum_{\delta=2}^{n} O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2).
\end{aligned}
$$

Thus

$$\begin{aligned} II =& II_1 + II_2 \\ =& \frac{2}{(C^{11})^2}\sum_{\alpha,\beta=2}^{n} C^{\alpha\beta}(C^{1\alpha}\rho_{\alpha\alpha})(C^{1\beta}\rho_{\beta\beta}) - \frac{2}{(C^{11})^3}\sum_{\alpha,\beta=2}^{n} C^{1\alpha}C^{1\beta}(C^{1\alpha}\rho_{\alpha\alpha})(C^{1\beta}\rho_{\beta\beta}) \\ &+ 2\sum_{\delta=2}^{n} C^{\delta\delta}\rho_{\delta\delta}^2 - \sum_{\delta=2}^{n}\frac{2(C^{1\delta})^2}{C^{11}}\rho_{\delta\delta}^2 + \sum_{\delta=2}^{n} O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2). \end{aligned} \tag{2.33}$$

And

$$\begin{aligned} \triangle\phi - \phi_t =& \frac{2}{(C^{11})^2}\sum_{\alpha,\beta=2}^{n} C^{\alpha\beta}(C^{1\alpha}\rho_{\alpha\alpha})(C^{1\beta}\rho_{\beta\beta}) - \frac{2}{(C^{11})^3}\sum_{\alpha,\beta=2}^{n} C^{1\alpha}C^{1\beta}(C^{1\alpha}\rho_{\alpha\alpha})(C^{1\beta}\rho_{\beta\beta}) \\ &+ 2\sum_{\delta=2}^{n} C^{\delta\delta}\rho_{\delta\delta}^2 - \sum_{\delta=2}^{n}\frac{2(C^{1\delta})^2}{C^{11}}\rho_{\delta\delta}^2 + \sum_{\delta=2}^{n} O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2) \\ =& \Pi_1 + \Pi_2. \end{aligned} \tag{2.34}$$

For the term Π_1,

$$\begin{aligned} \Pi_1 =& \frac{2}{(C^{11})^2}\sum_{\alpha,\beta=2}^{n} C^{\alpha\beta}(C^{1\alpha}\rho_{\alpha\alpha})(C^{1\beta}\rho_{\beta\beta}) - \frac{2}{(C^{11})^3}\sum_{\alpha,\beta=2}^{n} C^{1\alpha}C^{1\beta}(C^{1\alpha}\rho_{\alpha\alpha})(C^{1\beta}\rho_{\beta\beta}) \\ =& \frac{2}{(C^{11})^3}\sum_{\alpha,\beta=2}^{n}(C^{11}C^{\alpha\beta} - C^{1\alpha}C^{1\beta})(C^{1\alpha}\rho_{\alpha\alpha})(C^{1\beta}\rho_{\beta\beta}) \\ =& \frac{2}{(C^{11})^3}\sum_{\alpha,\beta=2}^{n}[(1-d_1^2)\delta_{\alpha\beta} - d_\alpha d_\beta](C^{1\alpha}\rho_{\alpha\alpha})(C^{1\beta}\rho_{\beta\beta}) \\ \geq& 0. \end{aligned}$$

The above formula is nonnegative, because the matrix $((1-d_1^2)\delta_{\alpha\beta} - d_\alpha d_\beta)_{\alpha,\beta\geq 2}$ is semi positive definite, due to $|\nabla d|^2 = 1$.

Next we set out to deal with the term Π_2,

$$\begin{aligned} \Pi_2 =& 2\sum_{\delta=2}^{n} C^{\delta\delta}\rho_{\delta\delta}^2 - \sum_{\delta=2}^{n}\frac{2(C^{1\delta})^2}{C^{11}}\rho_{\delta\delta}^2 + \sum_{\delta=2}^{n} O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2) \\ =& \frac{2}{C^{11}}\sum_{\delta=2}^{n}(C^{11}C^{\delta\delta} - (C^{1\delta})^2)\rho_{\delta\delta}^2 + \sum_{\delta=2}^{n} O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2) \\ =& 2\sum_{\delta=2}^{n}\frac{1-d_1^2-d_\delta^2}{C^{11}}\rho_{\delta\delta}^2 + \sum_{\delta=2}^{n} O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2) \\ =& 2\sum_{\delta=2}^{n} e_\delta\rho_{\delta\delta}^2 + \sum_{\delta=2}^{n} O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2), \end{aligned}$$

where $e_\delta = \frac{1-d_1^2-d_\delta^2}{C^{11}}$.

According to equation $\triangle w - w_t = f(t)$, we can obtain by Lemma 2.1 that $\triangle\rho = O(1)$. Joint with $\rho_{11} = (\frac{C^{1\delta}}{C^{11}})^2\rho_{\delta\delta} + O(\rho_1)$, we get

$$\sum_{\delta=2}^{n}(1 + (\frac{C^{1\delta}}{C^{11}})^2)\rho_{\delta\delta} = O(|\nabla\rho|).$$

Therefore

$$\rho_{22} = O(|\nabla\rho|) - \sum_{\delta=3}^{n}\frac{(C^{11})^2 + (C^{1\delta})^2}{(C^{11})^2 + (C^{12})^2}\rho_{\delta\delta}.$$

Substituting ρ_{22} into Π_2, we can get

$$\begin{aligned}\Pi_2 =& 2\sum_{\delta=2}^{n} e_\delta\rho_{\delta\delta}^2 + \sum_{\delta=2}^{n} O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2)\\ =& 2e_2(\sum_{\delta=3}^{n}\frac{(C^{11})^2 + (C^{1\delta})^2}{(C^{11})^2 + (C^{12})^2}\rho_{\delta\delta})^2 + 2\sum_{\delta=3}^{n} e_\delta\rho_{\delta\delta}^2 + \sum_{\delta=3}^{n} O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2).\end{aligned}$$

Now let us consider the quadratic form in Π_2, which is a quadratic form of $\rho_{33}, \rho_{44}, \cdots, \rho_{nn}$.

Let

$$\Lambda = 2e_2(\sum_{\delta=3}^{n}\frac{(C^{11})^2 + (C^{1\delta})^2}{(C^{11})^2 + (C^{12})^2}\rho_{\delta\delta})^2 + 2\sum_{\delta=3}^{n} e_\delta\rho_{\delta\delta}^2 = 2e_2(\sum_{\delta=3}^{n}\varepsilon_\delta\rho_{\delta\delta})^2 + 2\sum_{\delta=3}^{n} e_\delta\rho_{\delta\delta}^2.$$

Through observation, we know that

$$0 \le e_\delta \le 1,\ (2 \le \delta \le n),\ \sum_{\delta=2}^{n} e_\delta = n-2,$$

so at most one of $e_2, \cdots, e_n$ is zero, it is obvious that $0 < C_0 \le \varepsilon_\delta \le C_1, \delta = 3, \cdots, n$, the quadratic form Λ is positive definite.

Next, we give a positive controllable lower bound for the eigenvalues of this quadratic form.

We can regard Λ as a 3n-5 variables function, and its definition domain is

$$\begin{aligned}D = \{(e_2, e_3, \cdots, e_n, \varepsilon_3, \cdots, \varepsilon_n, \rho_{33}, \cdots, \rho_{nn}) | 0 \le e_\delta \le 1, \sum_{\delta=2}^{n} e_\delta = n-2,\\ 0 < C_0 \le \varepsilon_\delta \le C_1, \delta = 3, \cdots, n, \sum_{\delta=3}^{n}\rho_{\delta\delta}^2 = 1\}.\end{aligned}$$

It is easy to see that D is a compact set. The minimum value of Λ on D is denoted by λ_0, then the positive number λ_0 is a general positive lower bound of the eigenvalue of the quadratic form, that is

$$\Lambda \ge \lambda_0\sum_{\delta=3}^{n}\rho_{\delta\delta}^2.$$

Therefore, in light of $ax^2 + bx \ge -\frac{b^2}{4a}$, if $a > 0$ we can obtain

$$\bar{I} \ge \Pi_2 \ge \lambda_0\sum_{\delta=3}^{n}\rho_{\delta\delta}^2 + \sum_{\delta=3}^{n} O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2) \ge -k_1|\nabla\rho|^2. \tag{2.35}$$

Meanwhile, in consideration of $\phi = |\nabla\rho|^2 C^{11}$

$$\triangle\Phi - \Phi_t = \frac{\overline{I}}{\phi} + (\tau + 2\mu d)\Delta d + 2\mu - (\tau + 2\mu d)^2 \geq -\frac{k_1}{\overline{C}} - (\tau + 2\mu d)k_0 - (\tau + 2\mu d)^2 + 2\mu. \tag{2.36}$$

First, select μ to make

$$2\mu = \frac{k_1}{\overline{C}} + (\tau + 1)^2 + (\tau + 1)k_0 + 1.$$

Then, select $\sigma \leq \sigma_1$ to make $2\mu\sigma \leq 1$, then we have a contradiction $0 \geq \triangle\Phi - \Phi_t > 0$, so $|\nabla\rho|$ must be bounded.

