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Recognition of the symplectic simple group $PSp_4(p)$ by the order and degree prime-power graph

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Abstract: Let G be a finite group, $\mathrm{cd}(G)$ the set of all irreducible character degrees of G, and $\rho(G)$ the set of all prime divisors of integers in $\mathrm{cd}(G)$. For a prime p and a positive integer n, let n_p denote the p-part of n. The degree prime-power graph of G is a graph whose vertex set is $V(G) = \left\{p^{e_p(G)} \mid p \in \rho(G)\right\}$, where $p^{e_p(G)} = \max\left\{n_p \mid n \in \mathrm{cd}(G)\right\}$, and there is an edge between distinct numbers $x, y \in V(G)$ if xy divides some integer in $\mathrm{cd}(G)$. The authors have previously shown that some non-abelian simple groups can be uniquely determined by their orders and degree prime-power graphs. In this paper, the authors build on this work and demonstrate that the symplectic simple group $PSp_4(p)$ can be uniquely identified by its order and degree prime-power graph.

Keywords: degree prime-power graph; irreducible character; degree; maximal p-part; simple group; subnormal series
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1. Introduction

Throughout the following, all groups are assumed to be finite, and all characters are complex characters. Let G be a group, then $\mathrm{Irr}(G)$ denotes the set of all irreducible characters of G, and $\mathrm{cd}(G)$ denotes the set of all irreducible character degrees of G. For a positive integer n, we use $\pi(n)$ to denote the set of all prime divisors of n. For convenience, we write $\pi(G) = \pi(|G|)$, and $\rho(G) = \bigcup_{\chi\in\mathrm{Irr}(G)} \pi(\chi(1))$. For a prime p, the p-part of n, denoted by n_p, is the maximum power of p such that $n_p \mid n$. CFSG is the abbreviation for the Classification Theorem of Finite Simple Groups.

The set $\mathrm{cd}(G)$ is very important for studying G. Many results have been obtained about the relationship between the set $\mathrm{cd}(G)$ and the structure of G. This paper's primary focus centers on finite non-abelian simple groups. In 2000, Huppert proposed a conjecture: "If S is a non-abelian simple group such that $\mathrm{cd}(G) = \mathrm{cd}(M)$, then $G \cong M \times A$, where A is an abelian group" (see [1]). Currently, this conjecture has not been fully proven. If Huppert's conjecture proves valid, it would follow naturally that all finite non-abelian simple groups could be uniquely determined by their orders and sets of irreducible character degrees. Interestingly, provided $|G| = |M|$ and fewer numbers in $\mathrm{cd}(G)$ require consideration, one can still deduce the same conclusion more efficiently. Researchers have shown that certain simple groups can be uniquely determined by their orders and at most three distinct irreducible character degrees (see [2–6] etc.).

Other researchers have studied the structure of a finite group with the character degree graph $\Delta(G)$. The vertex set of $\Delta(G)$ is $\rho(G)$. In $\Delta(G)$, two vertices are joined by an edge if the product of them divides some number in $\mathrm{cd}(G)$. The character degree graph has been widely studied, for example, in [7]. Clearly, $\Delta(G)$ contains much less information of $\mathrm{cd}(G)$. An interesting fact is that some simple groups (not all) can be uniquely determined by their orders and degree graphs.

In [8], Khosravi et al. proved that A_5, A_6, A_7, A_8, $L_3(3)$, $L_3(4)$, $L_2(64)$, $L_2(q)$ (where q is an odd prime or a square of an odd prime, and $q \geqslant 5$), and $L_2(2^\alpha)$ (where α is a positive integer such that $2^\alpha - 1$ or $2^\alpha + 1$ is a prime) can be uniquely determined by their orders and character degree graphs. However, in [9] Heydari and Ahanjideh gave a counter example of this characterization, that is, M_{12} and $A_4 \times M_{11}$ have the same order and character degree graph. To overcome this shortcoming, Chao Qin and Guiyun Chen defined a new graph by using the information of cd(G) in [10], which is called the degree prime-power graph $\Gamma(G)$.

Definition. *For every $p \in \rho(G)$, let $p^{e_p(G)} = \max\left\{\chi(1)_p \mid \chi \in \mathrm{Irr}(G)\right\}$ and $V(G) = \left\{p^{e_p(G)} \mid p \in \rho(G)\right\}$. Define the degree prime-power graph $\Gamma(G)$ of G as follows: $V(G)$ is the vertex set, and there is an edge between distinct numbers $x, y \in V(G)$ if xy divides some integer in $\mathrm{cd}(G)$. Denote the edge between distinct numbers $x, y \in V(G)$ by $x \sim y$, and the set of all edges of $\Gamma(G)$ by $E(G)$.*

Chen, Qin, Wang and et al. have proven that the group M_{12} can be uniquely determined by its order and degree prime-power graph. They have also shown that all sporadic simple groups, $PSL_2(p)$, $PSL_2(p-1)$ (where p is a Fermat prime), $PSL_2(p^2)$, and $PSL_2(p^3)$ can be uniquely determined by their orders and degree prime-power graphs (see [10–13]). In this paper, we continue this investigation for the symplectic simple group $PSp_4(p)$. The following theorem is our main result.

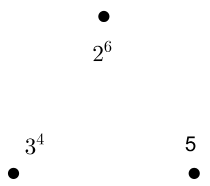
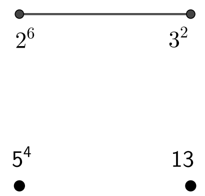
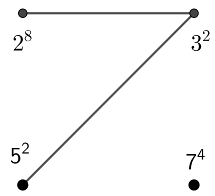
Main Theorem. *Let p be an odd prime, then $G \cong PSp_4(p)$ if and only if $|G| = |PSp_4(p)|$ and $\Gamma(G) = \Gamma(PSp_4(p))$.*

2. The degree prime-power graph of $PSp_4(p)$ when p is an odd prime

In this section, we present the degree prime-power graph of $PSp_4(p)$ when p is an odd prime, along with an important proposition.

By Atlas [14] and Magma, we obtain the order and the degree prime-power graph of $PSp_4(p)$ when $p = 3, 5, 7$ (see Table 1).

Table 1. The degree prime-power graph of $PSp_4(p)$ when $p = 3, 5, 7$.

