Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(2): 2619–2640.
DOI: 10.3934/math.2024129
Received: 09 October 2023
Revised: 19 December 2023
Accepted: 21 December 2023
Published: 26 December 2023

Research article

Twisted Rota-Baxter operators on Hom-Lie algebras

Senrong Xu[1], **Wei Wang**[2,*] **and Jia Zhao**[3]

[1] School of Mathematical Sciences, Jiangsu University, Zhenjiang, Jiangsu 212013, China
[2] Institute of Applied System Analysis, Jiangsu University, Zhenjiang, Jiangsu 212013, China
[3] School of Sciences, Nantong University, Nantong, Jiangsu 226019, China

* **Correspondence:** Email: weiwangmath@126.com.

Abstract: Uchino initiated the investigation of twisted Rota-Baxter operators on associative algebras. Relevant studies have been extensive in recent times. In this paper, we introduce the notion of a twisted Rota-Baxter operator on a Hom-Lie algebra. By utilizing higher derived brackets, we establish an explicit L_∞-algebra whose Maurer-Cartan elements are precisely twisted Rota-Baxter operators on Hom-Lie algebras. Additionally, we employ Getzler's technique of twisting L_∞-algebras to establish the cohomology of twisted Rota-Baxter operators. We demonstrate that this cohomology can be regarded as the Chevalley-Eilenberg cohomology of a specific Hom-Lie algebra with coefficients in an appropriate representation. Finally, we study the linear and formal deformations of twisted Rota-Baxter operators by using the cohomology defined above. We also show that the rigidity of a twisted Rota-Baxter operator can be derived from Nijenhuis elements associated with a Hom-Lie algebra.

Keywords: twisted Rota-Baxter operators; Hom-Lie algebras; Maurer-Cartan elements; L_∞-algebras; cohomologies; deformations
Mathematics Subject Classification: 17B38, 17B61, 17B56

1. Introduction

Rota-Baxter operators, which were initially introduced by Baxter in probability theory [1] and were later developed by Rota and Cartier in combinatorics [2, 3]. The significance of Rota-Baxter operators has been established in various research areas, such as Connes-Kreimer's algebraic method for renormalizing perturbative quantum field theory and dendriform algebras [4, 5]. Furthermore, the study of inverse scattering theory, integrable systems and quantum groups reveals a close association between Rota-Baxter operators on Lie algebras and the classical Yang-Baxter equation; see the book by Guo for more details [6].

In order to gain a deeper understanding of the classical Yang-Baxter equation, Kupershmidt introduced a broader concept of an $\mathcal{O}$-operator (also known as a relative Rota-Baxter operator) on a Lie algebra [7]. Recently, there has been the establishment of cohomologies and deformations of relative Rota-Baxter operators on different algebraic structures, including Lie algebras, Leibniz algebras, 3-Lie algebras, n-Lie algebras, Lie conformal algebras and others [8–12].

In addition, the close relationship with other operators also reflects the importance of (relative) Rota-Baxter operators. One of them is the Reynolds operator, known as a time-average operator in fluid dynamics, which was initially presented by Reynolds in his renowned work on fluctuation theory in 1895 [13], and subsequently named by Kampé de Fériet to provide a comprehensive analysis of Reynolds operators in general [14]. Reynolds operators were also widely used in functional analysis, invariant theory and have a close relation with algebra endomorphisms, derivations, rational G-modules, geometry and operads [15–19].

Motivated by the twisted Poisson structures introduced and studied in [20, 21], Uchino introduced a twisted version of Rota-Baxter operators on associative algebras, known as twisted Rota-Baxter operators (or generalized Reynolds operators), and examined its correlation with NS-algebras [22]. Based on Uchino's work, Das conducted an additional investigation into the cohomology and deformations of twisted Rota-Baxter operators on associative algebras and Lie algebras [23, 24]. Note that twisted Rota-Baxter operators can be seen as extensions of Reynolds operators [22, Example 3.5]. Hou and Sheng employed the terminology of a generalized Reynolds operator instead of a twisted Rota-Baxter operator on 3-Lie algebras [25]. In [26], Gharbi et al. delved into the investigation of generalized Reynolds operators on Lie triple systems, while also introducing NS-Lie triple systems as the fundamental framework of generalized Reynolds operators.

In this paper, we consider twisted Rota-Baxter operators on Hom-Lie algebras. Hartwig et al. were the first to introduce Hom-Lie algebras in 2006 for the purpose of studying the deformation of the Witt and the Virasoro algebras [27], which can be traced back to q-deformations of algebras of vector fields in the field of physics. Since then, other algebras of the Hom type (e.g., Hom-associative algebras, Hom-Leibniz algebras), as well as their n-ary generalizations, have been widely studied both in mathematics and mathematical physics [28–32]. Additionally, it is noteworthy noting that Wang and his collaborators investigated twisted Rota-Baxter operators on 3-Hom-Lie algebras and Reynolds operators on Hom-Leibniz algebras by using cohomology and deformation theory [33, 34].

This paper is organized as follows. Section 2 provides an overview of the concepts and properties related to Hom-Lie algebras, including representation and cohomology. In Section 3, we delve into the topic of twisted Rota-Baxter operators on Hom-Lie algebras, exploring their connection to Reynolds operators and derivations. Furthermore, we present a novel approach to the construction of twisted Rota-Baxter operators on Hom-Lie algebras by using R-admissible 1-cocycles. Moving on to Section 4, we construct a new L_∞-algebra, whose Maurer-Cartan elements correspond precisely to twisted Rota-Baxter operators on Hom-Lie algebras. With this foundation, we define the cohomology of twisted Rota-Baxter operators by using the technique of constructing twisting L_∞-algebras pioneered by Getzler. Additionally, we establish an intriguing relationship between the cohomology of a twisted Rota-Baxter operator and the Chevalley-Eilenberg cohomology of a specific Hom-Lie algebra with coefficients in an appropriate representation. Section 5 is devoted to the realm of linear and formal deformations of twisted Rota-Baxter operators, demonstrating that the linear term in such deformations of a twisted Rota-Baxter operator R manifests as a 1-cocycle in the cohomology of R. Finally, we introduce Nijenhuis elements as a means of characterizing the rigidity of twisted Rota-Baxter operators.

In this paper, all vector spaces, linear maps and tensor products are assumed to be over a field $\mathbb{K}$ of characteristic 0.

2. Preliminaries

In this section, we recall several fundamental concepts, including the representation and cohomology of Hom-Lie algebras. The material can be found in the literature [31, 35, 36].

Definition 2.1. A **Hom algebra** is a triple $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ consisting of a vector space $\mathfrak{g}$, a bilinear map (bracket) $[\cdot,\cdot] : \wedge^2\mathfrak{g} \to \mathfrak{g}$ and a linear map $\alpha : \mathfrak{g} \to \mathfrak{g}$ such that $\alpha([x, y]) = [\alpha(x), \alpha(y)]$. Moreover, if a Hom algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ also satisfies the following Hom-Jacobi identity:

$$[[x, y], \alpha(z)] + [[y, z], \alpha(x)] + +[[z, x], \alpha(y)] = 0, \ \forall x, y, z \in \mathfrak{g}, \tag{2.1}$$

then $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ will be called a **Hom-Lie algebra**.

Definition 2.2. A **morphism** of Hom-Lie algebras $\phi : (\mathfrak{g}_1, [\cdot,\cdot], \alpha) \to (\mathfrak{g}_2, [[\cdot,\cdot]], \beta)$ is a linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

$$\begin{aligned} \phi([x, y]) &= [[\phi(x), \phi(y)]], \ \forall x, y \in \mathfrak{g}_1, \\ \phi \circ \alpha &= \beta \circ \phi. \end{aligned}$$

In particular, if ϕ is invertible, we say that $(\mathfrak{g}_1, [\cdot,\cdot], \alpha)$ and $(\mathfrak{g}_2, [[\cdot,\cdot]], \beta)$ are isomorphic.

Definition 2.3. A linear map $D : \mathfrak{g} \to \mathfrak{g}$ on a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ is called an α^k-derivation if it satisfies that

$$\begin{aligned} D \circ \alpha &= \alpha \circ D, \\ D([x, y]) &= [D(x), \alpha^k(y)] + [\alpha^k(x), D(y)] \end{aligned}$$

for all $x, y \in \mathfrak{g}$, where k is a nonnegative integer.

In the sequel, an α^0-derivation on a Hom-Lie algebra will be called a derivation for simplicity.

Definition 2.4. A **representation** of a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ on a vector space V with respect to $A \in \mathfrak{gl}(V)$ is a linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ such that, for any $x, y \in \mathfrak{g}$, it holds that

$$\rho(\alpha(x)) \circ A = A \circ \rho(x), \tag{2.2}$$

$$\rho([x, y]) \circ A = \rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x). \tag{2.3}$$

We denote a representation of a Hom-Lie algebra $\mathfrak{g}$ with respect to A by (V, ρ, A).

Example 2.5. *Let* $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ *be a Hom-Lie algebra. Define* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *by* $\mathrm{ad}(x)(y) = [x, y]$ *for all* $x, y \in \mathfrak{g}$*; sometimes, we may write* $\mathrm{ad}(x)$ *as* ad_x*. Then,* $(\mathfrak{g}, \mathrm{ad}, \alpha)$ *is a representation of* $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ *with respect to* α*, which is called the adjoint representation of* $(\mathfrak{g}, [\cdot,\cdot], \alpha)$.

