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*Research article*

# A new family of positively based algebras $\mathcal{H}_n$

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**Abstract:** In this paper, we introduce a new family of algebras $\mathcal{H}_n$, which are generated by three generators $x, y, z$, with the following relations: (1) $x^{2n} = 1,\ y^2 = xy + y,\ xy = yx$; and (2) $z^2 = z,\ xz = zx = z,\ zy = 2z$. First, it shows that $\mathcal{H}_n$ is a positively based algebra. Then, all the indecomposable modules of $\mathcal{H}_n$ are constructed. Additionally, it shows that the dimension of each indecomposable $\mathcal{H}_n$-module is at most 2. Finally, all the left (right) cells and left (right) cell modules of $\mathcal{H}_n$ are described, and the decompositions of the decomposable left cell modules are also obtained.

**Keywords:** positively based algebra; indecomposable module; cell module
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## 1. Introduction

An algebra $A$ is called a positively based algebra if it has a positive basis. By the latter, we mean that a set of bases with all the structure constants with respect to this basis are non-negative real numbers. The concept of a positive basis can be traced back to the work of Schur on the centralizer algebra of a transitive permutation representation of a finite group in [16, 17]. Since then, variations of the positively based algebras have been discovered in many different fields (see [1, 3, 15]). Examples of positively based algebras include the group algebras, semigroup algebras, table algebras, and Hecke algebras corresponding to Coxeter groups with respect to the Kazhdan-Lusztig basis. As an interesting infinite dimensional example, Thurston proved that the Kauffman bracket skein algebra for a compact oriented surface has a natural positive basis in [19], which generalized the positivity conjecture in cluster algebra. In [14], Mazorchuk et al. defined the cell 2-representations of finitary 2-categories. It turns out that the cell 2-representation is a based module over some finite-dimensional positively based algebras on the level of the Grothendieck group. In addition, in the perspective of representation theory, Green algebra of a bialgebra is a central instance of positively based algebras. For example, Cao et al. [4] studied the cell modules of the Green algebra of the generalized Taft Hopf algebras, in which the approach depends on the Green ring.

In [10], Lin and Yang introduced a class of positively based algebras $A_{n,d}$ over the subring $\mathbb{K}$ of the complex field $\mathbb{C}$; all indecomposable modules and cell modules of $A_{n,d}$ are classified and constructed. It is noted that $A_{n,d}$ can be regarded as the Green algebra of weak Hopf algebras $\mathfrak{w}_{n,d}^1$ constructed from the generalized Taft algebra (see [18]). In the present paper, we introduce the $\mathbb{K}$-algebra $\mathcal{H}_n$, which is generated by $x, y, z$ with the relations (1) and (2); since we can construct a set of positive bases of $\mathcal{H}_n$, we can conclude that $\mathcal{H}_n$ is a positively based algebra. The algebra $\mathcal{H}_n$ can be viewed as the extension of the Green ring of the weak Hopf algebra $\mathfrak{w}H_{4n}$, which is constructed from neither the pointed nor the semisimple Hopf algebra $H_{4n}$ (see [5]). Recall that $A_{n,d}$ in [10, Definition 2.1] is generated by $x, y, z$ with the following relations:

$$
\begin{aligned}
&(1') \qquad x^{nd} = 1,\ xy = yx;\\
&(2') \qquad (1 + x^n - y) \sum_{i=0}^{[\frac{d-1}{2}]} (-1)^i \binom{d-1-i}{i} x^{ni} y^{d-1-2i} = 0,\ \text{for } d \geq 2;\\
&(3') \qquad z^2 = z,\ xz = zx = z,\ yz = 2z.
\end{aligned}
$$

We see that the generating relations of $A_{n,d}$ and $\mathcal{H}_n$ are strongly different. For instance, the representation type of $A_{n,d}$ depends on $d$, and it is of finite representation type if $d \leq 4$, of tame type if $d = 5$, and of wild type if $d \geq 6$. However, the algebra $\mathcal{H}_n$ is only of finite representation type. Up to now, there is a limited number of works that classifies the representations of this family of positively based algebras. This provides a good chance to understand the representation theory of $\mathcal{H}_n$. In this paper, as the analog of the method in [10], we first construct a set of positive basis of $\mathcal{H}_n$ and show that $\mathcal{H}_n$ is a positively based algebra. Then, we make efforts to classify all the indecomposable modules of $\mathcal{H}_n$ by utilizing the representation theory of a quiver. It is easily seen that the algebra $\mathcal{H}_n$ is of finite representation type, and thus all the indecomposable modules of $\mathcal{H}_n$ are constructed. Finally, all the left (right) cells and left cell (right) modules of $\mathcal{H}_n$ are constructed. Moreover, the decompositions of the left cell modules are provided.

Let us describe the arrangement of this paper. In Section 2, we introduce the algebra $\mathcal{H}_n$ by generators and generating relations, and show that $\mathcal{H}_n$ is a positively based algebra. In Section 3, we focus on constructing all indecomposable modules of $\mathcal{H}_n$ and we see that $\mathcal{H}_n$ is of finite representation type. In Section 4, we classify all the left (right) cells and left (right) cell modules of $\mathcal{H}_n$.

## 2. Preliminaries

Throughout the paper and unless otherwise stated, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$, stand for the field of complex numbers, the field of real numbers, the ring of integers, and the set of natural numbers, respectively. Fixing an integer $n \geq 1$, we always suppose that $\mathbb{K}$ is an unital subring of $\mathbb{C}$ containing the primitive $2n$-th root of the unity:

$$\eta = \cos\frac{\pi}{n} + \mathrm{i}\sin\frac{\pi}{n}.$$

Let $A$ be a $\mathbb{K}$-algebra of finite free rank $n$ with a basis $\mathfrak{B} = \{a_i \mid 1 \leq i \leq n\}$. For any $1 \leq i, j, k \leq n$, if

$$a_i \cdot a_j = \sum_{k=1}^{n} \gamma_{i,j}^{(k)} a_k,$$

where $\gamma_{i,j}^{(k)} \in \mathbb{R} \geq 0$, then $\mathfrak{B}$ is said to be a positive basis of $A$, and $A$ is called a positively based algebra.

Now, we define the algebra $\mathcal{H}_n$ and describe its basic properties.

**Definition 2.1.** *The $\mathbb{K}$-algebra $\mathcal{H}_n$ is generated by $x, y, z$ with the following relations*:

*(1)* $x^{2n} = 1$, $y^2 = xy + y$, $xy = yx$;
*(2)* $z^2 = z$, $xz = zx = z$, $zy = 2z$.