Groups	$PSp_4(3)$	$PSp_4(5)$	$PSp_4(7)$
Order	$2^6 \cdot 3^4 \cdot 5$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	$2^8 \cdot 3^2 \cdot 5^4 \cdot 7^4$
$\Gamma(\cdot)$			

If $p > 7$, then the order of $PSp_4(p)$ is

$$|PSp_4(p)| = \frac{1}{2}p^4\left(p^2-1\right)\left(p^4-1\right) = p^4\,(p-1)^2\,(p+1)^2\left(\frac{p^2+1}{2}\right).$$

The character degrees of $PSp_4(p)$ were determined by Shahabi and Mohtadifar in [15], that is

$$\begin{aligned}\mathrm{cd}\left(\mathrm{PSp}_4(p)\right) = \Big\{&1, \frac{1}{2}p\left(p^2+1\right), \frac{1}{2}p(p+1)^2, p^4, \frac{1}{2}p(p-1)^2,\ p^4-1,\ (p-1)^2\left(p^2+1\right),\\ &(p-1)^2\,(p+1)^2,\ (p+1)\left(p^2+1\right),\ (p-1)\left(p^2+1\right),\ p(p+1)\left(p^2+1\right),\\ &p(p-1)\left(p^2+1\right),\ \frac{1}{2}\left(p^4-1\right),\ p\left(p^2+1\right),\ \frac{1}{2}\left(p^2+1\right),\ \frac{1}{2}p^2\left(p^2+1\right),\\ &\frac{1}{2}(p-1)^2\left(p^2+1\right),\ (p-1)^2\left(p^2+1\right),\ \frac{1}{2}(p+1)^2\left(p^2+1\right),\\ &\frac{1}{2}(p+\epsilon)\left(p^2+1\right), \frac{1}{2}p(p+\epsilon)\left(p^2+1\right)\Big\}\end{aligned}$$

where $\epsilon = (-1)^{(p-1)/2}$. We see that $PSp_4(p)$ has irreducible characters of the following degrees:

$$p^4,\quad (p-1)^2\,(p+1)^2\,,\quad (p-1)^2(p^2+1),\quad \frac{(p+1)^2}{2}(p^2+1). \tag{2.1}$$

Note that

$$(p-1, p+1) = 2, \quad \left(p-1, \frac{p^2+1}{2}\right) = \left(p+1, \frac{p^2+1}{2}\right) = 1,$$

and we have $|PSp_4(p)|_2 = \left((p-1)^2\,(p+1)^2\right)_2$, moreover, $2^{e_2(PSp_4(p))} = |PSp_4(p)|_2$ by (2.1). Similarly, for every $r \in \rho(PSp_4(p)) = \pi(PSp_4(p))$, one has

$$r^{e_r(PSp_4(p))} = |PSp_4(p)|_r.$$

That means the vertex set of $\Gamma(PSp_4(p))$ is

$$V(PSp_4(p)) = \left\{|PSp_4(p)|_r \,\middle|\, r \in \pi(PSp_4(p))\right\}.$$

Observing $\mathrm{cd}(PSp_4(p))$, the edges of $\Gamma(PSp_4(p))$ can be described as

(1) p^4 is an isolated vertex,

(2) let $r_1, r_2 \in \pi(PSp_4(p)) \setminus \{2,\ p\}$. From (2.1) there is an edge

$$r_1^{e_{r_1}(PSp_4(p))} \sim r_2^{e_{r_2}(PSp_4(p))}, \text{ i.e., } |PSp_4(p)|_{r_1} \sim |PSp_4(p)|_{r_2},$$

(3) for every $r \in \pi\left((p-1)^2(p+1)^2\right) \setminus \{2\}$, there is an edge $|PSp_4(p)|_2 \sim |PSp_4(p)|_r$,

(4) for every $r \in \pi\left(\frac{(p+1)^2}{2}\right)$, there is not an edge $|PSp_4(p)|_2 \sim |PSp_4(p)|_r$.

Apart from the above cases, the following holds.

Proposition 2.1. *If $p \geqslant 7$ and $r \in \pi(PSp_4(p)) \setminus \{p, 3\}$, then there is always an edge $|PSp_4(p)|_3 \sim |PSp_4(p)|_r$ in $\Gamma(PSp_4(p))$.*

Proof. By the above statements about $\Gamma(PSp_4(p))$, we only need to prove

$$3 \in \pi\left((p-1)^2(p+1)^2\right).$$

This is trivial, because we can obtain $3 \,\big|\, (p-1)(p+1)$ by $3 \nmid p$. □

Example. The orders of $PSp_4(11)$ and $PSp_4(13)$ respectively are

$$|PSp_4(11)| = 11^4(11-1)^2(11+1)^2\left(\frac{11^2+1}{2}\right) = 11^4 \cdot \left(2^6 \cdot 3^2 \cdot 5^2\right) \cdot 61,$$
$$|PSp_4(13)| = 13^4(13-1)^2(13+1)^2\left(\frac{13^2+1}{2}\right) = 13^4 \cdot \left(2^6 \cdot 3^2 \cdot 7^2\right) \cdot 5 \cdot 17.$$

Using the description above, we obtain the degree prime-power graphs of $PSp_4(11)$ and $PSp_4(13)$ (see Figure 1).

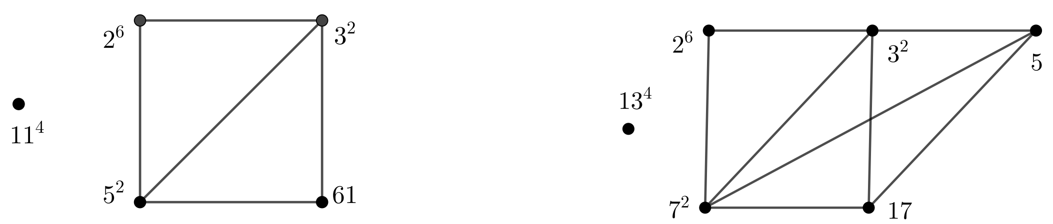


Figure 1. The degree prime-power graphs of $PSp_4(11)$ and $PSp_4(13)$.

3. The subnormal series of a non-solvable group

In this section, we study a subnormal chain of non-solvable groups through some lemmas. These lemmas will play an important role in the proof of the main theorem.

Lemma 3.1. *[10, Lemma 2.5] Let G be a non-solvable group. If T/S is a non-abelian chief factor of G, then there is a normal series $1 \leq H < K \leq G$ such that $K/H \cong T/S$ and $G/K \lesssim \mathrm{Out}(T/S)$.*

Lemma 3.2. *[10, Corollary 2.1] Let G be a non-solvable group. Then there is a subnormal series*

$$G = G_0 \geqslant G_1 > G_2 \geqslant \cdots > G_{2k-2} \geqslant G_{2k-1} > G_{2k} \geqslant 1, \quad (k \geqslant 1) \tag{3.1}$$

such that G_{2k} is solvable, G_{2i-1}/G_{2i} is a non-abelian chief factor of G_{2i-2}, and

$$G_{2i-2}/G_{2i-1} \lesssim \mathrm{Out}\,(G_{2i-1}/G_{2i})$$

for each $1 \leqslant i \leqslant k$.

For convenience, we denote the subnormal series in Lemma 3.2 by sn_{2k}, or sn_{2k}^H if the subnormal subgroups in the series are written as H_i, and denote the j-th term in sn_{2k} by $sn_{2k}(j)$, where $0 \leqslant j \leqslant 2k$. Generally, sn_{2k} is not unique for G.

Lemma 3.3. *Let G be a finite non-solvable group, and M be a non-abelian composition factor of G. If, for any sn_{2k} and any $1 \leqslant i \leqslant k$, the quotient group $sn_{2k}(2i-2)/sn_{2k}(2i-1)$ does not have a non-abelian composition factor which is isomorphic to M, then there exists a subnormal series*

$$sn_{2t}^H: \quad G = H_0 \geqslant H_1 > H_2 \geqslant \cdots > H_{2t-2} \geqslant H_{2t-1} > H_{2t} \geqslant 1, \quad (t \geqslant 1) \tag{3.2}$$

such that $H_{2t-1}/H_{2t} \cong M^m$ for some integer m.