Next, we recall the cohomology of Hom-Lie algebras. Let (V, ρ, A) be a representation of the Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$. Denote the space of p-cochains by

$$C^p_{HLie}(\mathfrak{g}, V) = \begin{cases} \{v \in V \mid Av = v\}, & p = 0, \\ \{f \in \mathrm{Hom}(\wedge^p \mathfrak{g}, V) \mid A \circ f = f \circ \alpha^{\otimes p}\}, & p \geq 1, \end{cases} \tag{2.4}$$

where $A \circ f = f \circ \alpha^{\otimes p}$ means that

$$A(f(x_1, \cdots, x_p)) = f(\alpha(x_1), \cdots, \alpha(x_p)),\ \forall x_1, \cdots, x_p \in \mathfrak{g}. \tag{2.5}$$

For any $x_1, \cdots, x_{p+1} \in \mathfrak{g}$, define the coboundary operator $d_\rho : C^p_{HLie}(\mathfrak{g}, V) \to C^{p+1}_{HLie}(\mathfrak{g}, V)$, $p \geq 1$ by

$$\begin{aligned}(d_\rho f)(x_1, \cdots, x_{p+1}) = &\sum_{j=1}^{p+1} (-1)^{j+1} \rho(\alpha^{p-1}(x_j)) f(x_1, \cdots, \widehat{x_j}, \cdots, x_{p+1}) \\ &+ \sum_{j<k} (-1)^{j+k} f([x_j, x_k], \alpha(x_1), \cdots, \widehat{\alpha(x_j)}, \cdots, \widehat{\alpha(x_k)}, \cdots, \alpha(x_{p+1})),\end{aligned} \tag{2.6}$$

and when $p = 0$, define $d_\rho : C^0_{HLie}(\mathfrak{g}, V) \to \mathrm{Hom}(\mathfrak{g}, V)$ by

$$d_\rho(v)(x) = \rho(x)v,\ \forall\ v \in C^0_{HLie}(\mathfrak{g}, V), x \in \mathfrak{g}. \tag{2.7}$$

Since

$$A \circ (d_\rho(v)) = (d_\rho(v)) \circ \alpha,$$

we deduce that d_ρ is a map from $C^0_{HLie}(\mathfrak{g}, V)$ to $C^1_{HLie}(\mathfrak{g}, V)$, indeed. Thus, we have that $d_\rho \circ d_\rho = 0$ and, hence, $\left(\oplus_{p=0}^{+\infty} C^p_{HLie}(\mathfrak{g}, V), d_\rho\right)$ is a cochain complex. Denote the set of p-cocycles by $Z^p_{HLie}(\mathfrak{g}, V)$ and the set of p-coboundaries by $B^p_{HLie}(\mathfrak{g}, V)$. Then, the corresponding p-th **cohomology group** is

$$H^p_{HLie}(\mathfrak{g}, V) = Z^p_{HLie}(\mathfrak{g}, V)/B^p_{HLie}(\mathfrak{g}, V).$$

In view of (2.6), a 1-cochain $f \in C^1_{HLie}(\mathfrak{g}, V)$ is a 1-cocycle on $\mathfrak{g}$ with coefficients in (V, ρ, A) if f satisfies

$$0 = (d_\rho f)(x, y) = \rho(x)f(y) - \rho(y)f(x) - f([x, y]),\ \forall x, y \in \mathfrak{g}, \tag{2.8}$$

and a 2-cochain $\Phi \in C^2_{HLie}(\mathfrak{g}, V)$ is a 2-cocycle if Φ satisfies

$$\begin{aligned}0 = (d_\rho \Phi)(x, y, z) = &\rho(\alpha(x))\Phi(y, z) - \rho(\alpha(y))\Phi(x, z) + \rho(\alpha(z))\Phi(x, y) \\ &- \Phi([y, z], \alpha(x)) + \Phi([x, z], \alpha(y)) - \Phi([x, y], \alpha(z)),\ \forall x, y, z \in \mathfrak{g}.\end{aligned} \tag{2.9}$$

3. Twisted Rota-Baxter operators on Hom-Lie algebras

In this section, we introduce the notion of twisted Rota-Baxter operators on Hom-Lie algebras. We establish the relation between Reynolds operators and derivations. We also show that a linear map is a twisted Rota-Baxter operator if and only if its graph is a subalgebra of the Φ-twisted semi-direct Hom-Lie algebra. Moreover, we provide a method for constructing twisted Rota-Baxter operators by using R-admissible 1-cocycles.

First, by a direct check, we have the following result.

Proposition 3.1. *Let* $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ *be a Hom-Lie algebra and* (V, ρ, A) *a representation of* $\mathfrak{g}$*. Given a 2-cocycle* $\Phi \in C^2_{HLie}(\mathfrak{g}, V)$*, there exists a Hom-Lie algebra structure on the direct sum* $\mathfrak{g} \oplus V$ *that is defined by*

$$[x_1 + v_1, x_2 + v_2]_\Phi = [x_1, x_2] + \rho(x_1)v_2 - \rho(x_2)v_1 + \Phi(x_1, x_2), \tag{3.1}$$

$$(\alpha \oplus A)(x_1 + v_1) = \alpha(x_1) + Av_1, \quad \forall x_1, x_2 \in \mathfrak{g}, v_1, v_2 \in V. \tag{3.2}$$

This Hom-Lie algebra is called the ***Φ-twisted semi-direct Hom-Lie algebra*** *and will be denoted by* $(\mathfrak{g} \ltimes_\Phi V, \alpha \oplus A)$.

A twisted Rota-Baxter operator on a Hom-Lie algebra is defined as follows, the weight of which is a 2-cocycle instead of a scalar, as in the classical case.

Definition 3.2. Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra and (V, ρ, A) a representation of $\mathfrak{g}$. A linear map $R : V \to \mathfrak{g}$ is called a **twisted Rota-Baxter operator** on $\mathfrak{g}$ associated with a 2-cocycle Φ with respect to (V, ρ, A) (of weight Φ) if

$$\alpha \circ R = R \circ A, \tag{3.3}$$

$$[Rv_1, Rv_2] = R(\rho(Rv_1)v_2 - \rho(Rv_2)v_1 + \Phi(Rv_1, Rv_2)), \ \forall v_1, v_2 \in V. \tag{3.4}$$

Remark 3.3. A twisted Rota-Baxter operator is also called a generalized Reynolds operator; see [25, 26] for more details, where the authors considered it on Lie triple systems and 3-Lie algebras, respectively. Furthermore, it was also named a relative cocycle weighted Reynolds operator by Guo and Zhang in the setting of pre-Lie algebras [37].

Example 3.4. *Any Rota-Baxter operator or relative Rota-Baxter operator of weight 0 is a twisted Rota-Baxter operator with* $\Phi = 0$.

Example 3.5. *Let* $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ *be a Hom-Lie algebra and* (V, ρ, A) *a representation of* $\mathfrak{g}$*. Assume that a linear map* $f \in C^1_{HLie}(\mathfrak{g}, V)$ *is invertible. Set* $\Phi = -d_\rho f$ *and* $R = f^{-1}$*. Then,* Φ *is a 2-cocycle. Since*

$$\Phi(Rv_1, Rv_2) = (-d_\rho f)(Rv_1, Rv_2) = -\rho(Rv_1)f(Rv_2) + \rho(Rv_2)f(Rv_1) + f[Rv_1, Rv_2],$$

we obtain that R is a twisted Rota-Baxter operator of weight $\Phi = -d_\rho f$.

Example 3.6. *Let* $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ *be a Hom-Lie algebra and* $(\mathfrak{g}, \mathrm{ad}, \alpha)$ *the adjoint representation. Set* $\Phi = -[\cdot,\cdot]$*; then, a linear transformation* $R : \mathfrak{g} \to \mathfrak{g}$ *defined by* (3.3) *and* (3.4) *is called a* ***Reynolds operator****; more specifically, R satisfies that*

$$\alpha \circ R = R \circ \alpha, \tag{3.5}$$

$$[Rx_1, Rx_2] = R([Rx_1, x_2] + [x_1, Rx_2] - [Rx_1, Rx_2]), \ \forall x_1, x_2 \in \mathfrak{g}. \tag{3.6}$$

Note that the authors of [34] defined Reynolds operators on Hom-Leibniz algebras. Furthermore, twisted Rota-Baxter operators on Hom-Lie algebras are extensions of both twisted Rota-Baxter operators and 0-weighted Rota-Baxter operators on Lie algebras.

Next, we establish the connection between derivations and Reynolds operators on Hom-Lie algebras. First, a derivation can induce a Reynolds operator on a Hom-Leibniz algebra [34]. Specifically, we have the following:

Proposition 3.7. *Assume that $D : \mathfrak{g} \to \mathfrak{g}$ is a derivation on a Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$. If $(D + \mathrm{Id}) : \mathfrak{g} \to \mathfrak{g}$ has an inverse, then $(D + \mathrm{Id})^{-1}$ is a Reynolds operator.*

Conversely, a derivation on a Hom-Lie algebra can be derived from a Reynolds operator.

Proposition 3.8. *Assume that $R : \mathfrak{g} \to \mathfrak{g}$ is a Reynolds operator on the Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$. If R has an inverse, then $(R^{-1} - \mathrm{Id}) : \mathfrak{g} \to \mathfrak{g}$ is a derivation on $(\mathfrak{g}, [\cdot, \cdot], \alpha)$.*

Proof. Suppose that $R : \mathfrak{g} \to \mathfrak{g}$ is an invertible Reynolds operator. Then, $\alpha \circ R = R \circ \alpha$. Moreover, thanks to (3.6), we obtain

$$R^{-1}[x_1, x_2] = [x_1, R^{-1}x_2] + [R^{-1}x_1, x_2] - [x_1, x_2]$$

for any $x_1, x_2 \in \mathfrak{g}$, which is equivalent to

$$(R^{-1} - \mathrm{Id})[x_1, x_2] = [(R^{-1} - \mathrm{Id})x_1, x_2] + [x_1, (R^{-1} - \mathrm{Id})x_2].$$

This completes the proof. □

In the sequel, Φ denotes a 2-cocycle, and a twisted Rota-Baxter operator is always endowed with the weight Φ unless otherwise specified elsewhere.

Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra and (V, ρ, A) a representation of $\mathfrak{g}$. Suppose that $R : V \to \mathfrak{g}$ is a linear map which satisfies that $\alpha \circ R = R \circ A$. Then, we call the set $Gr(R) = \{Rv + v | v \in V\}$ the graph of R.