It is easy to see that $\mathcal{H}_n$ is noncommutative, and it is noted that when $\mathbb{K} = \mathbb{C}$, the algebra $\mathcal{H}_n$ is just the Green algebra of the weak Hopf algebra $\mathfrak{w}H_{4n}$ (see [5]).

In the following, we construct a set of positive bases of $\mathcal{H}_n$ to show that $\mathcal{H}_n$ is a positively based algebra. For this purpose, we list one lemma as follows.

**Lemma 2.2.** *(1) The $\mathbb{K}$-space $\mathcal{W}_i$ $(0 \leq i \leq 2n-1)$ with a basis $\{v_i\}$ is an $\mathcal{H}_n$-module with the action of $\mathcal{H}_n$ :*

$$x \cdot v_i = \eta^i v_i, \quad y \cdot v_i = (1 + \eta^i) v_i, \quad z \cdot v_i = 0.$$

*(2) The $\mathbb{K}$-space $\mathcal{V}_1$ with a basis $\{\theta_1, \theta_2\}$ is an $\mathcal{H}_n$-module with the action of $\mathcal{H}_n$ :*

$$\begin{aligned} x \cdot \theta_1 &= -\theta_1, \quad y \cdot \theta_1 = 0, \quad z \cdot \theta_1 = 0, \\ x \cdot \theta_2 &= -\theta_2, \quad y \cdot \theta_2 = \theta_1, \quad z \cdot \theta_2 = 0. \end{aligned}$$

*(3) The $\mathbb{K}$-space $\mathcal{V}_2$ with a basis $\{\xi_1, \xi_2\}$ is an $\mathcal{H}_n$-module with the action of $\mathcal{H}_n$ :*

$$\begin{aligned} x \cdot \xi_1 &= \xi_1, \quad y \cdot \xi_1 = \xi_2, \quad z \cdot \xi_1 = \xi_1, \\ x \cdot \xi_2 &= \xi_2, \quad y \cdot \xi_2 = 2\xi_2, \quad z \cdot \xi_2 = 2\xi_1. \end{aligned} \tag{2.1}$$

*(4) The $\mathbb{K}$-space $\mathcal{V}_3$ with a basis $\left\{\mu_j \mid 0 \leq j \leq 2n-1\right\}$ is an $\mathcal{H}_n$-module with the action of $\mathcal{H}_n$ :*

$$x \cdot \mu_j = \mu_{j+1(\mathrm{mod} 2n)}, \quad y \cdot \mu_j = 0, \quad z \cdot \mu_j = 0.$$

*Proof.* In the following, we only prove the statement (3); the proofs of the other statements are similar.

First, it is easy to see that

$$\underbrace{(x \cdot (x \cdots (x}_{2n} \cdot \xi_i) \cdots)) = \xi_i. \quad (i = 1, 2),$$

Then, the straightforward verification shows that

$$y \cdot (y \cdot \xi_1) = 2\xi_2 = x \cdot (y \cdot \xi_1) + y \cdot \xi_1, \quad y \cdot (y \cdot \xi_2) = 4\xi_2 = x \cdot (y \cdot \xi_2) + y \cdot \xi_2,$$

and

$$y \cdot (x \cdot \xi_1) = \xi_2 = x \cdot (y \cdot \xi_1), \quad y \cdot (x \cdot \xi_2) = 2\xi_2 = x \cdot (y \cdot \xi_2).$$

Moreover, it is obvious that

$$z \cdot (z \cdot \xi_1) = \xi_1 = z \cdot \xi_1, \quad z \cdot (z \cdot \xi_2) = 2\xi_1 = z \cdot \xi_2,$$

and

$$x \cdot (z \cdot \xi_1) = \xi_1 = z \cdot \xi_1 = z \cdot (x \cdot \xi_1),$$

$$x \cdot (z \cdot \xi_2) = 2\xi_1 = z \cdot \xi_1 = z \cdot (x \cdot \xi_1).$$

Finally, we see that

$$z \cdot (y \cdot \xi_1) = 2\xi_1 = 2z \cdot \xi_1, \quad z \cdot (y \cdot \xi_2) = 4\xi_1 = 2z \cdot \xi_2.$$

It follows that the generators $x, y$ and $z$ acting on $\mathcal{V}_2$ keep the defining relations.
Hence, the actions of $x, y$ and $z$ on $\mathcal{V}_2$ define an $\mathcal{H}_n$-module. □

**Proposition 2.3.** *The set* $\left\{x^i y^j, y^k z \mid 0 \leq i \leq 2n-1, j, k = 0, 1\right\}$ *forms a basis of* $\mathcal{H}_n$.

*Proof.* By the defining relations of $\mathcal{H}_n$, one can see that $\mathcal{H}_n$ are spanned by the following:

$$\left\{x^i y^j, y^k z \mid 0 \leq i \leq 2n-1, j, k = 0, 1\right\}.$$

Therefore, it is sufficient to prove that the set

$$\left\{x^i y^j, y^k z \mid 0 \leq i \leq 2n-1, j, k = 0, 1\right\}$$

consists of linearly independent elements.

Now, we assume the following:

$$\sum_{i=0}^{2n-1} a_i x^i + \sum_{j=0}^{2n-1} b_j x^j y + \sum_{k=0}^{1} c_k y^k z = 0. \tag{I}$$

Acting on $\mu_0$ by the both sides of (I), we have $\sum\limits_{i=0}^{2n-1} a_i \mu_i = 0$ and

$$a_i = 0, \quad 0 \leq i \leq 2n-1.$$

It yields that

$$\sum_{j=0}^{2n-1} b_j x^j y + \sum_{k=0}^{1} c_k y^k z = 0. \tag{II}$$

Acting on $\{\nu_0, \cdots, \nu_{n-1}, \theta_2, \nu_{n+1}, \cdots \nu_{2n-1}\}$ by the both sides of (II), we have

$$\left(\sum_{j=0}^{2n-1} b_j x^j y + \sum_{k=0}^{1} c_k y^k z\right) \cdot \theta_2 = \left(\sum_{j=0}^{2n-1} b_j x^j\right) \cdot \theta_1 = \sum_{j=0}^{2n-1} (-1)^j b_j \theta_1 = 0$$

and

$$\left(\sum_{j=0}^{2n-1} b_j x^j y + \sum_{k=0}^{1} c_k y^k z\right) \cdot \nu_i = \left(\sum_{j=0}^{2n-1} b_j x^j\right) \cdot \left(1 + \eta^i\right) \nu_i = \left(1 + \eta^i\right) \sum_{j=0}^{2n-1} \eta^{ij} b_j \nu_i = 0.$$

Hence, obtain the following:

$$\sum_{j=0}^{2n-1} (-1)^j b_j = 0 \quad \text{and} \quad \sum_{j=0}^{2n-1} \eta^{ij} b_j = 0 \quad (i \neq n),$$

since $1 + \eta^i \neq 0$.

