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Research article

The discrete convolution for fractional cosine-sine series and its application in convolution equations

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Abstract: The fractional sine series (FRSS) and the fractional cosine series (FRCS) were defined. Three types of discrete convolution operations for FRCS and FRSS were introduced, along with a detailed investigation into their corresponding convolution theorems. The interrelationship between these convolution operations was also discussed. Additionally, as an application of the presented results, two forms of discrete convolution equations based on the proposed convolution theorems were examined, resulting in explicit solutions for these equations. Furthermore, numerical simulations were provided to demonstrate that our proposed solution can be easily implemented with low computational complexity.

Keywords: fractional sine series (FRSS); fractional cosine series (FRCS); discrete convolution; convolution equations
Mathematics Subject Classification: 44A35, 45E10, 47A30

1. Introduction

Convolution plays an important role in various fields, including integral evaluation, numerical analysis, solving differential and integral equations, mathematical physics [1–9] and filter design for signal processing [10–14]. The importance of different convolutions lies in their diverse properties and applications that they are able to exhibit [15–24]. Therefore, it is worthwhile and interesting to continually investigate convolution and associated theorems due to their high potential for revealing new properties and concrete applications

The discrete Fourier transforms [25] and the discrete fractional transform [26] have emerged as significant tools in digital signal processing, given that practical data is predominantly processed at discrete samples. Extensive research has been conducted on the discrete Fourier cosine convolution and the discrete Fourier sine convolution associated with the Fourier cosine and sine series, resulting in numerous valuable theoretical achievements including Titchmarsh's type theorem, discrete Young's type theorem and Watson's type theorem [27–30], as well as the application of discrete fractional calculus [31,32]. However, while the theory of discrete convolution for fractional cosine and sine series is still in its early stages, it provides a driving force toward formulating a discrete fractional cosine and sine series. Additionally, compared to the Fourier transform, which lacks free parameters, the fractional transform offers enhanced flexibility and processing capabilities. Therefore, exploring the define fractional cosine series (FRCS) and fractional sine series (FRSS), along with their corresponding convolution theorems and applications, is both theoretically intriguing and practically beneficial.

The convolution equation has found significant applications in diverse fields such as engineering mechanics, energy transfer and membrane or axis oscillation, particularly when addressing problems related to the electrostatic field and magnetic field synthesis as well as the digital signal processing (see [3, 4, 7, 15–24, 32, 33]). Obtaining solutions for these convolution equations is a meaningful issue in equation theory that has been extensively studied by many researchers.

Taking this opportunity, our primary objective is to introduce the concept of FRCS and FRSS while exploring their fundamental properties and applications in convolution equations. This paper makes a threefold contribution: First, we extend the Fourier sine series (FSS) and the Fourier cosine series (FCS) into the fractional domain by providing definitions for the FRSS and the FRCS. Second, we propose three types of convolution operations for FRSS and FRCS, derive corresponding convolution theorems and discuss the relationship between two discrete convolution operations. The classical Fourier series results can be viewed as a special case of the derived results in this paper. Third, the application of the discrete fractional cosine-sine series convolution is discussed, and solutions for two types of convolution equations are analyzed with numerical simulations demonstrating that our proposed solution is easily implementable with low computational complexity.

The structure of this paper is outlined as follows: Section 2 reviews the convolution operations for Fourier sine and cosine series, along with their associated convolution theorems. In Section 3, three distinct convolution operations for FRSS and FRCS are presented in detail together with derived convolution theorems. Section 4 investigates the relationship between two discrete convolution operations, while Section 5 discusses two types of convolution equations. Finally, conclusions are summarized in Section 6.

2. Preliminaries

In this section, we mainly review the Fourier cosine and sine series, discrete convolution and the corresponding convolution theorem relevant to this study.

The discrete Fourier cosine convolution $(x \underset{F_{cDT}}{*} y)(n)$ [29] and the discrete Fourier sine convolution $(x \underset{F_{sDT}}{*} y)(n)$ [30] of two series $x(n)$ and $y(n)$ on $\mathbb{N}$ are defined as follows:

$$(x \underset{F_{cDT}}{*} y)(n) = \sum_{m=1}^{\infty} x(m)[y(n+m) + y(|n-m|)] + x(0)y(n), \quad n \geq 0 \tag{2.1}$$

and

$$(x \underset{F_{sDT}}{*} y)(n) = \sum_{m=1}^{\infty} x(m)[y(|n-m|) - y(n+m)], \quad n \geq 0, \tag{2.2}$$

respectively. The following two convolution theorems fulfill

$$F_{cDT}\{(x \underset{F_{cDT}}{*} y)(n)\}(w) = 2F_{cDT}\{x(n)\}(w)F_{cDT}\{y(n)\}(w) \tag{2.3}$$

and

$$F_{sDT}\{(x \underset{F_{sDT}}{*} y)(n)\}(w) = 2F_{sDT}\{x(n)\}(w)F_{cDT}\{y(n)\}(w), \tag{2.4}$$

respectively, where F_{cDT} denotes the FCS

$$F_{cDT}x(n)(\omega) = \frac{x(0)}{2} + \sum_{n=1}^{\infty} x(n)\cos(n\omega), \quad \omega \in [0, \pi], \tag{2.5}$$

and F_{sDT} denotes the FSS

$$F_{sDT}x(n)(\omega) = \sum_{n=0}^{\infty} x(n)\sin(n\omega), \quad \omega \in [0, \pi]. \tag{2.6}$$

Let $\ell_p(\mathbb{N})$, $1 \le p < \infty$, denote the space of sequences $x = \{x(n)\}$ equipped with a norm

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x(n)|^p\right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty \tag{2.7}$$

and

$$\|x\|_\infty = \sup_{n\ge0} |x_n| < \infty, \tag{2.8}$$

where $\ell_1(\mathbb{N})$ is equipped with a norm

$$\|x\| = (\sum_{n=1}^{\infty} |x(n)|),$$

and $\ell_p^o(\mathbb{N})$ is its subspace with $x(0) = 0$.

