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Research article

Triangular algebras with nonlinear higher Lie n-derivation by local actions

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Abstract: This paper was devoted to the study of the so-called nonlinear higher Lie n-derivation of triangular algebras $\mathcal{T}$, where n is a nonnegative integer greater than two. Under some mild conditions, we proved that every nonlinear higher Lie n-derivation by local actions on the triangular algebras is of a standard form. As an application, we gave a characterization of higher Lie n-derivation by local actions on upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras, respectively.

Keywords: higher Lie n-derivation; higher derivations; local action; triangular algebras; nest algebras
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1. Introduction

In this paper, we assume that $\mathcal{A}$ is an unital algebra over commutative ring $\mathcal{R}$ and $\mathcal{Z}(\mathcal{A})$ is the center of $\mathcal{A}$. The main purpose of this paper is to study the structure of nonlinear higher Lie n-derivation by local actions on triangular algebras. To achieve this goal, we first introduce some definitions related to nonlinear higher Lie n-derivation by local actions.

Let $\mathcal{N}$ be the set of all nonnegative integers and $\Delta = \{\delta_m\}_{m\in\mathcal{N}}$ be a family of $\mathcal{R}$-linear (resp. nonlinear) mapping $\delta_m : \mathcal{A} \to \mathcal{A}$ on $\mathcal{A}$ such that $\delta_0 = id_{\mathcal{A}}$. Δ is called:
a) a (resp. nonlinear) higher derivation if

$$\delta_m(xy) = \sum_{i+j=m} \delta_i(x)\delta_j(y) \tag{1.1}$$

for all $x, y \in \mathcal{A}$ and for each $n \in \mathcal{N}$;
b) a (resp. nonlinear) higher Lie n-derivation if

$$\delta_m(P_n(x_1, \cdots, x_n)) = \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(x_1), \delta_{i_2}(x_2), \cdots, \delta_{i_n}(x_n)), \tag{1.2}$$

where $P_n(x_1, \cdots, x_n) = [P_{n-1}(x_1, \cdots, x_{n-1}), x_n]$ is a polynomial defined by induction with variables $x_1, \cdots, x_n$ for all $x_1, \cdots, x_n \in \mathcal{A}$; the symbol $[x_1, x_2] = x_1x_2 - x_2x_1$ is called the Lie product. If Eq (1.2) holds only under condition $x_1 \cdots x_n = 0$ for all $x_1, \cdots, x_n \in \mathcal{A}$ and for each $m, n \in \mathcal{N}$, $\Delta = \{\delta_m\}_{m\in\mathcal{N}}$ is said to be higher Lie n-derivation by local actions. That is, Δ is said to be higher Lie n-derivation by local actions if every mapping δ_m satisfies the equation

$$\delta_m(P_n(x_1, \cdots, x_n)) = \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(x_1), \delta_{i_2}(x_2), \cdots, \delta_{i_n}(x_n)) \tag{1.3}$$

for all $x_1, \cdots, x_n \in \mathcal{A}$ with $x_1 \cdots x_n = 0$ and $m, n \in \mathcal{N}$. It should be noted that the study of derivatives satisfying local properties originated from papers [1, 2]. For $n = 2, 3$ and any positive integer m in (1.2) (resp. (1.3)), Δ is referred to as Lie high derivation (resp. by local actions) and Lie triple high derivation (resp. by local actions), respectively. If $m = 1$ and any positive integer n in Eq (1.2) (resp. (1.3)), then the mapping δ_1 is called Lie n-derivation (resp. by local actions). Therefore, in the sense of Herstein Lie type mapping, higher Lie n-derivation is a natural extension of Herstein Lie type mappings. From the structure of maps (1.1)–(1.3), it can be seen that the sum of the higher derivations and the central map of annihilation $P_n(x_1, \cdots, x_n)$ is still higher Lie n-derivation (resp. by local actions), for all $x_1, \cdots, x_n \in \mathcal{A}$ (resp. with $x_1 \cdots x_n = 0$). If every higher Lie n-derivation has this decomposition form, it is said that higher Lie n-derivation has a **standard form**. Under this framework, some special situations of (nonlinear) higher Lie n-derivation have been studied by many scholars, (nonlinear) Lie triple derivation in paper [3, 4], (nonlinear) Lie higher derivations in papers [5–7], Lie higher derivations studied in paper [8], higher derivations [9, 10] etc.