Thus,

$$\begin{pmatrix} 1 & \cdots & 1 & 1 & 1 & \dots & 1 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \cdots & (\eta^{n-1})^{n-1} & (\eta^{n-1})^{n} & (\eta^{n-1})^{n+1} & \dots & (\eta^{2n-1})^{i} \\ 1 & \cdots & (-1)^{n-1} & (-1)^{n} & (-1)^{n+1} & \dots & -1 \\ 1 & \cdots & (\eta^{n+1})^{n-1} & (\eta^{n+1})^{n} & (\eta^{n+1})^{n+1} & \dots & (\eta^{2n-1})^{i} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \cdots & (\eta^{2n-1})^{n-1} & (\eta^{2n-1})^{n} & (\eta^{2n-1})^{n+1} & \dots & (\eta^{2n-1})^{2n-1} \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_{n-1} \\ b_n \\ b_{n+1} \\ \vdots \\ b_{2n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

It is easy to see that the determinant of the coefficient matrix is nonzero; thus, we have the following:

$$b_j = 0 \text{ for } 0 \leq j \leq 2n-1.$$

Moreover, it yields that

$$\sum_{k=0}^{1} c_k y^k z = 0. \qquad \text{(III)}$$

Acting on $\{\xi_1\}$ by the both sides of (III), we have

$$\sum_{k=0}^{1} c_k y^k z \cdot \xi_1 = c_0 \xi_1 + c_1 \xi_2 = 0,$$

and $c_0 = c_1 = 0$.

In summary,

$$\left\{x^i y^j, y^k z \mid 0 \leq i \leq 2n-1, j, k = 0, 1\right\}$$

is a linearly independent set. Hence, it is a basis of $\mathcal{H}_n$.

The proof is finished. □

Set $L_i = x^i$, $M_i = x^i y$ for $0 \leq i \leq 2n-1$, $N_0 = z$ and $N_1 = yz$. Let

$$\mathcal{B} = \{L_i, M_i, N_k \mid 0 \leq i \leq 2n-1, k = 0, 1\}.$$

Then, $\mathcal{B}$ is a basis of $\mathcal{H}_n$.

**Proposition 2.4.** *For $0 \leq i, j \leq 2n-1$, we have the following*:

*(1)* $L_i \cdot L_j = L_j \cdot L_i = L_{i+j(\mathrm{mod}2n)}$;
*(2)* $L_i \cdot M_j = M_j \cdot L_i = M_{i+j(\mathrm{mod}2n)}$;
*(3)* $M_i \cdot M_j = M_j \cdot M_i = M_{i+j(\mathrm{mod}2n)} + M_{i+j+1(\mathrm{mod}2n)}$;
*(4)* $N_0 \cdot N_0 = N_0 \cdot L_i = L_i \cdot N_0 = N_0$;
*(5)* $N_0 \cdot N_1 = N_0 \cdot M_i = 2N_0$;
*(6)* $N_1 \cdot N_0 = M_i \cdot N_0 = N_1 \cdot L_i = L_i \cdot N_1 = N_1$;
*(7)* $N_1 \cdot N_1 = N_1 \cdot M_i = M_i \cdot N_1 = 2N_1$.

*Proof.* For $0 \leq i, j \leq 2n-1$, by a straightforward verification, we have

$$\begin{aligned} L_i \cdot L_j &= x^i x^j = x^{i+j(\mathrm{mod} 2n)} \\ &= L_{i+j(\mathrm{mod} 2n)} = L_j \cdot L_i \end{aligned}$$

and

$$\begin{aligned} L_i \cdot M_j &= x^i x^j y = x^{i+j(\mathrm{mod} 2n)} y \\ &= M_{i+j(\mathrm{mod} 2n)} = M_j \cdot L_i, \end{aligned}$$

since $x^{2n} = 1, xy = yx$.

Hence, (1) and (2) hold.

(3) By $y^2 = xy + y$, we have the following:

$$\begin{aligned} M_i \cdot M_j &= x^i y x^j y = x^i x^j y^2 = x^i x^j (xy + y) \\ &= x^{i+j(\mathrm{mod} 2n)} y + x^{i+j+1(\mathrm{mod} 2n)} y \\ &= M_{i+j(\mathrm{mod} 2n)} + M_{i+j+1(\mathrm{mod} 2n)} \\ &= M_j \cdot M_i. \end{aligned}$$

(4) By $z^2 = z$, we have

$$L_i \cdot N_0 = N_0 \cdot L_i = x^i z = z x^i = z = N_0$$

and

$$N_0 \cdot N_0 = z^2 = z = N_0.$$

(5) By $zy = 2z$, we have

$$\begin{aligned} N_0 \cdot N_1 &= zyz = 2z^2 = 2z \\ &= 2N_0 \end{aligned}$$

and

$$\begin{aligned} N_0 \cdot M_i &= z x^i y = zy = 2z \\ &= 2N_0. \end{aligned}$$

(6) First, we have

$$N_1 \cdot N_0 = yzz = yz^2 = yz = N_1$$

and

$$N_1 \cdot L_i = yzx^i = yz = N_1.$$

Moreover, we have

$$M_i \cdot N_0 = x^i yz = yx^i z = yz = N_1$$

and

$$L_i \cdot N_1 = x^i yz = yx^i z = yz = N_1.$$

Therefore, $N_1 \cdot N_0 = M_i \cdot N_0 = N_1 \cdot L_i = L_i \cdot N_1 = N_1$.

(7) First, we have

$$N_1 \cdot N_1 = yzyz = 2yz^2 = 2yz = 2N_1$$

and

$$N_1 \cdot M_i = yzx^i y = yzy = 2yz = 2N_1,$$

since $zy = 2z$.

Moreover, we have the following:

$$\begin{aligned} M_i \cdot N_1 &= x^i yyz = x^i(xy + y)z \\ &= x^i xyz + x^i yz = x^{i+1}yz + x^i yz \\ &= yx^{i+1}z + yx^i z = yz + yz \\ &= 2N_1. \end{aligned}$$

Hence, $N_1 \cdot N_1 = N_1 \cdot M_i = M_i \cdot N_1 = 2N_1$. □

**Theorem 2.5.** *The algebra $\mathcal{H}_n$ is a positively based algebra.*

*Proof.* By Proposition 2.4, one can easily check that

$$\mathcal{B} = \{L_i, M_i, N_k \mid 0 \leq i \leq 2n - 1, k = 0, 1\}$$

is a set of positive bases of $\mathcal{H}_n$.