Theorem 3.9. *Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra and (V, ρ, A) a representation of $\mathfrak{g}$. A linear map $R : V \to \mathfrak{g}$ is a twisted Rota-Baxter operator if and only if the graph $Gr(R) = \{Rv + v | v \in V\}$ is a subalgebra of the Φ-twisted semi-direct Hom-Lie algebra $(\mathfrak{g} \ltimes_\Phi V, \alpha \oplus A)$.*

Proof. Set $v_1, v_2 \in V$. Then, we have

$$[Rv_1 + v_1, Rv_2 + v_2]_\Phi = [Rv_1, Rv_2] + \rho(Rv_1)(v_2) - \rho(Rv_2)v_1 + \Phi(Rv_1, Rv_2).$$

Hence, the graph $Gr(R) = \{Rv + v | v \in V\}$ is a subalgebra of $\mathfrak{g} \ltimes_\Phi V$ if and only if

$$[Rv_1, Rv_2] = R(\rho(Rv_1)(v_2) - \rho(Rv_2)v_1 + \Phi(Rv_1, Rv_2)),$$

which is precisely (3.4). The proof is finished. □

The following corollary is straightforward given $Gr(R) \cong V$ as vector spaces.

Corollary 3.10. *Suppose that $R : V \to \mathfrak{g}$ is a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ with respect to the representation (V, ρ, A). Then, $(V, [\cdot, \cdot]_R, A)$ is a Hom-Lie algebra, called the **sub-adjacent Hom-Lie algebra** of R, where the operation $[\cdot, \cdot]_R$ is given by*

$$[u, v]_R = \rho(Ru)v - \rho(Rv)u + \Phi(Ru, Rv). \tag{3.7}$$

Moreover, R is a homomorphism of Hom-Lie algebras from $(V, [\cdot, \cdot]_R, A)$ to $(\mathfrak{g}, [\cdot, \cdot], \alpha)$.

We will now present a method for constructing twisted Rota-Baxter operators on Hom-Lie algebras by introducing the concept of an R-admissible 1-cocycle. Consider a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$, a representation (V, ρ, A) of $\mathfrak{g}$ and a linear map $\theta \in C^1_{HLie}(\mathfrak{g}, V)$. Define $\Omega_\theta : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$ by $\Omega_\theta = \begin{pmatrix} \mathrm{Id} & 0 \\ \theta & \mathrm{Id} \end{pmatrix}$. Then, Ω_θ is invertible. Note that $\Phi - d_\rho\theta$ is a 2-cocycle.

Lemma 3.11. *Consider the above notations. Then, Ω_θ serves as an isomorphism between the Φ-twisted semi-direct Hom-Lie algebra $(\mathfrak{g} \ltimes_\Phi V, \alpha \oplus A)$ and the $(\Phi - d_\rho\theta)$-twisted semi-direct Hom-Lie algebra $(\mathfrak{g} \ltimes_{\Phi - d_\rho\theta} V, \alpha \oplus A)$.*

Proof. Set $x_1, x_2 \in \mathfrak{g}$ and $v_1, v_2 \in V$. Then,

$$\begin{aligned}
&[\Omega_\theta(x_1 + v_1), \Omega_\theta(x_2 + v_2)]_{(\Phi - d_\rho\theta)} = [x_1 + \theta(x_1) + v_1, x_2 + \theta(x_2) + v_2]_{(\Phi - d_\rho\theta)} \\
\overset{(3.1)}{=}\; & [x_1, x_2] + \rho(x_1)(\theta(x_2) + v_2) - \rho(x_2)(\theta(x_1) + v_1) + (\Phi - d_\rho\theta)(x_1, x_2) \\
\overset{(2.8)}{=}\; & [x_1, x_2] + \rho(x_1)(\theta(x_2) + v_2) - \rho(x_2)(\theta(x_1) + v_1) + \Phi(x_1, x_2) + \theta([x_1, x_2]) - \rho(x_1)\theta(x_2) + \rho(x_2)\theta(x_1) \\
=\; & [x_1, x_2] + \theta([x_1, x_2]) + \rho(x_1)v_2 - \rho(x_2)v_1 + \Phi(x_1, x_2) \\
=\; & \Omega_\theta([x_1 + v_1, x_2 + v_2]_\Phi),
\end{aligned}$$

as required. □

Proposition 3.12. *Let $R : V \to \mathfrak{g}$ be a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ with respect to a representation (V, ρ, A). Suppose that $\theta \in C^1_{HLie}(\mathfrak{g}, V)$. If the linear map $(\mathrm{Id}_V + \theta \circ R) : V \to V$ has an inverse, then the map $R \circ (\mathrm{Id}_V + \theta \circ R)^{-1}$ is a twisted Rota-Baxter operator of weight $(\Phi - d_\rho\theta)$.*

Proof. Thanks to Theorem 3.9, $Gr(R)$ is a subalgebra of the Φ-twisted semi-direct Hom-Lie algebra $(\mathfrak{g} \ltimes_\Phi V, \alpha \oplus A)$. In view of Lemma 3.11, $\Omega_\theta(Gr(R)) \subseteq (\mathfrak{g} \ltimes_{\Phi - d_\rho\theta} V, \alpha \oplus A)$ is also a subalgebra. Given that the linear map $(\mathrm{Id}_V + \theta \circ R) : V \to V$ has an inverse, by a direct check, we see that

$$\alpha \circ \left(R \circ (\mathrm{Id}_V + \theta \circ R)^{-1}\right) = \left(R \circ (\mathrm{Id}_V + \theta \circ R)^{-1}\right) \circ A,$$

and, hence, $\Omega_\theta(Gr(R))$ is the graph of $R \circ (\mathrm{Id}_V + \theta \circ R)^{-1} : V \to \mathfrak{g}$. Then, by Theorem 3.9, again, we deduce that the map $R \circ (\mathrm{Id}_V + \theta \circ R)^{-1}$ is a twisted Rota-Baxter operator of weight $(\Phi - d_\rho\theta)$, we have the conclusion. □

Definition 3.13. Assume that $R : V \to \mathfrak{g}$ is a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ with respect to the representation (V, ρ, A). An ***R*-admissible 1-cocycle** is a 1-cocycle $\theta \in Z^1_{HLie}(\mathfrak{g}, V)$ such that the map $(\mathrm{Id}_V + \theta \circ R) : V \to V$ is invertible.

The following corollary is straightforward due to Proposition 3.12 and Definition 3.13.

Corollary 3.14. *Assume that $R : V \to \mathfrak{g}$ is a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ with respect to the representation (V, ρ, A), and let $\theta : \mathfrak{g} \to V$ denote an R-admissible 1-cocycle. The composition of the map R with the inverse of $(\mathrm{Id}_V + \theta \circ R)$ forms a twisted Rota-Baxter operator. Denote this twisted Rota-Baxter operator by R_θ.*

With the help of Corollary 3.2, $(V, [\cdot,\cdot]_R)$ and $(V, [\cdot,\cdot]_{R_\theta})$ are Hom-Lie algebras. We conclude this section by pointing out that $(V, [\cdot,\cdot]_R)$ and $(V, [\cdot,\cdot]_{R_\theta})$ are isomorphic as Hom-Lie algebras.

Proposition 3.15. *Let $R : V \to \mathfrak{g}$ be a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ with respect to the representation (V, ρ, A) and θ an R-admissible 1-cocycle. Then, $(V, [\cdot,\cdot]_R) \cong (V, [\cdot,\cdot]_{R_\theta})$ denotes Hom-Lie algebras.*

Proof. It suffices to show that the invertible map $(\mathrm{Id}_V + \theta \circ R)$ is an isomorphism between $(V, [\cdot,\cdot]_R)$ and $(V, [\cdot,\cdot]_{R_\theta})$. For any $v_1, v_2 \in V$, it holds that

$$\begin{aligned}
& [(\mathrm{Id} + \theta \circ R)(v_1), (\mathrm{Id} + \theta \circ R)(v_2)]_{R_\theta} \\
= \;& \rho(Rv_1)(v_2 + \theta(Rv_2)) - \rho(Rv_2)(v_1 + \theta(Rv_1)) + \Phi(Rv_1, Rv_2) \\
\overset{(3.7)}{=} \;& [v_1, v_2]_R + \rho(Rv_1)(\theta(Rv_2)) - \rho(Rv_2)(\theta(Rv_1)) \\
\overset{(2.8)}{=} \;& [v_1, v_2]_R + \theta([Rv_1, Rv_2]) \\
= \;& [v_1, v_2]_R + \theta(R([v_1, v_2]_R)) \\
= \;& (\mathrm{Id} + \theta \circ R)([v_1, v_2]_R),
\end{aligned}$$

which finishes the proof. □

4. Cohomology of twisted Rota-Baxter operators on Hom-Lie algebras

In this section, we first recall the concept of an L_∞-algebra and the Nijenhuis-Richardson bracket for Hom-Lie algebras. Subsequently, we construct an L_∞-algebra whose Maurer-Cartan elements are given by twisted Rota-Baxter operators on Hom-Lie algebras. Following this, we delve into the study of twisting theory for Hom-Lie algebras and establish the cohomology of twisted Rota-Baxter operators. Furthermore, we demonstrate that this cohomology can be perceived as the Chevalley-Eilenberg cohomology of a specific Hom-Lie algebra with coefficients in an appropriate representation.

4.1. Maurer-Cartan characterization and the controlling L_∞-algebra of twisted Rota-Baxter operators

In this subsection, we construct an explicit L_∞-algebra whose Maurer-Cartan elements are twisted Rota-Baxter operators on Hom-Lie algebras by using Voronov's higher derived bracket [38]. Using Getzler's method in [39], we also establish a twisted L_∞-algebra which governs the deformations of twisted Rota-Baxter operators on Hom-Lie algebras.

An $(i, n-i)$-shuffle is a permutation $\sigma \in \mathbb{S}_n$ such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$. In the case in which $i = 0$ or $i = n$, we make the assumption that $\sigma = \mathrm{Id}$. $\mathbb{S}_{(i,n-i)}$ will represent the collection of all $(i, n-i)$-shuffles.

