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Research article

Note on fractal interpolation function with variable parameters

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Abstract: Fractal interpolation function (FIF) is a new method of constructing new data points within the range of a discrete set of known data points. Consider the iterated functional system defined through the functions $W_n(x, y) = (a_n x + e_n, \alpha_n(x)y + \psi_n(x))$, $n = 1, \ldots, N$. Then, we may define the generalized affine FIF f interpolating a given data set $\{(x_n, y_n) \in I \times \mathbb{R}, n = 0, 1, \ldots, N\}$, where $I = [x_0, x_N]$. In this paper, we discuss the smoothness of the FIF f. We prove, in particular, that f is θ-hölder function whenever ψ_n are. Furthermore, we give the appropriate upper bound of the maximum range of FIF in this case.

Keywords: iterated function system; generalized affine fractal interpolation function; hölder and Lipschitz functions
Mathematics Subject Classification: 28A80, 47H10, 65D05

1. Introduction and main results

In approximation theory, fractal interpolation is an alternative to classical interpolation used when studying irregular curves. The motivation to study fractal interpolation functions (FIFs in short) comes from the fact that most time series studied in practice often exhibit fluctuations or abrupt changes that fractal interpolants can intrinsically model. The results indicate that the use of fractal interpolation in many areas (financial applications for example) is promising. The concept of the FIF was first introduced by Barnsley [1] via an iterated functional system (IFS in short) on a compact subset of $\mathbb{R}$, which fundamentally acts as the pivot to construct fractals. Since then, this theory has become a useful and powerful tool in applied science and engineering [2–6]. Moreover, various and important properties of FIF have been proved, including smoothness, stability, and disturbance error (see for instance [7–12]).

Specifically, IFS is a collection of a complete metric space $(\mathbb{X}, d)$ with a finite set of continuous mappings $w_1, w_2 \ldots, w_N$, for $N \geq 2$. One can find that there exists a compact set $G = \bigcup_{n=1}^{N} w_n(G)$ referred to an invariant set or an attractor to the IFS. Moreover, Hutchinson's idea gives that the invariant compact set G is fully determined by the IFS, and also G is the limit of a sequence of sets that can be built by the members of the IFS (see for instance [13–19] for some extension of Hutchinson's framework). Recently, many researchers have been working on some extensions of the IFS framework (generalized contractions or more general spaces...). Fixed point theory plays a significant and vital role in the existence of invariant sets in different types of IFSs. To this end, many researchers have studied the existence of FIFs by using different results related to the fixed point theory [9, 10, 20–23].

A function g, defined on a set I, is said to be a Hölder continuous function with exponent θ (or shortly θ–Hölder function) when g satisfies

$$\left|g(x) - g(y)\right| \leq c\left|x - y\right|^{\theta}, \quad \forall x, y \in I,$$

for some positive constants c and $0 < \theta \leq 1$. This relation is called the Hölder condition and, when $\theta = 1$, the function g is said to be Lipschitz in I with Lipschitz constant c. Let $\Phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a function. Then g is said to be a Φ-Hölder function if

$$|g(y) - g(x)| \leq \Phi(|y - x|), \quad \forall x, y \in I.$$

We denote $\mathcal{H}_{\Phi}(I)$ the class of all Φ-Hölder functions on I [24, 25] : It is well known that the class of $\mathcal{H}_{\Phi}(I)$ is closed and convex with respect to the pointwise supremum and infimum [24]. We say that Φ satisfies the doubling condition if there exists $\xi \geq 1$, depends on Φ and called Φ-Hölder constant, such that

$$\Phi(bx) \leq \xi b\Phi(x) \ \text{ for } \ b \geq 1 \qquad \text{and} \qquad \Phi(bx) \leq \xi\Phi(x) \ \text{ for } \ b < 1.$$

Also, we denote by $\mathcal{H}_{\Phi}^{\mathrm{d}}(I)$ the family of Φ-Hölder functions such that Φ satisfies the doubling condition. The most important class of functions satisfies the doubling condition on $\mathbb{R}_+$ are the increasing and subadditive function, that is, $\Phi(x + y) \leq \Phi(x) + \phi(y)$, with $\Phi(0) = 0$.

For $n \in J := \{1, \ldots, N\}$, let $\alpha_n : I \longrightarrow \mathbb{R}$ be a Lipschitz function and $\psi_n : I \longrightarrow \mathbb{R}$ be a continuous function. In this paper, we consider the generalized affine FIF defined by

$$\begin{cases} L_n(x) = a_n x + e_n \\ F_n(x, y) = \alpha_n(x)y + \psi_n(x), \end{cases} \quad n \in J, \tag{1.1}$$

where, the real positive numbers a_n and e_n are determined by condition (2.1) and such that conditions (2.2) and (2.3) hold. This system is extensively studied when the functions $\{\alpha_n\}_n$ are constants (they are called vertical scaling factors) [7, 8, 12, 26–29]. Further, the IFS can be selected suitably so that the corresponding FIF shares the quality of smoothness or non-smoothness. This depends on the choice of the vertical scaling factors and the functions ψ_n [26, 27, 30]. Then, choosing the appropriate vertical scale factors and functions ψ_n remind fundamental and they can fit the real rough curve precisely. Consider the case where the vertical scaling factor parameters are constants, then Chen [28], Chand and Kapoor [26, 27] studied the smoothness of a class of FIFs and the smoothness of coalescence hidden variable FIFs, respectively, using the techniques of operator approximation. Moreover, and in the case where ψ_n, $n \in J$, are Lipschitz functions defined on I Yang and Yu [30] investigated the smoothness of a class of FIFs with variable parameters using new techniques. In this paper, we consider more general cases by letting $\psi_n \in \mathcal{H}_\Phi^{\mathsf{d}}(I)$, $n \in J$. Our smoothness results are obtained by evaluating $|f(x) - f(y)|$ for $x, y \in I$. To this end, we study in Section 3, the effect of the choice of the function ψ_n on the FIF denoted by f, when $\psi_n \in \mathcal{H}_\Phi^{\mathsf{d}}(I)$. More precisely, we will prove the following result.

Theorem 1.1. *Let f be the FIF generated by the IFS* (1.1) *and assume that $\psi_n \in \mathcal{H}_\Phi^{\mathsf{d}}(I)$. Let $\zeta := \max_n \|\psi_n\|_\infty$, $a = \min_n a_n$, $\alpha := \max_n \|\alpha_n\|_\infty$, $\xi_r = \xi a^{-r}$ and $C := \max_n C_n$ where C_n is the Lipschitz constant of α_n. For a given $x, y \in L_{n_1 n_2 \dots n_k}(I)$, we have*

$$|f(x) - f(y)| \le \sum_{r=1}^{k} \xi_r \alpha^{r-1} \Phi\Big(\big|x - y\big|\Big) + \sum_{r=2}^{k} \frac{\zeta \alpha^{r-2} C}{1 - a} \big|x - y\big| (a^{1-r} - 1) + 2\alpha^k \|f\|_\infty.$$

Note that Theorem 1.1 considers only the case when $x, y \in L_{n_1 n_2 \dots n_k}(I)$ (see definition in Section 3). However, for any $x, y \in I$, there exists k_0 and $\{a_{n_j}\}_j$, such that

$$\prod_{j=1}^{k_0+1} a_{n_j} \le |x - y] \le \prod_{j=1}^{k_0} a_{n_j}, \qquad n_j \in J.$$

It follows that $x, y \in L_{n_1 n_2 \dots n_{k_0}}(I)$ or x and y are belong to the adjacent two intervals with common boundary point denoted by z and then $|f(x) - f(y)| \le |f(x) - f(z)| + |f(z) - f(y)|$. The previous calculation may gives the useful upper bound of $|f(x) - f(y)|$ for all $x, y \in I$. Moreover, , the most widely Φ-Hölder functions are the θ-Hölder functions. To this end, let $\Phi_0(x) = \mathsf{s}|x|^\theta$, for some positive real number s and $\theta \in]0, 1]$. As an application of Theorem 1.1, we obtain the following result.