The theorem follows. □

## 3. The representations of $\mathcal{H}_n$

From this section, we always assume that $\mathbb{K}$ is a subfield of $\mathbb{C}$ containing $\eta$.

The aim of section is to construct all indecomposable modules of $\mathcal{H}_n$ with the help of the representation theory of a quiver. For this purpose, we first consider an algebra $\mathcal{A}$ over the field $\mathbb{K}$, which is generated by $x, y$ with the following relations:

$$x^{2n} = 1, \quad y^2 = xy + y, \quad xy = yx.$$

By a straightforward calculation, we see that the following system of equations

$$\begin{cases} x^{2n} = 1, \\ y^2 - xy - y = 0, \end{cases}$$

has $4n - 1$ distinct solutions given by the following:

$$\Gamma = \left\{(\eta^i, 0) \mid 0 \leq i \leq 2n - 1\right\} \cup \left\{(\eta^j, 1 + \eta^j) \mid 0 \leq j \leq 2n - 1, j \neq n\right\}.$$

For each solution $(x_1, x_2) \in \Gamma$, we can define a simple $\mathcal{A}$-module by

$$x \cdot v = x_1 v, \ y \cdot v = x_2 v.$$

Hence, we have the following Lemma.

**Lemma 3.1.** *(1) For $0 \leq i \leq 2n - 1$, there are $2n$ non-isomorphic $1$-dimensional $\mathcal{A}$-modules $S_i$ with the basis $\{v_i\}$. The action of $\mathcal{A}$ is given by the following:*

$$x \cdot v_i = \eta^i v_i, \quad y \cdot v_i = 0.$$

(2) *For $0 \leq i \leq 2n-1$ and $i \neq n$, there are $2n-1$ non-isomorphic 1-dimensional $\mathcal{A}$-modules $W_i$ with the basis $\{v_i\}$. The action of $\mathcal{A}$ is given by the following*:

$$x \cdot v_i = \eta^i v_i, \quad y \cdot v_i = (1+\eta^i)v_i.$$

*Proof.* It is obvious. □

In the following, we describe all indecomposable modules of $\mathcal{A}$.

Let $\mathcal{D}$ be a 2-dimensional vector space with the basis $\{v_1, v_2\}$. We define an action of $\mathcal{A}$ on $\mathcal{D}$ as follows:

$$\begin{aligned} x \cdot v_1 &= -v_1, \quad y \cdot v_1 = 0, \\ x \cdot v_2 &= -v_2, \quad y \cdot v_2 = v_1. \end{aligned}$$

Then, $\mathcal{D}$ is an indecomposable $\mathcal{A}$-module.

**Proposition 3.2.** *Any $d$-dimensional, non-simple, indecomposable $\mathcal{A}$-module $V$ is isomorphic to $\mathcal{D}$.*

*Proof.* Assume that $V$ is a $d$-dimensional, non-simple, indecomposable module with $d \geq 2$. Then, $\mathrm{End}V$ is local and $\mathrm{End}V/\mathrm{rad}\,\mathrm{End}V \cong \mathbb{K}$. Accordingly, we can assume that the matrices of $x$ and $y$ acting on some suitable basis of $V$ are $X = \eta^i E_d$ and

$$Y = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}_{d\times d}, \lambda \in \mathbb{K},$$

respectively, since $x^{2n} = 1$ and $xy = yx$.

One can easily see that matrices $X$ and $Y$ satisfy the following:

$$\left(Y - \left(1+\eta^i\right)E_d\right)Y = 0.$$

It follows that $i = n$, $d = 2$ and $\lambda = 1 + \eta^i = 0$. Thus, we have $V \cong \mathcal{D}$. □

In the following, set

$$e_i = \frac{1}{2n}\sum_{k=0}^{2n-1} \eta^{-ik} x^k$$

for $0 \leq i \leq 2n-1$.

One can easily check that $\{e_0, e_1, ..., e_{2n-1}\}$ is a set of central idempotents of $\mathcal{H}_n$, and

$$z(e_0 - z) = (e_0 - z)z = 0, \quad ze_i = e_i z = 0 \text{ for } 1 \leq i \leq 2n-1.$$

Therefore, we have the following:

$$\mathcal{H}_n = \mathcal{H}_n z \oplus \mathcal{H}_n(e_0 - z) \oplus \mathcal{H}_n e_1 \oplus \cdots \oplus \mathcal{H}_n e_{2n-1}.$$

Now, we can deduce the simple modules and indecomposable modules of $\mathcal{H}_n$ by Lemma 3.1 and Proposition 3.2:

(1) For $0 \leq i \leq 2n-1$, let $S_i$ be a 1-dimensional $\mathcal{H}_n$-module with the basis $\{v_i\}$; the action of $\mathcal{H}_n$ is given by

$$x \cdot v_i = \eta^i v_i, \quad y \cdot v_i = 0, \quad z \cdot v_i = 0.$$

(2) For $0 \leq i \leq 2n-1$ and $i \neq n$, let $\mathcal{W}_i$ be a 1-dimensional $\mathcal{H}_n$-module with the basis $\{v_i\}$; the action of $\mathcal{H}_n$ is given by

$$x \cdot v_i = \eta^i v_i, \quad y \cdot v_i = (1+\eta^i)v_i, \quad z \cdot v_i = 0.$$

(3) The indecomposable $\mathcal{H}_n$-module $\mathcal{V}_1$ is produced by extending $\mathcal{D}$, which is given in Lemma 2.2.

**Corollary 3.3.** *$\mathcal{V}_1 \cong \mathcal{H}_n e_n$ is the projective cover of $S_n$.*

*Proof.* First, by a straightforward calculation, we have

$$x^i e_n = (-1)^i e_n,\ x^i y e_n = (-1)^i y e_n,\ z e_n = 0,\ yz e_n = 0.$$

It follows that $\{ye_n, e_n\}$ is a basis of $\mathcal{H}_n e_n$; the action of $\mathcal{H}_n$ is given by

$$\begin{aligned} &x \cdot ye_n = -ye_n, \quad y \cdot ye_n = 0, \quad z \cdot ye_n = 0,\\ &x \cdot e_n = -e_n, \quad y \cdot e_n = ye_n, \quad z \cdot e_n = 0. \end{aligned}$$

Hence, it is easy to obtain $\mathcal{V}_1 \cong \mathcal{H}_n e_n$ and $S_n = \mathrm{top}\mathcal{V}_1$ by Lemma 2.2.