Namely,

$$|\nabla w| \leq C(\Omega, n, u_0, ||\varphi||_{C^3(\overline{\Omega})}), \ \forall (x, t) \in \Omega_\sigma \times [0, T']. \tag{2.37}$$

Since the bound is independent of T', Proposition 2.4 is proved. By the uniform estimate of u_t, we can deduce the estimate of uniform bound of w_t, Combining with Propositions 2.3 and 2.4, we then get the uniform $C^{k,\alpha}$ estimate for $k \in Z^+$ and $0 < \alpha < 1$ by the Schauder theory. □

Proof of Lemma 2.2. We continue to prove Lemma 2.2. For $n \in Z^+$, denoted by $w_n = w|_{\overline{\Omega}\times[0,n]}$, suppose $A_n = \sup\limits_{\overline{\Omega}\times[0,n]} |w_n|$ which is obtained at the point (x_n, t_n). For $(x, s) \in \overline{\Omega} \times [0, 1]$, Let $g_n(x, s) = w_n(x, s + t_n - 1)$, Then $g_n(x, s)$ suits

$$\begin{cases} \dfrac{\partial g_n}{\partial s} - \Delta g_n = -\dfrac{f(s + t_n - 1)}{A_n} & \text{in } \ \Omega \times [0, 1], \\ g_n(x, 0) = w_n(x, t_n - 1) & \text{on } \ \overline{\Omega} \times \{0\}, \\ \dfrac{\partial g_n}{\partial \beta} = \dfrac{\varphi(x)}{A_n} & \text{on } \ \partial\Omega \times [0, 1]. \end{cases} \tag{2.38}$$

Since we have obtained the uniform C^1 estimate of $w_n(x, t)$ independent of $t \in [0, n]$, $g_n(x, s)$ also has the uniform estimate of gradient independent of n and s. Therefore, (for convenience, we set $g_n(x, 0) = w_n(x, t_n - 1) \stackrel{\Delta}{=} a_n(x)$), $\{a_n(x)\}$ and its derivative sequence are uniformly bounded. Thus, from the Arzela-Ascoli theorem, $g_n(x, 0)$ has convergent subsequences. Without losing generality, we suppose that $g_n(x, 0)$ converges to a continuous function $g_0(x)$ defined on $\overline{\Omega}$ satisfying $g_0(x_0) = 0$ and $\sup\limits_{x\in\Omega} |g_0(x)| \leq 1$.

From the relationship between g_n and w_n, we can obtain the uniform $C^{k,\alpha}$ estimate of g_n on $\overline{\Omega}\times[0, 1]$. So we choose a subsequence of g_n converges in the sense of $C^{k,\alpha}$ ($k \in Z^+$ and $0 < \alpha < 1$) to g on $\overline{\Omega} \times [0, 1]$. Clearly, we get

$$\begin{cases} \dfrac{\partial g}{\partial s} - \Delta g = 0 & \text{in } \ \Omega \times [0, 1], \\ g(x, 0) = g_0(x) & \text{on } \ \overline{\Omega} \times \{0\}, \\ \dfrac{\partial g}{\partial \beta} = 0 & \text{on } \ \partial\Omega \times [0, 1]. \end{cases} \tag{2.39}$$

Because of $\frac{\partial g}{\partial s} - \Delta g = 0$, g obtains the maximum value on $\Omega \times \{0\}$ or $\partial\Omega \times [0, 1]$, but $\frac{\partial g}{\partial \beta} = 0$ shows that it can only be achieved at $\Omega\times\{0\}$, however g reaches the maximum at $s = 1$. It is a contradiction by the maximum principle and Hopf Lemma for the parabolic differential equations. Thus, we complete the proof of Lemma 2.2. □

Theorem 2.5. $\forall\ T > 0$, *supposing that u is a smooth solution to* (2.1)*, then we have the estimate,*

$$\|u_t(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla u(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla^2 u(\cdot,t)\|_{C(\overline{\Omega})} \le C, \tag{2.40}$$

where C is a constant independent of t and T.

Proof. From the definition of v, v satisfies the following equation

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = -u_t(x_0,t) & \text{in } \ \Omega \times (0,T), \\ v(x,0) = u_0(x) - u_0(x_0) & \text{on } \ \overline{\Omega} \times \{0\}, \\ \dfrac{\partial v}{\partial \beta} = \varphi(x) & \text{on } \ \partial\Omega \times (0,T). \end{cases} \tag{2.41}$$

From Lemma 2.2 we have $|v| \le A_0$, the step similar to Propositions 2.3 and 2.4 deduces

$$\|\nabla v(\cdot,t)\|_{C(\overline{\Omega})} \le C.$$

Schauder theory then deduces

$$\|\nabla^2 v(\cdot,t)\|_{C(\overline{\Omega})} \le C.$$

Since $v(x,t) = u(x,t) - u(x_0,t)$, we get

$$\|\nabla u(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla^2 u(\cdot,t)\|_{C(\overline{\Omega})} \le C.$$

Combining with Lemma 2.1, we finish the proof of Theorem 2.5. □

3. Long time behavior for the Fully Nonlinear equation

In this part, we consider

$$\begin{cases} u_t = F(u_{ij}) & \text{in } \ \Omega \times [0,T), \\ u(x,0) = u_0(x) & \text{on } \ \Omega \times \{0\}, \\ \dfrac{\partial u}{\partial \beta} = \varphi(x) & \text{on } \ \partial\Omega \times [0,T), \end{cases} \tag{3.1}$$

where $\Omega \subset R^n$ is a smooth bounded domain, $\varphi(x), u_0(x) \in C^\infty(\overline{\Omega})$, so that $u_{0,\beta} = \varphi(x)$ on $\partial\Omega$. Moreover, we assume that F satisfies (F_1)–(F_4).

Lemma 3.1. *Let $\Omega \subset R^n$ $(n \ge 2)$ be a bounded domain with smooth boundary. Assuming that $u(x,t)$ is a smooth solution to* (3.1)*, there is a constant $C_0 = C_0(u_0) > 0$ such that $\forall (x,t) \in \Omega \times (0,\infty)$,*

$$|u_t|(x,t) \le C_0.$$

Proof. Let F_u^{ij} denote $\frac{\partial}{\partial r_{ij}}|_{r=\nabla^2 u}F(r)$ and $L = F_u^{ij}\partial_{ij} - \partial_t$, take the derivative of t on both sides of $u_t = F(\nabla^2 u)$, we have

$$u_{tt} = F_u^{ij} u_{ijt},$$

then $L(u_t^2) = 2\sum_{i,j=1}^{n} F_u^{ij} u_{ti}u_{tj} + 2F_u^{ij}u_t u_{tij} - 2u_t u_{tt} \geq 2\sum_{i,j=1}^{n} F_u^{ij} u_{ti}u_{tj} \geq 0$, from the weak maximum principle, we get

$$\sup_{\Omega\times(0,T)} |u_t|^2 = \sup_{\Omega\times\{0\}\cup\partial\Omega\times(0,T)} |u_t|^2.$$

Since, $(u_t^2)_\beta = 2u_t u_{t\beta} = 2u_t u_{\beta t} = 0$. Hopf lemma makes it impossible for the maximum to occur on $\partial\Omega \times (0, T)$, then

$$\sup_{\Omega\times(0,T)} |u_t|^2 = \sup_{\Omega\times\{0\}} |u_t|^2 = \sup_{\Omega} \left|F(\nabla^2 u_0)\right|^2.$$

□

Let $v(x, t) = u(x, t) - u(x_0, t)$ where $x_0 \in \Omega$. Similar to Section 2, we first give a time-independent bound for $|v|$ by a blow-up technique. Then from the C^0 estimate of v, we get the bound of $\|v\|_{C^2}$. Naturally, it follows the estimates of $|\nabla u|$ and $|\nabla^2 u|$. Finally, we get the convergence result according to the method of [1].

Lemma 3.2. *Let $\Omega \subset R^n$ $(n \geq 2)$ be a bounded domain with smooth boundary. If $u(x, t)$ is a smooth solution to* (3.1)*, $v(x, t)$ is defined as above, then there is a constant $A_0 > 0$, independent of T , such that*

$$\|v\|_{L^\infty(\Omega\times[0,T))} \leq A_0. \tag{3.2}$$

Proof. Let $A = \|v\|_{C^0(\Omega\times[0,T))}$. Without loss of generality, we assume that $A \geq \delta = \delta(u_0) > 0$, (otherwise the solution to (3.1) *is a constant). Assume A is unbounded, that is, $A \to \infty$, as $T \to \infty$. Let*

$$w(x, t) = \frac{v(x, t)}{A}.$$

Obviously w satisfies $w(x_0, t) = 0, t \in [0, T), |w|_{C^0(\overline{\Omega}\times[0,T))} = 1$, and

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{1}{A}F(A\nabla^2 w) = -\dfrac{u_t(x_0, t)}{A} & in \quad \Omega \times [0, T), \\ w(x, 0) = \dfrac{(u_0(x) - u_0(x_0))}{A} & on \quad \overline{\Omega} \times \{0\}, \\ \dfrac{\partial w}{\partial \beta} = \dfrac{1}{A}\varphi(x) & on \quad \partial\Omega \times [0, T). \end{cases} \tag{3.3}$$

In order to prove the above estimate, we need the following propositions.