In the author's knowledge system, the study of the structure of higher Lie n-derivation by local actions satisfying Eq (1.3) over rings or algebras has attracted many scholars to study among the many extensions of Lie-n derivation (see [11–19]. In 2011, Ji and Qi [11] studied the structural form of linear Lie derivation by local actions on triangular algebras (in (1.3), where $(m, n) = (1, 2)$), and proved that each Lie derivation by local action has a standard form. Subsequently, Lin [12] extended this to three Lie higher derivation by local actions (in (1.3), m was an arbitrary positive integer and n=2) and obtained that each Lie derivation by local action has a standard form. At the same time, in recent years, the authors and collaborators have found that many scholars have studied the structural problems of some special cases of nonlinear higher Lie n-derivation by local actions defined by equation (1.3) on rings or algebras. Liu in [15] worked the structure of Lie triple derivations by local actions satisfying the condition $x_1x_2x_3 \in \Omega = \{0, p\}$ (in Relation (1.3) with $n = 3$ and $m = 1$), where p is a fixed nontrivial projection of factor von Neumann algebra M with dimension greater than one. He showed that every Lie triple derivations by local actions be of **standard form**, for von Neumann algebra with no central abelian projections $\mathcal{M}$. In 2021, Zhao [16] considered the structure of nonlinear Lie triple derivations by local actions (in Relation (1.3) with $n = 3$ and $m = 1$) on triangular algebra. He confirmed that every nonlinear Lie triple derivations by local actions be of *standard form*. On the basis of his work [16], the first authors and collaborators extended the structure of nonlinear Lie triple derivations [16] to nonlinear Lie triple higher derivations by local actions [17] (condition: For arbitrary $m \in \mathcal{N}$ and $n = 3$ in (1.3)), and proved that each nonlinear Lie triple higher derivation by local actions has a *standard form* (see Eq (1.4)) under the same conditions as Zhao [16]. Inspired by the above results [16, 17], it is natural to consider the structure form of the higher Lie n-derivation by local actions (the case: $m \in \mathcal{N}$ and $n > 2$ in Eq (1.3)) on triangular algebras. The results of this paper generalize Zhao [16] and Liang [17] to a more general form: For arbitrary $m \in \mathcal{N}$ and $n \geq 3$. It should be noted that we are temporarily unable to find a method to prove the structural form of nonlinear Lie higher derivations by local actions on triangular algebras. This is also an open problem left over in this article.

In this paper we established the higher Lie n-derivation by local actions ($n > 2$) on triangular algebras. Let $\mathcal{T}$ be a triangular algebra over a commutative ring $\mathcal{R}$. Under some mild conditions, we prove that if a family $\Delta = \{\delta_m\}_{m\in\mathcal{N}}$ of *nonlinear* mappings δ_m on $\mathcal{T}$ satisfies the condition (1.3), then there exists an additive higher derivation $D = \{\chi_m\}_{m\in\mathcal{N}}$ and a nonlinear mapping $h_m : \mathcal{T} \to \mathcal{T}$ on $\mathcal{T}$ vanishing all $P_n(x_1, \cdots, x_n)$ for all $x_1, \cdots, x_n \in \mathcal{T}$ with $x_1 \cdots x_n = 0$, such that

$$\delta_m(x) = \chi_m(x) + h_m(x).$$

Next, we immediately apply our results to typical examples of triangular algebra: Upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras. At the same time, our conclusion generalizes the conclusions of papers [16, Theorems 2.1 and 2.2] and [17, Theorems 1 and 3].

2. Triangular algebras

In this part, we introduce some basic theories of triangular algebra. In 2001, triangular algebra was first introduced by Cheung [20].

Based on algebra A with identity 1_A and algebra B with identity 1_B, defined on ring $\mathcal{R}$ and faithful (A, B)-bimodule M, for $a \in A$, $aM = \{0\}$ implies $a = 0$ and for $b \in B$, $Mb = \{0\}$ implies $b = 0$. Cheung introduced a set

$$\mathcal{T} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \middle\| \; \forall\, a \in A, m \in M, b \in B \right\}.$$

According to the addition and multiplication properties of matrices, set $\mathcal{T}$ is an associative and noncommutative $\mathcal{R}$-algebra. This algebra is called triangular algebra. The most classical examples of triangular algebras are upper triangular matrix algebras, nest algebras and block upper triangular matrix algebras (see [7, 20, 21] for details). Furthermore, the center $\mathcal{Z}(\mathcal{T})$ of $\mathcal{T}$ is (see [20, 22])

$$\mathcal{Z}(\mathcal{T}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle\| \; am = mb, \; \forall\, m \in M \right\}. \tag{$\heartsuit$}$$

Let us define two natural $\mathcal{R}$-linear projections $\pi_A : \mathcal{T} \to A$ and $\pi_B : \mathcal{T} \to B$ by

$$\pi_A : \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \longmapsto a \quad \text{and} \quad \pi_B : \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \longmapsto b.$$

It follows from simple calculation that $\pi_A(\mathcal{Z}(\mathcal{T}))$ is a subalgebra of $\mathcal{Z}(A)$ and that $\pi_B(\mathcal{Z}(\mathcal{T}))$ is a subalgebra of $\mathcal{Z}(B)$. Additionally, there exists a unique algebraic isomorphism $\tau\colon \pi_A(\mathcal{Z}(\mathcal{T})) \longrightarrow \pi_B(\mathcal{Z}(\mathcal{T}))$ such that $am = m\tau(a)$ for all $a \in \pi_A(\mathcal{Z}(\mathcal{T}))$ and for all $m \in M$.