Definition 4.1. ([40]) A $\mathbb{Z}$-graded vector space $\mathfrak{g} = \oplus_{k\in\mathbb{Z}} \mathfrak{g}^k$ with a collection $(k \geq 1)$ of linear maps $l_k : \otimes^k \mathfrak{g} \to \mathfrak{g}$ of degree 1 is called an L_∞**-algebra** if, for all homogeneous elements $x_1, \cdots, x_n \in \mathfrak{g}$, it holds that

(i) (**graded symmetry**) for any $\sigma \in \mathbb{S}_n$,

$$l_n(x_{\sigma(1)}, \cdots, x_{\sigma(n)}) = \varepsilon(\sigma) l_n(x_1, \cdots, x_n),$$

(ii) (**generalized Jacobi identity**) for any $n \geq 1$,

$$\sum_{i=1}^{n} \sum_{\sigma \in \mathbb{S}_{(i,n-i)}} \varepsilon(\sigma) l_{n-i+1}\left(l_i\left(x_{\sigma(1)}, \cdots, x_{\sigma(i)}\right), x_{\sigma(i+1)}, \cdots, x_{\sigma(n)}\right) = 0,$$

where $\varepsilon(\sigma) := \varepsilon(\sigma; x_1, \cdots, x_n) \in \{-1, 1\}$ is the Koszul sign.

Definition 4.2. A **Maurer-Cartan element** of an L_∞-algebra $(\mathfrak{g} = \oplus_{k\in\mathbb{Z}}\mathfrak{g}^k, \{l_i\}_{i=1}^{+\infty})$ is an element $x \in \mathfrak{g}^0$ such that $\sum_{n=1}^{+\infty} \frac{1}{n!} l_n(x, \cdots, x)$ converges to 0, that is, x obeys the Maurer-Cartan equation

$$\sum_{n=1}^{+\infty} \frac{1}{n!} l_n(x, \cdots, x) = 0. \tag{4.1}$$

Before proceeding further, let us recall the higher derived brackets due to Voronov, which can be utilized for the construction of explicit L_∞-algebras.

Definition 4.3. ([38]) A **V-data** consists of a quadruple $(L, \mathfrak{h}, \mathcal{P}, \Delta)$, where the following holds:

- $(L, [\cdot,\cdot])$ is a graded Lie algebra.
- $\mathfrak{h}$ is an abelian graded Lie subalgebra of $(L, [\cdot,\cdot])$.
- $\mathcal{P} : L \to L$ is a projection, i.e., $\mathcal{P} \circ \mathcal{P} = \mathcal{P}$, where $\mathfrak{h}$ is the image and the kernel is a graded Lie subalgebra of $(L, [\cdot,\cdot])$.
- Δ is an element of $\ker\mathcal{P}$ with degree 1 satisfying that $[\Delta, \Delta] = 0$.

Theorem 4.4. ([38]) *Assume $(L, \mathfrak{h}, \mathcal{P}, \Delta)$ to be a V-data. Then, $(\mathfrak{h}, \{l_i\}_{i=1}^{+\infty})$ forms an L_∞-algebra, where*

$$l_i(a_1, \cdots, a_i) = \mathcal{P}\underbrace{[\cdots[[}_{i}\Delta, a_1], a_2], \cdots, a_i] \quad \textit{for homogeneous } a_1, \cdots, a_i \in \mathfrak{h}. \tag{4.2}$$

*We call $\{l_i\}_{i=1}^{+\infty}$ the **higher derived brackets** of the V-data $(L, \mathfrak{h}, \mathcal{P}, \Delta)$.*

Let $\mathfrak{g}$ be a vector space and $\alpha : \mathfrak{g} \to \mathfrak{g}$ a linear map. Denote by $V_\alpha^p(\mathfrak{g}) = C_\alpha^{p+1}(\mathfrak{g}, \mathfrak{g})$, $p \geq 0$ the space of all linear maps $P : \mathfrak{g}^{\otimes(p+1)} \to \mathfrak{g}$ satisfying that $\alpha \circ P = P \circ \alpha^{\otimes^{p+1}}$, that is,

$$\alpha(P(x_1, \cdots, x_{p+1})) = P(\alpha(x_1), \cdots, \alpha(x_{p+1})) \text{ for all } x_i \in \mathfrak{g}. \tag{4.3}$$

Set $C_\alpha^0(\mathfrak{g}, \mathfrak{g}) = \mathfrak{g}$. Recall from [28, 30] that the graded space $V_\alpha^*(\mathfrak{g}) = \oplus_{p\geq -1} C_\alpha^{p+1}(\mathfrak{g}, \mathfrak{g})$ carries a graded Lie algebra structure $[\cdot,\cdot]_\alpha : V_\alpha^p(\mathfrak{g}) \times V_\alpha^q(\mathfrak{g}) \to V_\alpha^{p+q}(\mathfrak{g})$ (called Nijenhuis-Richardson bracket), defined by

$$[P, Q]_\alpha = (-1)^{pq}\{P, Q\}_\alpha - \{Q, P\}_\alpha \text{ for all } P \in V_\alpha^p(\mathfrak{g}), Q \in V_\alpha^q(\mathfrak{g}), \tag{4.4}$$

where $\{P, Q\}_\alpha \in V_\alpha^{p+q}(\mathfrak{g})$ is given by

$$\{P, Q\}_\alpha(x_1, \cdots, x_{p+q+1}) = \sum_{\sigma\in\mathbb{S}_{(q+1,p)}} (-1)^{|\sigma|} P(Q(x_{\sigma(1)}, \cdots, x_{\sigma(q+1)}), \alpha^q(x_{\sigma(q+2)}), \cdots, \alpha^q(x_{\sigma(p+q+1)})),$$

and the above notation $|\sigma|$ denotes the signature of the permutation σ.

Now, let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a Hom algebra. For simplicity, we shall use $\mu : \wedge^2\mathfrak{g} \to \mathfrak{g}$ to denote the bilinear bracket $[\cdot,\cdot]$, and a Hom algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ can be rewritten as $(\mathfrak{g}, \mu, \alpha)$. Note that $\mu \in V_\alpha^1(\mathfrak{g})$. Thus, a Hom algebra $(\mathfrak{g}, \mu, \alpha)$ becomes a Hom-Lie algebra if and only if $[\mu, \mu]_\alpha = 0$, that is, μ is a Maurer-Cartan element of the graded Lie algebra $(V_\alpha^*(\mathfrak{g}), [\cdot,\cdot]_\alpha)$.

Let $\mathfrak{g}$ and V be vector spaces with linear maps $\alpha : \mathfrak{g} \to \mathfrak{g}$ and $A : V \to V$. Suppose that $\mu : \wedge^2\mathfrak{g} \to \mathfrak{g}$, $\rho : \mathfrak{g} \to \mathrm{End}(V)$ and $\Phi : \wedge^2\mathfrak{g} \to V$ are linear maps. Define $\mu + \rho + \Phi \in \mathrm{Hom}(\wedge^2(\mathfrak{g} \oplus V), \mathfrak{g} \oplus V)$ by

$$(\mu + \rho + \Phi)(x_1 + v_1, x_2 + v_2) = [x_1, x_2] + \rho(x_1)v_2 - \rho(x_2)v_1 + \Phi(x_1, x_2), \ \forall x_1, x_2 \in \mathfrak{g}, v_1, v_2 \in V.$$

Proposition 4.5. *The map μ defines a Hom-Lie algebra structure on the pair $(\mathfrak{g}, \alpha)$, the map ρ defines a representation of the Hom-Lie algebra $(\mathfrak{g}, \mu, \alpha)$ on the pair (V, A) and the map Φ defines a 2-cocycle with respect to the representation (V, ρ, A) if and only if $\mu + \rho + \Phi$ is a Maurer-Cartan element of the graded Lie algebra $(V^*_{\alpha\oplus A}(\mathfrak{g} \oplus V), [\cdot, \cdot]_{\alpha\oplus A})$.*

Proof. Due to (4.3), $\mu + \rho + \Phi \in V^1_{\alpha\oplus A}(\mathfrak{g} \oplus V)$ if and only if

$$(\alpha \oplus A)((\mu + \rho + \Phi)(x_1 + v_1, x_2 + v_2)) = (\mu + \rho + \Phi)((\alpha \oplus A)(x_1 + v_1), (\alpha \oplus A)(x_2 + v_2))$$

for all $x_1, x_2 \in \mathfrak{g}$ and $v_1, v_2 \in V$, that is,

$$\alpha(\mu(x_1, x_2)) = \mu(\alpha(x_1), \alpha(x_2)),\ \rho(\alpha(x_1))(Av_2) = A(\rho(x_1)v_2),\ \text{and } A(\Phi(x_1, x_2)) = \Phi(\alpha(x_1), \alpha(x_2)).$$

In addition, the map $\mu + \rho + \Phi$ is a Maurer-Cartan element if and only if

$$[\mu+\rho+\Phi, \mu+\rho+\Phi]_{\alpha\oplus A} = -2\{\mu+\rho+\Phi, \mu+\rho+\Phi\}_{\alpha\oplus A}(x_1+v_1, x_2+v_2, x_3+v_3) = 0$$

for $x_1, x_2, x_3 \in \mathfrak{g}$ and $v_1, v_2, v_3 \in V$. Equivalently,

$$\begin{aligned}
\mu(\mu(x_1, x_2), \alpha(x_3)) + \mu(\mu(x_2, x_3), \alpha(x_1)) + \mu(\mu(x_3, x_1), \alpha(x_2)) &= 0,\ \forall x_1, x_2, x_3 \in \mathfrak{g}.\\
\rho(\mu(x_1, x_2)(Av_3) - \rho(\alpha(x_1))\rho(x_2)v_3 + \rho(\alpha(x_2))\rho(x_1)v_3 &= 0,\ \forall x_1, x_2 \in \mathfrak{g}, v_3 \in V.\\
\Phi(\mu(x_1, x_2), \alpha(x_3)) + \Phi(\mu(x_2, x_3), \alpha(x_1)) + \Phi(\mu(x_3, x_1), \alpha(x_2))&\\
-\rho(\alpha(x_1))\Phi(x_2, x_3) - \rho(\alpha(x_2))\Phi(x_3, x_1) - \rho(\alpha(x_3))\Phi(x_1, x_2) &= 0,\ \forall x_1, x_2, x_3 \in \mathfrak{g}.
\end{aligned}$$