Proposition 3.3. *If $w \in C^{3,2}(\Omega \times [0, T))$ satisfies $|w| \leq M$ for a normal number M and*

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{1}{A}F(A\nabla^2 w) = f(t) & in \quad \Omega \times [0, T), \\ w(x, 0) = \dfrac{(u_0(x) - u_0(x_0))}{A} = w_0(x) & on \quad \overline{\Omega} \times \{0\}. \end{cases} \tag{3.4}$$

Then $\forall\ \Omega' \subset\subset \Omega$,

$$\sup_{\Omega'\times[0,T)} |\nabla w| \leq C(\lambda,\ \mu_0,\ \mu_1,\ M,\ w_0,\ dist(\Omega',\ \partial\Omega),\ |f|_{L^\infty([0,T))}).$$

Remark. *One can refer to [17] for the proof of this proposition. Note that* $f = -\frac{u_t(x_0,t)}{A}, M = 1$ *in problem* (3.3)*, we get*

$$\sup_{\Omega'\times[0,T)} |Dw| \leq C(\lambda, \mu_0, \mu_1, dist(\Omega', \partial\Omega), u_0). \tag{3.5}$$

Proposition 3.4. *If* $\Omega \subset R^n$ $(n \geq 2)$ *is a bounded domain with smooth boundary. Assuming that* $w \in C^{3,2}(\overline{\Omega} \times [0,T))$ *is a solution to* (3.3)*, there exists a constant* $C = C(\Omega, n, u_0, \varphi, \lambda, \mu_0, \mu_1)$ *such that for* $\sigma \leq \sigma_1$,

$$\sup_{\Omega_\sigma\times[0,T)} |\nabla w| \leq C. \tag{3.6}$$

Proof. For $0 < T' < T$, we will complete the proof on $\Omega_\sigma \times [0, T']$ and show that the bound is independent of T'.

Let

$$\Phi = \log\phi + \tau d + \mu d^2,$$

$$\phi = |\nabla\rho|^2 - (\sum_{i=1}^n \rho_i d_i)^2 = \sum_{i,j=1}^n (\delta_{ij} - d_i d_j)\rho_i\rho_j \triangleq \sum_{i,j=1}^n C^{ij}\rho_i\rho_j,$$

where τ, μ are positive constants to be determined later, $\rho = w - \frac{\varphi' d}{\cos\theta}$, and $\varphi' = \frac{\varphi(x)}{A} = \frac{\partial w}{\partial\beta} = \frac{\partial w}{\partial\nu}\beta_n + \sum_{l=1}^{n-1}\beta_l w_l$.

Assume Φ gets the maximum value at (x_0, t_0) on $\Omega_\sigma \times [0, T']$.

Case 1. $t_0 = 0$. we get

$$|\nabla w|^2(x_0, 0) \leq C(\Omega, n, u_0).$$

Case 2. $x_0 \in \partial\Omega_\sigma \bigcap \Omega$. In this case, the estimate follows from interior gradient estimate in Proposition 3.3.

Case 3. $x_0 \in \partial\Omega$. Similar to the process of Proposition 2.4, we can choose the appropriate τ to guarantee this case does not occur.

Case 4. $x_0 \in \Omega_\sigma$, and $t_0 > 0$.

Select a particular coordinate, so that $\rho_1 = |\nabla\rho|$, $\rho_l = 0$ $(l = 2, 3, \cdots, n)$ and (ρ_{ij}) $(2 \leq i, j \leq n)$ is diagonal. We assume that $|\nabla w|$ is large enough at this point so that $|\nabla\rho|$, $|\nabla w|$ are equivalent.

Through a process similar to Proposition 2.4, we have

$$C^{11} \geq \widetilde{C}(\sigma_1,\ c_0,\ |\varphi|_{C^1(\Omega)},\ |u_0|_{C^1(\Omega)}) > 0.$$

At the maximum point (x_0, t_0), we have

$$0 = \Phi_i = \frac{(C^{kl}\rho_k\rho_l)_i}{\phi} + \tau d_i + 2\mu d d_i = \frac{\phi_i}{\phi} + \tau d_i + 2\mu d d_i,$$

thus it can be seen

$$\frac{\phi_i}{\phi} = -\tau d_i - 2\mu d d_i,$$

$$C^{kl}\rho_{ki}\rho_l = -\frac{\phi}{2}(\tau + 2\mu d)d_i - \frac{C^{kl}{}_{,i}}{2}\rho_k\rho_l.$$

When $i = 1$, it follows

$$C^{11}\rho_{11} + \sum_{\delta=2}^{n} C^{\delta 1}\rho_{\delta 1} = -\frac{1}{2}C^{11}{}_{,1}\rho_1 - \frac{\phi}{2\rho_1}(\tau + 2\mu d)d_1.$$

When $\delta > 1$, we obtain

$$C^{11}\rho_{1\delta} + C^{1\delta}\rho_{\delta\delta} = -\frac{1}{2}C^{11}{}_{,\delta}\rho_1 - \frac{\phi}{2\rho_1}(\tau + 2\mu d)d_\delta.$$

Thus,

$$\rho_{1\delta} = -\frac{C^{1\delta}}{C^{11}}\rho_{\delta\delta} - \frac{C^{11}{}_{,\delta}}{2C^{11}}\rho_1 - \frac{(\tau + 2\mu d)d_\delta}{2}\rho_1 = -\frac{C^{1\delta}}{C^{11}}\rho_{\delta\delta} + O(|\nabla\rho|),$$

and

$$\begin{aligned}\rho_{11} =&(\frac{C^{1\delta}}{C^{11}})^2\rho_{\delta\delta} + \frac{C^{\delta 1}C^{11}{}_{,\delta}}{2(C^{11})^2}\rho_1 + \frac{C^{\delta 1}\rho_1(\tau + 2\mu d)d_\delta}{2C^{11}} - \frac{C^{11}{}_{,1}}{2C^{11}}\rho_1 - \frac{\rho_1(\tau + 2\mu d)d_1}{2}\\ =&(\frac{C^{1\delta}}{C^{11}})^2\rho_{\delta\delta} + O(|\nabla\rho|).\end{aligned}$$

At the same time, at this point we have

$$0 \le \Phi_t = \frac{\phi_t}{\phi} = \frac{2C^{kl}\rho_k\rho_{lt}}{\phi},$$

and

$$0 \ge \Phi_{ij} = \frac{\phi_{ij}}{\phi} - (\tau + 2\mu d)^2 d_i d_j + (\tau + 2\mu d)d_{ij} + 2\mu d_i d_j.$$

Then,

$$0 \ge F^{ij}\Phi_{ij} - \Phi_t = \frac{F^{ij}\phi_{ij} - \phi_t}{\phi} + (\tau + 2\mu d)F^{ij}d_{ij} + [2\mu - (\tau + 2\mu d)^2]F^{ij}d_i d_j.$$

First, we come to calculate $F^{kl}\phi_{kl} - \phi_t$,

$$\begin{aligned}F^{kl}\phi_{kl} - \phi_t =&2C^{ij}F^{kl}\rho_{ikl}\rho_j - 2C^{ij}\rho_j\rho_{it} + 2C^{ij}F^{kl}\rho_{ik}\rho_{jl} + 4F^{kl}C^{ij}{}_{,k}\rho_{il}\rho_j + F^{kl}C^{ij}{}_{,kl}\rho_i\rho_j\\ =&I + II + III + IV,\end{aligned}$$

where

$$\begin{aligned}I =&2C^{ij}F^{kl}\rho_{ikl}\rho_j - 2C^{ij}\rho_j\rho_{it} = 2C^{ij}[F^{kl}\rho_{ikl} - \rho_{it}]\rho_j\\ =&2C^{ij}[-F^{kl}(\frac{\varphi' d}{\cos\theta})_{ikl}]\rho_j = O(|\nabla w|),\\ IV =&F^{kl}C^{ij}{}_{,kl}\rho_i\rho_j = O(|\nabla w|^2),\\ III =&4F^{kl}C^{ij}{}_{,k}\rho_{il}\rho_j = 4F^{kl}C^{i1}{}_{,k}\rho_{il}\rho_1 = 4\rho_1F^{k1}C^{11}{}_{,k}\rho_{11} + 4\rho_1F^{k1}C^{\delta 1}{}_{,k}\rho_{1\delta}\\ &+ 4\rho_1F^{k\delta}C^{11}{}_{,k}\rho_{1\delta} + 4\rho_1F^{k\delta}C^{\delta 1}{}_{,k}\rho_{\delta\delta} = III_1 + III_2 + III_3 + III_4,\end{aligned}$$

and

$$\begin{aligned}III_1 =&4\rho_1F^{k1}C^{11}{}_{,k}\rho_{11} = 4\rho_1F^{k1}C^{11}{}_{,k}[(\frac{C^{1\delta}}{C^{11}})^2\rho_{\delta\delta} + O(\rho_1)]\\ =& O(|\nabla w|)\rho_{\delta\delta} + O(|\nabla w|^2),\\ III_2+III_3 =& 4\rho_1F^{k1}C^{\delta 1}{}_{,k}\rho_{1\delta} + 4\rho_1F^{k\delta}C^{11}{}_{,k}\rho_{1\delta} = 4\rho_1(F^{k1}C^{\delta 1}{}_{,k} + F^{k\delta}C^{11}{}_{,k})\rho_{1\delta}\\ =& 4\rho_1(F^{k1}C^{\delta 1}{}_{,k} + F^{k\delta}C^{11}{}_{,k})(-\frac{C^{1\delta}}{C^{11}}\rho_{\delta\delta} + O(|\nabla\rho|))\\ =& O(|\nabla w|)\rho_{\delta\delta} + O(|\nabla w|^2),\end{aligned}$$