Corollary 1.1. *Let f be the FIF generated by the IFS* (1.1) *such that α_n are constant parameters, $\psi_n \in \mathcal{H}_{\Phi_0}^{\mathsf{d}}(I)$. Assume that $\alpha := \max_n |\alpha_n| < a = \min_n a_n$ then the function f is a θ-Hölder on I, that is, there exists a positive constant d' such that*

$$|f(x) - f(y)| \le d'|x - y|^\theta, \qquad x, y \in I.$$

2. Preliminaries

2.1. Iterated function systems

Let $(\mathbb{X}, d)$ be a complete metric space. We define $\mathcal{H}(\mathbb{X})$ to be the set of all nonempty complex subsets of $\mathbb{X}$ and $g : \mathbb{X} \longrightarrow \mathbb{X}$. The mapping g will said to be a contraction if there exists $c \in [0, 1)$ such that

$$d\,(g(x), g(y)) \le c\, d(x, y), \qquad \forall x, y \in \mathbb{X}.$$

We define, on the set $\mathcal{H}(\mathbb{X})$, the Hausdorff metric d_H defined as

$$d(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y) \text{ and } d(B, A) = \sup_{x \in B} \inf_{y \in A} d(x, y), \qquad A, B \in \mathcal{H}(\mathbb{X}),$$

where $d_H(A, B) = \max\{d(A, B), d(B, A)\}$. It is well known [31], that the space $(\mathcal{H}(\mathbb{X}), d_H)$ is complete, and compact whenever $\mathbb{X}$ is compact. Now, we consider the IFS $\mathbb{I} = \{\mathbb{X}\,,\ w_n\ ;\ n \in J\}$, where $w_n : \mathbb{X} \longrightarrow \mathbb{X}$ is a continuous mapping for $n \in J$, and the Hutchinson operator W as a selfmapping of $\mathcal{H}(\mathbb{X})$ by

$$W(A) = \bigcup_{n=1}^{N} w_n(A)\,, \quad \forall\, A \in \mathcal{H}(\mathbb{X}).$$

A set $G \in \mathcal{H}(\mathbb{X})$ is said to be an attractor of the IFS if it satisfies $G = \bigcup_{n=1}^{N} w_n(G)$ that is $W(G) = G$. In fact, the IFS admits always at least one attractor [1]. Moreover, if the IFS is hyperbolic, that is each w_n is a contraction, then we can prove that the operator W is a contraction mapping on $(\mathcal{H}(\mathbb{X}), d_H)$ [1, 31].

2.2. *Fractal interpolation function*

The FIF can be defined as an interpolant function such that its graph is a fractal, or also as fixed point of maps using the notion of IFS. More precisely, let $I = [x_0, x_N]$ be a real compact interval and let $\Delta = \Big\{(x_n, y_n) \in I \times \mathbb{R}\ ;\ n \in J_0 := \{0, 1, \dots, N\}\Big\}$ be a set of data, where $x_0 < x_1 < \cdots < x_N$, $y_i \in [\mathsf{a}, \mathsf{b}]$, with $-\infty < \mathsf{a} < \mathsf{b} < \infty$. For $n \in J$, set $I_n = [x_{n-1}, x_n]$ and let $L_n : I \longrightarrow I_n$ be a contractive homeomorphism such that

$$\begin{aligned} L_n(x_0) = x_{n-1}, \qquad & L_n(x_N) = x_n, \\ |L_n(x) - L_n(x')| \le l|x - x'|, \qquad & \forall\, x, x' \in I, \end{aligned} \tag{2.1}$$

for some $0 \le l < 1$. We consider N continuous mappings $F_n : K := I \times [\mathsf{a}, \mathsf{b}] \longrightarrow \mathbb{R}$ satisfying

$$F_n(x_0, y_0) = y_{n-1}, \qquad F_n(x_N, y_N) = y_n, \tag{2.2}$$

$$|F_n(x, y) - F_n(x, y')| \le |\mathsf{r}_n||y - y'|, \qquad \forall x \in I,\ y, y' \in [\mathsf{a}, \mathsf{b}], \tag{2.3}$$

for some $\mathsf{r}_n \in (-1, 1), n \in J$. Now, we define the mapping $W_n : K \longrightarrow I_n \times \mathbb{R}$, as

$$W_n(x, y) = \big(L_n(x), F_n(x, y)\big), \quad \forall (x, y) \in K, n \in J.$$

It is well known that the IFS $\{K, W_n : n \in J\}$ has a unique attractor G. Moreover G is the graph of continuous function $f : I \longrightarrow \mathbb{R}$ that passes through all interpolation points (x_n, y_n), $n \in J$. This function is called FIF corresponding to the points (x_n, y_n), $n \in J$. It is a self-affine function since each affine transformation W_n maps the entire graph of the function to its section within the corresponding interpolation interval [1].

Let $\mathcal{G} = \{g : I \longrightarrow \mathbb{R}$, such that g is continuous, $g(x_0) = x_0$ and $g(x_N) = x_N\}$. Then, $(\mathcal{G}, \rho)$ is a complete metric space, where ρ is a metric defined by

$$\rho(g, h) = \|g - h\|_\infty = \max\{|g(x) - h(x)| : x \in I\}, \quad \forall g, h \in \mathcal{G}.$$

Therefore, Read-Bajraktarevic operator T, defined on $(\mathcal{G}, \rho)$ by

$$T(g(x)) = F_n\Big(L_n^{-1}(x), g(L_n^{-1}(x))\Big), \quad x \in I_n, n \in J$$

is a contraction mapping. Indeed, using (2.3), we obtain

$$\|T(f) - T(g)\| \leq \alpha\|f - g\|_\infty,$$

where $\alpha := \max_n |\alpha_n|$. Hence T possesses a unique fixed point f on $\mathcal{G}$ and then the FIF is the unique function satisfying the following functional relation

$$f(x) = F_n\Big(L_n^{-1}(x), f(L_n^{-1}(x))\Big), \qquad \forall x \in I_n, n \in J. \tag{2.4}$$