The proof is finished. □

**Lemma 3.4.** *$\mathcal{H}_n z \cong \mathcal{V}_2$ is a 2-dimensional indecomposable projective $\mathcal{H}_n$-module.*

*Proof.* First, it is easy to have $\mathcal{H}_n z$ is projective, since $\mathcal{H}_n = \mathcal{H}_n z \oplus \mathcal{H}_n(1-z)$. Moreover, we have

$$x^i z = z,\ x^i yz = yx^i z = yz,\ z^2 = z,\ yzz = yz.$$

It follows that $\{z, yz\}$ is a basis of $\mathcal{H}_n z$; the action of $\mathcal{H}_n$ is given by

$$\begin{aligned} &x \cdot z = z, \quad y \cdot z = yz, \quad z \cdot z = z,\\ &x \cdot yz = yz, \quad y \cdot yz = 2yz, \quad z \cdot yz = 2z. \end{aligned}$$

By Lemma 2.2, it is easy to obtain that $\mathcal{H}_n z \cong \mathcal{V}_2$.

Suppose that $\mathcal{V}_2 = A_1 \oplus A_2$, where $A_1$ and $A_2$ are nonzero. One has $0 \neq \varphi = k_1 z + k_2 yz \in A_1$ for some $k_1 \neq 0$ or $k_2 \neq 0$. Then,

$$y \cdot \varphi = (k_1 + 2k_2)yz \in A_1, \quad z \cdot \varphi = (k_1 + 2k_2)z \in A_1.$$

If $k_1 + 2k_2 = 0$, we get that $\varphi = k_2(2z - yz) \in A_1$ with $k_2 \neq 0$ and $2z - yz \in A_1$. Accordingly, the decomposition $yz = (yz - 2z) + 2z$ implies that $2z \in A_2$ and $y \cdot z = yz \in A_2$. Hence, $A_2 = \mathcal{V}_2$, which is a contradiction. This implies that $k_1 + 2k_2 \neq 0$, $yz \in A_1$ and $z \in A_1$. Thus, we have $A_1 = \mathcal{V}_2$, which is also a contradiction.

Therefore, $\mathcal{H}_n z \cong \mathcal{V}_2$ is an indecomposable projective module. □

Assume $S = \mathbb{K}v$ is a 1-dimensional $\mathcal{H}_n$-module defined by

$$x \cdot v = v,\ y \cdot v = 2v,\ z \cdot v = v. \tag{3.1}$$

Reviewing the action of $\mathcal{H}_n$ on $\mathcal{V}_2$, we let $\xi_1' = 2\xi_1 - \xi_2$, $\xi_2' = \xi_2$; then the action of $\mathcal{H}_n$ can be written as follows:

$$\begin{aligned} &x \cdot \xi_1' = \xi_1', \quad y \cdot \xi_1' = 0, \quad\ \ z \cdot \xi_1' = 0, \\ &x \cdot \xi_2' = \xi_2', \quad y \cdot \xi_2' = 2\xi_2', \quad z \cdot \xi_2' = \xi_1' + \xi_2'. \end{aligned} \tag{3.2}$$

We get that $S = \mathrm{top}\mathcal{V}_2$ and the exact sequence

$$0 \to S_0 \to \mathcal{V}_2 \to S \to 0.$$

It concludes the following theorem.

**Theorem 3.5.** *The following is a complete set of all simple $\mathcal{H}_n$-modules up to isomorphism:*

*(1) One non-projective simple module $S_n$ with the projective cover $\mathcal{V}_1$;*
*(2) One non-projective simple module $S$ with the projective cover $\mathcal{V}_2$;*
*(3) $4n-2$ projective simple modules $S_i$ and $\mathcal{W}_i$, where $0 \leq i \leq 2n-1$ and $i \neq n$.*

We also need more preparations to list all the indecomposable $\mathcal{H}_n$-modules.

**Lemma 3.6.** $\mathcal{H}_n(e_0 - z) = S_0 \oplus \mathcal{W}_0$.

*Proof.* For $0 \leq i \leq 2n-1$, we have

$$x^i(e_0 - z) = x^i e_0 - x^i z = e_0 - z,$$

$$x^i y(e_0 - z) = yx^i e_0 - yx^i z = ye_0 - yz,$$

and

$$z(e_0 - z) = ze_0 - z^2 = 0, \quad yz(e_0 - z) = yz - yz^2 = 0.$$

Set $v_1 = 2e_0 - 2z - ye_0 + yz$, $v_2 = ye_0 - yz$; then, $\{v_1, v_2\}$ is a basis of $\mathcal{H}_n(e_0 - z)$ and

$$\begin{aligned} &x \cdot v_1 = v_1, \quad y \cdot v_1 = 0, \quad\ \ z \cdot v_1 = 0, \\ &x \cdot v_2 = v_2, \quad y \cdot v_2 = 2v_2, \quad z \cdot v_2 = 0. \end{aligned}$$

By Lemma 2.2, $\mathcal{H}_n(e_0 - z) = S_0 \oplus \mathcal{W}_0$.

The result follows. □

**Lemma 3.7.** *For $1 \leq i \leq 2n-1$, we have the following*:

*(1)* $\mathrm{Hom}_{\mathcal{H}_n}(\mathcal{H}_n e_i, \mathcal{H}_n e_0) = 0$;
*(2)* $\mathrm{Hom}_{\mathcal{H}_n}(\mathcal{H}_n e_0, \mathcal{H}_n e_i) = 0$.

*Proof.* (1) By [2, Lemma 4.2], we have

$$\mathrm{Hom}_{\mathcal{H}_n}(\mathcal{H}_n e_i, \mathcal{H}_n e_0) = e_i \mathcal{H}_n e_0 = 0,$$

since $e_i e_0 = e_0 e_i = 0$, and $xy = yx, xz = zx$ for $1 \leq i \leq 2n-1$.

The proof of (2) is similar to (1). □

**Lemma 3.8.** *(1)* $\mathrm{Hom}_{\mathcal{H}_n}(\mathcal{V}_2, \mathcal{W}_0) = 0$, $\mathrm{Hom}_{\mathcal{H}_n}(\mathcal{V}_2, S_n) = 0$;
*(2)* $\mathrm{Hom}_{\mathcal{H}_n}(S_0, \mathcal{V}_2) \neq 0$, $\mathrm{Hom}_{\mathcal{H}_n}(\mathcal{W}_0, \mathcal{V}_2) = 0$.

*Proof.* (1) By Lemmas 3.4 and 3.8, we have

$$\mathrm{Hom}_{\mathcal{H}_n}(\mathcal{V}_2, \mathcal{W}_0) = \mathrm{Hom}_{\mathcal{H}_n}(\mathcal{H}_n z, \mathcal{W}_0) = z\mathcal{W}_0 = 0.$$

Similarly,

$$\mathrm{Hom}_{\mathcal{H}_n}(\mathcal{V}_2, S_0) = S_0 = 0.$$

(2) By (3.2), $S_0$ is the socle of $\mathcal{V}_2$. Thus,

$$\mathrm{Hom}_{\mathcal{H}_n}(S_0, \mathcal{V}_2) \neq 0.$$

For $\mathcal{W}_0$, the result is obvious. □

According to Lemmas 3.6–3.8, one can easily see that $\mathcal{V}_2 \oplus S_0$ is a block of $\mathcal{H}_n$. Meanwhile, the indecomposable modules of other blocks are either 1-dimensional or 2-dimensional. Therefore, we only need to consider the indecomposable modules of the block $C = \mathcal{V}_2 \oplus S_0$.