3. The convolution for FRCS and FRSS

In this section, we present the definitions of FRSS and FRCS, followed by three novel types of convolution operations for FRSS and FRCS. We also explore the relationships among these defined convolution operations and derive corresponding convolution theorems for FRCS and FRSS

Definition 3.1. *Let $\alpha \in \mathbb{R}$, a sequence $\{x(n)\} \in \ell_1(\mathbb{N})$ and the FRCS and the FRSS be defined as*

$$S_c^{\alpha}x(n)(w) = 2\sum_{n=0}^{\infty} x(n)K^{\varphi}(n, w)\cos(\csc\varphi \cdot nw), \quad w \in [0, \pi] \tag{3.1}$$

and

$$S_s^{\alpha}x(n)(w) = \sum_{n=0}^{\infty} x(n)K^{\varphi}(n, w)\sin(\csc\varphi \cdot nw), \quad w \in [0, \pi], \tag{3.2}$$

respectively, where $K^{\varphi}(n, w)$ is given by

$$K^{\varphi}(n,\omega)=\begin{cases} A_{\varphi}e^{j\frac{n^2+w^2}{2}\cot\varphi}, & \varphi\neq k\pi,\\ \delta(n-w), & \varphi=2k\pi,\\ \delta(n+w), & \varphi=(2k-1)\pi,\end{cases}$$

and

$$A_{\varphi}=\sqrt{\frac{1-j\cot\varphi}{2\pi}},\ \ \varphi=\frac{\pi}{2}\alpha.$$

Remark 3.1. *When $\alpha = 2k-1$, $k \in \mathbb{Z}$, the FCS in (3.1) and the FSS in (3.2) reduce to the classical FCS in (2.5) and the FSS in (2.6), except for certain individual terms.*

Definition 3.2. *Let $x(n), y(n) \in \ell_1(\mathbb{N})$ and the convolution operation $\underset{c,\alpha}{*}$ for FRCS is defined as follows*

$$(x\underset{c,\alpha}{*}y)(n)=T_{n,\varphi}\sum_{m=0}^{\infty}\widetilde{x}(m)[\widetilde{y}(|n-m|)+\widetilde{y}(n+m)], \tag{3.3}$$

where

$$A_{\varphi}=\sqrt{\frac{1-j\cot\varphi}{2\pi}},\ \ \widetilde{x}(n)=x(n)e^{M_{n,\varphi}},\ \ \widetilde{y}(n)=y(n)e^{M_{n,\varphi}},$$
$$T_{n,\varphi}=A_{\varphi}e^{-M_{n,\varphi}},\qquad M_{n,\varphi}=j\frac{n^2}{2}\cot\varphi,\qquad \varphi=\frac{\pi}{2}\alpha,\ \alpha\in\mathbb{R}.$$

Definition 3.3. *Let $x(n), y(n) \in \ell_1(\mathbb{N})$ and the convolution operation $\underset{s,s,c,\alpha}{*}$ for FRSS is defined as follows*

$$(x\underset{s,s,c,\alpha}{*}y)(n)=T_{n,\varphi}\sum_{m=0}^{\infty}\widetilde{x}(m)[\widetilde{y}(|n-m|)-\widetilde{y}(n+m)]. \tag{3.4}$$

Definition 3.4. *Let $x(n), y(n) \in \ell_1(\mathbb{N})$ and the convolution operation $\overset{\omega}{\underset{c,s,c,\alpha}{*}}$ with the weight function*

$$\omega=\sin(\csc\varphi\cdot w)$$

for FRCS is defined as follows

$$\begin{aligned}(x\overset{\omega}{\underset{c,s,c,\alpha}{*}}y)(n)=&\frac{T_{n,\varphi}}{4}\sum_{m=0}^{\infty}\widetilde{x}(m)[\widetilde{y}(|n+m-1|)+\widetilde{y}(|n-m+1|)\\&-\widetilde{y}(n+m+1)-\widetilde{y}(|n-m-1|)].\end{aligned} \tag{3.5}$$

Remark 3.2. *When $\alpha = 2k-1$, $k \in \mathbb{Z}$, the convolution operations defined in Definitions 3.2 and 3.3 can be simplified to the convolution operations presented in (2.1) and (2.2), respectively. Additionally, the convolution operation described in Definition 3.4 can be reduced to the Fourier weighted convolution operation, except for certain individual terms.*

The following convolution theorems for FRCS and FRSS are then derived in detail. Furthermore, the relationship between $\overset{\omega}{\underset{c,s,c,\alpha}{*}}$ and $\underset{s,s,c,\alpha}{*}$ is provided.

Theorem 3.1. *Let $x(n), y(n) \in \ell_1(\mathbb{N})$, then*

$$(x \underset{c,\alpha}{*} y)(n) \in \ell_1(\mathbb{N}).$$

$S_c^\alpha x$ and $S_c^\alpha y$ denote the FRCS for $x(n)$ and $y(n)$. The convolution theorem for FRCS is obtained as follows

$$S_c^\alpha \left[(x \underset{c,\alpha}{*} y)(n)\right](w) = (S_c^\alpha x)(w)(S_c^\alpha \overline{y})(w), \quad w \in [0, \pi], \tag{3.6}$$

where

$$(S_c^\alpha \overline{y})(w) = (S_c^\alpha y)(w) e^{-j\frac{w^2}{2}\cot\varphi}.$$