The most widely studied FIFs are defined by the following system

$$\begin{cases} L_n(x) = a_n x + e_n, \\ F_n(x, y) = \alpha_n y + \psi_n(x), \end{cases} \quad n \in J,$$

where the real constants a_n and e_n are determined by condition (2.1), ψ_n are some continuous functions such that conditions (2.2) and (2.3) hold, $\alpha_n \in (-1, 1)$ are free parameters, called vertical scaling factors of the transformations W_n, and have an important consequences on the properties of the FIF. Indeed, if we consider the case of equally spaced interpolation points, we obtain smooth or non-smooth fractal function depending on the scaling factors choice. More precisely, we have the box dimension D of the graph of the FIF defined by [31]

$$D := 1 + \frac{\log\Big(\sum_{n=1}^{N} |\alpha_n|\Big)}{\log(N)}. \tag{2.5}$$

In particular, if $\alpha_1 = \cdots = \alpha_N = \alpha$ then $D = 2 + \log_N |\alpha|$. Nevertheless, there are questions about optimal choice of the vertical scaling factors α_n, $n \in J$, so that the obtained curves fit as closely as possible the real values. There are different ways to measure the quality of fit of the interpolation, for example one can use the normalized mean squared error [22] (see also [32, 33]).

In Figures 1–4, we plot the FIF associated to the interpolation points

$$\Delta = \Big\{(0, 9), (0.2, 11), (0.4, 15), (0.6, 8), (0.8, 12), (1.0, 10)\Big\}.$$

However, different vertical scaling factors are employed in each construction. As, we can see, we obtain different shape of graph of FIF even, here, the vertical scaling factors were carefully selected, so that the box-counting dimension of each graph is equal to $D = 1.3988$. Hence, the self-similarity of the fractal interpolation curve depends on the choice of the vertical scaling factors. To this end, considering more general case by choosing a variable parameters ($\alpha_n(x)$ instead of constant parameters α_n) provide a wide variety of systems for different approximations problems [30]. In the present work, we consider the IFS, with variable parameters [30], defined by (1.1). In this case, the FIF will be called generalized affine FIF and denoted by f^α where $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_N)$ (or simply by f if there is no ambiguity).

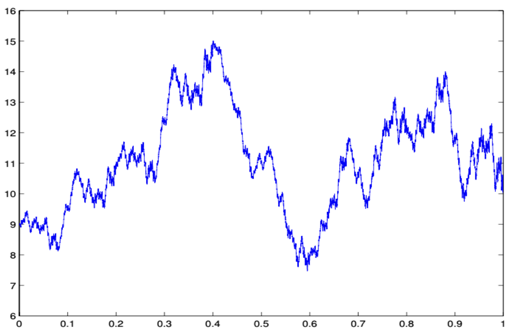

Figure 1. Scaling factors $\alpha = (-0.3, -0.4, -0.3, 0.4, 0.5)$.

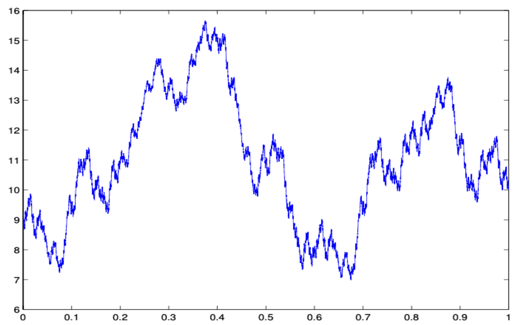

Figure 2. Scaling factors $\alpha = (-0.4, 0.3, -0.4, -0.4, 0.4)$.

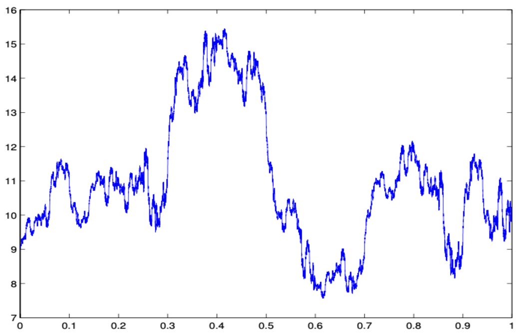

Figure 3. Scaling factors $\alpha = (0.3, -0.5, 0.4, -0.3, -0.5)$.

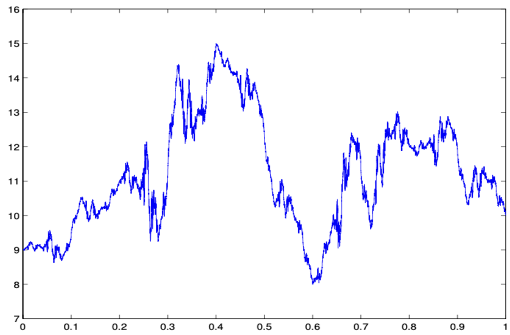

Figure 4. Scaling factors $\alpha = (-0.2, -0.6, 0.3, 0.5, 0.3)$.

3. Fractal interpolation function defined using Φ-Hölder functions

In this section we consider the generalized affine FIF generated by the IFS defined by (1.1) in Section 1 We will assume, for $n \in J$, that the functions $\psi_n \in \mathcal{H}^d_\Phi(I)$ and $\alpha_n : I \longrightarrow \mathbb{R}$ are Lipschitz functions, with Lipschitz constant C_n, such that $\alpha := \max_n \|\alpha_n\|_\infty < 1$, where $\|\alpha_n\|_\infty := \sup\{\alpha_n(x); x \in I, n \in J\}$. Now, for $x \in I$, let

$$\begin{cases} L_{n_1 n_2 \dots n_k}(x) := L_{n_1} \circ L_{n_2} \circ \cdots \circ L_{n_k}(x) \\ L_{n_1 n_2 \dots n_k}(I) := L_{n_1} \circ L_{n_2} \circ \cdots \circ L_{n_k}(I), \end{cases}$$

where $n_j \in J$, $k \geq 1$, $j \in \{1, \dots, k\}$. We define also, for $j = 1, \dots, k-1$, a shift operator σ^j by $\sigma^j(n_1 n_2 \dots n_k) = n_{j+1} \dots n_k$ and

$$L_{\sigma^j(n_1 n_2 \dots n_k)}(x) = L_{n_{j+1} \dots n_k}(x), \qquad 1 \leq j \leq k-1,$$

while $L_{\sigma^k(n_1 n_2 \dots n_k)}(x) = x$. In this paper, we consider the following convention $\prod_{j=1}^0 S_j(x) = 1$ for any family of functions $\{S_j\}_j$.

3.1. *Proof of Theorem 1.1*

First, we will prove the next lemma which will be useful in the proof of Theorem 1.1.