thus,

$$III = O(|\nabla w|)\rho_{\delta\delta} + O(|\nabla w|^2).$$

For the second term

$$\begin{aligned}
II &= 2F^{kl}C^{ij}\rho_{ik}\rho_{jl} = 2F^{kl}C^{1i}\rho_{ik}\rho_{1l} + 2F^{kl}C^{i\delta}\rho_{ik}\rho_{\delta l} = II_1 + II_2,\\
II_1 &= 2F^{kl}C^{1i}\rho_{ik}\rho_{1l} = F^{kl}(-C^{i1}{}_{,k}\rho_i - \frac{\phi}{\rho_1}(\tau + 2\mu d)d_k)\rho_{1l}\\
&= F^{k1}(-C^{11}{}_{,k}\rho_1 - C^{11}\rho_1(\tau + 2\mu d)d_k)\rho_{11} + F^{k\delta}(-C^{11}{}_{,k}\rho_1 - C^{11}\rho_1(\tau + 2\mu d)d_k)\rho_{1\delta}\\
&= II_{11} + II_{12},
\end{aligned}$$

and

$$\begin{aligned}
II_{11} &= -F^{k1}(C^{11}{}_{,k}\rho_1 + C^{11}\rho_1(\tau + 2\mu d)d_k)\rho_{11}\\
&= -F^{k1}(C^{11}{}_{,k}\rho_1 + C^{11}\rho_1(\tau + 2\mu d)d_k)((\frac{C^{1\delta}}{C^{11}})^2\rho_{\delta\delta} + O(\rho_1)) = O(|\nabla w|)\rho_{\delta\delta} + O(|\nabla w|^2),\\
II_{12} &= F^{k\delta}(-C^{11}{}_{,k}\rho_1 - C^{11}\rho_1(\tau + 2\mu d)d_k)\rho_{1\delta}\\
&= F^{k\delta}(-C^{11}{}_{,k}\rho_1 - C^{11}\rho_1(\tau + 2\mu d)d_k)(-\frac{C^{1\delta}}{C^{11}}\rho_{\delta\delta} + O(|\nabla\rho|)) = O(|\nabla w|)\rho_{\delta\delta} + O(|\nabla w|^2),
\end{aligned}$$

therefore,

$$II_1 = O(|\nabla w|)\rho_{\delta\delta} + O(|\nabla w|^2).$$

For II_2, we have

$$\begin{aligned}
II_2 =&2F^{kl}C^{i\delta}\rho_{ik}\rho_{\delta l} = 2F^{kl}C^{1\delta}\rho_{1k}\rho_{\delta l} + 2F^{kl}C^{\alpha\delta}\rho_{\alpha k}\rho_{\delta l}\\
=&2F^{1l}C^{1\delta}\rho_{11}\rho_{\delta l} + 2F^{\alpha l}C^{1\delta}\rho_{1\alpha}\rho_{\delta l} + 2F^{1l}C^{\alpha\delta}\rho_{\alpha 1}\rho_{\delta l} + 2F^{\alpha l}C^{\alpha\delta}\rho_{\alpha\alpha}\rho_{\delta l}\\
=&II_{21} + II_{22} + II_{23} + II_{24},
\end{aligned}$$

where

$$\begin{aligned}
II_{21} =&2F^{1l}C^{1\delta}\rho_{11}\rho_{\delta l} = 2F^{11}C^{1\delta}\rho_{11}\rho_{1\delta} + 2F^{1\delta}C^{1\delta}\rho_{11}\rho_{\delta\delta}\\
=&2F^{11}[-\frac{(C^{1\delta})^2}{C^{11}}\rho_{\delta\delta} + O(|\nabla\rho|)] \times [(\frac{C^{1\delta}}{C^{11}})^2\rho_{\delta\delta} + O(|\nabla\rho|)] + 2F^{1\delta}C^{1\delta}\rho_{\delta\delta}[(\frac{C^{1\alpha}}{C^{11}})^2\rho_{\alpha\alpha} + O(|\nabla\rho|)]\\
=&-\frac{2}{(C^{11})^3}F^{11}\sum_{\alpha,\beta=2}^{n}(C^{1\alpha})^2(C^{1\beta})^2\rho_{\alpha\alpha}\rho_{\beta\beta} + \frac{2}{(C^{11})^2}\sum_{\alpha,\beta=2}^{n}(C^{1\alpha})^2F^{1\beta}C^{1\beta}\rho_{\alpha\alpha}\rho_{\beta\beta}\\
&+\sum_{\delta=2}^{n}O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2),\\
II_{22} =&2F^{\alpha l}C^{1\delta}\rho_{1\alpha}\rho_{\delta l} = 2F^{\alpha 1}C^{1\delta}\rho_{1\alpha}\rho_{1\delta} + 2F^{\alpha\delta}C^{1\delta}\rho_{1\alpha}\rho_{\delta\delta}\\
=&2F^{\alpha 1}C^{1\delta}[-\frac{C^{1\alpha}}{C^{11}}\rho_{\alpha\alpha} + O(|\nabla\rho|)][-\frac{C^{1\delta}}{C^{11}}\rho_{\delta\delta} + O(|\nabla\rho|)] + 2F^{\alpha\delta}C^{1\delta}\rho_{\delta\delta}[-\frac{C^{1\alpha}}{C^{11}}\rho_{\alpha\alpha} + O(|\nabla\rho|)]\\
=&\frac{2}{(C^{11})^2}\sum_{\alpha,\beta=2}^{n}F^{1\alpha}C^{1\alpha}(C^{1\beta})^2\rho_{\alpha\alpha}\rho_{\beta\beta} - \frac{2}{C^{11}}\sum_{\alpha,\beta=2}^{n}F^{\alpha\beta}C^{1\alpha}C^{1\beta}\rho_{\alpha\alpha}\rho_{\beta\beta}\\
&+\sum_{\delta=2}^{n}O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2),
\end{aligned}$$

$$\begin{aligned}
II_{23} =&2F^{1l}C^{\alpha\delta}\rho_{\alpha 1}\rho_{\delta l} = 2F^{11}C^{\alpha\delta}\rho_{1\alpha}\rho_{1\delta} + 2F^{1\delta}C^{\alpha\delta}\rho_{\alpha 1}\rho_{\delta\delta}\\
=&2F^{11}C^{\alpha\delta}[-\frac{C^{1\alpha}}{C^{11}}\rho_{\alpha\alpha} + O(|\nabla\rho|)][-\frac{C^{1\delta}}{C^{11}}\rho_{\delta\delta} + O(|\nabla\rho|)] + 2F^{1\delta}C^{\alpha\delta}\rho_{\delta\delta}(-\frac{C^{1\alpha}}{C^{11}}\rho_{\alpha\alpha} + O(|\nabla\rho|))\\
=&\frac{2F^{11}}{(C^{11})^2}\sum_{\alpha,\beta=2}^{n} C^{\alpha\beta}C^{1\alpha}C^{1\beta}\rho_{\alpha\alpha}\rho_{\beta\beta} - \frac{2}{C^{11}}\sum_{\alpha,\beta=2}^{n} F^{1\beta}C^{\alpha\beta}C^{1\alpha}\rho_{\alpha\alpha}\rho_{\beta\beta}\\
&+\sum_{\delta=2}^{n} O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2),\\
II_{24} =&2F^{\alpha l}C^{\alpha\delta}\rho_{\alpha\alpha}\rho_{\delta l} = 2F^{\alpha 1}C^{\alpha\delta}\rho_{\alpha\alpha}\rho_{\delta 1} + 2F^{\alpha\delta}C^{\alpha\delta}\rho_{\alpha\alpha}\rho_{\delta\delta}\\
=&2F^{\alpha 1}C^{\alpha\delta}\rho_{\alpha\alpha}[-\frac{C^{1\delta}}{C^{11}}\rho_{\delta\delta} + O(|\nabla\rho|)] + 2F^{\alpha\delta}C^{\alpha\delta}\rho_{\alpha\alpha}\rho_{\delta\delta}\\
=&-\frac{2}{C^{11}}\sum_{\alpha,\beta=2}^{n} F^{1\alpha}C^{\alpha\beta}C^{1\beta}\rho_{\alpha\alpha}\rho_{\beta\beta} + 2\sum_{\alpha,\beta=2}^{n} F^{\alpha\beta}C^{\alpha\beta}\rho_{\alpha\alpha}\rho_{\beta\beta} + \sum_{\delta=2}^{n} O(|\nabla\rho|)\rho_{\delta\delta},
\end{aligned}$$