Proof. We first prove the existence of the convolution operation $(x \underset{c,\alpha}{*} y)(n)$

$$\begin{aligned}
\sum_{n=0}^{\infty} |(x \underset{c,\alpha}{*} y)(n)| \leq & |T_{n,\varphi}| \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |\widetilde{x}(m)[\widetilde{y}(|n-m|) + \widetilde{y}(n+m)]| \\
\leq & 2|T_{n,\varphi}| \sum_{m=0}^{\infty} |\widetilde{x}(m)| \sum_{n=0}^{\infty} |\widetilde{y}(n)| \\
= & 2|T_{n,\varphi}| ||x||_1 \cdot ||y||_1;
\end{aligned} \tag{3.7}$$

therefore,

$$(x \underset{c,\alpha}{*} y)(n) \in \ell_1(\mathbb{N}).$$

Next, we prove the convolution Theorem 3.1 and from the Definition 3.1, we have

$$\begin{aligned}
(S_c^\alpha x)(w)(S_c^\alpha y)(w) = & 4\sum_{a=0}^{\infty} x(a) K^\varphi(a, w) \cos(\csc\varphi \cdot wa) \cdot \sum_{b=0}^{\infty} y(b) K^\varphi(b, w) \cos(\csc\varphi \cdot wb) \\
= & 4N_{w,\varphi} A_\varphi^2 \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} e^{j\frac{a^2+b^2}{2}\cot\varphi} x(a) y(b) \cdot \cos(\csc\varphi \cdot wa)\cos(\csc\varphi \cdot wb) \\
= & 2N_{w,\varphi} A_\varphi^2 \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} e^{j\frac{a^2+b^2}{2}\cot\varphi} x(a) y(b) \cdot [\cos(\csc\varphi \cdot w(a+b)) + \cos(\csc\varphi \cdot w(a-b))] \\
= & 2N_{w,\varphi} A_\varphi^2 \left[\sum_{n=0}^{\infty}\sum_{m=n}^{\infty} e^{j\frac{n^2+(m-n)^2}{2}\cot\varphi} \cdot x(n) y(m-n) \cos(\csc\varphi \cdot wm)\right. \\
& + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{j\frac{n^2+(m+n)^2}{2}\cot\varphi} \cdot x(n) y(m+n) \cos(\csc\varphi \cdot wm) \\
& \left. + \sum_{n=0}^{\infty}\sum_{m=0}^{n} e^{j\frac{n^2+(m-n)^2}{2}\cot\varphi} \cdot x(n) y(n-m) \cos(\csc\varphi \cdot wm)\right],
\end{aligned} \tag{3.8}$$

where

$$N_{w,\varphi} = e^{jw^2\cot\varphi}.$$

One finds

$$\begin{aligned}(S_c^\alpha x)(w)(S_c^\alpha y)(w) =&2N_{w,\varphi}A_\varphi^2\sum_{n=0}^{\infty}x(n)\cdot\left[\sum_{m=0}^{\infty}e^{j\frac{n^2+(m-n)^2}{2}\cot\varphi}y(|m-n|)\cos(\csc\varphi\cdot wm)\right.\\&\left.+\sum_{m=0}^{\infty}e^{j\frac{n^2+(m+n)^2}{2}\cot\varphi}y(m+n)\cos(\csc\varphi\cdot wm)\right]\\=&2N_{w,\varphi}A_\varphi^2\sum_{m=0}^{\infty}\cos(\csc\varphi\cdot wm)\cdot\left\{\sum_{n=0}^{\infty}x(n)\left[e^{j\frac{n^2+(m-n)^2}{2}\cot\varphi}y(|m-n|)\right.\right.\\&\left.\left.+e^{j\frac{n^2+(m+n)^2}{2}\cot\varphi}y(m+n)\right]\right\}.\end{aligned}\tag{3.9}$$

From the Definition 3.2, we have

$$\begin{aligned}(S_c^\alpha x)(w)(S_c^\alpha y)(w) =&2e^{j\frac{w^2}{2}\cot\varphi}\cdot\sum_{m=0}^{\infty}[(x\underset{c,\alpha}{*}y)(m)]K^{\varphi}(m,w)\cos(\csc\varphi\cdot wm)\\=&e^{j\frac{k^2}{2}\cot\varphi}S_c^\alpha[(x\underset{c,\alpha}{*}y)(n)](w).\end{aligned}\tag{3.10}$$

The proof is completed. □

Theorem 3.2. *Let $x(n), 5y(n) \in \ell_1(\mathbb{N})$, then*

$$(x\underset{s,s,c,\alpha}{*}y)(n)\in\ell_1(\mathbb{N}).$$

$S_s^\alpha x$, $S_c^\alpha y$ denote the FRSS for $x(n)$ and FRCS for $y(n)$, respectively. The convolution theorem for FRSS is satisfied

$$S_s^\alpha[(x\underset{s,s,c,\alpha}{*}y)(n)](w) = (S_s^\alpha x)(w)(S_c^\alpha\overline{y})(w),\quad w\in[0,\pi],\tag{3.11}$$

where

$$(S_c^\alpha\overline{y})(w) = (S_c^\alpha y)(w)e^{-j\frac{w^2}{2}\cot\varphi}.$$

Proof. The proof of Theorem 3.2 is similar to that of Theorem 3.1 and it is omitted. □

Remark 3.3. *When $\alpha = 2k-1$, $k\in\mathbb{Z}$, it is easy to verify that the convolution theorem for the FCS and the FSS in (2.3) and (2.4) are both special cases of Theorems 3.1 and 3.2 in (3.6) and (3.11), respectively.*

Theorem 3.3. *Let $x(n), y(n) \in \ell_1(\mathbb{N})$, then*

$$(x\overset{\omega}{\underset{c,s,c,\alpha}{*}}y)(n)\in\ell_1(\mathbb{N}).$$

$S_s^\alpha x$, $S_c^\alpha y$ denote the FRSS for $x(n)$ and the FRCS for $y(n)$, respectively, then the convolution theorem with a weight function

$$\omega = \sin(\csc\varphi\cdot w)$$

for FRCS is obtained

$$S_c^\alpha[(x\overset{\omega}{\underset{c,s,c,\alpha}{*}}y)(n)](w) = \sin(\csc\varphi\cdot w)(S_s^\alpha x)(w)(S_c^\alpha\overline{y})(w),\quad w\in[0,\pi],\tag{3.12}$$

where

$$(S_c^\alpha\overline{y})(w) = (S_c^\alpha y)(w)e^{-j\frac{w^2}{2}\cot\varphi}.$$