Proof. If M is the unique (up to an isomorphism) non-abelian composition factor of G. By Lemma 3.2, the conclusion is obvious. Now suppose that the non-abelian composition factor of G is not unique (up to an isomorphism).

Assume M_1 is a non-abelian composition factor of G and M_1 is not isomorphic to M. Suppose T/K is a non-abelian chief factor of G and $T/K \cong M_1^{m_1}$. By Lemma 3.1, there is a normal series

$$G = H_0 \geqslant H_1 > H_2 \geqslant 1$$

such that $H_1/H_2 \cong T/K \cong M_1^{m_1}$ and $H_0/H_1 \lesssim \mathrm{Out}(H_1/H_2)$. By this assumption, M is not a non-abelian composition factor of H_0/H_1. Then, M is a non-abelian composition factor of H_2.

If M is the unique (up to an isomorphism) non-abelian composition factor of H_2, then the conclusion holds by Lemma 3.2. If M is not unique (up to an isomorphism) non-abelian composition factor of H_2, then we assume that M_2 is a non-abelian composition factor of H_2 and M_2 is not isomorphic to M. Similar to the above reason, there is a normal series

$$H_2 \geqslant H_3 > H_4 \geqslant 1$$

such that $H_3/H_4 \cong M_2^{m_2}$ is a non-abelian chief factor of H_2, and $H_2/H_3 \lesssim \mathrm{Out}(H_3/H_4)$. By this assumption, M is a non-abelian composition factor of H_4.

Repeating the above process, we obtain a subnormal series

$$G = H_0 \geqslant H_1 > H_2 \geqslant \cdots > H_{2r-2} \geqslant H_{2r-1} > H_{2r} \tag{3.3}$$

such that

(1) M is the unique (up to an isomorphism) non-abelian composition factor of H_{2r};

(2) $H_{2i-1}/H_{2i} \cong M_i^{m_i}$ is a non-abelian chief factor of H_{2i-2}, and $H_{2i-2}/H_{2i-1} \lesssim \mathrm{Out}\,(H_{2i-1}/H_{2i})$ where M_i is not isomorphic to M and $i = 1, \cdots, r$.

By Lemma 3.2, there is a subnormal series

$$H_{2r} \geqslant H_{2r+1} > H_{2r+2} \geqslant \cdots > H_{2t-2} \geqslant H_{2t-1} > H_{2t} \geqslant 1 \tag{3.4}$$

such that H_{2t} is solvable, $H_{2i-1}/H_{2i} \cong M^{m_i}$ is a non-abelian chief factor of H_{2i-2}, and $H_{2i-2}/H_{2i-1} \lesssim \mathrm{Out}\,(H_{2i-1}/H_{2i})$ for each $r+1 \leqslant i \leqslant t$. Therefore, the proof is complete by (3.3) and (3.4). □

Lemma 3.4. *Let G be a group, and p be an odd prime. Suppose that*

(1) $|G| \leqslant p^{10}$, and

(2) G has a non-abelian composition factor M satisfying $p \,\big|\, |M|$.

Then, there exists a subnormal series sn_{2k} such that

$$sn_{2k}(2k-1)/sn_{2k}(2k) \cong M^m,$$

where m is an integer. Specifically, if $sn_{2k}(2k) = 1$, then there exists a subnormal subgroup of G which is isomorphic to M.

Proof. By Lemma 3.3, we only need to prove that for any sn_{2t}^H and any $1 \leqslant i \leqslant t$ the quotient group $sn_{2t}^H(2i-2)/sn_{2t}^H(2i-1)$ does not have a non-abelian composition factor which is isomorphic to M.

Otherwise, suppose there is a sn_{2t}^H such that $sn_{2t}^H(2i-2)/sn_{2t}^H(2i-1)$ has a non-abelian composition factor which is isomorphic to M for some $1 \leqslant i \leqslant t$. Note that $sn_{2t}^H(2i-1)/sn_{2t}^H(2i)$ is a non-abelian chief factor of $sn_{2t}^H(2i-2)$, and it is a direct product of isomorphic non-abelian simple groups. Let

$$sn_{2t}^H(2i-1)/sn_{2t}^H(2i) = (M_1)^m,$$

where M_1 is a non-abelian simple group. Hence, we have

$$\mathrm{Out}\left(sn_{2t}^H(2i-1)/sn_{2t}^H(2i)\right) = \mathrm{Out}\,(M_1) \wr S_m.$$

For convenience, denote $sn_{2t}^H(2i-2)/sn_{2t}^H(2i-1)$ as K, then $K \lesssim \mathrm{Out}\,(M_1) \wr S_m$. Without loss of generality, assume $K \leqslant \mathrm{Out}\,(M_1) \wr S_m$. Then

$$\begin{aligned} K\Big/K \cap (\mathrm{Out(M_1)})^m &\cong K \cdot (\mathrm{Out(M_1)})^m \Big/ (\mathrm{Out(M_1)})^m \\ &\leqslant (\,\mathrm{Out}\,(M_1) \wr S_m)\Big/ (\mathrm{Out(M_1)})^m \cong S_m. \end{aligned}$$

Since M_1 is a non-abelian simple group, we know $(\mathrm{Out(M_1)})^m$ is solvable. So, $K \cap (\mathrm{Out(M_1)})^m$ is solvable. Because $K = sn_{2t}^H(2i-2)/sn_{2t}^H(2i-1)$ has a non-abelian composition factor which is isomorphic to M, we have

$$M \lesssim K\Big/K \cap (\mathrm{Out(M_1)})^m \lesssim S_m, \text{ then } |M| \,\Big|\, |S_m|.$$

Hence, $p \,\Big|\, |S_m|$ and $p \leqslant m$. Therefore

$$\begin{aligned} p^{10} \geqslant |G| \geqslant &\left|sn_{2t}^H(2i-1)/sn_{2t}^H(2i)\right| = \left|(M_1)^m\right| = |M_1|^m \\ \geqslant &|M_1|^p \geqslant 60^p. \end{aligned}$$

This is a contradiction, because $p^{10} \geqslant 60^p$ is impossible. □

4. The proof of the main theorem

For the main theorem, the necessity is obvious. It is enough to prove the sufficiency. To achieve this, we need the following series of lemmas.

Lemma 4.1. *Let G be a finite group and K be a subnormal subgroup of G. If $V(G) = \{\,|G|_p \,\big|\, p \in \pi(G)\}$, then the following statements hold.*

(1) G is non-solvable.

(2) Every minimal subnormal subgroup of G is a non-abelian simple group.

(3) 1 is the unique solvable subnormal subgroup of G. Furthermore, for any sn_{2k}, we have $sn_{2k}(2k) = 1$.

(4) If $K \neq 1$, then $V(K) = \{\,|K|_p \,\big|\, p \in \pi(K)\}$.

(5) If there are $p, r \in \pi(K)$ such that $|G|_p \sim |G|_r$ is an edge of $\Gamma(G)$, then $|K|_p \sim |K|_r$ is an edge of $\Gamma(K)$.