Regarding the center of algebra $\mathcal{A}$, we need to make the following notes. In 2012, Benkovic and Eremita [23] introduced the following useful condition: For an arbitrary $\mathcal{R}$-algebra $\mathcal{A}$:

$$[x, \mathcal{A}] \in \mathcal{Z}(\mathcal{A}) \Longrightarrow x \in \mathcal{Z}(\mathcal{A}), \quad \forall x \in \mathcal{A}. \tag{$\diamond$}$$

This amounts to saying that

$$[[x,\mathcal{A}],\mathcal{A}] = 0 \Longrightarrow [x,\mathcal{A}] = 0 \in \mathcal{Z}(\mathcal{A}), \quad \forall x \in \mathcal{A}.$$

Note that ($\diamond$) is equivalent to the condition that there does not exist nonzero central inner derivations on $\mathcal{A}$. The usual examples of algebras satisfying ($\diamond$) are upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras (see [24]).

3. Nonlinear Lie n-derivations: Local actions

This section examines the structure of Lie n-derivations on triangular algebras by local operations at zero product. To put it more specifically, we demonstrate that any nonlinear Lie n-derivations by local actions at zero product have a standard form under mild conditions after first demonstrating that the nonlinear Lie n-derivations by local actions at zero product are an additive mapping of module $\mathcal{Z}(\mathcal{T})$. The information will provide the generalized version matching to [16, Theorems 2.1 and 2.2].

Theorem 3.1. *Let $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ be a triangular algebra satisfying $\pi_A(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(A)$ and $\pi_B(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(B)$). Suppose that a mapping $L_n : \mathcal{T} \to \mathcal{T}$ $(n \geq 3)$ is a nonlinear map satisfying*

$$L_n(P_n(x_1, x_2, \cdots, x_n)) = \sum_{i=1}^{n} P_n(x_1, x_2, \cdots, \delta_1(x_i), \cdots, x_n)$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{T}$ with $x_1 x_2 \cdots x_n = 0$, then for every $n \in \mathcal{N}$,

$$L_n(x + y) - L_n(x) - L_n(y) \in \mathcal{Z}(\mathcal{T})$$

for all $x, y \in \mathcal{T}$.

For convenience, let us write $A_{11} = A$, $A_{22} = B$ and $A_{12} = M$, then triangular algebra $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ can be rewritten by $\mathcal{T} = \left[\begin{smallmatrix} A_{11} & A_{12} \\ O & A_{22} \end{smallmatrix}\right]$.

In order to facilitate readers' understanding, we will divide the proof process into the following lemmas.

Lemma 3.1. *[16, Claim 1] Let $a_{ii} \in A_{ii}, i \in \{1, 2\}$. If $a_{11} m_{12} = m_{12} a_{22}$ for all $m_{12} \in A_{12}$, then $a_{11} \oplus a_{22} \in \mathcal{Z}(\mathcal{T})$.*

Lemma 3.2. $L_n(0) = 0$.

In particular, take $x_i = 0$ in formula (1.1) for $i \in \{1, 2, \cdots, n\}$.

Lemma 3.3. *Let $a_{ij} \in A_{ij}$, for $1 \leq i \leq j \leq 2$, then*

1) $L_n(a_{11} + a_{12}) - L_n(a_{11}) - L_n(a_{12}) \in \mathcal{Z}(\mathcal{T})$;
2) $L_n(a_{22} + a_{12}) - L_n(a_{22}) - L_n(a_{12}) \in \mathcal{Z}(\mathcal{T})$.

Proof. **1)** Let $a_{11} \in A_{11}$ and $a_{12}, c_{12} \in A_{12}$. Denote $T = L_n(a_{11} + a_{12}) - L_n(a_{11}) - L_n(a_{12})$. It is clear that the elements a_{11}, a_{12}, c_{12} and idempotents p_1, p_2 satisfy the relations $a_{12}(a_{11} + c_{12}) p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}} = 0 =$

$a_{12}a_{11}p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}} = 0 = a_{12}c_{12}p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}$, then we have

$$\begin{aligned}
L_n(a_{11}a_{12}) &= L_n(P_n(a_{12}, a_{11} + c_{12}, p_1, \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}})) \\
&= P_n(L_n(a_{12}), a_{11} + c_{12}, p_1, \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}) + P_n(a_{12}, L_n(a_{11} + c_{12}), p_1, \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}) \\
&+ P_n(a_{12}, a_{11} + c_{12}, L_n(p_1), \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}) \\
&+ \sum_{i=4}^{n} P_n(a_{12}, a_{11} + c_{12}, p_1, p_2, \cdots, \underbrace{L_n(p_2)}_{i-th\ component}, \cdots, p_2),
\end{aligned}$$

and on the other hand, we obtain

$$\begin{aligned}
L_n(a_{11}a_{12}) &= L_n(P_n(a_{12}, a_{11}, p_1, \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}})) + L_n(P_n(a_{12}, c_{12}, p_1, \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}})) \\
&= P_n(L_n(a_{12}), a_{11} + c_{12}, p_1, \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}) + P_n(a_{12}, L_n(a_{11}) + L_n(c_{12}), p_1, \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}}) \\
&+ P_n(a_{12}, a_{11} + c_{12}, L_n(p_1), \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}}) \\
&+ \sum_{i=4}^{n} P_n(a_{12}, a_{11} + c_{12}, p_1, p_2, \cdots, \underbrace{L_n(p_2)}_{i-th\ component}, \cdots, p_2).
\end{aligned}$$