Proof. The existence of the convolution operation $(x \overset{\omega}{\underset{c,s,c,\alpha}{*}} y)(n)$ is similar to that of Theorem 3.1. Next, we will prove the convolution Theorem 3.3 and based on Definitions 3.1 and 3.4, we have

$$\begin{aligned}
&\sin(\csc\varphi\cdot k)(S_s^\alpha x)(w)(S_c^\alpha y)(w)\\
&=2\sin(\csc\varphi\cdot w)\sum_{a=0}^{\infty}x(a)K^{\varphi}(a,w)\sin(\csc\varphi\cdot wa)\cdot\sum_{b=0}^{\infty}y(b)K^{\varphi}(b,w)\cos(\csc\varphi\cdot wb)\\
&=\frac{N_{w,\varphi}A_\varphi^2}{2}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}e^{j\frac{a^2+b^2}{2}\cot\varphi}x(a)y(b)\cdot\{\cos[w(\csc\varphi\cdot(a+b-1))]-\cos[w(\csc\varphi\cdot(a+b+1))]\\
&\quad+\cos[w(\csc\varphi\cdot(a-b-1))]-\cos[w(\csc\varphi\cdot(a-b+1))]\}\\
&=-\frac{N_{w,\varphi}A_\varphi^2}{2}(F_1+F_2-F_3-F_4),
\end{aligned}\tag{3.13}$$

where

$$\begin{aligned}
F_1&=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}e^{j\frac{a^2+b^2}{2}\cot\varphi}x(a)y(b)\cdot\cos[w(\csc\varphi\cdot(a+b+1))],\\
F_2&=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}e^{j\frac{a^2+b^2}{2}\cot\varphi}x(a)y(b)\cdot\cos[w(\csc\varphi\cdot(a-b+1))],\\
F_3&=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}e^{j\frac{a^2+b^2}{2}\cot\varphi}x(a)y(b)\cdot\cos[w(\csc\varphi\cdot(a-b-1))],\\
F_4&=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}e^{j\frac{a^2+b^2}{2}\cot\varphi}x(a)y(b)\cdot\cos[w(\csc\varphi\cdot(a+b-1))].
\end{aligned}\tag{3.14}$$

Let $a=n$ and $a+b+1=m$ and we obtain

$$\begin{aligned}
F_1&=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}e^{j\frac{a^2+b^2}{2}\cot\varphi}x(a)y(b)\cos[w(\csc\varphi\cdot(a+b+1))]\\
&=\sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}e^{j\frac{n^2+(m-n-1)^2}{2}\cot\varphi}x(n)y(m-n-1)\cdot\cos(\csc\varphi\cdot wm).
\end{aligned}\tag{3.15}$$

Similar to (3.15), we obtain

$$\begin{aligned}
F_2&=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}e^{j\frac{a^2+b^2}{2}\cot\varphi}x(a)y(b)\cos[w(\csc\varphi\cdot(a-b+1))]\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}e^{j\frac{n^2+(m+n+1)^2}{2}\cot\varphi}x(n)y(m+n+1)\cdot\cos(\csc\varphi\cdot wm)\\
&\quad+\sum_{n=0}^{\infty}\sum_{m=0}^{n+1}e^{j\frac{n^2+(n-m+1)^2}{2}\cot\varphi}x(n)y(n-m+1)\cdot\cos(\csc\varphi\cdot wm),
\end{aligned}\tag{3.16}$$

and from (3.15) and (3.16), we can derive

$$\begin{aligned} F_1 + F_2 = &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{j\frac{n^2+(m+n+1)^2}{2}\cot\varphi} x(n)y(m+n+1)\cdot\cos(\csc\varphi\cdot wm) \\ &+ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{j\frac{n^2+(m-n-1)^2}{2}\cot\varphi}\cdot x(n)y(|m-n-1|)\cos(\csc\varphi\cdot wm). \end{aligned} \tag{3.17}$$

In a manner similar to (3.15)–(3.17), we have

$$\begin{aligned} F_3 + F_4 = &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{j\frac{n^2+(m+n-1)^2}{2}\cot\varphi} x(n)y(|m+n-1|)\cdot\cos(\csc\varphi\cdot wm) \\ &+ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{j\frac{n^2+(m-n+1)^2}{2}\cot\varphi}\cdot x(n)y(|m-n+1|)\cos(\csc\varphi\cdot wm), \end{aligned} \tag{3.18}$$

and by (3.17) and (3.18), we can achieve

$$\begin{aligned} \sin(\csc\varphi\cdot w)(S_s^{\alpha}x)(w)(S_c^{\alpha}y)(w) =& \frac{N_{w,\varphi}A_{\varphi}^2}{2}(F_3+F_4-F_1-F_2) \\ =& 2e^{j\frac{w^2}{2}\cot\varphi}\sum_{m=0}^{\infty}(x \overset{\omega}{\underset{c,s,c,\alpha}{*}} y)(m)K^{\varphi}(m,w)\cos(\csc\varphi\cdot wm) \\ =& e^{j\frac{w^2}{2}\cot\varphi} S_c^{\alpha}[(x \overset{\omega}{\underset{c,s,c,\alpha}{*}} y)(m)](w). \end{aligned} \tag{3.19}$$

The proof is completed. □

4. Convolution relation of the fractional cosine and sine series

In this subsection, the convolution relation of $\overset{\omega}{\underset{c,s,c,\alpha}{*}}$ and $\underset{s,s,c,\alpha}{*}$ will be presented as follows.