Proof. Statements (1) and (2) are Lemma 2.2 in [10]. Statement (3) is a corollary of (2).

For (4) and (5), without loss of generality we suppose K is a normal subgroup of G. Suppose that $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)_p = |G|_p$ where $p \in \pi(K)$. Let $\theta \in \mathrm{Irr}(K)$ be a constituent of χ_K. Using Corollary 11.29 in [16], we have $\frac{\chi(1)}{\theta(1)} \Big| |G/K|$. Furthermore,

$$\frac{\chi(1)_p}{\theta(1)_p} \Bigg| \frac{|G|_p}{|K|_p}.$$

Then, $\theta(1)_p = |K|_p$ by $\chi(1)_p = |G|_p$. Hence, $|K|_p \in V(K)$. This means statement (4) holds.

Let $p, r \in \pi(K)$ such that $|G|_p \sim |G|_r$ is an edge of $\Gamma(G)$, then there exists $\chi \in \mathrm{Irr}(G)$ satisfying $(|G|_p \cdot |G|_r)\Big|\chi(1)$. If $\phi \in \mathrm{Irr}(K)$ is a constituent of χ_K , then $\frac{\chi(1)}{\theta(1)} \Big| |G/K|$ by Corollary 11.29 in [16]. Hence, $(|K|_p \cdot |K|_r)\Big|\phi(1)$, i.e., $|K|_p \sim |K|_r$ is an edge of $\Gamma(K)$. Statement (5) holds. □

Lemma 4.2. *The following statements about the exceptional group of Lie type ${}^2B_2\left(2^{2n+1}\right)$ ($n \geqslant 1$) hold.*

(1) ${}^2B_2\left(2^{2n+1}\right)$ is the only simple group whose order is prime to 3.

(2) Let $p \geqslant 5$ be a prime. If

$$p \,\Big|\, |{}^2B_2\left(2^{2n+1}\right)| \ \text{ and } \ |{}^2B_2\left(2^{2n+1}\right)| \,\Big|\, |PSp_4(p)|,$$

then $n = 1$ and $p = 13$.

Proof. (1) Obviously, for any alternating group A_n ($n \geqslant 5$), its order is divisible by 3. According to Atlas [14], any sporadic simple group's order is divisible by 3 as well as the order of ${}^2F_4(2)'$.

If M is a simple group of Lie type over $GF(q)$ with $M \neq {}^2F_4(2)'$ and ${}^2B_2\left(2^{2n+1}\right)$ ($n \geqslant 1$), then $q(q^2-1) \,\big|\, |M|$. Notice that $3 \,\big|\, q(q^2-1)$, and we have $3 \,\big|\, |M|$.

The order of ${}^2B_2\left(2^{2n+1}\right)$ is

$$\left|{}^2B_2\left(2^{2n+1}\right)\right| = 2^{2n+1}(2^{4n+2}+1)(2^{2n+1}-1).$$

Since

$$
\begin{aligned}
2^{2n+1}-1 &= (3-1)^{2n+1}-1 \equiv -2 \qquad \text{mod } 3,\\
2^{4n+2}-1 &= (3-1)^{4n+2}+1 \equiv \ 2 \qquad \text{mod } 3,
\end{aligned}
$$

the order of ${}^2B_2\left(2^{2n+1}\right)$ is not divisible by 3. This completes the proof of (1) by CFSG.

(2) Suppose

$$q = 2^{2n+1}, \quad a = q + \sqrt{2q} + 1, \quad b = q - \sqrt{2q} + 1, \quad \epsilon_i = 1 \text{ or } -1, \ (i = 1, \ 2).$$

It is not hard to prove that $q^2 + 1 = ab$, $\left(q^2 + 1, q - 1\right) = 1$, and $(a, b) = 1$. Hence,

$$|{}^2B_2\left(2^{2n+1}\right)| = |{}^2B_2\,(q)\,| = q^2(q-1)(q^2+1) = q^2(q-1)ab.$$

By $p\,\Big|\,|{}^2B_2\left(2^{2n+1}\right)|$, we have

$$p\,\big|\,(q-1), \ \ p\,\big|\,a \ \text{ or } \ p\,\big|\,b.$$

From $|{}^2B_2\left(2^{2n+1}\right)|\,\Big|\,|PSp_4(p)|$, it follows that $q^2\,\Big|\,(p+1)^2(p-1)^2$, i.e., $q\,\big|\,(p+1)(p-1)$. Since $(p+1, p-1) = 2$, we get

$$q\,\big|\,2(p+1) \ \text{ or } \ q\,\big|\,2(p-1).$$

(i) If $p\,\big|\,(q-1)$ and $q\,\big|\,2(p+\epsilon_1)$, then there are positive integers k and m such that

$$q - 1 = pk \ \text{ and } \ 2(p+\epsilon_1) = qm.$$

It follows that

$$2(p+\epsilon_1) = (pk+1)m, \ \text{ i.e., } \ (mk-2)p = -m + 2\epsilon_1.$$

Hence $p\,\big|\,-m+2\epsilon_1$.

If $-m + 2\epsilon_1 = 0$, then $m = 2$ and $\epsilon_1 = 1$, moreover, $q = p+1$, i.e., $p = q-1$. Therefore,

$$|PSp_4(p)| = (q-1)^4(q-2)^2q^2\frac{1}{2}\left(q^2-2q+2\right).$$

By $|{}^2B_2\left(2^{2n+1}\right)|\,\Big|\,|PSp_4(p)|$, we obtain

$$q^2+1\,\Big|\,(q-1)^3(q-2)^2\frac{1}{2}\left(q^2-2q+2\right). \tag{4.1}$$

A short calculation reveals that

$$q^2+1 = (q-2)(q+2)+5, \ \text{ and } q^2-2q+2 = (q^2+1) - 2(q-1).$$

Then, $\left(q^2+1, q-2\right)\Big|5$ and $\left(q^2+1, q^2-2q+2\right) = \left(q^2+1, q-1\right) = 1$. Consequently, by (4.1) we know that $q^2+1\,\Big|\,(q-2)^2$, and q^2+1 is a power of 5. This leads to $\pi\left(|{}^2B_2\left(2^{2n+1}\right)|\right) = \{2, 5, p\}$. This is a contradiction, because $|{}^2B_2\left(2^{2n+1}\right)|$ is not a K_3-simple group, where the so-called K_3-simple group is a non-abelian simple group with only three prime factors of its order.

Now we assume $-m+2\epsilon_1 \neq 0$. By $p \mid -m+2\epsilon_1$, it follows that $m \geqslant p+2\epsilon_1 \geqslant 3$. Then, $2(p+\epsilon_1) = mq \geqslant 3(pk+1)$, i.e., $0 < (3k-2)p \leqslant 2\epsilon_1 - 3 < 0$. This is a contradiction.