The above two equations can be lead to $P_n(a_{12}, T, p_1, \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}) = 0$, which is $p_1Tp_1a_{12} = a_{12}p_2Tp_2$. It follows from Lemma 3.1 that

$$p_1Tp_1 \oplus p_2Tp_2 \in \mathcal{Z}(\mathcal{T}). \tag{3.1}$$

Let's prove that $p_1Tp_2 = 0$. Since the elements a_{11}, a_{12} and idempotents p_1, p_2 satisfy the relation $p_2(a_{11}+a_{12})p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}} = 0 = p_2a_{11}p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}} = 0 = p_2a_{12}p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}$, we study the form of elements $L_n(a_{12})$ form two perspectives, namely,

$$\begin{aligned}
L_n(a_{12}) &= L_n(P_n(p_2, a_{11} + a_{12}, p_1, \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}})) \\
&= P_n(L_n(p_2), a_{11} + a_{12}, p_1, \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}) + P_n(p_2, L_n(a_{11} + a_{12}), p_1, \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}) \\
&+ P_n(p_2, a_{11} + a_{12}, L_n(p_1), \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}) \\
&+ \sum_{i=4}^{n} P_n(p_2, a_{11} + a_{12}, p_1, p_2, \cdots, \underbrace{L_n(p_2)}_{i-th\ component}, \cdots, p_2)
\end{aligned}$$

and

$$\begin{aligned}
L_n(a_{12}) &= L_n(P_n(p_2, a_{11}, p_1, \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}})) + L_n(P_n(p_2, a_{12}, p_1, \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}})) \\
&= P_n(L_n(p_2), a_{11} + a_{12}, p_1, \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}) + P_n(p_2, L_n(a_{11}) + L_n(a_{12}), p_1, \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}) \\
&+ P_n(p_2, a_{11} + a_{12}, L_n(p_1), \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}) \\
&+ \sum_{i=4}^{n} P_n(p_2, a_{11} + a_{12}, p_1, p_2, \cdots, \underbrace{L_n(p_2)}_{i-th\ component}, \cdots, p_2)
\end{aligned}$$

for all $a_{11} \in A_{11}, a_{12} \in A_{12}$. It follows from the above two equations that $P_n(p_2, T, p_1, \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}) = 0$, which is $p_1 T p_2 = 0$. Combining with relation (3.1), we have

$$L_n(a_{11} + a_{12}) - L_n(a_{11}) - L_n(a_{12}) \in \mathcal{Z}(\mathcal{T})$$

for all $a_{ij} \in A_{ij}, 1 \leq i \neq j \leq 2$.

Through a similar calculation process, we can conclude that (2) holds. □

Lemma 3.4. *Let $a_{12}, c_{12} \in A_{12}$, then $L_n(a_{12} + c_{12}) = L_n(a_{12}) + L_n(c_{12})$.*

Proof. Combining $(-a_{12} - p_1)(p_2 + c_{12})p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}} = 0$ with Lemmas 3.2 and 3.3, we have

$$\begin{aligned}
L_n(a_{12} + c_{12}) &= L_n(P_n(-a_{12} - p_1, p_2 + c_{12}, p_1, p_2, \cdots, p_2)) \\
&= P_n(L_n(-a_{12}) + L_n(-p_1), p_2 + c_{12}, p_1, p_2, \cdots, p_2) \\
&+ P_n(-a_{12} - p_1, L_n(p_2) + L_n(c_{12}), p_1, p_2, \cdots, p_2) \\
&+ P_n(-a_{12} - p_1, p_2 + c_{12}, L_n(p_1), p_2, \cdots, p_2) \\
&+ \sum_{i=4}^{n} P_n(-a_{12} - p_1, p_2 + c_{12}, p_1, p_2, \cdots, \underbrace{L_n(p_2)}_{i-th\ component}, \cdots, p_2) \\
&= L_n(P_n(-a_{12}, p_2, p_1, p_2, \cdots, p_2)) + L_n(P_n(-a_{12}, c_{12}, p_1, p_2, \cdots, p_2)) \\
&+ L_n(P_n(-p_1, p_2, p_1, p_2, \cdots, p_2)) + L_n(P_n(-p_1, c_{12}, p_1, p_2, p_2, \cdots, p_2)) \\
&= L_n(a_{12}) + L_n(c_{12})
\end{aligned}$$

for all $a_{12}, c_{12} \in A_{12}$. □

Lemma 3.5. *Let $a_{ii} \in A_{ii}, i \in \{1, 2\}$, then $\delta_1(a_{ii} + b_{ii}) = \delta_1(a_{ii}) + \delta_1(b_{ii}) + Z_{a_{ii},b_{ii}}$ for some central element $Z_{a_{ii},b_{ii}} \in \mathcal{Z}(\mathcal{T})$.*

Proof. We will only prove the case with $i = 1$. The proof of the case $i = 2$ can be proved through a similar process.