**Proposition 3.9.** *The quiver of the block C is as follows*:

$$\underset{0}{\circ} \xrightarrow{\alpha_1} \underset{1}{\circ}\ .$$

*Proof.* On the one hand, $\mathcal{V}_2$ is the projective cover of $S$, and $S_0$ is projective by Lemma 3.4. Therefore,

$$\mathrm{Ext}^1_{\mathcal{H}_n}(S_0, S) = 0.$$

On the other hand, we have the following extension of $S_0$ by $S$:

$$0 \to S_0 \to \mathcal{V}_2 \to S \to 0.$$

Assume that

$$0 \to S_0 \to Q \to S \to 0$$

is another extension of $S_0$ by $S$, where $Q$ is indecomposable of the basis $\{v_1, v_2\}$; the action of $\mathcal{H}_n$ is given by

$$\begin{aligned} &x \cdot v_1 = v_1, \quad y \cdot v_1 = 0, \qquad z \cdot v_1 = 0, \\ &x \cdot v_2 = v_2, \quad y \cdot v_2 = 2v_2, \quad z \cdot v_2 = kv_1 + v_2, \end{aligned}$$

with $k \neq 0$.

Now, let $h : \mathcal{V}_2 \to Q$, which is given by $\xi_1' \mapsto kv_1$, $\xi_2' \mapsto v_2$. Then, $h$ is an isomorphism. It follows that

$$\dim \mathrm{Ext}^1_{\mathcal{H}_n}(S, S_0) = 1.$$

The result follows. □

**Theorem 3.10.** *The algebra $\mathcal{H}_n$ has $4n + 2$ pairwise, non-isomorphic, indecomposable modules*:

$$\left\{S_i, \mathcal{W}_j, S \;\mid\; 0 \leq i, j \leq 2n-1, j \neq n\right\} \cup \{\mathcal{V}_1, \mathcal{V}_2\}.$$

*Proof.* By Proposition 3.9, the quiver of block $C$ is as follows:

$$\underset{0}{\circ} \xrightarrow{\alpha_1} \underset{1}{\circ}\ .$$

It is easy to see that this block has non-isomorphic, indecomposable modules $S_0$, $S$ and $\mathcal{V}_2$. By Theorem 3.5, we get $4n + 2$ pairwise, non-isomorphic, indecomposable modules of $\mathcal{H}_n$ as the aforementioned list. □

## 4. The left cell modules of $\mathcal{H}_n$

Let $A$ be a positively based algebra with a fixed positive basis $\mathcal{B} = \{a_i \mid i \in I\}$ with the identity $a_1$ of $A$. For $i, j \in I$, set

$$i \star j = \left\{k \mid \gamma_{i,j}^{(k)} > 0\right\}.$$

This defines an associative multi-valued operation on the set $I$ and turns the latter set into a finite multi-semigroup, see [9, Subsection 3.7]. If there is an $s \in I$ such that $j \in s \star i$, then we denoted it by $i \leq_L j$. For $i, j \in I$, if $i \leq_L j$ and $j \leq_L i$, we denote it by $i \sim_L j$, which is an equivalent relation. The associated equivalence class is referred to as a left cell. Moreover, we write $i <_L j$, provided that $i \leq_l j$ and $i \nsim_L j$.

Assume that $\mathcal{L}$ is a left cell in $I$, and let $\overline{\mathcal{L}}$ be the union of all left cells $\mathcal{L}'$ in $I$ such that $\mathcal{L}' \geq \mathcal{L}$. Set $\underline{\mathcal{L}} = \overline{\mathcal{L}} \backslash \mathcal{L}$. Let $M_{\mathcal{L}}$ be the vector space spanned by $a_j$, where $j \in \overline{\mathcal{L}}$, and $N_{\mathcal{R}}$ be the vector space spanned by $a_j$ with $j \in \underline{\mathcal{L}}$. According to [8, Proposition 1], $M_{\mathcal{L}}$ and $N_{\mathcal{L}}$ are submodules of ${}_AA$ and $N_{\mathcal{L}} \subset M_{\mathcal{L}}$. It allows us to define the cell module $C_{\mathcal{R}} = M_{\mathcal{L}}/N_{\mathcal{L}}$. Especially, if $\overline{\underline{\mathcal{L}}} = \emptyset$, denote it by $N_{\mathcal{L}} = 0$. For the study of cells and cell modules, the readers can refer to [6–8, 11–13].

Now, we investigate the left (right) cells and left (right) cell modules of $\mathcal{H}_n$. By Theorem 2.5,

$$\mathcal{B} = \left\{L_i, M_j, N_k \mid 0 \leq i, j \leq 2n - 1, k = 0, 1\right\}$$

is a positive basis of $\mathcal{H}_n$.

**Proposition 4.1.** *The algebra $\mathcal{H}_n$ has three left cells $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$, as listed in the following*;

*(1)* $\mathcal{L}_1 = \{i \mid i \text{ is the index of } L_i, i \in \mathbb{Z}_{2n}\}$;
*(2)* $\mathcal{L}_2 = \left\{j \mid j \text{ is the index of } M_j, j \in \mathbb{Z}_{2n}\right\}$;
*(3)* $\mathcal{L}_3 = \{k \mid k \text{ is the index of } N_k, k = 0, 1\}$.

*Proof.* (1) Set $i, i' \in \mathbb{Z}_{2n}$ and $i < i'$. We obtain $L_{i'-i} \cdot L_i = L_{i'}$ by Proposition 2.4. It follows that $i' \in (i' - i) \star i$, which implies that $i \leq_L i'$. Similarly, we have $L_{2n-i'+i} \cdot L_{i'} = L_i$; then, $i \in (2n - i' + i) \star i'$ and $i' \leq_L i$. The equivalent relation $i \sim_L i'$ holds and $\mathcal{L}_1$ is a left cell.

(2) Set $j, j' \in \mathbb{Z}_{2n}$ and $j < j'$. We have $j' \in (j' - j) \star j$ and $j \leq_L j'$, since $M_{j'-j} \cdot M_j = M_{j'} + M_{j'+1}$. Similarly, we obtain that $j \in (2n + j - j') \star j'$ and $j' \leq_L j$. Hence, $\mathcal{L}_2$ is a left cell.