Owing to Definition 2.1, Definition 2.4 and (2.9), the conclusion follows. □

Proposition 4.6. *Let (V, ρ, A) be a representation of a* Hom-Lie *algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ and $\Phi \in C^2_{HLie}(\mathfrak{g}, V)$ a 2-cocycle with respect to (V, ρ, A). Thus we have a V-data $(L, \mathfrak{h}, \mathcal{P}, \Delta)$, as follows:*

- *the graded Lie algebra $(L, [\cdot, \cdot])$ is given by $(V^*_{\alpha\oplus A}(\mathfrak{g} \oplus V), [\cdot, \cdot]_{\alpha\oplus A})$;*
- *the abelian graded Lie subalgebra $\mathfrak{h}$ is defined by*

$$\mathfrak{h} = C^*_{HLie}(V, \mathfrak{g}) = \oplus_{p\geq 1} C^p_{HLie}(V, \mathfrak{g}),\ \text{where } C^p_{HLie}(V, \mathfrak{g}) = \{f \in \mathrm{Hom}(\wedge^p V, \mathfrak{g}) | \alpha \circ f = f \circ A^{\otimes^p}\};$$

- *$\mathcal{P} : L \to L$ is the projection onto the space $\mathfrak{h}$;*
- *$\Delta = \mu + \rho + \Phi$.*

Therefore, we get an L_∞-algebra $(C^*_{HLie}(V, \mathfrak{g}), l_2, l_3)$, where

$$\begin{aligned}
l_2(P, Q) &= \mathcal{P}[[\mu + \rho + \Phi, P]_{\alpha\oplus A}, Q]_{\alpha\oplus A},\\
l_3(P, Q, S) &= \mathcal{P}[[[\mu + \rho + \Phi, P]_{\alpha\oplus A}, Q]_{\alpha\oplus A}, S]_{\alpha\oplus A}
\end{aligned}$$

for $P \in C^p_{HLie}(V, \mathfrak{g})$, $Q \in C^q_{HLie}(V, \mathfrak{g})$ and $S \in C^s_{HLie}(V, \mathfrak{g})$.

Proof. First note that $\Delta = \mu + \rho + \Phi \in \ker \mathcal{P}$ with degree 1 and $[\Delta, \Delta]_{\alpha\oplus A} = 0$ due to Proposition 4.5. Thus, we have that $(L, \mathfrak{h}, \mathcal{P}, \Delta)$ is a V-data. Define the higher derived brackets $\{l_i\}_{i=1}^{+\infty}$ as (4.2). Then, for any $P \in C^p_{HLie}(V, \mathfrak{g})$, $Q \in C^q_{HLie}(V, \mathfrak{g})$ and $S \in C^s_{HLie}(V, \mathfrak{g})$ we have

$$[\mu + \rho + \Phi, P]_{\alpha\oplus A} \in \ker \mathcal{P},$$

and, hence, $l_1 = 0$. Similarly, we obtain that $l_k = 0$ for $k \geq 4$. Therefore, the graded vector space $C^*_{HLie}(V, \mathfrak{g})$ is an L_∞-algebra with nontrivial l_2, l_3, and the other higher derived brackets are trivial. □

With the aid of Proposition 4.6, we have the main theorem in this subsection.

Theorem 4.7. *Let (V,ρ,A) be a representation of a Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot],\alpha)$ and $\Phi\in C^2_{HLie}(\mathfrak{g},V)$ a 2-cocycle with respect to (V,ρ,A). Then, a linear map $R:V\to\mathfrak{g}$ is a twisted Rota-Baxter operator if and only if R is a Maurer-Cartan element of the L_∞-algebra $(C^*_{HLie}(V,\mathfrak{g}),l_2,l_3)$.*

Proof. Set $v_1,v_2\in V$. By direct computation, we have

$$l_2(R,R)(v_1,v_2)=\mathcal{P}[[\mu+\rho+\Phi,R]_{\alpha\oplus A},R]_{\alpha\oplus A}(v_1,v_2)=2([Rv_1,Rv_2]-R(\rho(Rv_1)v_2)+R(\rho(Rv_2)v_1)),$$
$$l_3(R,R,R)(v_1,v_2)=\mathcal{P}[[[\mu+\rho+\Phi,R]_{\alpha\oplus A},R]_{\alpha\oplus A},R]_{\alpha\oplus A}(v_1,v_2)=-6R(\Phi(Rv_1,Rv_2)).$$

Therefore, according to Definition 4.2 and (4.3), R is a Maurer-Cartan element of $(C^*_{HLie}(V,\mathfrak{g}),l_2,l_3)$ if and only if $\alpha\circ R=R\circ\alpha$ and

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{1}{n!}l_n(R,\cdots,R)(u,v)&=\frac{1}{2!}l_2(R,R)(u,v)+\frac{1}{3!}l_3(R,R,R)(u,v)\\&=[Ru,Rv]-R(\rho(Ru)v)+R(\rho(Rv)u)-R(\Phi(Ru,Rv))\\&=0,\end{aligned}$$

which is equivalent to $R:V\to\mathfrak{g}$ being a twisted Rota-Baxter operator on the Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot],\alpha)$ with respect to the representation (V,ρ,A). This completes the proof. □

4.2. Cohomology of twisted Rota-Baxter operators

In this subsection, we establish the cohomology of twisted Rota-Baxter operators on Hom-Lie algebras by utilizing twisted L_∞-algebra structures from a given L_∞-algebra and a Maurer-Cartan element, as introduced by Getzler [39].

Let ω be a Maurer-Cartan element of an L_∞-algebra $(\mathfrak{g}=\oplus_{k\in\mathbb{Z}}\mathfrak{g}^k,\{l_i\}_{i=1}^{+\infty})$. Define a series of twisted linear maps $l_k^\omega:\otimes^k\mathfrak{g}\to\mathfrak{g}$ of degree 1, $k\geq 1$ by

$$l_k^\omega(x_1,\cdots,x_k)=\sum_{n=0}^{+\infty}\frac{1}{n!}l_{n+k}(\underbrace{\omega,\cdots,\omega}_{n},x_1,\cdots,x_k),\ \forall x_1,\cdots,x_k\in\mathfrak{g}.$$

Theorem 4.8. ([39]) *Keeping the notations as above, $(\mathfrak{g},\{l_k^\omega\}_{k=1}^{+\infty})$ is an L_∞-algebra, obtained from $\mathfrak{g}$ by twisting with the Maurer-Cartan element ω. Furthermore, $\omega+\omega'$ is a Maurer-Cartan element of $(\mathfrak{g},\{l_i\}_{i=1}^{+\infty})$ if and only if ω' is a Maurer-Cartan element of the twisted L_∞-algebra $(\mathfrak{g},\{l_k^\omega\}_{k=1}^{+\infty})$.*

Applying Theorem 4.8 to the L_∞-algebra $(C^*_{HLie}(V,\mathfrak{g}),l_2,l_3)$, we get the following proposition.

Proposition 4.9. *Let $R:V\to\mathfrak{g}$ be a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot],\alpha)$ with respect to the representation (V,ρ,A). Then, $C^*_{HLie}(V,\mathfrak{g})$ carries a twisted L_∞-algebra structure, as follows:*

$$l_1^R(P)=l_2(R,P)+\frac{1}{2}l_3(R,R,P),$$
$$l_2^R(P,Q)=l_2(P,Q)+l_3(R,P,Q),$$

$$l_3^R(P,Q,S) = l_3(P,Q,S),$$
$$l_k^R = 0$$

for all $k \geq 4$, $P \in C^p_{HLie}(V,\mathfrak{g})$, $Q \in C^q_{HLie}(V,\mathfrak{g})$ and $S \in C^s_{HLie}(V,\mathfrak{g})$.

Proof. In view of Theorem 4.7, R is a Maurer-Cartan element of the L_∞-algebra $(C^*_{HLie}(V,\mathfrak{g}), l_2, l_3)$. Then, the result follows due to Theorem 4.8. □

Assume that $R : V \to \mathfrak{g}$ is a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ with respect to the representation (V, ρ, A). Denote the twisted L_∞-algebra in the above proposition by $(C^*_{HLie}(V,\mathfrak{g}), l_1^R, l_2^R, l_3^R)$. Therefore, we obtain that the twisted Rota-Baxter operator R generates a differential $l_1^R : C^p_{HLie}(V,\mathfrak{g}) \to C^{p+1}_{HLie}(V,\mathfrak{g})$, $p \geq 1$. Define the set of p-cochains by

$$C^p_R(V,\mathfrak{g}) = \begin{cases} \{x \in \mathfrak{g} \mid \alpha(x) = x\}, & p = 0, \\ C^p_{HLie}(V,\mathfrak{g}), & p \geq 1. \end{cases} \tag{4.5}$$

Define $d_R = l_1^R$ for all $p \geq 1$. Moreover, if $p = 0$, define $d_R : C^0_R(V,\mathfrak{g}) \to C^1_R(V,\mathfrak{g})$ by

$$d_R(x)(v) = [Rv, x] + R\rho(x)v - R\Phi(Rv, x), \ \forall x \in C^0_R(V,\mathfrak{g}), v \in V, \tag{4.6}$$

which is well defined since $\alpha(x) = x$. Thus, we have that $d_R \circ d_R = 0$; hence, $\left(\oplus_{p=0}^{+\infty} C^p_R(V,\mathfrak{g}), d_R\right)$ is a cochain complex. Then, the corresponding cohomology groups are

$$H^p_R(V,\mathfrak{g}) = \frac{Z^n_R(V,\mathfrak{g})}{B^p_R(V,\mathfrak{g})} = \frac{\left\{f \in C^p_R(V,\mathfrak{g}) \mid d_R f = 0\right\}}{\left\{d_R g \mid g \in C^{p-1}_R(V,\mathfrak{g})\right\}} \text{ for all } p \geq 0,$$

which represents **the cohomology of the twisted Rota-Baxter operator** R.