then,

$$\begin{aligned}
II_2 =&-\frac{2}{C^{11}}\sum_{\alpha,\beta=2}^{n} C^{\alpha\beta}F^{1\alpha}C^{1\beta}\rho_{\alpha\alpha}\rho_{\beta\beta} + 2\sum_{\alpha,\beta=2}^{n} F^{\alpha\beta}C^{\alpha\beta}\rho_{\alpha\alpha}\rho_{\beta\beta}\\
&+\frac{2F^{11}}{(C^{11})^2}\sum_{\alpha,\beta=2}^{n} C^{\alpha\beta}C^{1\alpha}C^{1\beta}\rho_{\alpha\alpha}\rho_{\beta\beta} - \frac{2}{C^{11}}\sum_{\alpha,\beta=2}^{n} F^{1\beta}C^{\alpha\beta}C^{1\alpha}\rho_{\alpha\alpha}\rho_{\beta\beta}\\
&+\frac{2}{(C^{11})^2}\sum_{\alpha,\beta=2}^{n} F^{1\alpha}C^{1\alpha}(C^{1\beta})^2\rho_{\alpha\alpha}\rho_{\beta\beta} - \frac{2}{C^{11}}\sum_{\alpha,\beta=2}^{n} F^{\alpha\beta}C^{1\alpha}C^{1\beta}\rho_{\alpha\alpha}\rho_{\beta\beta}\\
&-\frac{2}{(C^{11})^3}F^{11}\sum_{\alpha,\beta=2}^{n}(C^{1\alpha})^2(C^{1\beta})^2\rho_{\alpha\alpha}\rho_{\beta\beta} + \frac{2}{(C^{11})^2}\sum_{\alpha,\beta=2}^{n} F^{1\beta}(C^{1\alpha})^2C^{1\beta}\rho_{\alpha\alpha}\rho_{\beta\beta}\\
&+\sum_{\delta=2}^{n} O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2),
\end{aligned}$$

thus, we have

$$\begin{aligned}
II_2 =&-\frac{4}{C^{11}}\sum_{\alpha,\beta=2}^{n} F^{1\alpha}C^{\alpha\beta}C^{1\beta}\rho_{\alpha\alpha}\rho_{\beta\beta} + 2\sum_{\alpha,\beta=2}^{n} F^{\alpha\beta}C^{\alpha\beta}\rho_{\alpha\alpha}\rho_{\beta\beta}\\
&+\frac{2F^{11}}{(C^{11})^2}\sum_{\alpha,\beta=2}^{n} C^{\alpha\beta}C^{1\alpha}C^{1\beta}\rho_{\alpha\alpha}\rho_{\beta\beta} - \frac{2}{(C^{11})^3}F^{11}\sum_{\alpha,\beta=2}^{n}(C^{1\alpha})^2(C^{1\beta})^2\rho_{\alpha\alpha}\rho_{\beta\beta}\\
&+\frac{4}{(C^{11})^2}\sum_{\alpha,\beta=2}^{n} F^{1\alpha}C^{1\alpha}(C^{1\beta})^2\rho_{\alpha\alpha}\rho_{\beta\beta} - \frac{2}{C^{11}}\sum_{\alpha,\beta=2}^{n} F^{\alpha\beta}C^{1\alpha}C^{1\beta}\rho_{\alpha\alpha}\rho_{\beta\beta}\\
&+\sum_{\delta=2}^{n} O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2).
\end{aligned}$$

We mainly deal with the quadratic term in II_2,

$$\begin{aligned}\Pi = &-\frac{4}{C^{11}}\sum_{\alpha,\beta=2}^{n} F^{1\alpha}C^{\alpha\beta}C^{1\beta}\rho_{\alpha\alpha}\rho_{\beta\beta} + 2\sum_{\alpha,\beta=2}^{n} F^{\alpha\beta}C^{\alpha\beta}\rho_{\alpha\alpha}\rho_{\beta\beta}\\ &+\frac{2F^{11}}{(C^{11})^2}\sum_{\alpha,\beta=2}^{n} C^{\alpha\beta}C^{1\alpha}C^{1\beta}\rho_{\alpha\alpha}\rho_{\beta\beta} - \frac{2}{(C^{11})^3}F^{11}\sum_{\alpha,\beta=2}^{n}(C^{1\alpha})^2(C^{1\beta})^2\rho_{\alpha\alpha}\rho_{\beta\beta}\\ &+\frac{4}{(C^{11})^2}\sum_{\alpha,\beta=2}^{n} F^{1\alpha}C^{1\alpha}(C^{1\beta})^2\rho_{\alpha\alpha}\rho_{\beta\beta} - \frac{2}{C^{11}}\sum_{\alpha,\beta=2}^{n} F^{\alpha\beta}C^{1\alpha}C^{1\beta}\rho_{\alpha\alpha}\rho_{\beta\beta}.\end{aligned}$$

Simplify the above formula, we get

$$\Pi = \sum_{\alpha,\beta=2}^{n}\frac{2[F^{\alpha\beta}(C^{11})^2 + F^{11}C^{1\alpha}C^{1\beta} - F^{1\alpha}C^{1\beta}C^{11} - F^{1\beta}C^{1\alpha}C^{11}][C^{11}C^{\alpha\beta} - C^{1\alpha}C^{1\beta}]}{(C^{11})^3}\rho_{\alpha\alpha}\rho_{\beta\beta},$$

where $C^{11}C^{\alpha\beta} - C^{1\alpha}C^{1\beta} = (1-d_1^2)\delta_{\alpha\beta} - d_\alpha d_\beta$.

To deal with the above quadratic form, let us make the following preparations:

Definition 3.5. *Suppose A, B are two m-order symmetric matrices, its Hadamard product is defined as $A \circ B = (a_{ij}\cdot b_{ij})_{m\times m}$, that is, the element product at the corresponding position is defined as the element at the corresponding position of the Hadamard product matrix.*

Theorem 3.6. *If A and B are two m order positive semi-definite matrices, $A \circ B$ is also a m order positive semi-definite matrix; If A and B are two m order positive definite matrices, $A \circ B$ is also a m order positive definite matrix .*

Corollary 3.7. *If $A \geq \lambda E,\ B \geq 0$, then $A \circ B \geq \lambda E \circ B$.*

With the above knowledge about matrices, let's look at the two matrices contained in Π, one is

$$A = \sum_{\alpha,\beta=2}^{n}(F^{\alpha\beta}(C^{11})^2 + F^{11}C^{1\alpha}C^{1\beta} - F^{1\alpha}C^{1\beta}C^{11} - F^{1\beta}C^{1\alpha}C^{11}),$$

and the other is

$$B = \sum_{\alpha,\beta=2}^{n}((1-d_1^2)\delta_{\alpha\beta} - d_\alpha d_\beta).$$

Because $|\nabla d|^2 = 1$, it's easy to see that matrix B is positive semi-definite.

Let's consider symmetric matrix A.

Remark that F^{ij} is positive definite and by the assumption we know that $\lambda E \leq F^{ij} \leq \Lambda E$, for any $X = (x_2, x_3, \cdots, x_n)$, we have

$$\begin{aligned}&\sum_{\alpha,\beta=2}^{n}(F^{\alpha\beta}(C^{11})^2 + F^{11}C^{1\alpha}C^{1\beta} - F^{1\alpha}C^{1\beta}C^{11} - F^{1\beta}C^{1\alpha}C^{11})x_\alpha x_\beta\\ &\quad= (C^{11})^2\sum_{\alpha,\beta=2}^{n}F^{\alpha\beta}x_\alpha x_\beta + F^{11}(\sum_{\alpha=2}^{n}C^{1\alpha}x_\alpha)^2 - 2C^{11}(\sum_{\alpha=2}^{n}F^{1\alpha}x_\alpha)(\sum_{\alpha=2}^{n}C^{1\alpha}x_\alpha)\\ &\quad\geq (C^{11})^2[\sum_{\alpha,\beta=2}^{n}F^{\alpha\beta}x_\alpha x_\beta - \frac{1}{F^{11}}(\sum_{\alpha=2}^{n}F^{1\alpha}x_\alpha)^2] = (C^{11})^2\sum_{\alpha,\beta=2}^{n}(F^{\alpha\beta} - \frac{F^{1\alpha}F^{1\beta}}{F^{11}})x_\alpha x_\beta.\end{aligned}$$

We want to show that the matrix $(F^{\alpha\beta} - \frac{F^{1\alpha}F^{1\beta}}{F^{11}})_{2\le\alpha,\beta\le n}$ is positive definite and its eigenvalues are bounded from below by λ. In fact, since $\lambda E \le F^{ij} \le \Lambda E$, we have that the matrix $(F^{ij}) - diag\{0,\lambda,\lambda,\cdots,\lambda\}$ is positive semi-definite. However, according to a series of elementary transformations we can deduce that $(F^{ij}) - diag\{0,\lambda,\lambda,\cdots,\lambda\}$ is congruent with $\begin{pmatrix} F^{11} & 0 \\ 0 & F^{\alpha\beta} - \lambda\delta_{\alpha\beta} - \frac{F^{1\alpha}F^{1\beta}}{F^{11}} \end{pmatrix}$. Therefore, $(F^{\alpha\beta} - \frac{F^{1\alpha}F^{1\beta}}{F^{11}})_{2\le\alpha,\beta\le n}$ is positive definite and its eigenvalues are bounded from below by λ.