Let $a_{11}, b_{11} \in A_{11}, w_{12} \in A_{12}$. Denote $U = \delta_1(a_{11}+b_{11}) - \delta_1(a_{11}) - \delta_1(b_{11})$. It follows from $w_{12}p_1(a_{11}+b_{11})\underbrace{p_2\cdots p_2}_{n-3\text{ copies}} = w_{12}p_1a_{11}\underbrace{p_2\cdots p_2}_{n-3\text{ copies}} = w_{12}p_1b_{11}\underbrace{p_2\cdots p_2}_{n-3\text{ copies}} = 0$ that

$$\begin{aligned} L_n((a_{11}+b_{11})w_{12}) &= L_n(P_n(w_{12}, p_1, (a_{11}+b_{11}), p_2, \cdots, p_2)) \\ &= P_n(L_n(w_{12}), p_1, (a_{11}+b_{11}), p_2, \cdots, p_2) \\ &+ P_n(w_{12}, L_n(p_1), (a_{11}+b_{11}), p_2, \cdots, p_2) \\ &+ P_n(w_{12}, p_1, L_n(a_{11}+b_{11}), p_2, \cdots, p_2) \\ &+ \sum_{i=4}^{n} P_n(w_{12}, p_1, (a_{11}+b_{11}), p_2, \cdots, \underbrace{L_n(p_2)}_{i-th\ component}, \cdots, p_2) \end{aligned}$$

and

$$\begin{aligned} L_n((a_{11}+b_{11})w_{12}) &= L_n(a_{11}w_{12}) + L_n(b_{11}w_{12}) \\ &= L_n(P_n(w_{12}, p_1, a_{11}, p_2, \cdots, p_2)) \\ &+ L_n(P_n(w_{12}, p_1, b_{11}), p_2, \cdots, p_2)) \\ &= P_n(L_n(w_{12}), p_1, (a_{11}+b_{11}), p_2, \cdots, p_2) \\ &+ P_n(w_{12}, L_n(p_1), (a_{11}+b_{11}), p_2, \cdots, p_2) \\ &+ P_n(w_{12}, p_1, L_n(a_{11}) + L_n(b_{11}), p_2, \cdots, p_2) \\ &+ \sum_{i=4}^{n} P_n(w_{12}, p_1, (a_{11}+b_{11}), p_2, \cdots, \underbrace{L_n(p_2)}_{i-th\ component}, \cdots, p_2). \end{aligned}$$

By observing the two equations above, we have $P_n(w_{12}, p_1, U, p_2, \cdots, p_2) = 0$, which is $w_{12}U = Uw_{12}$. It follows from Lemma 3.1 that

$$p_1Up_1 \oplus p_2Up_2 \in \mathcal{Z}(\mathcal{T}). \tag{3.2}$$

In the rest of the lemma, we prove that the equation $p_1Up_2 = 0$ holds.

Since the equations $(a_{11}+b_{11})\underbrace{p_2\cdots p_2}_{n-1\text{ copies}} = a_{11}\underbrace{p_2\cdots p_2}_{n-1\text{ copies}} = b_{11}\underbrace{p_2\cdots p_2}_{n-1\text{ copies}} = 0$ hold, then we have

$$\begin{aligned} 0 &= L_n(P_n(a_{11}+b_{11}, \underbrace{p_2, \cdots, p_2}_{n-1\text{ copies}})) \\ &= P_n(L_n(a_{11}+b_{11}), \underbrace{p_2, \cdots, p_2}_{n-1\text{ copies}}) + \sum_{i=2}^{n} P_n(a_{11}+b_{11}, \underbrace{p_2, \cdots, \underbrace{L_n(p_2)}_{i-th\ component}, \cdots, p_2}_{n-1\text{ copies}}) \end{aligned}$$

and

$$
\begin{aligned}
0 &= L_n(P_n(a_{11}, \underbrace{p_2, \cdots, p_2}_{n-1 \text{ copies}})) + L_n(P_n(b_{11}, \underbrace{p_2, \cdots, p_2}_{n-1 \text{ copies}})) \\
&= P_n(L_n(a_{11}) + L_n(b_{11}), \underbrace{p_2, \cdots, p_2}_{n-1 \text{ copies}}) \\
&+ \sum_{i=2}^{n} P_n(a_{11} + b_{11}, \underbrace{p_2, \cdots, \underbrace{L_n(p_2)}_{i-th\ component}, \cdots, p_2}_{n-1 \text{ copies}}).
\end{aligned}
$$

By observing the two equations above, we have $P_n(U, \underbrace{p_2, \cdots, p_2}_{n-1 \text{ copies}}) = 0$, which is $p_1 U p_2 = 0$. Combining with Eq (3.2), we have

$$
L_n(a_{11} + b_{11}) - L_n(a_{11}) - L_n(b_{11}) \in \mathcal{Z}(\mathcal{T})
$$

for all $a_{11}, b_{11} \in A_{11}$.