For the last part of this subsection, we give a description showing that the above twisted L_∞-algebra governs the deformations of twisted Rota-Baxter operators on Hom-Lie algebras.

Theorem 4.10. *Let $R : V \to \mathfrak{g}$ be a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ with respect to the representation (V, ρ, A). Then, for a linear map $R' : V \to \mathfrak{g}$, $R + R'$ becomes a twisted Rota-Baxter operator if and only if R' is a Maurer-Cartan element of the twisted L_∞-algebra $(C^*_{HLie}(V,\mathfrak{g}), l_1^R, l_2^R, l_3^R)$.*

Proof. According to Definition 4.2, R' is a Maurer-Cartan element of $(C^*_{HLie}(V,\mathfrak{g}), l_1^R, l_2^R, l_3^R)$ if and only if

$$l_1^R(R') + \frac{1}{2!} l_2^R(R', R') + \frac{1}{3!} l_3^R(R', R', R') = 0.$$

By Proposition 4.9, the above formula is equivalent to

$$l_2(R, R') + \frac{1}{2} l_2(R', R') + \frac{1}{2} l_3(R, R, R') + \frac{1}{2} l_3(R, R', R') + \frac{1}{6} l_3(R', R', R') = 0. \tag{4.7}$$

Since R is a twisted Rota-Baxter operator, by Theorem 4.7, we have

$$\frac{1}{2!} l_2(R, R) + \frac{1}{3!} l_3(R, R, R) = 0. \tag{4.8}$$

Collecting the two equalities (4.7) and (4.8) gives that (4.7) is equivalent to

$$\frac{1}{2!}l_2(R+R',R+R')+\frac{1}{3!}l_3(R+R',R+R',R+R')=0,$$

that is, $R+R'$ is a Maurer-Cartan element of $(C^*_{HLie}(V,\mathfrak{g}),l_2,l_3)$. By Theorem 4.7, again, it is equivalent to $R+R'$ being a twisted Rota-Baxter operator; thus, we have the conclusion. □

4.3. Cohomology of twisted Rota-Baxter operators as Chevalley-Eilenberg cohomology

In this subsection, we offer an alternative understanding of the cohomology of twisted Rota-Baxter operators. It turns out that this cohomology can be perceived as the Chevalley-Eilenberg cohomology of a specific Hom-Lie algebra with coefficients in an appropriate representation.

Recall that $(V,[\cdot,\cdot]_R)$ is the sub-adjacent Hom-Lie algebra of R (see Corollary 3.10). First, we construct the representation of $(V,[\cdot,\cdot]_R)$ as follows.

Lemma 4.11. *Let $R:V\to\mathfrak{g}$ be a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot],\alpha)$ with respect to the representation (V,ρ,A). Define $\rho_R:V\to\mathfrak{gl}(\mathfrak{g})$ by*

$$\rho_R(v)(x)=[Rv,x]+R\rho(x)v-R\Phi(Rv,x),\ \forall x\in\mathfrak{g},v\in V. \tag{4.9}$$

Then, $(\mathfrak{g},\rho_R,\alpha)$ is a representation of the Hom-Lie algebra $(V,[\cdot,\cdot]_R,A)$.

Proof. Set $v_1,v_2\in V$ and $x\in\mathfrak{g}$. By a direct check, we get that $\rho_R(Av_1)\circ\alpha=\alpha\circ\rho_R(v_1)$. Moreover, since Φ is a 2-cocycle, by (2.2) and (2.3), (3.3) and (3.4), as well as (2.9), we have

$$\begin{aligned}
&\rho_R([v_1,v_2]_R)\alpha(x)-\rho_R(Av_1)\rho_R(v_2)(x)+\rho_R(Av_2)\rho_R(v_1)(x)\\
=&\rho_R\Big(\rho(Rv_1)v_2-\rho(Rv_2)v_1+\Phi(Rv_1,Rv_2)\Big)\alpha(x)-\rho_R(Av_1)([Rv_2,x]+R\rho(x)v_2-R\Phi(Rv_2,x))\\
&+\rho_R(Av_2)([Rv_1,x]+R\rho(x)v_1-R\Phi(Rv_1,x))\\
=&[R\rho(Rv_1)v_2,\alpha(x)]+R\rho\alpha(x)\rho(Rv_1)v_2-R\Phi(R\rho(Rv_1)v_2,\alpha(x))-[R\rho(Rv_2)v_1,\alpha(x)]\\
&-R\rho(\alpha(x)\rho(Rv_2)v_1+R\Phi(R\rho(Rv_2)v_1,\alpha(x))+[R\Phi(Rv_1,Rv_2),\alpha(x)]+R\rho\alpha(x)\Phi(Rv_1,Rv_2)\\
&-R\Phi(R(\Phi(Rv_1,Rv_2)),\alpha(x))-[R(Av_1),[Rv_2,x]]-R\rho([Rv_2,x])Av_1+R\Phi(R(Av_1),[Rv_2,x])\\
&-[R(Av_1),R\rho(x)v_2]-R\rho(R\rho(x)v_2)Av_1+R\Phi(R(Av_1),R\rho(x)v_2)+[R(Av_1),R\Phi(Rv_2,x)]\\
&+R\rho(R\Phi(Rv_2,x))Av_1-R\Phi(R(Av_1),R\Phi(Rv_2,x))+[R(Av_2),[Rv_1,x]]+R\rho([Rv_1,x])Av_2\\
&-R\Phi(R(Av_2),[Rv_1,x])+[R(Av_2),R\rho(x)v_1]+R\rho(R\rho(x)v_1)Av_2-R\Phi(R(Av_2),R\rho(x)v_1)\\
&-[R(Av_2),R\Phi(Rv_1,x)]-R\rho(R\Phi(Rv_1,x))Av_2+R\Phi(R(Av_2),R\Phi(Rv_1,x))\\
=&-([\alpha(Rv_1),[Rv_2,x]]+[\alpha(Rv_2),[x,Rv_1]]+[\alpha(x),[Rv_1,Rv_2]])\\
&+R\Big(\rho([Rv_1,x])Av_2-\rho\alpha(Rv_1)\rho(x)v_2+\rho\alpha(x)\rho(Rv_1)v_2\Big)\\
&-R\Big(\rho([Rv_2,x])Av_1-\rho\alpha(Rv_2)\rho(x)v_1+\rho\alpha(x)\rho(Rv_2)v_1\Big)+(d_\rho\Phi)(x,Rv_1,Rv_2)\\
=&\,0.
\end{aligned}$$

Therefore, $(\mathfrak{g},\rho_R,\alpha)$ is a representation of the Hom-Lie algebra $(V,[\cdot,\cdot]_R,A)$. □

The above lemma allows us to consider the Chevalley-Eilenberg cohomology of the Hom-Lie algebra $(V,[\cdot,\cdot]_R,A)$ with coefficients in the representation $(\mathfrak{g},\rho_R,\alpha)$. Let $\delta_{CE}: C^p_{HLie}(V,\mathfrak{g})\to C^{p+1}_{HLie}(V,\mathfrak{g}),(p\geq 1)$ be the corresponding coboundary operator of the Hom-Lie algebra $(V,[\cdot,\cdot]_R,A)$ with coefficients in the representation $(\mathfrak{g},\rho_R,\alpha)$, where $C^p_{HLie}(V,\mathfrak{g})$ is given in Proposition 4.6. More precisely, $\delta_{CE}: C^p_{HLie}(V,\mathfrak{g})\to C^{p+1}_{HLie}(V,\mathfrak{g})$ is given by

$$\begin{aligned}(\delta_{CE}f)(v_1,\cdots,v_{p+1})=&\sum_{i=1}^{p+1}(-1)^{i+1}[R(A^{p-1}v_i),f(v_1,\cdots,\widehat{v_i},\cdots,v_{p+1})]\\&+\sum_{i=1}^{p+1}(-1)^{i+1}R\rho(f(v_1,\cdots,\widehat{v_i},\cdots,v_{p+1}))(A^{p-1}v_i)\\&-\sum_{i=1}^{p+1}(-1)^{i+1}R(\Phi(R(A^{p-1}v_i),f(v_1,\cdots,\widehat{v_i},\cdots,v_{p+1})))\\&+\sum_{i<j}(-1)^{i+j}f(\rho(Rv_i)v_j-\rho(Rv_j)v_i+\Phi(Rv_i,Rv_j),Av_1,\cdots,\widehat{v_i},\cdots,\widehat{v_j},\cdots,Av_{p+1})\end{aligned}$$

for any $f\in C^p_{HLie}(V,\mathfrak{g})$ and $v_1,\cdots,v_{p+1}\in V$. If $p=0$, according to (2.7), the coboundary map $\delta_{CE}: C^0_{HLie}(V,\mathfrak{g})\to C^1_{HLie}(V,\mathfrak{g})$ is given by

$$\delta_{CE}(x)(v)=\rho_R(v)(x)=[Rv,x]+R\rho(x)v-R\Phi(Rv,x),\ \forall x\in\mathfrak{g},v\in V;\tag{4.10}$$

in view of (4.6), we obtain that

$$\delta_{CE}(x)=d_R(x),\ \forall x\in\mathfrak{g}.\tag{4.11}$$

Denote the Chevalley-Eilenberg cohomology group correspondng to the cochain complex $\left(\oplus_{p=0}^{+\infty}C^p_{HLie}(V,\mathfrak{g}),\delta_{CE}\right)$ by $H^*_{CE}(V,\mathfrak{g})$.

Furthermore, comparing the coboundary operator δ_{CE} given above with the twisted linear map l_1^R introduced in Proposition 4.9, we obtain the following result.