Lemma 3.1. *Let $k \geq 1$, for all $x, y \in L_{n_1 n_2 \dots n_k}(I)$, $n_j \in J$ and $j = 1, \dots, k$, there exist $\bar{x}, \bar{y} \in I$ such that*

(1) $x = \left(\prod_{j=1}^{k} a_{n_j}\right)\bar{x} + \sum_{r=1}^{k}\left(\prod_{j=1}^{r-1} a_{n_j}\right)e_{n_r}$ *and* $y = \left(\prod_{j=1}^{k} a_{n_j}\right)\bar{y} + \sum_{r=1}^{k}\left(\prod_{j=1}^{r-1} a_{n_j}\right)e_{n_r}$.
(2) Let $l \in \{1, \ldots, k\}$*, then*

$$\left|L_{\sigma^l(n_1n_2\ldots n_k)}(\bar{x}) - L_{\sigma^l(n_1n_2\ldots n_k)}(\bar{y})\right| = \left(\prod_{j=1}^{l} a_{n_j}^{-1}\right)\left|x - y\right| \tag{3.1}$$

and, there exits a positive constant $\xi_l == \xi a^{-l}$*, such that*

$$\left|\psi_{n_l}(L_{\sigma^l(n_1n_2\ldots n_k)}(\bar{x})) - \psi_{n_l}(L_{\sigma^l(n_1n_2\ldots n_k)}(\bar{y}))\right| \leq \xi_l \Phi\left(\left|x - y\right|\right). \tag{3.2}$$

Proof. Using a successive iteration and induction (see [30], Lemma 3.1, and [34], Lemma 1) we have, for all $n_j \in J,\ j = 1, \ldots, k$,

$$L_{n_1n_2\ldots n_k}(x) = \left(\prod_{j=1}^{k} a_{n_j}\right)x + \sum_{r=1}^{k}\left(\prod_{j=1}^{r-1} a_{n_j}\right)e_{n_r}. \tag{3.3}$$

Since for every $x, y \in I$ there exist $\bar{x}, \bar{y} \in I$ such that $x = L_{n_1n_2\cdots n_k}(\bar{x})$ and $y = L_{n_1n_2\cdots n_k}(\bar{y})$, the first assertion follows. Now, for $l \in \{1, \ldots, k\}$, we have

$$\begin{aligned}
L_{\sigma^l(n_1n_2\ldots n_k)}(\bar{x}) &= \left(\prod_{j=1}^{k-l} a_{n_{l+j}}\right)\bar{x} + \sum_{r=1}^{k-l}\left(\prod_{j=1}^{r-1} a_{n_{l+j}}\right)e_{n_{l+r}} \\
&= \left(\prod_{j=1}^{k-l} a_{n_{l+j}}\right)\left(\prod_{j=1}^{k} a_{n_j}^{-1}\right)\left[x - \sum_{r=1}^{k}\left(\prod_{j=1}^{r-1} a_{n_j}\right)e_{n_r}\right] + \sum_{r=1}^{k-l}\left(\prod_{j=1}^{r-1} a_{n_{l+j}}\right)e_{n_{l+r}} \\
&= \left(\prod_{j=1}^{l} a_{n_j}^{-1}\right)\left[x - \sum_{r=1}^{k}\left(\prod_{j=1}^{r-1} a_{n_j}\right)e_{n_r}\right] + \sum_{r=1}^{k-l}\left(\prod_{j=1}^{r-1} a_{n_{l+j}}\right)e_{n_{l+r}}.
\end{aligned}$$

Similarly, we have $L_{\sigma^l(n_1n_2\ldots n_k)}(\bar{y}) = \left(\prod_{j=1}^{l} a_{n_j}^{-1}\right)\left[y - \sum_{r=1}^{k}\left(\prod_{j=1}^{r-1} a_{n_j}\right)e_{n_r}\right] + \sum_{r=1}^{k-l}\left(\prod_{j=1}^{r-1} a_{n_{l+j}}\right)e_{n_{l+r}}$ and, as a consequence, we get (3.1). In addition, since Φ satisfies the doubling condition, there exists a constant ξ such that

$$\begin{aligned}
\left|\psi_{n_l}(L_{\sigma^l(n_1n_2\ldots n_k)}(\bar{x})) - \psi_{n_l}(L_{\sigma^l(n_1n_2\ldots n_k)}(\bar{y}))\right| &\leq \Phi\left(\left|L_{\sigma^l(n_1n_2\ldots n_k)}(\bar{x}) - L_{\sigma^l(n_1n_2\ldots n_k)}(\bar{y})\right|\right) \\
&\leq \Phi\left(\prod_{j=1}^{l} a_{n_j}^{-1}\left|x - y\right|\right) \\
&\leq \xi_l \Phi\left(\left|x - y\right|\right).
\end{aligned}$$

□

Now, we will give te prove of the Theorem 1.1. For this, let $x, y \in L_{n_1n_2\ldots n_k}(I)$. Then, by Lemma 3.1, there exist $\bar{x}, \bar{y} \in I$ such that $x = \left(\prod_{j=1}^{k} a_{n_j}\right)\bar{x} + \sum_{r=1}^{k}\left(\prod_{j=1}^{r-1} a_{n_j}\right)e_{n_r}$ and $y = \left(\prod_{j=1}^{k} a_{n_j}\right)\bar{y} +$

$\sum_{r=1}^{k}\Big(\prod_{j=1}^{r-1}a_{n_j}\Big)e_{n_r}$. Moreover, using [30, Lemma 3.2], we get for $r \geq 2$,

$$\begin{aligned}
&\Big|\prod_{l=1}^{r-1}\alpha_{n_l}(L_{\sigma^l(n_1n_2\ldots n_k)}(\bar{x})) - \prod_{l=1}^{r-1}\alpha_{n_l}(L_{\sigma^l(n_1n_2\ldots n_k)}(\bar{y}))\Big| \\
\leq &\sum_{l=1}^{r-1}\alpha^{r-2}C\Big|L_{\sigma^l(n_1n_2\ldots n_k)}(\bar{x}) - L_{\sigma^l(n_1n_2\ldots n_k)}(\bar{y})\Big| \\
\leq &\sum_{l=1}^{r-1}\alpha^{r-2}C\Big(\prod_{j=1}^{l}a_{n_j}^{-1}\Big)\big|x-y\big| \\
\leq &\alpha^{r-2}C\big|x-y\big|\sum_{l=1}^{r-1}a^{-l} = \frac{\alpha^{r-2}C}{1-a}\big|x-y\big|(a^{1-r}-1).
\end{aligned} \tag{3.4}$$

Now, since f is the FIF generated by the system (1.1), we obtain, using the successive iteration and induction,

$$\begin{aligned}
f(x) = f(L_{n_1n_2\ldots n_k}(\bar{x})) =&\Big[\prod_{j=1}^{k}\alpha_{n_j}(L_{\sigma^j(n_1n_2\ldots n_k)}(\bar{x}))\Big]f(\bar{x}) \\
&+ \sum_{r=1}^{k}\Big[\prod_{j=1}^{r-1}\alpha_{n_j}(L_{\sigma^j(n_1n_2\ldots n_k)}(\bar{x}))\Big]\psi_{n_r}(L_{\sigma^r(n_1n_2\ldots n_k)}(\bar{x})).
\end{aligned} \tag{3.5}$$