So, $(F^{\alpha\beta}(C^{11})^2 + F^{11}C^{1\alpha}C^{1\beta} - F^{1\alpha}C^{1\beta}C^{11} - F^{1\beta}C^{1\alpha}C^{11}) \ge ((C^{11})^2\lambda\delta_{\alpha\beta})$ and then by the corollary we have that

$$\begin{aligned}\Pi \ge& \frac{2\lambda}{C^{11}}\sum_{\alpha,\beta=2}^{n}((1-d_1^2)\delta_{\alpha\beta} - d_\alpha d_\beta)\delta_{\alpha\beta}\rho_{\alpha\alpha}\rho_{\beta\beta}\\ =&2\lambda\sum_{\alpha=2}^{n}\frac{(1-d_1^2)-d_\alpha^2}{C^{11}}\rho_{\alpha\alpha}^2 \triangleq 2\lambda\sum_{\alpha=2}^{n}e_\alpha\rho_{\alpha\alpha}^2.\end{aligned}$$

According to the first equation in (3.3), we can get $a_{ij}\rho_{ij} = O(1)$, where $\lambda\delta_{ij} \le a_{ij} \le \Lambda\delta_{ij}$. Reuse $\rho_{11} = (\frac{C^{1\delta}}{C^{11}})^2\rho_{\delta\delta} + O(\rho_1)$, there is

$$\sum_{\delta=2}^{n}(a_{\delta\delta} + a_{11}(\frac{C^{1\delta}}{C^{11}})^2 - 2a_{1\delta}\frac{C^{1\delta}}{C^{11}})\rho_{\delta\delta} = O(|\nabla\rho|).$$

Write

$$\gamma_\delta = a_{\delta\delta} + a_{11}(\frac{C^{1\delta}}{C^{11}})^2 - 2a_{1\delta}\frac{C^{1\delta}}{C^{11}}.$$

Thus

$$0 < \frac{\lambda^2}{\Lambda} \le \gamma_\delta \le \Lambda(1 + (\frac{1}{\widetilde{C}})^2 + \frac{2}{\widetilde{C}}).$$

Therefore

$$\rho_{22} = O(|\nabla\rho|) - \sum_{\delta=3}^{n}\frac{\gamma_\delta}{\gamma_2}\rho_{\delta\delta}.$$

Then

$$\begin{aligned}\Pi \ge& 2\lambda\sum_{\alpha=2}^{n}e_\alpha\rho_{\alpha\alpha}^2 = 2\lambda[e_2\rho_{22}^2 + \sum_{\alpha=3}^{n}e_\alpha\rho_{\alpha\alpha}^2]\\ =&2\lambda[e_2(\sum_{\delta=3}^{n}\frac{\gamma_\delta}{\gamma_2}\rho_{\delta\delta})^2 + \sum_{\alpha=3}^{n}e_\alpha\rho_{\alpha\alpha}^2] + \sum_{\delta=3}^{n}O(|\nabla\rho|)\rho_{\delta\delta} + O(|\nabla\rho|^2).\end{aligned}$$

Consider the quadratic form in brackets in the above formula, which is about the quadratic form of $\rho_{33}, \rho_{44}, \cdots, \rho_{nn}$,

$$\Theta = e_2(\sum_{\delta=3}^{n}\frac{\gamma_\delta}{\gamma_2}\rho_{\delta\delta})^2 + \sum_{\delta=3}^{n}e_\delta\rho_{\delta\delta}^2.$$

Since the coefficients $e_2, e_3, \cdots, e_n$ satisfy

$$0 \le e_\delta \le 1,\ \delta = 2, 3\cdots, n,\ \sum_{\delta=2}^{n}e_\delta = n-2,$$

so, at most one of $e_2,\cdots,e_n$ is zero, and considering the condition about γ_δ, so this quadratic form is positive definite.

Next, we give a positive controllable lower bound for the eigenvalues of this quadratic form.

We can regard Θ as a 3n-4 variables function, and its definition domain is

$$D=\{(e_2,e_3,\cdots,e_n,\gamma_2,\cdots,\gamma_n,\rho_{33},\cdots,\rho_{nn})|0\le e_\delta\le 1,\sum_{\delta=2}^n e_\delta=n-2,$$
$$0<\frac{\lambda^2}{\Lambda}\le\gamma_\delta\le\Lambda(1+(\frac{1}{\overline{C}})^2+\frac{2}{\overline{C}}),\sum_{\delta=3}^n\rho_{\delta\delta}^2=1\}.$$

It is easy to see that D is a compact set, so, the minimum value of Θ on D is written as λ_0, then the positive mumber λ_0 is a general positive lower bound of the eigenvalue of the quadratic form , that is

$$\Theta=e_2(\sum_{\delta=3}^n\frac{\gamma_\delta}{\gamma_2}\rho_{\delta\delta})^2+\sum_{\delta=3}^n e_\delta\rho_{\delta\delta}^2\ge\lambda_0\sum_{\delta=3}^n\rho_{\delta\delta}^2.$$

Therefore, on the basis of $ax^2+bx\ge-\frac{b^2}{4a}$, if $a>0$ we can obtain

$$II\ge 2\lambda\lambda_0\sum_{\delta=3}^n\rho_{\delta\delta}^2+\sum_{\delta=3}^n O(|\nabla\rho|)\rho_{\delta\delta}+O(|\nabla\rho|^2)\ge-k_1|\nabla\rho|^2.$$

In consideration of $\phi=|\nabla\rho|^2C^{11}$, and supposing $d_{ij}\ge-k_2\delta_{ij}$ we have

$$\begin{aligned}0\ge&F^{ij}\Phi_{ij}-\Phi_t=\frac{F^{ij}\phi_{ij}-\phi_t}{\phi}+(\tau+2\mu d)F^{ij}d_{ij}+[2\mu-(\tau+2\mu d)^2]F^{ij}d_id_j\\ \ge&-\frac{k_1}{C^{11}}-(\tau+2\mu d)k_2n\Lambda-(\tau+2\mu d)^2\Lambda+2\mu\lambda.\end{aligned}$$

First, select μ to make

$$2\mu\lambda=\frac{k_1}{\overline{C}}+\Lambda(\tau+1)^2+(\tau+1)k_2n\Lambda+1.$$

Then, select $\sigma\le\sigma_1$ to make $2\mu\sigma\le 1$, hence we have a contradiction $0\ge F^{ij}\Phi-\Phi_t>0$, so then $|\nabla\rho|$ must be bounded.

Then

$$|Dw|^2(x,t)\le C(\lambda,\mu_0,\mu_1,u_0,\|\varphi\|_{C^3(\overline{\Omega})},n,\Omega),\ \forall(x,t)\in\overline{\Omega}_\sigma\times[0,T']. \tag{3.7}$$

Since the bound is independent of T', the proof of Proposition 3.4 is completed. □

Proposition 3.8. *If* $w\in C^{4,2}(\Omega\times[0,T))$ *satisfies* $\|w\|_{C^1(\Omega\times[0,T))}\le M_1$ $(M_1>0)$ *and*

$$\begin{cases}\dfrac{\partial w}{\partial t}-\dfrac{1}{A}F(A\nabla^2w)=f(t) & in\quad \Omega\times[0,T),\\ w(x,0)=w_0(x) & in\quad \Omega.\end{cases} \tag{3.8}$$

Then $\forall\ \Omega'\subset\subset\Omega$,

$$\sup_{\Omega'\times[0,T)}\left|\nabla^2w\right|\le C(\lambda,\ \mu_0,\ \mu_1,\ M_1,\ w_0,\ dist(\Omega',\ \partial\Omega),\ \|f\|_{L^\infty([0,T))}).$$

Remark. *One can refer to [17] for the proof of this proposition.*

Proposition 3.9. *If* $\Omega \subset R^n$ $(n \geq 2)$ *is a bounded domain with smooth boundary. Assuming that* $w \in C^{4,2}(\overline{\Omega} \times [0,T))$ *is a solution to* (3.3)*, there is a constant* $C = C(\Omega, n, u_0, \varphi, \lambda, \mu_0, \mu_1)$ *, such that for* $\sigma \leq \sigma_1$,

$$\sup_{\Omega_\sigma \times [0,T)} |\nabla^2 w| \leq C(1 + \sup_{\partial\Omega \times [0,T)} |w_{\beta\beta}|). \tag{3.9}$$

Proof. For $0 < T' < T$, we will give the bound of $|\nabla^2 w|$ on $\Omega_\sigma \times [0, T']$ independent of T'.

Let

$$H(x, t, \xi) = e^{\alpha d}(w_{\xi\xi} + Bw_\xi^2),$$

where $\alpha, B\,(> 0)$ to be determined later, and $\xi \in S^{n-1}$ is a fixed unit vector, we can assume that $|w_{\xi\xi}| \geq 1$, otherwise, there is nothing to do. We first set the following differential inequality.

$$\sum_{i,j=1}^n F^{ij} H_{ij} - H_t \geq 0 \quad \text{mod} \quad \nabla H \quad \text{on} \quad \Omega_\sigma \times (0, T']. \tag{3.10}$$

In fact,

$$\begin{aligned} 0 = H_i =& \alpha d_i H + e^{\alpha d}(w_{\xi\xi i} + B(w_\xi^2)_i), \\ H_t =& e^{\alpha d}(w_{\xi\xi t} + B(w_\xi^2)_t), \\ H_{ij} =& (\alpha d_{ij} - \alpha^2 d_i d_j)H + e^{\alpha d}(w_{\xi\xi ij} + B(w_\xi^2)_{ij}). \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{i,j=1}^n F^{ij} H_{ij} - H_t \\ =& \sum_{i,j=1}^n F^{ij}(\alpha d_{ij} - \alpha^2 d_i d_j)H + e^{\alpha d}(\sum_{i,j=1}^n F^{ij} w_{\xi\xi ij} - w_{\xi\xi t}) + Be^{\alpha d}(\sum_{i,j=1}^n F^{ij}(w_\xi^2)_{ij} - (w_\xi^2)_t) \\ =& I + II + III, \end{aligned}$$

where

$$\begin{aligned} |I| \leq& \mu_1(\alpha \widetilde{C}^2 + \alpha^2)e^{\alpha d}|w_{\xi\xi}| + C_0(\alpha, \mu_1, n, \Omega), \\ II \geq& 0, \\ III =& 2Be^{\alpha d} \sum_{i,j=1}^n F^{ij} w_{\xi i} w_{\xi j} + 2Be^{\alpha d} w_\xi (\sum_{i,j=1}^n F^{ij} w_{\xi ij} - w_{\xi t}) \\ \geq& 2Be^{\alpha d} \lambda \sum_{i=1}^n |w_{\xi i}|^2. \end{aligned}$$

From Cauchy inequality, we have $|w_{\xi\xi}|^2 = |\sum_{i=1}^n w_{\xi i}\xi^i|^2 \leq \sum_{i=1}^n w_{\xi i}^2$, and then according to the hypothesis $|w_{\xi\xi}| \geq 1$, we get

$$III \geq 2Be^{\alpha d} \lambda |w_{\xi\xi}|.$$

Then if we take $B = \frac{1}{2\lambda}(\mu_1(\alpha \widetilde{C}^2 + \alpha^2) + C_0)$, so (3.10) is proved.