Theorem 4.12. *Assume that $R: V\to\mathfrak{g}$ is a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot],\alpha)$ with respect to the representation (V,ρ,A). Then, we have*

$$\delta_{CE}f=l_1^Rf,\ \forall f\in C^p_{HLie}(V,\mathfrak{g}),\ p\geq 1.$$

Proof. Set $\mu+\rho+\Phi=\Delta$. For any $f\in C^p_{HLie}(V,\mathfrak{g})$ and $v_1,\cdots,v_{p+1}\in V$, by (4.4), we have

$$\begin{aligned}&l_2(R,f)(v_1,\cdots,v_{p+1})=\mathcal{P}[[\Delta,R]_{\alpha\oplus A},f]_{\alpha\oplus A}(v_1,\cdots,v_{p+1})\\&=\mathcal{P}\Big((-1)^{p-1}\{[\Delta,R]_{\alpha\oplus A},f\}_{\alpha\oplus A}(v_1,\cdots,v_{p+1})-\{f,[\Delta,R]_{\alpha\oplus A}\}_{\alpha\oplus A}(v_1,\cdots,v_{p+1})\Big)\\&=(-1)^{p-1}\mathcal{P}\{\{\Delta,R\}_{\alpha\oplus A},f\}_{\alpha\oplus A}(v_1,\cdots,v_{p+1})-(-1)^{p-1}\{\{R,\Delta\}_{\alpha\oplus A},f\}_{\alpha\oplus A}(v_1,\cdots,v_{p+1})\\&\quad-\{f,\{\Delta,R\}_{\alpha\oplus A}\}_{\alpha\oplus A}(v_1,\cdots,v_{p+1})+\{f,\{R,\Delta\}_{\alpha\oplus A}\}_{\alpha\oplus A}(v_1,\cdots,v_{p+1})\\&=\sum_{i=1}^{p+1}(-1)^{i+1}[R(A^{p-1}v_i),f(v_1,\cdots,\widehat{v_i},\cdots,v_{p+1})]\end{aligned}$$

$$+\sum_{i=1}^{p+1}(-1)^{i+1}R\rho(f(v_1,\cdots,\widehat{v_i},\cdots,v_{p+1}))(A^{p-1}v_i)$$
$$+\sum_{i<j}(-1)^{i+j}f(\rho(Rv_i)v_j-\rho(Rv_j)v_i,Av_1,\cdots,\widehat{v_i},\cdots,\widehat{v_j},\cdots,Av_{p+1}).$$

Similarly, by direct computation, we obtain

$$\frac{1}{2}l_3(R,R,f)(v_1,\cdots,v_{p+1})=-\sum_{i=1}^{p+1}(-1)^{i+1}R(\Phi(R(A^{p-1}v_i),f(v_1,\cdots,\widehat{v_i},\cdots,v_{p+1}))).$$

Therefore, we deduce that

$$\delta_{CE}f=l_2(R,f)+\frac{1}{2}l_3(R,R,f)=l_1^Rf.$$

This completes the proof. □

Combining the above theorem and (4.11), we arrive at the subsequent corollary.

Corollary 4.13. *Assume that $R:V\to\mathfrak{g}$ is a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot],\alpha)$ with respect to the representation (V,ρ,A). Then,*

$$H_R^p(V,\mathfrak{g})=H_{CE}^p(V,\mathfrak{g}),\ \forall p\geq 0.$$

5. Deformations of twisted Rota-Baxter operators on Hom-Lie algebras

In this section, we employ the established cohomology to investigate the linear and formal deformations of twisted Rota-Baxter operators on Hom-Lie algebras. We establish that equivalent linear deformations define the same cohomology class, as well as formal deformations. Furthermore, we characterize the rigidity of formal deformations based on Nijenhuis elements.

5.1. Linear deformations of twisted Rota-Baxter operators

Assume that $R:V\to\mathfrak{g}$ is a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot],\alpha)$ with respect to the representation (V,ρ,A). Recall from (4.5) that $C_R^0(V,\mathfrak{g})=\{x\in\mathfrak{g}\mid\alpha(x)=x\}$ and $C_R^1(V,\mathfrak{g})=C_{HLie}^1(V,\mathfrak{g})=\{f\in\mathrm{Hom}(V,\mathfrak{g})\mid\alpha\circ f=f\circ A\}$.

Definition 5.1. A **linear deformation** of a twisted Rota-Baxter operator R consists of $R_t=R+tR_1$ with $R_0=R$ such that, for all $t\in\mathbb{K}$, R_t is still a twisted Rota-Baxter operator.

Suppose that $R_t=R+tR_1$ is a linear deformation of R. Thus, we have that $\alpha\circ R_t=R_t\circ A$ and

$$[R_tv_1,R_tv_2]=R_t(\rho(R_tv_1)v_2-\rho(R_tv_2)v_1+\Phi(R_tv_1,R_tv_2)),\ \forall v_1,v_2\in V.$$

By direct computation, we have that $\alpha\circ R_1=R_1\circ A$ and

$$\begin{aligned}[Rv_1,R_1v_2]+[R_1v_1,Rv_2]=&R(\rho(R_1v_1)v_2-\rho(R_1v_2)v_1+\Phi(Rv_1,R_1v_2)+\Phi(R_1v_1,Rv_2))\\&+R_1(\rho(Rv_1)v_2-\rho(Rv_2)v_1+\Phi(Rv_1,Rv_2)).\end{aligned}$$

Equivalently, $R_1\in C_R^1(V,\mathfrak{g})$ and $d_RR_1=0$. Then, we get that R_1 is a 1-cocycle within the cohomology of R.

Definition 5.2. Assume that R and R' are two twisted Rota-Baxter operators on a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ with respect to the representation (V, ρ, A). A **morphism** of twisted Rota-Baxter operators from R to R' consists of a pair (ϕ, ψ) of the Hom-Lie algebra morphism $\phi : \mathfrak{g} \to \mathfrak{g}$ and a linear map $\psi : V \to V$ such that, for any $x \in \mathfrak{g}$, it holds that

$$\psi \circ \rho(x) = \rho(\phi(x)) \circ \psi,\ \psi \circ \Phi = \Phi \circ (\phi \otimes_{\mathbb{K}} \phi),\ \psi \circ A = A \circ \psi,\ \phi \circ R = R' \circ \psi.$$

Furthermore, (ϕ, ψ) will be called an **isomorphism** from R to R' if ϕ and ψ are invertible.

Definition 5.3. Assume that $R : V \to \mathfrak{g}$ is a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ with respect to the representation (V, ρ, A). Two linear deformations, $R_t = R + tR_1$ and $R_t^{'} = R + tR_1^{'}$, are said to be **equivalent** if there exists $x \in \mathfrak{g}$ such that $\alpha(x) = x$ and

$$(\phi_t = \mathrm{Id}_{\mathfrak{g}} + t\mathrm{ad}_x, \psi_t = \mathrm{Id}_V + t\rho(x) + t\Phi(x, R-))$$

is a morphism from R_t to $R_t^{'}$.

Suppose that $(\phi_t = \mathrm{Id}_{\mathfrak{g}} + t\mathrm{ad}_x, \psi_t = \mathrm{Id}_V + t\rho(x) + t\Phi(x, R-))$ is a morphism from R_t to $R_t^{'}$. Then, ϕ_t being a Hom-Lie algebra morphism means that

$$[[x, y], [x, z]] = 0,\ \forall y, z \in \mathfrak{g}. \tag{5.1}$$

The condition $\psi_t \circ \rho(x) = \rho(\phi_t(x)) \circ \psi_t$ is equivalent to

$$\begin{cases} \Phi(x, R\rho(y)v) = \rho(y)\Phi(x, Rv), \\ \rho([x, y])(\rho(x)v + \Phi(x, Rv)) = 0, \forall y \in \mathfrak{g}, v \in V. \end{cases} \tag{5.2}$$

The condition $\psi_t \circ \Phi_t = \Phi \circ (\phi_t \otimes_{\mathbb{K}} \phi_t)$ means that

$$\begin{cases} \rho(x)\Phi(y, z) + \Phi(x, R\Phi(y, z)) = \Phi([x, y], z) + \Phi(y, [x, z]), \\ \Phi([x, y], [x, z]) = 0,\ \forall y, z \in \mathfrak{g}. \end{cases} \tag{5.3}$$

Moreover, the formula $\psi_t \circ A = A \circ \psi_t$ is established automatically because $\alpha(x) = x$. Finally, the condition $\phi \circ R = R' \circ \psi$ is equivalent to

$$\begin{cases} R_1 v + [x, Rv] = R\rho(x)v + R\Phi(x, Rv) + R_1^{'}v, \\ [x, R_1 v] = R_1^{'}(\rho(x)v + \Phi(x, Rv)),\ \forall v \in V. \end{cases} \tag{5.4}$$

Note that the first condition of (5.4) implies that $R_1 - R_1^{'} = d_R(x)$. Consequently, we have the following result.

Theorem 5.4. *Let $R_t = R + tR_1$ and $R_t^{'} = R + tR_1^{'}$ be two equivalent linear deformations of a twisted Rota-Baxter operator R. Then, R_1 and $R_1^{'}$ are in the same cohomology class in $H_R^1(V, \mathfrak{g})$.*

Remark 5.5. A linear deformation $R_t = R + tR_1$ of a twisted Rota-Baxter operator R is called **trivial** if R_t is equivalent to the unaltered deformation $R_t^{'} = R$.

5.2. Formal deformations of twisted Rota-Baxter operators

In this subsection, we investigate formal deformations of twisted Rota-Baxter operators on Hom-Lie algebras.

Assume that $R : V \to \mathfrak{g}$ is a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ with respect to the representation (V, ρ, A). Denote by $V[[t]]$ the formal power series in t with coefficients in V. There exists a Hom-Lie algebra structure over the ring $\mathbb{K}[[t]]$ on $\mathfrak{g}[[t]]$ that is given by

$$[\sum_{i=0}^{+\infty} x_i t^i, \sum_{j=0}^{+\infty} y_j t^j] = \sum_{s=0}^{+\infty}\sum_{i+j=s} [x_i, y_j] t^s,\ \forall x_i, y_j \in \mathfrak{g}. \tag{5.5}$$

Moreover, there is a representation (denoted also by ρ) of the Hom-Lie algebra $\mathfrak{g}[[t]]$ that is given by

$$\rho(\sum_{i=0}^{+\infty} x_i t^i)(\sum_{j=0}^{+\infty} v_j t^j) = \sum_{s=0}^{+\infty}\sum_{i+j=s} \rho(x_i) v_j t^s,\ \forall x_i \in \mathfrak{g}, v_j \in V. \tag{5.6}$$

The 2-cocycle Φ can be extended to a 2-cocycle on the Hom-Lie algebra $\mathfrak{g}[[t]]$ with coefficients in $V[[t]]$; denote it by using the same notation Φ.