(3) By Proposition 2.4, it is obvious that $N_0 \cdot N_1 = 2N_0$ and $N_1 \cdot N_0 = N_1$. Thus, we have $1 \leq_L 0$ and $0 \leq_L 1$. Therefore, $\mathcal{L}_3$ is a left cell.

The proof is completed. □

**Corollary 4.2.** *For the left cells of $\mathcal{H}_n$, we have $\mathcal{L}_1 <_L \mathcal{L}_2 <_L \mathcal{L}_3$.*

*Proof.* For $i, j \in \mathbb{Z}_{2n}$, we have $M_j \cdot L_i = M_{i+j(\mathrm{mod}2n)}$ by Proposition 2.4. It is clear that $\mathcal{L}_1 <_L \mathcal{L}_2$.
Furthermore, by Proposition 2.4, it is easy to see that

$$N_0 \cdot M_j = N_0 + N_0, \quad N_1 \cdot M_j = N_1 + N_1.$$

Thus, $\mathcal{L}_2 <_L \mathcal{L}_3$ holds.
Consequently, we have $\mathcal{L}_1 <_L \mathcal{L}_2 <_L \mathcal{L}_3$. □

**Proposition 4.3.** *For the left cells $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$, the corresponding left cell modules $C_{\mathcal{L}_1}$, $C_{\mathcal{L}_2}$ and $C_{\mathcal{L}_3}$ are given as follows*:

*(1)* $C_{\mathcal{L}_1} = \mathrm{Span}\left\{\overline{L_i} \mid i \in \mathbb{Z}_{2n}\right\}$, *where* $\overline{L_i} = L_i + \mathcal{N}_{\mathcal{L}_1}$, $\mathcal{N}_{\mathcal{L}_1} = \mathrm{Span}\left\{M_j, N_k \mid j \in \mathbb{Z}_{2n}, k = 0, 1\right\}$;

*(2)* $C_{\mathcal{L}_2} = \mathrm{Span}\left\{\overline{M_j} \mid j \in \mathbb{Z}_{2n}\right\}$, *where* $\overline{M_j} = M_j + \mathcal{N}_{\mathcal{L}_2}$, *and*

$$\mathcal{N}_{\mathcal{L}_2} = \mathrm{Span}\left\{N_k \mid k = 0, 1\right\};$$

*(3)* $C_{\mathcal{L}_3} = \mathrm{Span}\left\{N_k \mid k = 0, 1\right\}$, *where* $\mathcal{N}_{\mathcal{L}_3} = \{0\}$.

*Proof.* (1) As is shown in Corollary 4.2, we have

$$\overline{\mathcal{L}_1} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3, \quad \underline{\mathcal{L}_1} = \mathcal{L}_2 \cup \mathcal{L}_3.$$

By Proposition 4.1, it follows that

$$\mathcal{M}_{\mathcal{L}_1} = \mathrm{Span}\left\{L_i,\ M_j,\ N_k \mid i, j \in \mathbb{Z}_{2n},\ k = 0, 1\right\},$$

and

$$\mathcal{N}_{\mathcal{L}_1} = \mathrm{Span}\left\{M_j,\ N_k \mid j \in \mathbb{Z}_{2n},\ k = 0, 1\right\}.$$

Hence,

$$C_{\mathcal{L}_1} = \mathcal{M}_{\mathcal{L}_1}/\mathcal{N}_{\mathcal{L}_1} = \mathrm{Span}\left\{\overline{S_i} \mid i \in \mathbb{Z}_{2n}\right\}.$$

(2) By Corollary 4.2, we have

$$\overline{\mathcal{L}_2} = \mathcal{L}_2 \cup \mathcal{L}_3, \quad \underline{\mathcal{L}_2} = \mathcal{L}_3.$$

It follows that

$$\mathcal{M}_{\mathcal{L}_2} = \mathrm{Span}\left\{M_j,\ N_k \mid j \in \mathbb{Z}_{2n},\ k = 0, 1\right\},$$

and

$$\mathcal{N}_{\mathcal{L}_2} = \mathrm{Span}\left\{N_k \mid k = 0, 1\right\}.$$

Therefore,

$$C_{\mathcal{L}_2} = \mathcal{M}_{\mathcal{L}_2}/\mathcal{N}_{\mathcal{L}_2} = \mathrm{Span}\left\{\overline{M_j} \mid j \in \mathbb{Z}_{2n}\right\}.$$

(3) By Proposition 4.1 and Corollary 4.2, we have

$$\overline{\mathcal{L}_3} = \mathcal{L}_3, \text{ and } \underline{\mathcal{L}_3} = \emptyset.$$

It follows that

$$\mathcal{M}_{\mathcal{L}_3} = \mathrm{Span}\left\{N_k \mid k = 0, 1\right\}, \text{ and } \mathcal{N}_{\mathcal{L}_3} = \{0\}.$$

Hence,

$$C_{\mathcal{L}_3} = \mathcal{M}_{\mathcal{L}_3}/\mathcal{N}_{\mathcal{L}_3} = \mathrm{Span}\left\{N_k \mid k = 0, 1\right\}.$$

The proof is finished. □

By Proposition 4.3, we see that $C_{\mathcal{L}_1} = \mathrm{Span}\left\{\overline{L_i} \mid i \in \mathbb{Z}_{2n}\right\}$ and write $\mu_i = \overline{L_i}$ with $0 \leq i \leq 2n-1$. Then, the action of $\mathcal{H}_n$ on $C_{\mathcal{L}_1}$ is defined by

$$x \cdot \mu_i = \mu_{i+1(\mathrm{mod} 2n)}, \quad y \cdot \mu_i = 0, \quad z \cdot \mu_i = 0.$$

Now, set

$$\chi_i = \frac{1}{2n}\left(\mu_0 + \eta^{-i}\mu_1 + \eta^{-2i}\mu_2 + \cdots + \eta^{-(2n-1)i}\mu_{2n-1}\right),$$

where $0 \leq i \leq 2n-1$.