(ii) If $p \mid q+\epsilon_2\sqrt{2q}+1$ and $q \mid 2(p+\epsilon_1)$, then there exist positive integers k and m such that

$$q+\epsilon_2\sqrt{2q}+1 = pk \ \text{ and } \ 2(p+\epsilon_1) = qm. \tag{4.2}$$

Then, $2(q+\epsilon_2\sqrt{2q}+1) = 2pk = (qm-2\epsilon_1)k$. Substituting $q = 2^{2n+1}$ into this equation yields

$$2^{n+1}\left(2^n+\epsilon_2-2^{n-1}mk\right) = -1-\epsilon_1 k. \tag{4.3}$$

If $1+\epsilon_1 k = 0$, then $k=1$, $\epsilon_1 = -1$ and $2^n+\epsilon_2-2^{n-1}m = 0$, i.e., $\epsilon_2 = 2^{n-1}m-2^n$. It follows that $n=1$ and $m = \epsilon_2+2 = 1$ or 3. Therefore, $q = 2^{2n+1} = 8$ and $2(p-1) = 8m = 8$ or 24, by (4.2). This means that $p = 5$ or 13.

If $1+\epsilon_1 k \neq 0$, then $2^{n+1} \mid 1+\epsilon_1 k$ by (4.3). We conclude that $k \geqslant 2^{n+1}-\epsilon_1$. Therefore,

$$2(q+\epsilon_2\sqrt{2q}+1) = (qm-2\epsilon_1)k \geqslant (qm-2\epsilon_1)(2^{n+1}-\epsilon_1).$$

By $q = 2^{2n+1}$, the above inequality can be rewritten as

$$2^{2n+1}+\epsilon_2 2^{n+1} \geqslant 2^{3n+1}m-\epsilon_1(2^{n+1}+2^{2n}m).$$

Dividing both sides of this inequality by 2^{n+1}, we get

$$2^{n-1}[(2^{n+1}-\epsilon_1)m-2] \leqslant \epsilon_1+\epsilon_2 \leqslant 2$$

which implies that $(2^{n+1}-\epsilon_1)m \leqslant 2+\frac{2}{2^{n-1}} \leqslant 4$. This means $2^{n+1}-\epsilon_1 \leqslant 4$, moreover,

$$2^{n+1} \leqslant 4+\epsilon_1 \leqslant 5, \ \text{ i.e., } \ n = 1.$$

Therefore, $q = 2^{2n+1} = 8$. By $p \mid q+\epsilon_2\sqrt{2q}+1$, we obtain that $p \mid q^2+1$. Then, $p = 5$ or 13.

Because $|{}^2B_2(8)| = 8^2(8^2+1)(8-1) = 2^6\cdot(5\cdot 13)\cdot 7$, $|PSp_4(5)| = 2^6\cdot 3^2\cdot 5^4\cdot 13$, $|PSp_4(13)| = 13^4\cdot\left(2^6\cdot 3^2\cdot 7^2\right)\cdot 5\cdot 17$, and $|{}^2B_2\left(2^{2n+1}\right)| \mid |PSp_4(p)|$, we obtain that p can only be 13. □

Lemma 4.3. *If M is a simple group of Lie type over $GF(q)$, where $q = p^f$, then $|M|_p$ is an isolated vertex in $\Gamma(M)$.*

Proof. If M is a simple group of Lie type over some $GF(q)$ with $M \neq {}^2F_4(2)'$, the result can be obtained directly by Lemma 2.4 in [17]. If $M = {}^2F_4(2)'$, then $|M|_2 = 2^{11}$ is an isolated vertex in $\Gamma(S)$ by the character table of ${}^2F_4(2)'$ in [14]. □

Lemma 4.4. *Let G be a group, and $p \geqslant 7$ be an odd prime. If*

$$|G| = |PSp_4(p)| = \frac{1}{2}p^4\left(p^2-1\right)\left(p^4-1\right), \ \textit{and} \ \ \Gamma(G) = \Gamma(PSp_4(p)),$$

then the following statements hold.

(1) Let M be a simple group of Lie type over $GF(q)$ where $q = r^f$. If $p \mid |M|$ and M is a subnormal subgroup of G, then $r = p$ or $M \cong {}^2B_2(8)$.

(2) If T is a non-abelian composition factor of G, then $p \nmid |\mathrm{Out}(T)$.

(3) For any sn_{2k} and any $1 \leqslant i \leqslant k$, the order of $\mathrm{Out}\,(sn_{2k}(2i-1)/sn_{2k}(2i))$ is not divisible by p, and $p \nmid |sn_{2k}(2i-2)/sn_{2k}(2i-1)|$.

Proof. (1) By the assumption, the vertex set of $\Gamma(G)$ is $V(G) = \{|G|_t \mid t \in \pi(G)\}$. Then, $V(M) = \{|M|_t \mid t \in \pi(M)\}$ by Lemma 4.1(4).

First, we suppose that $r \neq p$ and M is not isomorphic to ${}^2B_2\left(2^{2n+1}\right)$ ($n \geqslant 1$).

If $r = 3$, then there exists an edge $|G|_2 \sim |G|_3$ by Proposition 2.1. Because M is not isomorphic to ${}^2B_2\left(2^{2n+1}\right)$, we know $3\,\big|\,|M|$ and $|M|_3 \sim |M|_2$ is an edge of $\Gamma(M)$ by Lemma 4.1(5). This is a contradiction, because $|M|_r$ is an isolated vertex in $\Gamma(M)$ by Lemma 4.3.

If $r \neq 3$, then there exists an edge $|G|_r \sim |G|_3$ by Proposition 2.1. We can obtain a contradiction using the same method as before.

Second, assume $r \neq p$ and M is isomorphic to ${}^2B_2\left(2^{2n+1}\right)$ ($n \geqslant 1$). By Lemma 4.2(2), we see that M is isomorphic to ${}^2B_2\,(8)$ and $p = 13$. This completes the proof of (1).

(2) Assume T is a non-abelian composition factor of G, and $p\,\big|\,|\mathrm{Out}(T)$. Since p is an odd prime, T is not isomorphic to an alternating group, one of 26 sporadic simple groups or the Tits simple group ${}^2F_4(2)'$, because the outer automorphism groups of these simple groups are 1, C_2, or $C_2 \times C_2$. By CFSG, T is a simple group of Lie type over some $GF(q)$ except ${}^2F_4(2)'$, where $q = r^f$, and r is a prime. Therefore, $|\,\mathrm{Out(T)}\,| = dfg$ where d, f, and g are the orders of diagonal, field, and graph automorphisms of T, respectively. By Table 5 in [14], we know that $g\,\big|\,6$, so $p \nmid g$. Therefore, if $p\big||Out(T)|$, then $p\,\big|\,d$ or $p\,\big|\,f$.

If $p\,\big|\,d$, then T is isomorphic to $A_n(q)$ or ${}^2A_n(q)$ (i.e., $PSL_{n+1}(q)$ or $PSU_{n+1}(q)$) by Table 5 in [14]. Assume $T \cong A_n(q)$, then $d = (n+1, q-1)$ and

$$|T| = |A_n(q)| = \frac{1}{(n+1, q-1)} q^{n(n+1)/2} \prod_{i=1}^{n} (q^{i+1} - 1).$$

Hence, $n+1 \geqslant p \geqslant 7$, $p\,\big|\,q-1$, and $p\,\big|\,|T|$. By Lemma 3.4 and Lemma 4.1(3), there exists a subnormal subgroup $K \trianglelefteq\trianglelefteq G$ such that $K \cong T \cong A_n(q)$. It follows that $q = p^f$ by (1). It contradicts to $p\,\big|\,q-1$. Using the same method, one can demonstrate that $T \not\cong {}^2A_n(q)$.