□

Lemma 3.6. *Let $a_{ij} \in A_{ij}, 1 \leq i \leq j \leq 2$, then $L_n(a_{11}+a_{12}+a_{22}) = L_n(a_{11})+L_n(a_{12})+L_n(a_{22})+Z_{a_{11},a_{12},a_{22}}$.*

Proof. Let $a_{ij} \in A_{ij}, 1 \leq i \leq j \leq 2$. Denote $U = L_n(a_{11} + a_{12} + a_{22}) - L_n(a_{11}) - L_n(a_{12}) - L_n(a_{22})$. With the help of the fact that equation $(a_{11} + a_{12} + a_{22})c_{12}p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}} = 0$ holds, we have

$$
\begin{aligned}
& L_n(c_{12}a_{22} - a_{11}c_{12}) \\
&= L_n(P_n(a_{11} + a_{12} + a_{22}, c_{12}, p_1, \underbrace{p_2, \cdots p_2}_{n-3 \text{ copies}})) \\
&= P_n(L_n(a_{11} + a_{12} + a_{22}), c_{12}, p_1, \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}}) + P_n(a_{11} + a_{12} + a_{22}, L_n(c_{12}), p_1, \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}}) \\
&+ P_n(a_{11} + a_{12} + a_{22}, c_{12}, L_n(p_1), \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}}) + \sum_{i=4}^{n} P_n(a_{11} + a_{12} + a_{22}, c_{12}, p_1, \underbrace{p_2, \cdots, \underbrace{L_n(p_2)}_{i-th\ component}, \cdots, p_2}_{n-3 \text{ copies}})
\end{aligned}
$$

and

$$\begin{aligned}L_n(c_{12}a_{22}-a_{11}c_{12}) &= L_n(c_{12}a_{22})+L_n(-a_{11}c_{12})\\ &= L_n(P_n(a_{11},c_{12},p_1,\underbrace{p_2,\cdots p_2}_{n-3\text{ copies}}))+L_n(P_n(a_{12},c_{12},p_1,\underbrace{p_2,\cdots p_2}_{n-3\text{ copies}}))\\ &+ L_n(P_n(a_{22},c_{12},p_1,\underbrace{p_2,\cdots p_2}_{n-3\text{ copies}}))\\ &= P_n(L_n(a_{11})+L_n(a_{12})+L_n(a_{22}),c_{12},p_1,\underbrace{p_2,\cdots,p_2}_{n-3\text{ copies}})\\ &+ P_n(a_{11}+a_{12}+a_{22},L_n(c_{12}),p_1,\underbrace{p_2,\cdots,p_2}_{n-3\text{ copies}})\\ &+ P_n(a_{11}+a_{12}+a_{22},c_{12},L_n(p_1),\underbrace{p_2,\cdots,p_2}_{n-3\text{ copies}})\\ &+ \sum_{i=4}^{n} P_n(a_{11}+a_{12}+a_{22},c_{12},p_1,\underbrace{p_2,\cdots,\underbrace{L_n(p_2)}_{i-th\ component},\cdots,p_2}_{n-3\text{ copies}}).\end{aligned}$$

By observing the two equations above, we have $P_n(U,c_{12},p_1,\underbrace{p_2,\cdots,p_2}_{n-3\text{ copies}})=0$, which is $c_{12}p_2Up_2=p_1Up_1c_{12}$. We have

$$p_1Up_1\oplus p_2Up_2\in\mathcal{Z}(\mathcal{T}). \tag{3.3}$$

For the rest, we prove the conclusion: $p_1Up_2=0$. It is clear that $(a_{11}+a_{12}+a_{22})(-p_1)\underbrace{p_2\cdots p_2}_{n-2\text{ copies}}=0$.

We then obtain

$$\begin{aligned}L_n(a_{12}) &= L_n(P_n(a_{11}+a_{12}+a_{22},-p_1,p_2,\cdots,p_2))\\ &= P_n(L_n(a_{11}+a_{12}+a_{22}),-p_1,p_2,\cdots,p_2)+P_n(a_{11}+a_{12}+a_{22},L_n(-p_1),p_2,\cdots,p_2)\\ &+ \sum_{i=3}^{n} P_n(a_{11}+a_{12}+a_{22},-p_1,p_2,\cdots,\underbrace{L_n(p_2)}_{i-th\ component},\cdots,p_2)\end{aligned}$$

and

$$\begin{aligned}L_n(a_{12}) &= L_n(P_n(a_{11},-p_1,p_2,\cdots,p_2))+L_n(P_n(a_{12},-p_1,p_2,\cdots,p_2))+L_n(P_n(a_{22},-p_1,p_2,\cdots,p_2))\\ &= P_n(L_n(a_{11})+L_n(a_{12})+L_n(a_{22})),-p_1,p_2,\cdots,p_2)+P_n(a_{11}+a_{12}+a_{22},L_n(-p_1),p_2,\cdots,p_2)\\ &+ \sum_{i=3}^{n} P_n(a_{11}+a_{12}+a_{22},-p_1,p_2,\cdots,\underbrace{L_n(p_2)}_{i-th\ component},\cdots,p_2).\end{aligned}$$

By observing the two equations above, we have $P_n(U,-p_1,\underbrace{p_2,\cdots,p_2}_{n-2\text{ copies}})=0$, which is $p_1Up_2=0$.