As a consequence, we get

$$\begin{aligned}
|f(x)-f(y)| \leq \sum_{r=1}^{k}&\Big|\Big[\prod_{j=1}^{r-1}\alpha_{n_j}(L_{\sigma^j(n_1n_2\ldots n_k)}(\bar{x}))\Big]\psi_{n_r}(L_{\sigma^r(n_1n_2\ldots n_k)}(\bar{x})) \\
&-\Big[\prod_{j=1}^{r-1}\alpha_{n_j}(L_{\sigma^j(n_1n_2\ldots n_k)}(\bar{y}))\Big]\psi_{n_r}(L_{\sigma^r(n_1n_2\ldots n_k)}(\bar{y}))\Big| \\
&+\Big|\Big[\prod_{j=1}^{k}\alpha_{n_j}(L_{\sigma^j(n_1n_2\ldots n_k)}(\bar{x}))\Big]f(\bar{x}) - \Big[\prod_{j=1}^{k}\alpha_{n_j}(L_{\sigma^j(n_1n_2\ldots n_k)}(\bar{y}))\Big]f(\bar{y})\Big|.
\end{aligned}$$

Now, using (3.2) and (3.4), we obtain

$$\begin{aligned}
|f(x)-f(y)| \leq &\sum_{r=1}^{k}\Big[\Big|\prod_{j=1}^{r-1}\alpha_{n_j}(L_{\sigma^j(n_1n_2\ldots n_k)}(\bar{x}))\Big|\Big|\psi_{n_r}(L_{\sigma^r(n_1n_2\ldots n_k)}(\bar{x})) - \psi_{n_r}(L_{\sigma^r(n_1n_2\ldots n_k)}(\bar{y}))\Big| \\
&+\Big|\prod_{j=1}^{r-1}\alpha_{n_j}(L_{\sigma^j(n_1n_2\ldots n_k)}(\bar{x})) - \prod_{j=1}^{r-1}\alpha_{n_j}(L_{\sigma^j(n_1n_2\ldots n_k)}(\bar{y}))\Big|\Big]\psi_{n_r}(L_{\sigma^r(n_1n_2\ldots n_k)}(\bar{y}) \\
&+ 2\alpha^k\|f\|_\infty \\
\leq &\sum_{r=1}^{k}\xi_r\alpha^{r-1}\Phi\Big(\big|x-y\big|\Big) + \sum_{r=2}^{k}\frac{\zeta\alpha^{r-2}C}{1-a}\big|x-y\big|(a^{1-r}-1) + 2\alpha^k\|f\|_\infty,
\end{aligned}$$

when we have used (3.2) and (3.4).

Remark 3.1. *Let $x, y \in I$ and k_0 be an integer such that*

$$\prod_{j=1}^{k_0+1} a_{n_j} \leq |x-y] \leq \prod_{j=1}^{k_0} a_{n_j}, \qquad n_j \in \{1, \ldots, k_0\}. \tag{3.6}$$

In order to simplify, we may take $C = \alpha$ (this is the case, for example, when α_n are constant functions α, for all $n \in J$). Therefore, using Theorem 1.1, we have, for all $k \geq k_0$,

$$|f(x) - f(y)| \leq \sum_{r=1}^{k} \xi_r \alpha^{r-1} \Phi\Big(\big|x-y\big|\Big) + \sum_{r=2}^{k} \frac{\zeta\alpha^{r-1}}{1-a}\big|x-y\big|(a^{1-r}-1) + 2\alpha^k \|f\|_\infty.$$

Moreover, we can choose k large enough so that

$$|f(x) - f(y)| \ \leq \ \sum_{r=1}^{k} \xi_r \alpha^{r-1} \Phi\Big(\big|x-y\big|\Big) + \sum_{r=2}^{k} \frac{\zeta\alpha^{r-1}}{1-a}\big|x-y\big|a^{1-r},$$

that is, we may take $\Xi_k = 2\alpha^k \|f\|_\infty - \sum_{r=2}^{k} \frac{\zeta\alpha^{r-1}}{1-a}\big|x-y\big| \leq 0$. Indeed, let $A_1 := a^{k_0+1} \leq |x-y|$ by (3.6) and then

$$\begin{aligned}
\Xi_k = 2\alpha^k \|f\|_\infty - \sum_{r=2}^{k} \frac{\zeta\alpha^{r-1}}{1-a}\big|x-y\big| \ &\leq \ 2\alpha^k \|f\|_\infty - A_1 \sum_{r=2}^{k} \frac{\zeta\alpha^{r-1}}{1-a} \\
&\leq \ 2\alpha^k \|f\|_\infty - \frac{A_1\zeta\alpha}{(1-a)(1-\alpha)}\Big(1-\alpha^{k-1}\Big) \\
&\leq \ \alpha^k \Big[2\|f\|_\infty - \frac{A_1\zeta}{(1-a)(1-\alpha)\alpha^{k-1}} + \frac{A_1\zeta}{(1-a)(1-\alpha)}\Big)\Big].
\end{aligned}$$

Therefore, we only have to take k such that

$$\frac{A_1\zeta}{(1-a)(1-\alpha)\alpha^{k-1}} \geq 2\|f\|_\infty + \frac{A_1\zeta}{(1-a)(1-\alpha)} := \Xi_1$$

or

$$\alpha^{k-1} \leq \frac{A_1\zeta}{\Xi_1(1-a)(1-\alpha)}.$$

In particular, take $\Phi(x) = |x|^\theta$ for $\theta \in (0, 1]$. It follows, since we can choose $\xi = 1$ and then $\xi_r = a^{-r}$, that

$$\begin{aligned}
|f(x) - f(y)| &\leq \sum_{r=1}^{k} \xi_r \alpha^{r-1} \big|x-y\big|^\theta + \sum_{r=2}^{k} \frac{\zeta\alpha^{r-1}}{1-a}\big|x-y\big|a^{1-r} \\
&\leq \frac{\big|x-y\big|^\theta}{\alpha} \sum_{r=1}^{\infty} (\frac{\alpha}{a})^r + \frac{\zeta a}{(1-a)\alpha}\big|x-y\big| \sum_{r=2}^{\infty} (\frac{\alpha}{a})^r \\
&\leq \frac{1}{a-\alpha}\big|x-y\big|^\theta + \frac{\zeta\alpha}{(1-a)(a-\alpha)}\big|x-y\big|.
\end{aligned}$$

Example 3.1. *The nowhere differentiable Weierstrass function is given by*

$$f^{\phi}_{\lambda,l}(x) = \sum_{k=0}^{\infty} \lambda^k \phi(l^k x), \qquad x \in \mathbb{R}, \tag{3.7}$$

where $l \geq 2$ be an integer, $1/l < \lambda < 1$ and $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a $\mathbb{Z}$-periodic real analytic function. This function displays self-similarity on different scales (see Figure 5) and it's graph exhibits fractal-like behavior, with intricate and complex structure on all scales [35, 36].