Now, we assume $p\,\big|\,f$. It follows that $q = r^{kp} \geqslant 2^p \geqslant 2^7$. If T is not isomorphic to $A_1(q)$, then one can obtain

$$|T| \geqslant q^3(q+1)$$

by observing the order of a simple group of Lie type over $GF(q)$. Hence,

$$\frac{1}{2}p^4(p^2-1)(p^4-1) = |G| \geqslant |T| \geqslant q^3(q+1) \geqslant 2^{3p}(2^p+1).$$

This implies $p \leqslant 5$ by Maple. This contradicts $p \geqslant 7$.

If T is isomorphic to $A_1(q)$, then $|T| = \frac{1}{(2,q-1)} q(q^2-1)$. When $p \geqslant 7$, one can obtain the following inequality by Maple:

$$\frac{1}{2}p^4(p^2-1)(p^4-1) < \frac{1}{2} \cdot 3^p(3^{2p}-1). \tag{4.4}$$

However, since $(2, q-1) \leqslant 2$ and $q = r^{kp}$, we have

$$\frac{1}{2}p^4(p^2-1)(p^4-1) = |G| \geqslant |T| = \frac{1}{(2,q-1)}q(q^2-1) \geqslant \frac{1}{2}\cdot r^{kp}(r^{2kp}-1). \tag{4.5}$$

By inequalities (4.4) and (4.5), one can see that $r^k = 2$, i.e., $q = 2^p$. Then, $|T| = \frac{1}{(2,2^p-1)}2^p(2^{2p}-1) = 2^p(2^{2p}-1)$. Using Maple to solve the inequality $|G| \geqslant |T|$, we conclude that $p \leqslant 11$, i.e., $p = 7$ or 11. If $p = 7$, then $|T| = 2^7 \cdot 3 \cdot 43 \cdot 127$ and $|G| = 2^8 \cdot 3^2 \cdot 5^4 \cdot 7^4$. This is a contradiction to $|T| \,\big|\, |G|$. If $p = 11$, then $|T| = 2^{11}\cdot 3\cdot 23\cdot 89\cdot 683$ and $|G| = 11^4\cdot\left(2^6\cdot 3^2\cdot 5^2\right)\cdot 61$. This is a contradiction. Therefore, f is not divisible by p when $p \geqslant 7$. This completes the proof of (2).

(3) Suppose that there exists an sn_{2k} and i such that

$$p \,\Big|\, |\,\mathrm{Out}\,(sn_{2k}(2i-1)/sn_{2k}(2i))\,|.$$

Notice that $sn_{2k}(2i-1)/sn_{2k}(2i)$ is a non-abelian chief factor of $sn_{2k}(2i-2)$, and we have

$$\mathrm{Out}\,(sn_{2k}(2i-1)/sn_{2k}(2i)) = \mathrm{Out}(T) \wr S_m$$

where T is a non-abelian composition factor of G, and $sn_{2k}(2i-1)/sn_{2k}(2i)$ is a direct product of m copies of T. By the proof of Lemma 3.4, we know $p \nmid |S_m|$. Hence, $p \,\big|\, |\mathrm{Out}(T)|$. However, this is impossible by (2). Therefore, statement (3) is proved. □

Now we can present a proof of the main theorem based on the above lemmas.

Proof of Main Theorem. Obviously, it is enough to prove the sufficiency. We first prove the sufficiency when $p = 5$. If $p = 5$, then $|G| = |PSp_4(5)| = 2^6 \cdot 3^2 \cdot 5^4 \cdot 13$ and $\Gamma(G) = \Gamma(PSp_4(5))$. Hence,

$$V(G) = \{|G_r| \mid r \in \pi(G)\} = \{2^6, 3^2, 5^4, 13\}, \quad E(G) = \{2^6 \sim 3^2\}.$$

By Lemma 4.1, G is non-solvable and every minimal subnormal subgroup of G is a non-abelian simple group. Let N be a minimal normal subgroup of G. Then, N is a direct product of isomorphic non-abelian simple groups. It is enough to prove $N \cong PSp_4(5)$. By $|N| \,\big|\, |G|$, we know that N is isomorphic to one of $A_5, A_5 \times A_5, A_6, PSL_3(3), PSL_2(25), PSU_3(4), PSp_4(5)$ by [14]. If N is isomorphic to one of A_5, $A_5 \times A_5$, A_6, $PSL_3(3)$, $PSU_3(4)$, then $|N|_2 \sim |N|_3$ is not an edge of $\Gamma(N)$ by checking their character table in [14]. However, there exists an edge $|N|_2 \sim |N|_3$ in $\Gamma(N)$ by Lemma 4.1(5). This is a contradiction. If $N \cong PSL_2(25)$, then

$$G\big/(C_G(N) \times N) \lesssim \mathrm{Out}(N).$$

By $|N| = |PSL_2(25)| = 2^3 \cdot 3 \cdot 5^2 \cdot 13$ and $\mathrm{Out}(PSL_2(25)) \cong C_2 \times C_2$, we have $|C_G(N)| = 2^\alpha \cdot 3 \cdot 5^2$, where $\alpha = 1, 2$, or 3. Notice that $C_G(N) \lhd G$, and we know there exists a minimal subnormal subgroup of G, denoted by T, such that $T \leqslant C_G(N)$. Hence, T is isomorphic to A_5. By Lemma 4.1(5), there exists an edge $|T|_2 \sim |T|_3$ in $\Gamma(T)$, a contradiction. Therefore, $N \cong PSp_4(5)$, moreover, $G \cong PSp_4(5)$.

When $p \geqslant 7$, we will discuss all non-abelian composition factors of G whose order can be divisible by p, covering all possible cases. Our proof for this purpose will consist of 5 steps. Now, assume that T is a non-abelian composition factor of G satisfying $p \,\big|\, |T|$.

Step 1. Give information about the degree-power graph $\Gamma(T)$.

By Lemma 4.1(3) and Lemma 3.4, there exists a subnormal subgroup of G which is isomorphic to T. Let $r \in \pi(T)\setminus\{p,3\}$. Then there is an edge $|T|_3 \sim |T|_r$ in $\Gamma(T)$, by Lemma 4.1(5) and Proposition 2.1.

Step 2. Prove T is not isomorphic to an alternating group A_n.