Combining with Eq (3.3), we have

$$L_n(a_{11}+a_{12}+a_{22})=L_n(a_{11})+L_n(a_{12})+L_n(a_{22})+Z_{a_{11},a_{12},a_{22}}$$

for all $a_{ij}\in A_{ij}, 1\le i\le j\le 2$. □

Proof of Theorem 3.1: For arbitrary $x, y \in \mathcal{T}$, elements x and y have decomposition form $x = a_{11} + a_{12} + a_{22}$ and $y = b_{11} + b_{12} + b_{22}$, where $a_{ij}, b_{ij} \in A_{ij}, 1 \leq i \leq j \leq 2$. It follows from Lemmas 3.2–3.6 that there exists $C_l \in \mathcal{Z}(\mathcal{T}), l \in \{1, 2, 3, 4, 5\}$.

$$\begin{aligned} L_n(x+y) &= L_n(a_{11} + a_{12} + a_{22} + b_{11} + b_{12} + b_{22}) \\ &= L_n(a_{11} + b_{11}) + L_n(a_{12} + b_{12}) + L_n(a_{22} + b_{22}) + C_1 \\ &= L_n(a_{11}) + L_n(b_{11}) + L_n(a_{12}) + L_n(b_{12}) + L_n(a_{22}) + L_n(b_{22}) + C_1 + C_2 + C_3 \\ &= L_n(a_{11} + a_{12} + a_{22}) + L_n(b_{11} + b_{12} + b_{22}) + C_1 + C_2 + C_3 + C_4 + C_5. \end{aligned}$$

Therefore, we have

$$L_n(x+y) = L_n(x) + L_n(y) + C_0$$

for some $C_0 = C_1 + C_2 + C_3 + C_4 + C_5 \in \mathcal{Z}(\mathcal{T})$.

Based on the almost additivity of L_n, we present the main theorem of this part to the readers as follows.

Theorem 3.2. *Let* $\mathcal{T} = \left[\begin{smallmatrix} A_{11} & A_{12} \\ O & A_{22} \end{smallmatrix}\right]$ *be a triangular algebra satisfying*

i) $\pi_{A_{11}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A}_{11})$ and $\pi_{A_{22}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A}_{22})$;
ii) For any $a_{11} \in A_{11}$, if $[a_{11}, A_{11}] \in \mathcal{Z}(\mathcal{A}_{11})$, then $a_{11} \in \mathcal{Z}(\mathcal{A}_{11})$ or for any $a_{22} \in A_{22}$, if $[a_{22}, A_{22}] \in \mathcal{Z}(\mathcal{A}_{22})$, then $a_{22} \in \mathcal{Z}(\mathcal{A}_{22})$.

Suppose that a mapping $L_n : \mathcal{T} \to \mathcal{T} (n \geq 3)$ is a nonlinear mapping satisfying

$$L_n(P_n(x_1, x_2, \cdots, x_n)) = \sum_{i=1}^{n} P_n(x_1, x_2, \cdots, \underbrace{L_n(x_i)}_{i-th\ component}, \cdots, x_n)$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{T}$ with $\prod_{i=1}^{n} x_i = 0$, then for every $n \in \mathcal{N}$,

$$L_n(x) = d_n(x) + f_n(x)$$

for all $x \in \mathcal{T}$, where $d_n : \mathcal{T} \to \mathcal{T}$ is an additive derivation, and $f_n : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ is a nonlinear mapping such that $f_n(P_n(x_1, x_2, \cdots, x_n)) = 0$ for any $x_1, x_2, \cdots, x_n \in \mathcal{T}$ with $x_1 x_2 \cdots x_n = 0$.

In order to facilitate readers' understanding, we will divide the proof process into the following lemmas for explanation.

Lemma 3.7. *With notations as above, we have*

1) $L_n(a_{12}) \in A_{12}$;
2) $p_1 L_n(p_1) p_1 \oplus p_2 L_n(p_1) p_2 \in \mathcal{Z}(\mathcal{T})$ and $p_1 L_n(p_2) p_1 \oplus p_2 L_n(p_2) p_2 \in \mathcal{Z}(\mathcal{T})$;
3) $L_n(p_1) \in A_{12} + \mathcal{Z}(\mathcal{T})$ and $L_n(p_2) \in A_{12} + \mathcal{Z}(\mathcal{T})$.

Proof. Since the equation $a_{12} p_1 p_1 p_2 \cdots p_2 = 0$ holds, we have

$$\begin{aligned} L_n(a_{12}) &= L_n(P_n(a_{12}, p_1, p_1, p_2, \cdots, p_2)) \\ &= P_n(L_n(a_{12}), p_1, p_1, p_2, \cdots, p_2) + P_n(a_{12}, L_n(p_1), p_1, p_2, \cdots, p_2) \\ &+ P_n(a_{12}, p_1, L_n(p_1), p_2, \cdots, p_2) + \sum_{i=4}^{n} P_n(a_{12}, p_1, p_1, p_2, \cdots, \underbrace{L_n(p_2)}_{i-th\ component}, \cdots, p_2) \\ &= p_1 L_n(a_{12}) p_2 + 2 p_1 L_n(p_1) a_{12} - 2 a_{12} L_n(p_1) p_2 + (n-3) p_1 [a_{12}, L_n(p_2)] p_2. \end{aligned} \tag{3.4}$$

According to above relation (3.4), we have $L_n(a_{12}) \in A_{12}$. Multiplying by p_1 on the left side and p_2 on the right side of the above Eq (3.4), we can obtain

$$-2p_1[a_{12}, L_n(p_1)] + (n-3)p_1[a_{12}, L_n(p_2)]p_2 = 0 \tag{3.5}$$

for all $a_{12} \in A_{12}$.