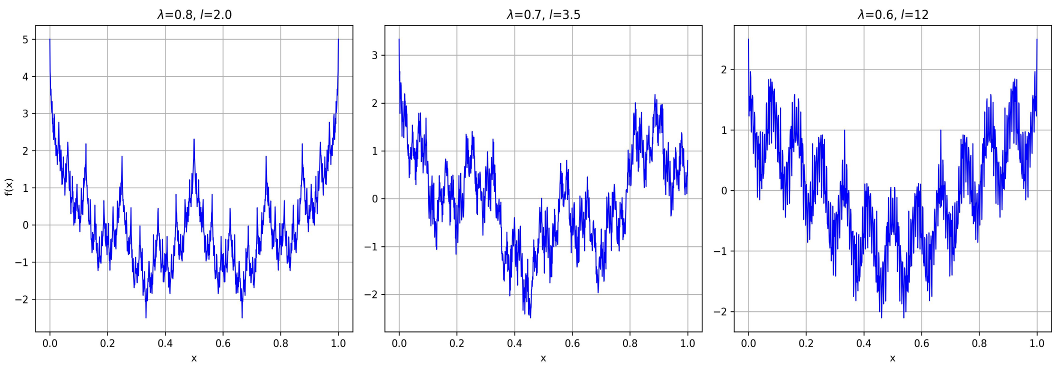

Figure 5. Weierstrass function for different choice of variables λ and l with $\phi(x) = \cos(2\pi x)$.

In this example, we consider the classical Weierstrass function f, that is, when $\phi(x) = \cos(2\pi x)$. Let $I = [0, 1]$, $N = l = 3$ and $\lambda = 1/2$. Now consider

$$W_n(x, y) = \Big(\frac{x+n-1}{3}, \alpha_n(x)y + \phi(\frac{x+n-1}{3})\Big), \qquad (x, y) \in I \times \mathbb{R}, \tag{3.8}$$

where $\alpha_n(x) = \frac{1}{2} + (-1)^{1+\lfloor n/2 \rfloor} \frac{\sin(2\pi x)}{4}$, for $n \in \{1, 2, 3\}$, where $\lfloor n/2 \rfloor$ means the integer part of x. In this case, we have Weierstrass function f is the FIF defined by $\{W_n\}_{n=1}^{3}$ [37]. Therefore, for $n = 1, 2, 3$, we have $\psi_n(x) = \cos(\frac{2\pi(x+n-1)}{3})$ and then, for $x, y \in I$, we have

$$\begin{aligned} \Big|\psi_n(x) - \psi_n(y)\Big| &= \Big| \sin\Big(\frac{2\pi}{3}x + \frac{2\pi}{3}n - \frac{\pi}{6}\Big) - \sin\Big(\frac{2\pi}{3}y + \frac{2\pi}{3}n - \frac{\pi}{6}\Big)\Big| \\ &\leq 2\Big| \sin\Big(\frac{\pi}{3}(x-y)\Big)\Big| \leq \frac{2\pi}{3}|x-y|, \end{aligned}$$

where, we have used the inequality $|\sin(x) - \sin(y)| \leq \sin(\frac{x-y}{2})\cos(\frac{x+y}{2})$. Therefore, the function ψ_n is Φ-Hölder function with $\Phi(x) = \frac{2\pi}{3}x$ ($\xi = 1$). In this case, we have

$$a = \frac{1}{3}, \qquad \xi_r = a^{-r}; \qquad C = \frac{1}{4}, \qquad \alpha \leq \frac{1}{4} \qquad \textit{and} \qquad \zeta = 1.$$

Now, applying Remark 3.1, for k large enough and a given $x, y \in L_{n_1 n_2 \ldots n_k}(I)$, we obtain

$$\begin{aligned} |f(x) - f(y)| &\leq \sum_{r=1}^{k} \xi_r \alpha^{r-1} \Phi\Big(|x-y|\Big) + \sum_{r=2}^{k} \frac{\zeta \alpha^{r-2} C}{1-a} \Big|x-y\Big| a^{1-r} \\ &\leq \frac{2\pi}{3\alpha}\Big|x-y\Big| \sum_{r=1}^{k} (\frac{\alpha}{a})^r + \frac{\zeta C}{(1-a)\alpha}\Big|x-y\Big| \sum_{r=1}^{k-1} (\frac{\alpha}{a})^r \\ &\leq \frac{\alpha}{a-\alpha}\Big|x-y\Big|\Big(\frac{8\pi}{3} + \frac{3}{2}\Big) = \Big(8\pi + \frac{9}{2}\Big)\Big|x-y\Big|. \end{aligned}$$

3.2. Proof of Corollary 1.1

In this section, we will prove some consequences of Theorem 1.1. For this, for each $n \in J$, let $\psi_n \in \mathcal{H}_\Phi^{\mathsf{d}}(I)$ and assume that $\varsigma := \sum_{r=1}^{\infty} \xi_r \alpha^r < \infty$. We define the function

$$\chi(x) = 2M_1\Big(\Phi(\big|x-y\big|) + |x-y|\Big),$$

where

$$M_1 = \max\Big\{\frac{\varsigma}{\alpha}, \frac{\varsigma\zeta C\alpha}{\xi(1-a)} + \frac{2}{a}\|f\|_\infty\Big\}.$$

As a consequence of Remark 3.1, we obtain, the following result.

Proposition 3.1. *Let f be the FIF generated by the IFS* (1.1) *such that $\psi_n \in \mathcal{H}_\Phi^{\mathsf{d}}(I)$ and α_n are Lipschitz functions for each $n \in J$. Assume that $\varsigma := \sum_{r=1}^{\infty} \xi_r \alpha^r < \infty$. Then f is χ-Hölder function on I.*

Proof. Let $x, y \in I$, then there exists k_0 such that (3.6) is satisfied. If $k_0 = 0$ then we prescribe $L_{n_1 n_2 \ldots n_{k_0}}(I) = I$. First, we consider the case when $x, y \in L_{n_1 n_2 \ldots n_{k_0}}(I)$, then, using the same notation as in Theorem 1.1, we have

$$\begin{aligned}
|f(x) - f(y)| &\leq \sum_{r=1}^{k_0} \xi_r \alpha^{r-1} \Phi\Big(\big|x-y\big|\Big) + \sum_{r=2}^{k_0} \frac{\zeta\alpha^{r-2}C}{1-a}\big|x-y\big|a^{1-r} + 2\alpha^{k_0}\|f\|_\infty \\
&\leq \sum_{r=1}^{k_0} \xi_r \alpha^{r-1} \Phi\Big(\big|x-y\big|\Big) + \frac{\zeta C\alpha}{\xi(1-a)}\big|x-y\big| \sum_{r=2}^{k_0} \xi_{r-1}\alpha^{r-1} + \frac{2\alpha^{k_0}\xi_{k_0+1}}{\xi}\|f\|_\infty |x-y| \\
&\leq \frac{\varsigma}{\alpha}\Phi\Big(\big|x-y\big|\Big) + \Big[\frac{\varsigma\zeta C\alpha}{\xi(1-a)} + \frac{2}{a}\|f\|_\infty\Big]|x-y|, \\
&\leq M_1\Big(\Phi(\big|x-y\big|) + |x-y|\Big) = \frac{1}{2}\chi\Big(\big|x-y\big|\Big).
\end{aligned}$$

where we have used the fact that $\xi_{k_0}\alpha^{k_0} < 1$. Now, we consider the other case, that is, when x, y do not belong to the same subinterval $L_{n_1 n_2 \ldots n_{k_0}}(I)$ but (3.6) holds. Then, clearly the reals numbers x and y must belong to the adjacent two intervals with common boundary point denoted by z. It follows that

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f(z)| + |f(z) - f(y)| \leq \frac{1}{2}\chi\Big(\big|x-z\big|\Big) + \frac{1}{2}\chi\Big(\big|y-z\big|\Big) \\
&\leq \chi\Big(\big|x-y\big|\Big).
\end{aligned}$$

□

In the following we will give the proof of the Corollary 1.1. For this, let f be the FIF generated by the IFS (1.1) and let $\Phi(x) := \Phi_0(x) = \mathsf{s}|x|^\theta$, for some positive real number s and $\theta \in]0, 1]$. We assume that, for each $n \in J$, $\psi_n \in \mathcal{H}_{\Phi_0}^{\mathsf{d}}(I)$. Again, we set $\zeta := \max_n \|\psi_n\|_\infty$, $a = \min_n a_n$, $\alpha := \max_n |\alpha_n|$ such that $\alpha < a$.