Theorem 4.1. *Let $x(n), y(n) \in \ell_1(\mathbb{N})$, and the convolution relation between $\overset{\omega}{\underset{c,s,c,\alpha}{*}}$ and $\underset{s,s,c,\alpha}{*}$ satisfies the following equation*

$$(x \overset{\omega}{\underset{c,s,c,\alpha}{*}} y)(n) = \frac{1}{4}(x \underset{s,s,c,\alpha}{*} y)(n+1) - \frac{1}{4}(x \underset{s,s,c,\alpha}{*} y)(n-1). \tag{4.1}$$

Proof. From Definition 3.4, we have

$$\begin{aligned} (x \overset{\omega}{\underset{c,s,c,\alpha}{*}} y)(n) =& \frac{A_{\varphi}e^{M_{n,\varphi}}}{4}\sum_{m=0}^{\infty}\widetilde{x}(m)[\widetilde{y}(|n-m+1|) - \widetilde{y}(n+m+1)] \\ &+ \frac{A_{\varphi}e^{M_{n,\varphi}}}{4}\sum_{m=0}^{\infty}\widetilde{x}(m)[\widetilde{y}(|n+m-1|) - \widetilde{y}(|n-m-1|)], \end{aligned} \tag{4.2}$$

and from Definition 3.3 and (4.2), we obtain

$$\frac{A_{\varphi}e^{M_{n,\varphi}}}{4}\sum_{m=0}^{\infty}\widetilde{x}(m)[\widetilde{y}(|n-m+1|) - \widetilde{y}(n+m+1)] = \frac{1}{4}(x \underset{s,s,c,\alpha}{*} y)(n+1). \tag{4.3}$$

Similarly to (4.3), we can achieve the following result

$$\begin{aligned}
&\frac{A_\varphi e^{M_{n,\varphi}}}{4}\sum_{m=0}^{\infty}\widetilde{x}(m)[\widetilde{y}(|n+m-1|)-\widetilde{y}(|n-m-1|)]\\
&=-\frac{A_\varphi e^{M_{n,\varphi}}}{4}\sum_{m=0}^{\infty}\widetilde{x}(m)[\widetilde{y}(|n-1-m|)-\widetilde{y}(n-1+m)]\\
&=-\frac{1}{4}(x\underset{s,s,c,\alpha}{*}y)(n-1).
\end{aligned}\tag{4.4}$$

From (4.3) and (4.4), we can obtain

$$(x\overset{\omega}{\underset{c,s,c,\alpha}{*}}y)(n)=\frac{1}{4}(x\underset{s,s,c,\alpha}{*}y)(n+1)-\frac{1}{4}(x\underset{s,s,c,\alpha}{*}y)(n-1).\tag{4.5}$$

The proof is completed. □

Based on Figure 1, it can be observed that the weighted convolution operation $\overset{\omega}{\underset{c,s,c,\alpha}{*}}$ defined in Definition 3.4 can be equivalently represented by the convolution operation $\underset{s,s,c,\alpha}{*}$ introduced in Definition 3.3.

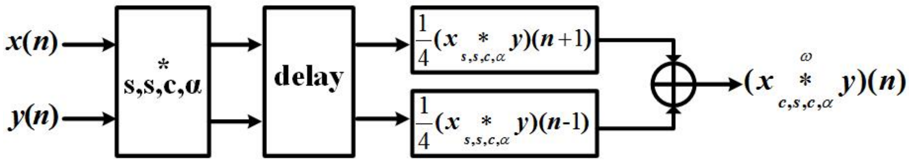

Figure 1. Convolution relationship between fractional cosine series and fractional sine series.

5. Application of the convolution theorems for FRCS and FRSS

The convolution equation is a crucial tool in various applications, including radiation energy transfer, film or shaft vibration problems, as well as the resolution of electrostatic field and magnetic field synthesis along with digital signal processing. The system of convolution equations represented by (5.1) and (5.2) also arises when addressing challenges in engineering mechanics and digital signal processing; thus, making it an active area of research interest. In this section, we will discuss two specific types of convolution equations

$$\begin{cases}
x(n)+\mu_1\sum\limits_{m=0}^{\infty}\widetilde{\eta}(m)y_{11}(n,m)=p(n),\\
y(n)+\mu_2\sum\limits_{m=0}^{\infty}\widetilde{x}(m)\psi_{21}(n,m)=q(n)
\end{cases}\tag{5.1}$$

and

$$z(n)+\mu\left[\sum_{m=0}^{\infty}\widetilde{\eta}(m)z_{12}(n,m)+\sum_{m=0}^{\infty}\widetilde{\psi}(m)z_{21}(n,m)\right]=p(n),\tag{5.2}$$

where

$$
\begin{aligned}
y_{11}(n,m) &= A_\varphi e^{j\frac{n^2}{2}\cot\varphi}\Big[y(|m-n|) - y(m+n)\Big],\\
z_{12}(n,m) &= A_\varphi e^{j\frac{n^2}{2}\cot\varphi}\Big[z(|m-n|) + z(m+n)\Big],\\
z_{21}(n,m) &= A_\varphi e^{j\frac{n^2}{2}\cot\varphi}\Big[z(|m-n+1|) + z(|m+n-1|) - z(|m-n-1|) - z(m+n+1)\Big],\\
\psi_{21}(n,m) &= A_\varphi e^{j\frac{n^2}{2}\cot\varphi}\Big[\psi(|m-n+1|) + \psi(|m+n-1|) - \psi(|m-n-1|) - \psi(m+n+1)\Big]
\end{aligned}
\tag{5.3}
$$