Suppose that the maximum point of H is (x_0, t_0, ξ_0), according to the maximum principle, it must occur on $\Omega_\sigma \times \{0\} \times S^{n-1}$, $(\partial\Omega_\sigma \bigcap \Omega) \times [0, T'] \times S^{n-1}$ or $\partial\Omega \times [0, T'] \times S^{n-1}$. Let's discuss it one by one in the following situations.

Case 1. $(x_0, t_0, \xi_0) \in \Omega_\sigma \times \{0\} \times S^{n-1}$. Then

$$w_{\xi_0\xi_0}(x_0, t_0) \leq \max\{H(x_0, 0, \xi_0), 0\} \leq C(u_0, \Omega).$$

Case 2. $(x_0, t_0, \xi_0) \in (\partial\Omega_\sigma \bigcap \Omega) \times [0, T'] \times S^{n-1}$. In this case, it is transformed into the interior estimate, and Proposition 3.8 guarantees the conclusion.

Case 3. $(x_0, t_0, \xi_0) \in \partial\Omega \times [0, T'] \times S^{n-1}$. Under this condition, we have

$$0 \geq H_\beta = \alpha\beta_n(w_{\xi_0\xi_0} + Bw_{\xi_0}^2) + w_{\xi_0\xi_0\beta} + 2Bw_{\xi_0}w_{\xi_0\beta}. \tag{3.11}$$

First,we suppose that $\xi_0 \cdot \nu = 0$.

Let's write $w_{ij}\tau^i\mu^j$ with $w_{\tau\mu}$, take the tangential derivatives on both sides of $w_\beta = \varphi' = \frac{\varphi}{A}$, and we have

$$\sum_{p,q=1}^{n}\sum_{k=1}^{n} C^{pq}(w_k\beta^k)_p\xi_0^q = \sum_{p=1}^{n}\sum_{q=1}^{n} C^{pq}(\varphi')_p\xi_0^q,$$

where $C^{pq} = \delta_{pq} - \nu^p\nu^q = \delta_{pq} - d_pd_q$ in Ω_σ. Thus

$$w_{\xi_0\beta} = (\varphi')_{\xi_0} - \sum_{k=1}^{n} w_k\beta^k{}_{,q}\xi_0^q.$$

It can be seen that there is a constant $\Lambda = \Lambda(\varphi, \widetilde{C}, \|\nabla w\|_{C^0(\overline{\Omega}\times[0,T))})$ such that

$$|w_{\xi_0\beta}| \leq \Lambda. \tag{3.12}$$

Taking double tangential derivative on both sides of $w_\beta = \varphi' = \frac{\varphi}{A}$, we get

$$\sum_{i,j,k,p,q=1}^{n} C^{jq}(C^{ip}(w_k\beta^k)_p)_q\xi_0^i\xi_0^j = \sum_{i,j,p,q=1}^{n} C^{jq}(C^{ip}\varphi'_p)_q\xi_0^i\xi_0^j,$$

thus

$$\begin{aligned} w_{\xi_0\xi_0\beta} = & \sum_{i,j,p,q=1}^{n} C^{jq}C^{ip}_{,q}\varphi'_p\xi_0^i\xi_0^j + \varphi'_{\xi_0\xi_0} - \sum_{k,p,q=1}^{n} \xi_0^p\xi_0^q(w_{kp}\beta_q^k + w_{kq}\beta_p^k + w_k\beta_{pq}^k) \\ & - \sum_{i,p,q,k=1}^{n} \xi_0^qC^{ip}_{,q}\xi_0^i(w_k\beta^k)_p. \end{aligned}$$

Therefore,

$$|w_{\xi_0\xi_0\beta} + 2Bw_{\xi_0}w_{\xi_0\beta}| \leq 2\widetilde{C}|\nabla^2 w| + C(\|\varphi\|_{C^2(\overline{\Omega})}, \widetilde{C}, \|\nabla w\|_{C^0(\overline{\Omega}\times[0,T))}, B).$$

Because w_t is bounded, operator F is uniformly elliptic, by classical theory of uniform elliptic differential equations, $\forall\ (x, t) \in \Omega_\sigma \times [0, T']$, we have

$$|\nabla^2 w| \leq C_0(\lambda, \mu_1, u_0)(1 + \sup_{\gamma\in S^{n-1}} w_{\gamma\gamma}^+).$$

Without loss of generality, we assume that $\sup_{\gamma\in S^{n-1}} w^+_{\gamma\gamma} = w_{\zeta\zeta} > 0$.

Choose a proper coordinate at x_0 : $\overrightarrow{e_1}, \cdots, \overrightarrow{e_{n-1}}, \overrightarrow{\beta}$, such that $\zeta = \sum_{i=1}^{n-1} a_i\overrightarrow{e_i} + a_n\overrightarrow{\beta}$, let $\zeta^\top = \sum_{i=1}^{n-1} a_i\overrightarrow{e_i}$, then $\zeta = \zeta^\top + a_n\overrightarrow{\beta}$, we then have by (3.12)

$$\begin{aligned}|\nabla^2 w| &\le C_0(1 + w_{\zeta\zeta})\\ &\le C_0(1 + w_{\zeta^\top\zeta^\top} + 2a_n w_{\zeta^\top\beta} + a_n^2 w_{\beta\beta})\\ &\le C_1(1 + 2\Lambda + H(x_0, t_0, \xi_0) + |w_{\beta\beta}|)\\ &\le C_1(1 + 2\Lambda + w_{\xi_0\xi_0} + B\|\nabla w\|^2_{C^0(\overline{\Omega}\times[0,T))} + |w_{\beta\beta}|).\end{aligned}$$

Then,

$$|w_{\xi_0\xi_0\beta} + 2Bw_{\xi_0}w_{\xi_0\beta}| \le 2C_1\widetilde{C}(1 + w_{\xi_0\xi_0} + |w_{\beta\beta}|) + C(\|\varphi\|_{C^2(\overline{\Omega})}, \widetilde{C}, B, \|\nabla w\|_{C^0(\overline{\Omega}\times[0,T))}).$$

Substitute the above inequality into (3.11), take $\alpha = 2C_1\widetilde{C} + 1$, and then we deduce

$$w_{\xi_0\xi_0}(x_0, t_0) \le C(1 + \sup_{\partial\Omega\times[0,T)} |w_{\beta\beta}|),$$

where $C = C(\lambda, \mu_1, u_0, \|\varphi\|_{C^2(\overline{\Omega})}, \widetilde{C}, B, \|\nabla w\|_{C^0(\overline{\Omega}\times[0,T))})$.

If $\xi_0 \cdot \nu \ne 0$, similar to the above discussion process, let $\xi_0 = \sum_{i=1}^{n-1} b_i\overrightarrow{e_i} + b_n\overrightarrow{\beta}$, and $\xi_0^\top = \sum_{i=1}^{n-1} b_i\overrightarrow{e_i}$, then $\xi_0 = \xi_0^\top + b_n\overrightarrow{\beta}$,
then we obtain

$$\begin{aligned}w_{\xi_0\xi_0} &= w_{\xi_0^\top\xi_0^\top} + 2b_n w_{\xi_0^\top\beta} + b_n^2 w_{\beta\beta}\\ &\le C(1 + |w_{\beta\beta}|).\end{aligned}$$

Combined with all the above, we come to the conclusion that

$$\sup_{\Omega_\sigma\times[0,T']} |\nabla^2 w| \le C(1 + \sup_{\partial\Omega\times[0,T)} |w_{\beta\beta}|),$$

where $C = C(\lambda, \mu_1, \Omega, n, \varphi, u_0, \|\nabla w\|_{C^0(\overline{\Omega}\times[0,T))})$ which is independent of T', so we finish the proof of Proposition 3.9. □

Proposition 3.10. *If $\Omega \subset R^n$ $(n \ge 2)$ is a bounded domain with smooth boundary. Assume that $w \in C^{4,2}(\overline{\Omega}\times[0,T))$ is a solution to (3.3), Then there is a constant $C = C(\Omega, n, u_0, \varphi, \lambda, \mu_0, \mu_1)$, such that*

$$\sup_{\partial\Omega\times[0,T)} |w_{\beta\beta}| \le C. \tag{3.13}$$

Proof. For any $0 < T' < T$, we use the barrier function to give $|w_{\beta\beta}|$ a bound independent of T' on $\partial\Omega \times [0, T']$, and then take a limit.