On the contrary, T is isomorphic to an alternating group A_n. By $p \,\Big|\, |T|$, we know $p \leqslant n$, moreover, $\frac{p!}{2} \,\Big|\, |T|$. Notice that $|T| \,\Big|\, |G|$, and we see that

$$p! \,\Big|\, p^4(p^2-1)(p^4-1).$$

By $\left(\frac{p+1}{e}\right)^p < p!$ (e is the natural constant), the following inequality holds:

$$\left(\frac{p+1}{e}\right)^p < p^4(p^2-1)(p^4-1).$$

Using Maple, it implies $p \leqslant 13$. If $p = 13$, then $p! = 2^{10}\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13$, $p^4(p^2-1)(p^4-1) = 2^7\cdot 3^2\cdot 5\cdot 7^2\cdot 13^4\cdot 17$. This contradicts $p! \,\Big|\, p^4(p^2-1)(p^4-1)$. If $p = 11$, then $p! = 2^8\cdot 3^4\cdot 5^2\cdot 7\cdot 11$, $p^4(p^2-1)(p^4-1) = 2^7\cdot 3^2\cdot 5^2\cdot 11^4\cdot 61$, a contradiction. If $p = 7$, then $|G| = 2^8\cdot 3^2\cdot 5^4\cdot 7^4$. Therefore, T only is isomorphic to A_7 or A_8 by $|T| \,\Big|\, |G|$. Observing the character tables of A_7 and A_8 in [14], there is not an edge $|T|_2 \sim |T|_3$ of $\Gamma(T)$. However, $|T|_2 \sim |T|_3$ should be an edge of $\Gamma(T)$ by Step 1. This is a contradiction.

Step 3. Prove T is not isomorphic to any sporadic simple group.

We assume that T is isomorphic to one of following simple groups:

$$M_{11},\ M_{12},\ M_{22},\ M_{23},\ M_{24},\ J_1,\ J_2,\ J_3,\ Co_1,\ Co_2,\ Co_3,\ Fi_{22},\ Fi_{23},\ Fi'_{24},$$
$$Suz,\ HS,\ McL,\ He,\ HN,\ Th,\ B,\ M,\ Ru.$$

Observing the degree prime-power graphs of the above 23 sporadic simple groups in [11], there is not an edge $|T|_2 \sim |T|_3$ in $\Gamma(T)$. This is a contradiction, by Step 1. Next, we analyze the remaining three sporadic simple groups: $O'N$, Ly, J_4.

If T is isomorphic to $O'N$, then $\pi(T) = \{2,3,5,7,11,19,31\}$. By Step 1, for every $r \in \pi(T)\setminus\{p,3\}$, there is an edge $|T|_3 \sim |T|_r$ in $\Gamma(T)$. This indicates that there are at least 5 edges at vertex $|T|_3$. However, by the character table of $O'N$, we know there are only 3 edges at vertex $|T|_3$, i.e., $|T|_3 \sim |T|_2$, $|T|_3 \sim |T|_5$ and $|T|_3 \sim |T|_{31}$. This is a contradiction. Using the same method, one can obtain a contradiction when T is isomorphic to Ly.

If $T \cong J_4$, then $|T| = 2^{21}\cdot 3^3\cdot 5\cdot 7\cdot 11^3\cdot 23\cdot 29\cdot 31\cdot 37\cdot 43$. Since $p \,\Big|\, |T|$, we obtain $p =7, 11, 23, 29, 31, 37$ or 43. Through calculations using the Magma, we have $|PSp_4(p)|_2 \Big| 2^{13}$ when $p \in \{7,11,23,29,31,37,43\}$. This contradicts $|T| \,\Big|\, |PSp_4(p)|$.

Therefore, T is not isomorphic to any sporadic simple group.

Step 4. Prove T is isomorphic to one of $A_1(p^3)(=PSL_2(p^3))$, $A_2(p)(=PSL_3(p))$, ${}^2A_2(p)(=PSU_3(p))$, and $B_2(p)(=PSp_4(p))$.

By Steps 2 and 3, T is a simple group of Lie type over some $GF(q)$ where $q = r^f$. Hence, by Lemma 4.4(1), we know $r = p$ or $T \cong {}^2B_2(8)$.

Assume $T \ncong {}^2B_2(8)$, then $r = p \geqslant 7$. Since $|G|_p = p^4$, we can obtain that T is isomorphic to one of

$$A_1(p^i)(= PSL_2(p^i)),\ \ i = 1,2,3,4;\ A_2(p)(= PSL_3(p)),$$
$$B_2(p)(= PSp_4(p)),\ \ {}^2A_2(p)(= PSU_3(p)).$$

Notice that

$$\begin{aligned}
&|A_1(p)| = \frac{1}{2}p(p^2-1),\ \ |A_1(p^2)| = \frac{1}{2}p^2(p^2-1)(p^2+1),\\
&|A_1(p^3)| = \frac{1}{2}p^3(p^6-1) = \frac{1}{2}p^3(p-1)(p^2+p+1)(p^3+1),\\
&|A_1(p^4)| = \frac{1}{2}p^4(p^4-1)(p^4+1),\ \ |A_2(p)| = \frac{p^3(p^2-1)}{(3,p-1)}(p^3-1),\\
&|{}^2A_2(p)| = \frac{p^3(p^2-1)}{(3,p-1)}(p^3+1).
\end{aligned}$$

By [18], the degrees of $A_1(p)$ and $A_1(p^2)$ are

$$\begin{aligned}
&\mathrm{cd}(A_1(p)) = \{1, p, \frac{p+\varepsilon}{2}, p-1, p+1\},\quad (\varepsilon = (-1)^{(p-1)/2}),\\
&\mathrm{cd}(A_1(p^2)) = \{1, p^2, \frac{p^2+1}{2}, p^2-1, p^2+1\}
\end{aligned}$$

respectively.

If $T \cong A_1(p^2)$, then we can obtain that $|T|_3 \sim |T|_s$ is not an edge of $\Gamma(T)$ for every $s \in \pi(\frac{p^2+1}{2})$, because $3 \,\big|\, (p^2-1)$ and $(p^2-1, p^2+1) = 2$. By Step 1, this is a contradiction.

If $T \cong A_1(p)$ and $\frac{p-1}{2}$ is an even number, then $\epsilon = -1$ and $\mathrm{cd}(T) = \{1, p, \frac{p+1}{2}, p-1, p+1\}$ where $\frac{p+1}{2} = \frac{p-1}{2} + 1$ is an odd number. Hence, for every $s \in \pi(\frac{p+1}{2})$ and $t \in \pi(p-1)$ there is not an edge $|T|_s \sim |T|_t$ in $\Gamma(T)$. Since $s, t \in \pi\left((p-1)^2(p+1)^2\right)$ and $PSp_4(p)$ has an irreducible character of degree $(p-1)^2(p+1)^2$, then $|G|_s \sim |G|_t$ is an edge of $\Gamma(G)$. By Step 1, T is isomorphic to a subnormal subgroup of G. Hence, $|T|_s \sim |T|_t$ is an edge of $\Gamma(T)$ by Lemma 4.1(5). So, it is a contradiction. Using the same method, there is a contradiction when $\frac{p-1}{2}$ is an odd number.

If $T \cong A_1(p^4)$, then $|T| > |G|$ obviously. This is a contradiction.

Assume $T \cong {}^2B_2(8)$, then $p = 13$ by Lemma 4.2. Hence, $|T| = 2^6\cdot 5\cdot 7\cdot 13$ and $|G| = |PSp_4(13)| = 13^4\cdot\left(2^6\cdot 3^2\cdot 7^2\right)\cdot 5\cdot 17$. Observing $|T|_2 = |G|_2$, we can conclude that G has only one non-abelian composition factor. By Lemma 3.2 and 4.1(2), there is a subnormal series

$$sn_2^G : G = G_0 \geqslant G_1 > G_2 = 1$$

such that $G_1/G_2 \cong T$ and $G_0/G_1 \lesssim \mathrm{Out}(T)$. This will lead to a contradiction. Since the outer automorphism group is C_3, we obtain $|G| = |G_1/G_2|\cdot|G_0/G_1| \leqslant |T|\cdot|C_3| = 2^6\cdot 5\cdot 7\cdot 13\cdot 3$. That is impossible.