Now, under our hypothesis, we note that $C = \alpha$, $\xi = 1$, $\xi_l = a^{-l}$ and

$$\varsigma = \sum_{r=1}^{\infty} \Big(\frac{\alpha}{a}\Big)^r = \frac{\alpha}{a-\alpha} < \infty.$$

Therefore, we deduce

$$\begin{aligned}
|f(x)-f(y)| &\leq 2\frac{\varsigma}{\alpha}\Phi\Big(\big|x-y\big|\Big)+2\Big[\frac{\varsigma\zeta C\alpha}{(1-a)}+\frac{2}{a}\|f\|_\infty\Big]|x-y| \\
&\leq \frac{2\,\alpha\,\mathsf{s}^\theta}{a-\alpha}\big|x-y\big|^\theta+\Big[\frac{2\,\zeta\alpha^3}{(a-\alpha)(1-a)}+\frac{4}{a}\|f\|_\infty\Big]|x-y| \\
&\leq \Big[\frac{2\,\alpha\,\mathsf{s}^\theta}{a-\alpha}+\frac{2\zeta\alpha^3}{(a-\alpha)(1-a)}+\frac{4}{a}\|f\|_\infty\Big]|x-y|^\theta
\end{aligned}$$

that is, the function f is a θ-Hölder on I.

Example 3.2. *In this example, we consider the Weierstrass function defined in Example 3.1 by* (3.7). *Let* $I = [0,1]$ *and the interpolating points* $x_0 = 0 < x_1 < \cdots < x_N = 1$ *such that* $x_n - x_{n-1} = 1/N$ $(N = l)$*. We consider the following system defined as*

$$\begin{cases} L_n(x) = \frac{x}{N} + \frac{n-1}{N}, \\ F_n(x,y) = \alpha y + \phi(\frac{x+n-1}{N}), \end{cases} \tag{3.9}$$

where $\alpha = \lambda$*. It is well known that the function f is a FIF [37]. Indeed, consider, for* $n \in J$*, the function*

$$W_n(x,y) = \Big(\frac{x+n-1}{N}, \alpha y + \phi(\frac{x+n-1}{N})\Big), \qquad (x,y) \in [0,1]\times\mathbb{R}.$$

It follows that

$$f(L_n(x)) = f(\frac{x+n-1}{N}) = \phi(\frac{x+n-1}{N}) + \alpha\sum_{k=0}^{\infty}\alpha^k\phi(N^k x) = \phi(\frac{x+n-1}{N}) + \alpha f(x)$$

and thus

$$C_f = \bigcup_{n=1}^{N} W_n(C_f).$$

Therefore, for $n \in J$*, we have* $\psi_n(x) = \cos\big(\frac{2\pi(x+n-1)}{N}\big)$*. It follows, as in Example 3.1, that* ψ_n *is* Φ*-Hölder function with* $\Phi(x) = \frac{2\pi}{N}x$ *and then we may choose* $\xi = 1$*. In addition, choose* $\alpha = \frac{1}{2N}$*, we get*

$$a = \frac{1}{N}, \qquad \xi_r = N^{-r}, \qquad C = \alpha = \frac{1}{2N} \qquad \text{and} \qquad \varsigma = \zeta = 1.$$

It follows, from Corollary 1.1, that

$$\begin{aligned}
|f(x)-f(y)| &\leq \Big[\frac{2\,\alpha}{a-\alpha}\frac{2\pi}{N}+\frac{2\zeta\alpha^3}{(a-\alpha)(1-a)}+\frac{4}{a}\|f\|_\infty\Big]|x-y| \\
&\leq \Big[\frac{4\pi}{N}+\frac{1}{2N(N-1)}+4N\|f\|_\infty\Big]|x-y|.
\end{aligned}$$

4. Upper bound of the maximum range of FIF

Let $I = [0,1]$, $P = \{(\frac{n}{N}, y_n) \in \mathbb{R}^2, n \in J\}$ be the interpolation points and $D = \{\frac{n}{N} \in I, n \in J_0\}$. We define

$$L^0(D) = D, \quad L(D) = \bigcup_{n=1}^{N} L_n(D), \qquad \text{and} \qquad L^k(D) = L \circ \cdots \circ L(D),$$

k times composition. In this section, an interesting case of the system (1.1) will studied. Indeed, in [38], the author observed that we can use the theory of FIF to generate a family of continuous functions having fractal property from a given continuous function and with different values of fractal dimension. Let $f \in C(I)$, the normed space of real valued endowed with the uniform norm continuous function on I, we define the following system

$$\begin{cases} L_n(x) = a_n x + e_n \\ F_n(x,y) = \alpha_n(x)y + f(L_n(x)) - \alpha_n(x)b(x), \end{cases} \tag{4.1}$$

where the real constants a_n and e_n are determined by condition (2.1), the functions $\alpha_n : I \longrightarrow \mathbb{R}$ are Lipschitz functions, with Lipschitz constant C_n such that $\alpha := \max_n \|\alpha_n\|_\infty < 1$ and $b \in C(I)$ such that $b(0) = f(0)$ and $b(1) = f(1)$. Then the FIF generated by (4.1) will be denoted by f^α which interpolates f at the nodes of the partition. Moreover, According to (2.4), the FIF f^α satisfies the fixed point equation [30, 38–40]

$$f^\alpha(x) = f(x) + \alpha_n(L_n^{-1}(x))(f^\alpha - b)(L_n^{-1}(x), \qquad \text{for all } x \in I_n, n \in J. \tag{4.2}$$

Now, we will assume through this section that f and b are Φ_1 and Φ_2 Hölder functions with Hölder constants ξ_f and ξ_b respectively.