Let

$$M_2 = \sup_{\Omega\times[0,T)} |\nabla^2 w|.$$

As before, we think about a function $G(x,t)=\sum_{i=1}^{n} w_i\beta^i-\frac{\varphi}{A}$ defined on $\Omega_\sigma\times[0,T']$, and we have

$$|G|<C(\|\nabla w\|_{C^0(\Omega\times[0,T))},u_0,\|\varphi\|_{C^0(\Omega)}):=\widehat{C}.$$

Suppose the barrier function is

$$H(x,t)=4\widehat{C}K(d-Kd^2)\pm G,$$

where

$$K\geq\frac{1}{2\sigma_1},\tag{3.14}$$

is a positive number to be determined. Clearly,

$$H=0\ \text{ on } \partial\Omega\times[0,T'].\tag{3.15}$$

Notice that if $K\sigma=\frac{1}{2}$, we get

$$H>0\ \text{ on }\ (\partial\Omega_\sigma\cap\Omega)\times[0,T'].\tag{3.16}$$

On $\Omega_\sigma\times\{0\}$, note that $G(x,0)$ is a function related only to $u_0(x)$ and we can suppose that

$$K\geq\widetilde{C}+\sqrt{\frac{\max_{\overline{\Omega}}|\Delta G(x,0)|}{4\widehat{C}}},\tag{3.17}$$

where $\widetilde{C}$ is from (1.6).

Now Let's compute $\Delta H(x,0)$ on $\Omega_\sigma\times\{0\}$. Combined with $K\sigma=\frac{1}{2}$, we get

$$\begin{aligned}\Delta H(x,0)=&4\widehat{C}K(\Delta d-2Kd\Delta d-2K)\pm\Delta G\\ \leq&4\widehat{C}K(\widetilde{C}-2K)\pm\Delta G\\ \leq&-4\widehat{C}K^2\pm\Delta G\leq 0.\end{aligned}$$

From the fact $H(x,0)\geq 0$ on $\partial\Omega_\sigma$ derived from (3.15) and (3.16), we derive that

$$H>0\quad\text{on}\quad\Omega_\sigma\times\{0\}.\tag{3.18}$$

Now we start to think about the function $H(x,t)$ on $\Omega_\sigma\times(0,T']$.

Set $F^{ij}=\frac{\partial}{\partial r_{ij}}|_{r=A\nabla^2 w}F(r)$, thus on $\Omega_\sigma\times(0,T']$,

$$\begin{aligned}\sum_{i,j=1}^{n}F^{ij}G_{ij}-G_t=&\sum_{i,j,k=1}^{n}F^{ij}w_{ijk}\beta^k-\sum_{i=1}^{n}w_{kt}\beta^k+\sum_{i,j,k=1}^{n}F^{ij}(w_{ik}\beta^k_j+w_{jk}\beta^k_i)-\sum_{i,j=1}^{n}\frac{1}{A}F^{ij}\varphi_{ij}\\ =&\sum_{i,j,k=1}^{n}F^{ij}(w_{ik}\beta^k_j+w_{jk}\beta^k_i)-\sum_{i,j=1}^{n}\frac{1}{A}F^{ij}\varphi_{ij},\end{aligned}$$

consequently,

$$|\sum_{i,j=1}^{n}F^{ij}G_{ij}-G_t|\leq C_2(\mu_1,\Omega,n,u_0,\|\varphi\|_{C^2(\Omega)})[1+M_2].$$

Hence, on $\Omega_\sigma \times (0, T']$

$$\begin{aligned}\sum_{i,j=1}^{n} F^{ij} H_{ij} - H_t =& 4\widehat{C}K \sum_{i,j=1}^{n} F^{ij}(d_{ij} - 2Kd_id_j - 2Kdd_{ij}) \pm (\sum_{i,j=1}^{n} F^{ij} G_{ij} - G_t)\\ \leq& 4\widehat{C}K(\mu_1\widetilde{C} - 2K\lambda) + C_2(1 + M_2)\\ \leq& -4\widehat{C}\lambda K^2 + C_2(1 + M_2) \leq 0,\end{aligned}$$

if we take

$$K \geq \frac{\mu_1\widetilde{C}}{\lambda} + \sqrt{\frac{C_2(1 + M_2)}{4\lambda\widehat{C}}}. \tag{3.19}$$

Combined with (3.14), (3.17) and (3.19), let

$$K = \frac{1}{2\sigma_1} + \frac{\mu_1\widetilde{C}}{\lambda} + \sqrt{\frac{C_2(1 + M_2)}{4\lambda\widehat{C}}} + \widetilde{C} + \sqrt{\frac{\max_{\overline{\Omega}} |\Delta G(x, 0)|}{4\widehat{C}}}, \tag{3.20}$$

and

$$\sigma = \frac{1}{2K}, \tag{3.21}$$

then we get

$$H_\beta \geq 0 \quad on \quad \partial\Omega \times [0, T'].$$

On the other side, we obtain

$$\begin{aligned}H_\beta =& 4\widehat{C}K\beta_n \pm G_\beta\\ =& 4\widehat{C}K\beta_n \pm (w_{kl}\beta^k\beta^l + w_k\beta^k_l\beta^l - \frac{1}{A}\varphi_l\beta^l).\end{aligned}$$

Therefore, from Proposition 3.9, $\forall\ (x, t) \in \Omega_\sigma \times [0, T']$, we gain

$$|w_{\beta\beta}| \leq C\sqrt{1 + M_2} \leq C\sqrt{1 + |w_{\beta\beta}|},$$

therefore,

$$|w_{\beta\beta}| \leq C.$$

then the proof of Proposition 3.10 is completed. □

Proof of Lemma 3.2. We continue to prove Lemma 3.2. It is almost similar to the proof process in the last part of Lemma 2.2, From conditions $(F_1), (F_2)$ and (F_4), we can deduce the following uniformly parabolic differential equation

$$\begin{cases}\dfrac{\partial g}{\partial s} - F_\infty(\nabla^2 g) = 0 & \text{in} \quad \Omega \times [0, 1],\\ g(x, 0) = g_0(x) & \text{on} \quad \overline{\Omega} \times \{0\},\\ \dfrac{\partial g}{\partial \beta} = 0 & \text{on} \quad \partial\Omega \times [0, 1),\end{cases} \tag{3.22}$$

where $g_0(x)$ is a continuous function defined on $\overline{\Omega}$ and $|g_0(x)| \leq 1$.

It can be inferred from $F_\infty(0) = 0$ that (3.22) can also be expressed as

$$\begin{cases} \dfrac{\partial g}{\partial s} - \displaystyle\sum_{i,j=1}^{n} \int_0^1 F_\infty^{ij}(t\nabla^2 g)dt \cdot g_{ij} = 0 & \text{in } \Omega \times [0,1], \\ g(x,0) = g_0(x) & \text{on } \overline{\Omega} \times \{0\}, \\ \dfrac{\partial g}{\partial \beta} = 0 & \text{on } \partial\Omega \times [0,1). \end{cases} \tag{3.23}$$

However, similar to the proof of Lemma 2.2, for $s \in [0,1]$, we have $g(x_0, s) = 0$ and for some $\overline{x} \in \overline{\Omega}$, $|g(\overline{x}, 1)| = 1$. This also runs counter to the maximum principle and Hopf Lemma of parabolic differential equations. Therefore, we receive (3.2) and finish the proof of Lemma 3.2. □

Theorem 3.11. *For any $T > 0$, if u is a smooth solution to* (3.1)*, thus we get the estimate,*

$$\|u_t(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla u(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla^2 u(\cdot,t)\|_{C(\overline{\Omega})} \leq C, \tag{3.24}$$

where C is a constant independent of t and T.

Proof. The equation for v is

$$\begin{cases} \dfrac{\partial v}{\partial t} - F(\Delta^2 v) = -u_t(x_0,t) & \text{in } \Omega \times (0,\infty), \\ v(x,0) = u_0(x) - u_0(x_0) & \text{on } \Omega \times \{0\}, \\ \dfrac{\partial v}{\partial \beta} = \varphi & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{3.25}$$

From Lemma 3.2 we gain $|v| \leq A_0$, A process similar to Propositions 3.3 and 3.4 deduces

$$\|\nabla v(\cdot,t)\|_{C(\overline{\Omega})} \leq C.$$

Schauder theory derives

$$\|\nabla^2 v(\cdot,t)\|_{C(\overline{\Omega})} \leq C.$$

Since $v(x,t) = u(x,t) - u(x_0,t)$, combining with Lemma 3.1, we conclude that

$$\|u_t(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla u(\cdot,t)\|_{C(\overline{\Omega})} + \|\nabla^2 u(\cdot,t)\|_{C(\overline{\Omega})} \leq C.$$

In this way, we have completed the proof of Theorem 3.11. □

4. Conclusions

Based on the conclusion of the above theorem, we have completed the proof of Theorem 1.3. On this basis, according to the Theorem 1.2, we ensure the validity of Theorem 1.1, thus obtaining the convergence conclusion of the equation solution discussed in this paper.

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author would like to thank Professor Peihe Wang for his guide and encouragement.
The author is supported by Shandong Provincial Natural Science Foundation ZR2020MA018.

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