and

$$
\begin{aligned}
&\eta(m) = (\eta_1 \underset{c,\alpha}{*} \eta_2)(m), \ \ \psi(m) = (\psi_1 \underset{s,s,c,\alpha}{*} \psi_2)(m),\\
&\widetilde{x}(m) = A_\varphi e^{j\frac{m^2}{2}\cot\varphi} x(m), \ \ \widetilde{\eta}(m) = A_\varphi e^{j\frac{m^2}{2}\cot\varphi}\eta(m), \ \ \widetilde{\psi}(m) = A_\varphi e^{j\frac{m^2}{2}\cot\varphi}\psi(m).
\end{aligned}
\tag{5.4}
$$

where $\mu, \mu_1, \mu_2 \in \mathbb{C}$, $\widetilde{\eta}, \eta, \eta_1, \eta_2, \widetilde{\psi}, \psi, \psi_1, \psi_2, p, q \in \ell_1(\mathbb{N})$, x, y and z are unknown sequences in $\ell_1(\mathbb{N})$, $m, n \in \mathbb{N}$.

Next, we will utilize the convolution operation along with the corresponding convolution theorem proposed in this article to investigate the solution of the convolution equation. First, we will examine a system of convolution Eq (5.1).

Theorem 5.1. *Let $x, y \in \ell_1(\mathbb{N})$, such that*

$$
1 - \mu_1\mu_2 e^{-j\frac{w^2}{2}\cot\varphi} S_c^\alpha(\eta \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi)(w) \neq 0,
$$

then the system of convolutions Eq (5.1) has a unique solution

$$
x(n) = p(n) + (p \underset{s,s,c,\alpha}{*} u)(n) - \mu_1(\eta \underset{s,s,c,\alpha}{*} q)(n) - \mu_1(\eta \underset{s,s,c,\alpha}{*} q) \underset{s,s,c,\alpha}{*} u)(n)
\tag{5.5}
$$

and

$$
y(n) = q(n) + (q \underset{c,\alpha}{*} u)(n) - \mu_2(p \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi)(n) - \mu_2((p \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi) \underset{c,\alpha}{*} u)(n),
\tag{5.6}
$$

where $u \in \ell_1(\mathbb{N})$, and the following equality holds

$$
(S_c^\alpha u)(w) = \frac{\mu_1\mu_2 S_c^\alpha(\eta \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi)(w)}{1 - \mu_1\mu_2 e^{-j\frac{w^2}{2}\cot\varphi} S_c^\alpha(\eta \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi)(w)}.
\tag{5.7}
$$

Proof. By applying the FRSS and FRCS to both sides of Eq (5.1) and simultaneously utilizing (5.3), we can obtain

$$
\begin{cases}
(S_s^\alpha x)(w) + \mu_1 e^{-j\frac{w^2}{2}\cot\varphi}(S_s^\alpha \eta)(w)(S_c^\alpha y)(w) = (S_s^\alpha p)(w),\\
\mu_2 e^{-j\frac{w^2}{2}\cot\varphi}\sin(\csc\varphi \cdot w)(S_s^\alpha x)(w)(S_c^\alpha \psi)(w) + (S_c^\alpha y)(w) = (S_c^\alpha q)(w)
\end{cases}
\tag{5.8}
$$

from

$$
1 - \mu_1\mu_2 e^{-j\frac{w^2}{2}\cot\varphi} S_c^\alpha(\eta \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi)(w) \neq 0.
$$

By applying the Wiener-Levi's theorem [34] and (5.7), we can deduce

$$\begin{aligned}(S_s^\alpha x)(w) &= \frac{(S_s^\alpha p)(w) - \mu_1 S_s^\alpha(\eta \underset{s,s,c,\alpha}{*} q)(w)}{1-\mu_1\mu_2 e^{-j\frac{w^2}{2}\cot\varphi} S_c^\alpha(\eta \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi)(w)} \\ &= ((S_s^\alpha p)(w) - \mu_1 S_s^\alpha(\eta \underset{s,s,c,\alpha}{*} q)(w)) \cdot (1 + e^{-j\frac{w^2}{2}\cot\varphi} S_c^\alpha(u)(w)) \\ &= (S_s^\alpha p)(w) + S_s^\alpha(p \underset{s,s,c,\alpha}{*} u)(w) - \mu_1 S_s^\alpha(\eta \underset{s,s,c,\alpha}{*} q)(w) - \mu_1 S_s^\alpha((\eta \underset{s,s,c,\alpha}{*} q) \underset{s,s,c,\alpha}{*} u)(w).\end{aligned} \tag{5.9}$$

Hence, we have

$$x(n) = p(n) + (p \underset{s,s,c,\alpha}{*} u)(n) - \mu_1(\eta \underset{s,s,c,\alpha}{*} q)(n) - \mu_1((\eta \underset{s,s,c,\alpha}{*} q) \underset{s,s,c,\alpha}{*} u)(n). \tag{5.10}$$

Similarly, we have

$$\begin{aligned}(S_c^\alpha y)(w) &= \frac{(S_c^\alpha q)(w) - \mu_2 S_c^\alpha(p \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi)(w)}{1-\mu_1\mu_2 e^{-j\frac{w^2}{2}\cot\varphi} S_c^\alpha(\eta \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi)(w)} \\ &= ((S_c^\alpha q)(w) - \mu_2 S_c^\alpha(p \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi)(w)) \cdot (1 + e^{-j\frac{w^2}{2}\cot\varphi} S_c^\alpha(u)(w)) \\ &= (S_c^\alpha q)(w) + S_c^\alpha(q \underset{c,\alpha}{*} u)(w) - \mu_2 S_c^\alpha(p \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi)(w) - \mu_2 S_c^\alpha((p \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi) \underset{c,\alpha}{*} u)(w);\end{aligned} \tag{5.11}$$

therefore, we can achieve

$$y(n) = q(n) + (q \underset{c,\alpha}{*} u)(n) - \mu_2(p \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi)(n) - \mu_2((p \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi) \underset{c,\alpha}{*} u)(n). \tag{5.12}$$

The proof is completed. □

The solutions of Eq (5.1) are illustrated in Figure 2, where the fractional order is $\alpha = 1.14$ and $\mu_1 = \mu_2 = 0.5$. The functions used in this paper are $p(n) = \sin n$, $q(n) = n$, $\eta(n) = \cos n$ and $\psi(n) = n^2$.