Consider the following formal power series:

$$R_t = \sum_{i=0}^{+\infty} R_i t^i,\ \forall R_i \in C^1_R(V, \mathfrak{g}). \tag{5.7}$$

Since $R_t \in \mathrm{Hom}(V, \mathfrak{g})[[t]] = \mathrm{Hom}(V, \mathfrak{g}[[t]])$, we may extend it to a $\mathbb{K}[[t]]$-module map from $V[[t]]$ to $\mathfrak{g}[[t]]$, and we still denote it by R_t.

Definition 5.6. A **formal deformation** of a twisted Rota-Baxter operator R consists of a formal power series $R_t = \sum_{i=0}^{+\infty} R_i t^i$, with $R_0 = R$ such that, for all $t \in \mathbb{K}$, R_t remains as a twisted Rota-Baxter operator.

Lemma 5.7. *$R_t = \sum\limits_{i=0}^{+\infty} R_i t^i$ is a formal deformation of R if and only if*

$$\sum_{i+j=n}^{+\infty} [R_i v_1, R_j v_2] = \sum_{i+j=n}^{+\infty} R_i(\rho(R_j v_1) v_2 - \rho(R_j v_2) v_1) + \sum_{i+j+k=n}^{+\infty} R_i \Phi(R_j v_1, R_k v_2) t^{i+j+k},\ \forall v_1, v_2 \in V, n \geq 0. \tag{5.8}$$

Proof. Straightforward. □

For $n = 0$, (5.8) gives that $R = R_0$ is a twisted Rota-Baxter operator. For $s = 1$, it follows that

$$\begin{aligned}&[Rv_1, R_1 v_2] + [R_1 v_1, R v_2] = \\ &R(\rho(R_1 v_1) v_2 - \rho(R_1 v_2) v_1) + R_1(\rho(R v_1) v_2 - \rho(R v_2) v_1) + R\Phi(R v_1, R_1 v_2) + R\Phi(R_1 v_1, R v_2) + R_1 \Phi(R v_1, R v_2),\end{aligned}$$

which implies that $d_R R_1 = 0$; hence, R_1 is exactly a 1-cocycle of the cohomology of the twisted Rota-Baxter operator R. Moreover, by direct computation, we have the following.

Proposition 5.8. *Let $R_t = \sum\limits_{i=0}^{+\infty} R_i t^i$ be a formal deformation of a twisted Rota-Baxter operator R. If $R_i = 0, 1 \leq i < n$, then R_n is a 1-cocycle with respect to the cohomology of R, that is, $R_n \in Z^1_R(V, \mathfrak{g})$.*

A 1-cochain R_n is called the n-**infinitesimal** of R_t if $R_i = 0$ for all $1 \leq i < n$. In particular, the 1-cocycle R_1 is called the **infinitesimal** (or 1-infinitesimal) of R_t. Due to Proposition 5.8, the n-infinitesimal R_n is a 1-cocycle.

Definition 5.9. Assume that $R : V \to \mathfrak{g}$ is a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ with respect to the representation (V, ρ, A). Two formal deformations $R_t = R + \sum\limits_{i \geq r}^{+\infty} R_i t^i$ and $R_t^{'} = R + \sum\limits_{i \geq r}^{+\infty} R_i^{'} t^i$ $(r \geq 1)$ are called **equivalent** if there exist $x \in \mathfrak{g}$, $\phi_i \in \mathfrak{gl}(\mathfrak{g})$ and $\psi_i \in \mathfrak{gl}(V), i \geq r+1$ such that

$$(\phi_t = \mathrm{Id}_{\mathfrak{g}} + t^r \mathrm{ad}_x + \sum_{i=r+1}^{+\infty} \phi_i t^i, \psi_t = \mathrm{Id}_V + t^r \rho(x) + t^r \Phi(x, R-) + \sum_{i=r+1}^{+\infty} \psi_i t^i)$$

is a morphism from R_t to $R_t^{'}$.

Particularly, if $r = 1$, owing to Definition 5.9, two formal deformations $R_t = \sum\limits_{i=0}^{+\infty} R_i t^i$ and $R_t^{'} = \sum\limits_{i=0}^{+\infty} R_i^{'} t^i$ are equivalent if there exist $x \in \mathfrak{g}$, $\phi_i \in \mathfrak{gl}(\mathfrak{g})$ and $\psi_i \in \mathfrak{gl}(V), i \geq 2$ such that

$$(\phi_t = \mathrm{Id}_{\mathfrak{g}} + t\mathrm{ad}_x + \sum_{i=2}^{+\infty} \phi_i t^i, \psi_t = \mathrm{Id}_V + t\rho(x) + t\Phi(x, R-) + \sum_{i=2}^{+\infty} \psi_i t^i)$$

is a morphism from R_t to $R_t^{'}$. Then, by extracting the coefficients of t from both sides of $\phi_t \circ R_t = R_t^{'} \circ \psi_t$, we get

$$R_1 v - R_1^{'} v = [Rv, x] + R\rho(x)v + R\Phi(x, Rv) = d_R(x)(v), \ \forall\, v \in V;$$

thus, we have the following result:

Proposition 5.10. *Assume that $R : V \to \mathfrak{g}$ is a twisted Rota-Baxter operator on a* Hom-Lie *algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ with respect to the representation (V, ρ, A). If two formal deformations $R_t = \sum\limits_{i=0}^{+\infty} R_i t^i$ and $R_t^{'} = \sum\limits_{i=0}^{+\infty} R_i^{'} t^i$ of R are equivalent, then their infinitesimals R_1 and $R_1^{'}$ are in the same cohomology class.*

At the end of this subsection, we investigate the rigidity of a twisted Rota-Baxter operator based on Nijenhuis elements.

Definition 5.11. A twisted Rota-Baxter operator R is called **rigid** if any formal deformation of R is equivalent to the unaltered deformation $R_t^{'} = R$.

Definition 5.12. Assume that $R : V \to \mathfrak{g}$ is a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ with respect to the representation (V, ρ, A). An element $x \in \mathfrak{g}$ with $\alpha(x) = x$ is said to be a **Nijenhuis element** if x satisfies that

$$[x, [Rv, x] + R\rho(x)v + R\Phi(x, Rv)] = 0, \ \forall v \in V$$

and (5.1)–(5.3) hold. Denote the set of Nijenhuis elements associated with R by $Nij(R)$.

By the above definition and Remark 5.5, any trivial linear deformation induces a Nijenhuis element. Moreover, we give a sufficient condition to characterize the rigidity of a twisted Rota-Baxter operator, as follows.

Theorem 5.13. *Let $R : V \to \mathfrak{g}$ be a twisted Rota-Baxter operator on a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ with respect to the representation (V, ρ, A). If $Z^1_R(V, \mathfrak{g}) = d_R(Nij(R))$, then R is rigid.*

Proof. Suppose that $R_t = \sum_{i=0}^{+\infty} R_i t^i$ is a formal deformation of R. Thanks to Proposition 5.8, R_1 is a 1-cocycle of R. Hence, there exists a Nijenhuis element $x \in \mathfrak{g}$ such that $R_1 = d_R(x)$. Set

$$\phi_t = \mathrm{Id}_{\mathfrak{g}} + t\mathrm{ad}_x, \quad \psi_t = \mathrm{Id}_V + t\rho(x) + t\Phi(x, R-).$$

Define $R'_t = \phi_t \circ R_t \circ \psi_t^{-1}$. Since x is a Nijenhuis element, we deduce that (ϕ_t, ψ_t) is a morphism from R_t to R'_t; hence, R'_t is a formal deformation of R, which is equivalent to R_t. Furthermore, for $v \in V$, by direct computation, we obtain

$$\begin{aligned} R'_t v &= (\mathrm{Id}_{\mathfrak{g}} + t\mathrm{ad}_x) \circ R_t \circ (\mathrm{Id}_V - t\rho(x) - t\Phi(x, R-) + \text{power of } t^{\geq 2}) \\ &= (\mathrm{Id}_{\mathfrak{g}} + t\mathrm{ad}_x)(Rv - t(R\rho(x)v + R\Phi(x, Rv) + R_1 v) + \text{power of } t^{\geq 2}) \\ &= Rv + t(R_1 v - d_R(x)v) + \text{power of } t^{\geq 2} \\ &= Rv + \text{power of } t^{\geq 2}, \end{aligned}$$

which implies that the coefficient of t in the expression of R'_t is trivial. Continuing by induction, we finally have that R_t is equivalent to R. This completes the proof. □

6. Conclusions

In this article, we introduced the concept of a twisted Rota-Baxter operator on a Hom-Lie algebra and defined its cohomology by constructing a twisting L_∞-algebra associated with the twisted Rota-Baxter operator. We constructed a Chevalley-Eilenberg cohomology for a certain Hom-Lie algebra with coefficients in an appropriate representation. Surprisingly, this Chevalley-Eilenberg cohomology coincides with the cohomology of twisted Rota-Baxter operators. We also showed that the linear component in a linear or formal deformation of a twisted Rota-Baxter operator R is a 1-cocycle in the cohomology of R. At the end, we gave a sufficient condition to characterize the rigidity of formal deformations based on Nijenhuis elements.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

Acknowledgments

The authors would like to thank the anonymous referees for valuable comments that have helped to improved the article significantly.

Xu was partially supported by NSF of China (12201253) and NSF of Jiangsu Province (BK20220510).

Conflict of interest

The authors declare that they have no conflicts of interest.

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