Based on the above discussion, T is isomorphic to one of $A_1(p^3)(= PSL_2(p^3))$, $A_2(p)(= PSL_3(p))$, ${}^2A_2(p)(= PSU_3(p))$, and $B_2(p)(= PSp_4(p))$.

Step 5. Prove G is isomorphic to $PSp_4(p)$.

Take a subnormal series sn_k^G

$$sn_{2k}^G : G = G_0 \geqslant G_1 > G_2 \geqslant \cdots > G_{2k-2} \geqslant G_{2k-1} > G_{2k} \geqslant 1,\quad (k \geqslant 1).$$

By Lemmas 4.1(3) and 4.4(3), we have $G_{2k} = 1$ and $p \nmid |G_{2i-2}/G_{2i-1}|$ where $i = 1, 2, \cdots, k$. It follows that

$$p^4 \left| \prod_{i=1}^{k} |G_{2i-1}/G_{2i}|. \right. \tag{4.6}$$

Notice that every G_{2i-1}/G_{2i} is a product of isomorphic non-abelian simple groups, and we assume that the direct product factor of G_{2i-1}/G_{2i} is T_i for $i = 1, 2, \cdots, k$. Then there exists a T_{j_1} such that its order is divisible by p. By Step 4, T_{j_1} is isomorphic to one of $PSL_2(p^3)$, $PSL_3(p)$, $PSU_3(p)$, and $PSp_4(p)$. We claim T_{j_1} is isomorphic to $PSp_4(p)$. Otherwise, T_{j_1} is isomorphic to $PSL_2(p^3)$, $PSL_3(p)$, or $PSU_3(p)$. Then $|T_{j_1}|_p = p^3$, moreover $G_{2j_1-1}/G_{2j_1} \cong T_{j_1}$ by $|G|_p = p^4$. Hence, there is a T_{j_2} such that its order is divisible by p, and $|T_{j_2}|_p$ can only be p^1. This contradicts the result in Step 4. Therefore, G has a non-abelian composition factor that is isomorphic to $PSp_4(p)$. This means $G \cong PSp_4(p)$. This completes the proof of Main Theorem. □

5. Conclusions

We have primarily investigated whether the symplectic simple group $PSp_4(p)$ can be uniquely determined by its order and degree prime-power graph. In the main theorem, we provide a positive answer to this question. In Lemma 3.3, a sufficient condition for adjusting the positions of specific non-abelian composition factors in the subnormal series sn_{2k} is provided. This condition plays a crucial role in the proof of the main theorem. We believe that this condition is valuable for studying the characterization of more non-abelian simple groups. Based on the results of this paper and [10–13], we conjecture that most non-abelian simple groups can be uniquely determined by their orders and degree prime-power graphs.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. B. Huppert, Some simple groups which are determined by the set of their character degrees I, *Illinois J. Math.*, **44** (2000), 828–842. http://dx.doi.org/10.1215/ijm/1255984694

2. H. Behravesh, M. Ghaffarzadeh, M. Ghasemi, S. Hekmatara, Recognition of Janko groups and some simple K_4-groups by the order and one irreducible character degree or character degree graph, *Int. J. Group Theory*, **10** (2021), 1–10. http://dx.doi.org/10.22108/IJGT.2019.113029.1502
3. S. Heydari, N. Ahanjideh, Characterization of some simple K_4-groups by some irreducible complex character degrees, *Int. J. Group Theory*, **5** (2016), 61–74. http://dx.doi.org/10.22108/IJGT.2016.8233
4. B. Khosravi, B. Khosravi, B. Khosravi, Z. Momen, A new characterization for the simple group $PSL(2, p^2)$ by order and some character degrees, *Czech. Math. J.*, **65** (2015), 271–280. http://dx.doi.org/10.1007/s10587-015-0173-6
5. H. Xu, G. Chen, Y. Yan, A new characterization of simple K_3-groups by their orders and large degrees of their irreducible characters, *Commun. Algebra*, **42** (2014), 5374–5380. http://dx.doi.org/10.1080/00927872.2013.842242
6. H. Xu, Y. Yan, G. Chen, A new characterization of Mathieu-groups by the order and one irreducible character degree, *J. Inequal. Appl.*, **2013** (2013), 209. http://dx.doi.org/10.1186/1029-242X-2013-209
7. M. Lewis, An overview of graphs associated with character degrees and conjugacy class sizes in finite groups, *Rocky Mountain J. Math.*, **38** (2008), 175–211. http://dx.doi.org/10.1216/RMJ-2008-38-1-175
8. B. Khosravi, B. Khosravi, B. Khosravi, Z. Momen, Recognition of some simple groups by character degree graph and order, *Math. Reports*, **18** (2016), 51–61.
9. S. Heydari, N. Ahanjideh, Some simple groups which are determined by their character degree graphs, *Sib. Elektron. Math. Re.*, **13** (2016), 1290–1299. http://dx.doi.org/10.17377/semi.2016.13.101
10. C. Qin, Y. Yan, K. Shum, G. Chen, Mathieu groups and its degree prime-power graphs, *Commun. Algebra*, **47** (2019), 4173–4180. http://dx.doi.org/10.1080/00927872.2019.1579342
11. C. Qin, Research on the sporadic simple groups and their degree prime-power graphs of irreducible character degrees, Ph.D thesis, Southwest University, 2019.
12. Z. Wang, C. Qin, H. Lv, Y. Yan, G. Chen, A new characterization of $L_2(p^2)$, *Open Math.*, **18** (2020), 907–915. http://dx.doi.org/10.1515/math-2020-0048
13. Z. Wang, G. Chen, A new characterization of $L_2(p^3)$, *Commun. Algebra*, **50** (2022), 4000–4008. http://dx.doi.org/10.1080/00927872.2022.2057509
14. J. Conway, R. Curtis, S. Norton, R. Parker, R. Wilson, *Atlas of finite groups*, New York: Oxford University Press, 1985.
15. M. Shahabi, H. Mohtadifar, The characters of finite projective symplectic group $PSp(4, q)$, In: *Groups St Andrews 2001 in Oxford*, Cambridge: Cambridge University Press, 2003, 496–527. http://dx.doi.org/10.1017/CBO9780511542787.018
16. I. Isaacs, *Character theory of finite groups*, New York: Academic Press, 1976.
17. H. Tong-Viet, The simple Ree groups ${}^2F_4(q^2)$ are determined by the set of their character degrees, *J. Algebra*, **339** (2011), 357–369, http://dx.doi.org/10.1016/j.jalgebra.2011.04.034

18. D. White, Character degrees of extensions of $PSL_2(q)$ and $SL_2(q)$, *J. Group Theory*, **16** (2013), 1–33. http://dx.doi.org/10.1515/jgt-2012-0026