A straightforward calculation shows that

$$x \cdot \chi_i = \eta^i \chi_i, \quad y \cdot \chi_i = 0, \quad z \cdot \chi_i = 0.$$

**Theorem 4.4.** *$C_{\mathcal{L}_1}$ is decomposable and*

$$C_{\mathcal{L}_1} \cong S_0 \oplus S_1 \oplus \cdots \oplus S_{2n-1}.$$

*Proof.* By Lemma 2.2, the 1-dimensional $\mathcal{H}_n$-module $\mathbb{K}\chi_i = S_i$. Hence, $S_i$ is a submodule of $C_{\mathcal{L}_1}$. We easily see that

$$\dim_{\mathbb{K}}(S_0 \oplus S_1 \oplus \cdots \oplus S_{2n-1}) = \dim_{\mathbb{K}} C_{\mathcal{L}_1},$$

and conclude

$$C_{\mathcal{L}_1} \cong S_0 \oplus S_1 \oplus \cdots \oplus S_{2n-1}.$$

The proof is completed. □

For the left cell module $C_{\mathcal{L}_2}$, we have $C_{\mathcal{L}_2} = \mathrm{Span}\left\{\overline{M_j} \mid j \in \mathbb{Z}_{2n}\right\}$ and denote $\nu_j = \overline{M_j}$ with $0 \leq j \leq 2n-1$. It is easy to see that

$$x \cdot \nu_j = \nu_{j+1(\mathrm{mod} 2n)}, \quad y \cdot \nu_j = \nu_j + \nu_{j+1(\mathrm{mod} 2n)}, \quad z \cdot \nu_j = 0.$$

Set

$$\omega_j = \frac{1}{2n}\left(\nu_0 + \eta^{-j}\nu_1 + \eta^{-2j}\nu_2 + \cdots + \eta^{-(2n-1)j}\nu_{2n-1}\right)$$

for $0 \leq j \leq 2n-1$.

Then, a straightforward calculation shows that

$$x \cdot \omega_j = \eta^j \omega_j, \quad y \cdot \omega_j = (1+\eta^j)\omega_j, \quad z \cdot \omega_j = 0.$$

**Theorem 4.5.** *$C_{\mathcal{L}_2}$ is decomposable and*

$$C_{\mathcal{L}_2} \cong \mathcal{W}_0 \oplus \cdots \oplus \mathcal{W}_{n-1} \oplus S_n \oplus \mathcal{W}_{n+1} \oplus \cdots \oplus \mathcal{W}_{2n-1}.$$

*Proof.* On the one hand, by Lemma 2.2, we see that the $\mathcal{H}_n$-module $\mathbb{K}\omega_j \cong \mathcal{W}_j$ when $j \neq n$, and $\mathbb{K}\omega_n \cong S_n$. Hence, $\mathcal{W}_j$ is a submodule of $C_{\mathcal{L}_2}$ for $0 \leq j \leq 2n-1$ and $j \neq n$.

On the other hand, it is easy to see that

$$\dim_{\mathbb{K}}(\mathcal{W}_0 \oplus \cdots \oplus \mathcal{W}_{n-1} \oplus S_n \oplus \mathcal{W}_{n+1} \oplus \cdots \oplus \mathcal{W}_{2n-1}) = \dim_{\mathbb{K}} C_{\mathcal{L}_2}.$$

Hence,

$$C_{\mathcal{L}_2} \cong \mathcal{W}_0 \oplus \cdots \oplus \mathcal{W}_{n-1} \oplus S_n \oplus \mathcal{W}_{n+1} \oplus \cdots \oplus \mathcal{W}_{2n-1}.$$

The proof is finished. □

**Theorem 4.6.** *$C_{\mathcal{L}_3} \cong \mathcal{V}_2$ is indecomposable.*

*Proof.* By Proposition 4.3, for the left cell module $C_{\mathcal{L}_3}$, we have $C_{\mathcal{L}_3} = \text{Span}\{N_k \mid k = 0, 1\}$ , on which the action of $\mathcal{H}_n$ is given by

$$\begin{aligned} x \cdot N_0 &= N_0, \quad y \cdot N_0 = N_1, \quad z \cdot N_0 = N_0, \\ x \cdot N_1 &= N_1, \quad y \cdot N_1 = 2N_1, \quad z \cdot N_1 = 2N_0. \end{aligned}$$

Hence, $C_{\mathcal{L}_3} \cong \mathcal{V}_2$ by Lemma 3.4.

The result follows. □

Finally, we give some remarks for the right cell modules of $\mathcal{H}_n$. For the positive basis $\mathcal{B}$, if there is an $s \in I$ such that $j \in i \star s$, then one can denote it by $i \leq_R j$. If $i \leq_R j$ and $j \leq_R i$, then one can denote it by $i \sim_R j$, which is an equivalent relation. The equivalence class is called a right cell. The similar statement to Proposition 4.1 shows that $\mathcal{H}_n$ has the following four right cells:

(1) $\mathcal{R}_1 = \{i \mid i \text{ is the index of } S_i, i \in \mathbb{Z}_{2n}\}$;
(2) $\mathcal{R}_2 = \left\{j \mid j \text{ is the index of } M_j, j \in \mathbb{Z}_{2n}\right\}$;
(3) $\mathcal{R}_3 = \{0 \mid 0 \text{ is the index of } N_0\}$;
(4) $\mathcal{R}_4 = \{1 \mid 1 \text{ is the index of } N_1\}$.

We see that $\mathcal{R}_1 <_R \mathcal{R}_2 <_R \mathcal{R}_3$ and $\mathcal{R}_1 <_R \mathcal{R}_2 <_R \mathcal{R}_4$. Consequently, we get the following:

(1) $C_{\mathcal{R}_1} = \text{Span}\{\mu_i \mid i \in \mathbb{Z}_{2n}\}$, on which the action of $\mathcal{H}_n$ is given by

$$\mu_i \cdot x = \mu_{i+1(\text{mod}2n)}, \quad \mu_i \cdot y = 0, \quad \mu_i \cdot z = 0.$$

(2) $C_{\mathcal{R}_2} = \text{Span}\left\{\nu_j \mid j \in \mathbb{Z}_{2n}\right\}$, on which the action of $\mathcal{H}_n$ is given by

$$\nu_j \cdot x = \nu_{j+1(\text{mod}2n)}, \quad \nu_j \cdot y = \nu_j + \nu_{j+1(\text{mod}2n)}, \quad \nu_j \cdot z = 0.$$

(3) $C_{\mathcal{R}_3} = \text{Span}\{\xi_1\}$, on which the action of $\mathcal{H}_n$ is given by

$$\xi_1 \cdot x = \xi_1, \quad \xi_1 \cdot y = 2\xi_1, \quad \xi_1 \cdot z = \xi_1.$$

(4) $C_{\mathcal{R}_4} \cong C_{\mathcal{R}_3}$.

## 5. Conclusions

In the paper, all indecomposable modules of $\mathcal{H}_n$, a family of positively based algebras, are constructed and classified. Also, their left cell modules are described. In our further study, we will focus on the family of positive based algebras associated to the Green algebras of the dual of the generalized Taft algebra. These results may help us to understand the general representation theory of a positive based algebra.

## Use of AI tools declaration

All authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

All authors declare no conflict of interest in this paper.

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