The solution to Eq (5.2) will be discussed below.

Theorem 5.2. *Let $y \in \ell_1(\mathbb{N})$, such that*

$$1 + \mu e^{-j\frac{w^2}{2}\cot\varphi}(S_c^\alpha(\eta_1 \underset{c,\alpha}{*} \eta_2)(w) + (S_c^\alpha(\psi_1 \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi_2)(w)) \neq 0,$$

then Eq (5.2) has a unique solution

$$z(n) = p(n) - (p \underset{c,\alpha}{*} v)(n) \tag{5.13}$$

and the following equality holds

$$(S_c^\alpha v)(w) = \frac{\mu(S_c^\alpha(\eta_1 \underset{c,\alpha}{*} \eta_2)(w) + S_c^\alpha(\psi_1 \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi_2)(w))}{1 + \mu e^{-j\frac{w^2}{2}\cot\varphi}(S_c^\alpha(\eta_1 \underset{c,\alpha}{*} \eta_2)(w) + S_c^\alpha(\psi_1 \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi_2)(w))}. \tag{5.14}$$

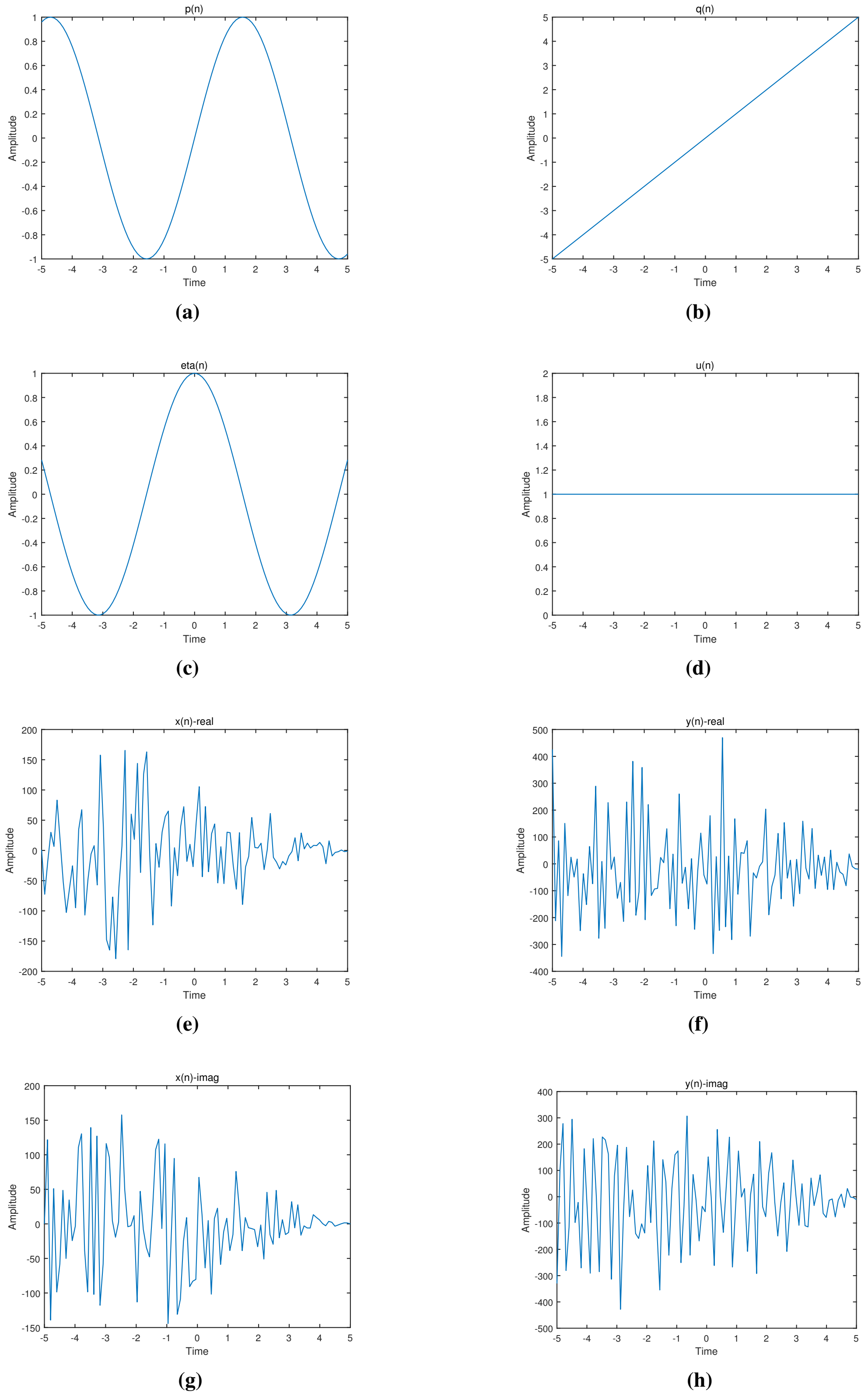

Figure 2. The solution of Eq (5.1). (a)–(d): the input function $p(n)$, $q(n)$, $\eta(n)$ and $u(n)$, respectively; (e): real part of solution $x(n)$; (f): real part of solution $y(n)$; (g): image part of solution $x(n)$; (h): image part of solution $y(n)$.