Lemma 4.1. *Let f^α be the FIF generated by the system* (4.1) *and assume that $\alpha = \max_n \|\alpha_n\|_\infty < 1$. Then, there exists a positive constant A_1 such that*

$$\left|f^\alpha(x) - y_{n-1}\right| \leq \frac{\alpha\Gamma_1 + \xi_f + \alpha\xi_b}{1-\alpha}, \qquad x \in I_n.$$

Proof. We define, for $k = 1, 2, \ldots,$

$$\Gamma_k = \max\{|f^\alpha(x) - y_0|, x \in L^{k-1}(D)\} \quad \text{and} \quad \gamma_k = \max_n \{|f^\alpha(x) - y_{n-1}|, x \in L^{k-1}(D) \cap I_n\}.$$

First, observe that

$$\begin{aligned} \Gamma_k &\leq \max_n \{|f^\alpha(x) - y_{n-1}|, x \in L^{k-1}(D) \cap I_n\} + \max_n \{|y_{n-1} - y_0|\} \\ &\leq \Gamma_1 + \gamma_k. \end{aligned} \tag{4.3}$$

For $x \in L^k(D) \cap I_n$, we have,

$$f^\alpha(x) = f(x) + \alpha_n(L_n^{-1}(x))(f^\alpha - b)(L_n^{-1}(x))$$

and then

$$\begin{aligned}\left|f^{\alpha}(x)-y_{n-1}\right| \le& \left|f(x)-f(\frac{n-1}{N})\right|+\alpha\left|f^{\alpha}(L_n^{-1}(x))-y_0\right|+\alpha\left|b(L_n^{-1}(x))-y_0\right|\\ \le&\Phi_1\Big(\left|x-\frac{n-1}{N}\right|\Big)+\alpha\Gamma_{k-1}+\alpha\Phi_2\Big(\left|L_n^{-1}(x)\right|\Big)\\ \le&\xi_f\Phi_1(1)+\alpha\Gamma_{k-1}+\alpha\xi_b\Phi_2(1)\\ \le&\xi_f+\alpha\Gamma_{k-1}+\alpha\xi_b.\end{aligned}$$

We denote by $A=\xi_f+\alpha\xi_b$ which nor depends on k. It follows, using (4.3), that

$$\begin{aligned}\gamma_{k+1} &\le \alpha\Gamma_k+A\le\alpha\gamma_k+\alpha\Gamma_1+A\\ &\le \beta\Big(\alpha\Gamma_{k-1}+A\Big)+\alpha\Gamma_1+A\\ &\le \alpha^2\gamma_{k-1}+\alpha^2\Gamma_1+\alpha\Gamma_1+\alpha A+A\\ &\vdots\\ &\le \sum_{j=1}^{k}\alpha^j\Gamma_1+\sum_{j=0}^{k-1}\alpha^jA\le\frac{\alpha\Gamma_1+A}{1-\alpha}.\end{aligned}$$

For any $x\in I_n$, there exits a sequence $\{x_j\}_j\in I_n\cap\Big(\bigcup_k L^k(D)\Big)$ such that $x_j\longrightarrow x$ and then $\lim_{j\to\infty}|f^{\alpha}(x_j)-y_{n-1}|=|f^{\alpha}(x)-y_{n-1}|$, by continuity of the function f^{α} . Therefore, we get

$$\left|f^{\alpha}(x)-y_{n-1}\right|\le\frac{\alpha\Gamma_1+\xi_f+\alpha\xi_b}{1-\alpha},\qquad x\in I_n.$$

□

Given a function S defined on I, we define the maximum range R_S of S as

$$R_S(I)=\sup_{s_1,s_2\in I}\left|S(s_1)-S(s_2)\right|.$$

Theorem 4.1. *Let f^{α} be the α-FIF the IFS* (4.1) *with interpolation points P. Assume that $\alpha<1$, then*

$$R_{\widetilde{f^{\alpha}}}(I)\le\min\Big\{N\,\frac{\alpha\Gamma_1+H_f+\alpha H_b}{1-\alpha},\frac{2}{1-\alpha}(\alpha\|b\|_\infty+\|f\|_\infty)\Big\}.$$

Proof. From Lemma 4.1, we have

$$\sup_{I_n}\left|f^{\alpha}(x)-y_{n-1}\right|\le\frac{\alpha\Gamma_1+H_f+\alpha H_b}{1-\alpha}.$$

Now, let $s_1,s_2\in I$, then there exists $n_1\le n_2\in J$ such that $s_1\in I_{n_1}$ and $s_2\in I_{n_2}$. It follows,

$$\begin{aligned}\left|f^{\alpha}(s_1)-f^{\alpha}(s_2)\right| &\le \left|f^{\alpha}(s_1)-y_{n_1-1}\right|+\left|y_{n_1-1}-y_{n_1}\right|+\cdots+\left|y_{n_2-1}-f^{\alpha}(s_2)\right|\\ &\le N\,\frac{\alpha\Gamma_1+H_f+\alpha H_b}{1-\alpha}.\end{aligned}$$

In the other hand, using (4.2), we obtain

$$R_{\widetilde{f^{\alpha}}}\le 2\|f^{\alpha}\|_\infty\le2\|f^{\alpha}-f\|_\infty+2\|f\|_\infty$$

$$\begin{aligned} &\leq \frac{2\alpha}{1-\alpha}\|f-b\|_\infty + 2\|f\|_\infty \\ &\leq \frac{2}{1-\alpha}(\alpha\|b\|_\infty + \|f\|_\infty). \end{aligned}$$

as required. □

Example 4.1. *Let $I = [0, 1]$ and $f(x) = x - x^2$. Observe that for any $x, y \in I$, we have*

$$|f(x) - f(y)| \leq |x - y| + |x^2 - y^2| \leq 3|x - y|$$

then the function f is Hölderian with exponent 1 and Hölder constant $H_f = 3$. In this example, we consider the following perturbed system

$$\begin{cases} L_n(x) = \frac{x}{N} + \frac{n-1}{N} \\ F_n(x, y) = \alpha y + f(L_n(x)) - \alpha b(x), \end{cases} \tag{4.4}$$

where $b(x) = f(x)/3$. It follows that

$$\|f^\alpha - f\|_\infty \leq \frac{\alpha}{1-\alpha}\|f - b\|_\infty \leq \frac{\alpha}{6(1-\alpha)}.$$

In particular if $\alpha = 1/6$, we obtain

$$\|f^\alpha - f\|_\infty \leq \frac{1}{30}.$$

Therefore, we have

$$\begin{aligned} R_{\widetilde{f^\alpha}}(I) &\leq \frac{2}{1-\alpha}(\alpha\|b\|_\infty + \|f\|_\infty) \\ &\leq \frac{1}{1-\alpha}(\alpha/12 + 1/4) \end{aligned}$$

and then $R_{\widetilde{f^\alpha}}(I) = \frac{19}{60}$ for $\alpha = 1/6$.

5. Conclusions

In the present work, a class of generalized affine FIFs with variable parameters, where ordinate scaling is substituted by real-valued control function, is investigated. Their smoothness is discussed according to the choice of ψ_n, $n \in J$. We prove, in particular, that the FIF is θ-hölder function whenever ψ_n are. Our study is limited to functions $\psi_n \in \mathcal{H}^{\mathrm{d}}_{\Phi}(I)$ and it is worth studying more general cases, for example when doubling condition is not satisfied. Furthermore, we note that the thechnique using in this paper does not allows to study more general case, for example where $F_n(x, y) = \varphi_n(y) + \psi_n(x)$ with φ_n are Matkowski contractions [22].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflit of interest.

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