Proof. Applying the FRCS to both sides of Eq (5.2), we have

$$(S_c^\alpha z)(w)(1+\mu e^{-j\frac{w^2}{2}\cot\varphi}(S_c^\alpha(\eta_1 \underset{c,\alpha}{*} \eta_2)(w) + S_c^\alpha(\psi_1 \overset{\omega}{\underset{c,s,c,\alpha}{*}} \psi_2)(w))) = (S_c^\alpha p)(w). \tag{5.15}$$

Owing to the Wiener-Levi's theorem [34], from (5.4) and (5.14), we can derive

$$(S_c^\alpha z)(w) = (S_c^\alpha p)(w) \cdot (1 - e^{-j\frac{w^2}{2}\cot\varphi}(S_c^\alpha v)(w)), \tag{5.16}$$

hence, we have

$$(S_c^\alpha z)(w) = (S_c^\alpha p)(w) - S_c^\alpha(p \underset{c,\alpha}{*} v)(w). \tag{5.17}$$

The proof of Theorem 5.2 is completed. □

The solution of Eq (5.2) is also depicted in Figure 3, where the fractional order is $\alpha = 1.26$, $p(n) = \sin n$ and $v(n) = n^3$.

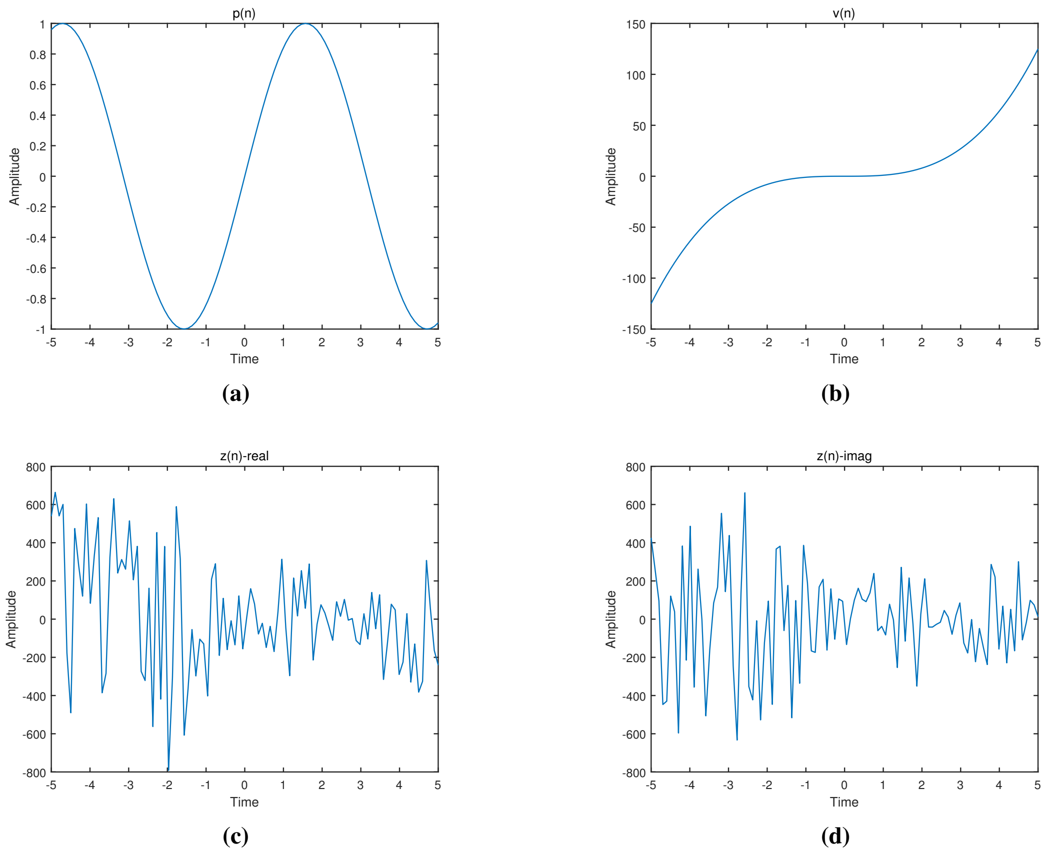

Figure 3. The solution of Eq (5.2). (a)-(b): represent the input function $p(n)$ and $v(n)$, respectively; (c): real part of solution $z(n)$; (d): image part of solution $z(n)$.

Next, let us analyze the computational complexity of the solution achieved in convolution integral Eqs (5.1) and (5.2) in detail. From (5.5), using the fast Fourier transform (FFT), $(p \underset{s,s,c,\alpha}{*} u)(n)$, $(\eta \underset{s,s,c,\alpha}{*} q)(n)$ and $(\eta \underset{s,s,c,\alpha}{*} q) \underset{s,s,c,\alpha}{*} u)(n)$ require $\frac{3}{2}Nlog_2^N$, $\frac{3}{2}Nlog_2^N$ and $2Nlog_2^N$ real number multiplications, respectively, for a discrete signal of size N. Therefore, the computational complexity of a solution $x(n)$ to Eq (5.1) can be obtained as $O(\frac{15}{2}Nlog_2^N)$, which is much lower than that of the computational complexity in fractional domain. Similarly, we can obtain the computational complexity of a solution $y(n)$ to Eq (5.1) and a solution $z(n)$ to Eq (5.2) as $O(\frac{15}{2}Nlog_2^N)$ and $O(\frac{5}{2}Nlog_2^N)$, respectively.

6. Conclusions

The FRCS and the FRSS, which are valuable tools in mathematics and signal processing, were introduced in this paper. Convolution operations for FRCS and FRSS were proposed, along with a detailed derivation of their corresponding convolution theorems. Furthermore, it was demonstrated that our derived results encompassed all classical findings regarding Fourier cosine and sine series as special cases. As an application, explicit solutions for two classes of convolution equations in the FRCS and FRSS domains were explored. Additionally, the computational complexity of solving these equations was also analyzed in detail.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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