On the other hand, with the help of $p_2 a_{12} p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}} = 0$, we have

$$\begin{aligned} L_n(a_{12}) &= L_n(P_n(p_2, a_{12}, p_1, p_2, \cdots, p_2)) \\ &= P_n(L_n(p_2), a_{12}, p_1, p_2, \cdots, p_2) + P_n(p_2, L_n(a_{12}), p_1, p_2, \cdots, p_2) \\ &+ P_n(p_2, a_{12}, L_n(p_1), p_2, \cdots, p_2) + \sum_{i=4}^{n} P_n(p_2, a_{12}, p_1, p_2, \cdots, \underbrace{L_n(p_2)}_{i-th\ component}, \cdots, p_2) \\ &= -p_1[L_n(p_2), a_{12}]p_2 + p_1 L_n(a_{12})p_2 - p_1[a_{12}, L_n(p_1)]p_2 + (n-3)p_1[a_{12}, L_n(p_2)]p_2. \end{aligned} \tag{3.6}$$

According to (3.6), we have

$$(n-4)p_1[a_{12}, L_n(p_2)]p_2 - p_1[a_{12}, L_n(p_1)]p_2 = 0. \tag{3.7}$$

Combining (3.5) with (3.6), we have

$$p_1[a_{12}, L_n(p_2)]p_2 = -p_1[a_{12}, L_n(p_1)]p_2. \tag{3.8}$$

Using equalities (3.7) with (3.8) and considering the $(n-1)$-torsion-free properties of rings $\mathcal{R}$, we conclude that

$$p_1[a_{12}, L_n(p_2)]p_2 = p_1[a_{12}, L_n(p_1)]p_2 = 0$$

for all $a_{12} \in A_{12}$.

With the help of Lemma 3.1 that

$$p_1 L_n(p_1)p_1 \oplus p_2 L_n(p_1)p_2 \in \mathcal{Z}(\mathcal{T}) \text{ and } p_1 L_n(p_2)p_1 \oplus p_2 L_n(p_2)p_2 \in \mathcal{Z}(\mathcal{T}),$$

the structural characteristics of triangular algebra lead to conclusions

$$L_n(p_1) = p_1 L_n(p_1)p_1 + p_1 L_n(p_1)p_2 + p_2 L_n(p_1)p_2 \in A_{12} + \mathcal{Z}(\mathcal{T}).$$

Furthermore, we claim that this lemma holds. □

Lemma 3.8. *With notations as above, for all $a_{ii} \in A_{ii}, i \in \{1, 2\}$, we have*

i) $L_n(a_{11}) \in A_{11} + A_{12} + \mathcal{Z}(\mathcal{T})$, where $p_2 L_n(a_{11})p_2 \in Z(A_{22})$;
ii) $L_n(a_{22}) \in A_{22} + A_{12} + \mathcal{Z}(\mathcal{T})$, where $p_1 L_n(a_{22})p_1 \in Z(A_{11})$;
iii) $p_1 L_n(p_1)p_2 + p_1 L_n(p_2)p_2 = 0$.

Proof. In fact, it is clear that if $a_{11}p_2\cdots p_2 = 0$, we have

$$\begin{aligned}0 &= L_n(P_n(a_{11}, p_2, \cdots, p_2))\\ &= P_n(L_n(a_{11}), p_2, \cdots, p_2) + P_n(a_{11}, L_n(p_2), p_2, \cdots, p_2)\\ &= p_1L_n(a_{11})p_2 + a_{11}L_n(p_2)p_2,\end{aligned}$$

which implies that

$$p_1L_n(a_{11})p_2 + a_{11}L_n(p_2)p_2 = 0 \tag{3.9}$$

for all $a_{11} \in A_{11}$. Through similar calculations, combining with relation that if $a_{22}p_1p_1p_2, \cdots p_2 = 0$, we can show that

$$\begin{aligned}0 &= L_n(P_n(a_{22}, p_1, p_1, p_2, \cdots, p_2))\\ &= P_n(L_n(a_{22}), p_1, p_1, p_2, \cdots, p_2) + P_n(a_{22}, L_n(p_1), p_1, p_2, \cdots, p_2)\\ &= p_1L_n(a_{22})p_2 + p_1L_n(p_2)a_{22},\end{aligned}$$

which implies

$$p_1L_n(a_{22})p_2 + p_1L_n(p_2)a_{22} = 0 \tag{3.10}$$

for all $a_{22} \in A_{22}$.

Now, we consider the following equation

$$\begin{aligned}0 = L_n(P_n(p_1, p_2, \cdots, p_2)) &= P_n(L_n(p_1), p_2, \cdots, p_2) + P_n(p_1, L_n(p_2), \cdots, p_2)\\ &= p_1L_n(p_1)p_2 + p_1L_n(p_2)p_2.\end{aligned}$$

That is,

$$p_1L_n(p_1)p_2 + p_1L_n(p_2)p_2 = 0. \tag{3.11}$$

In the light of equations $a_{22}a_{11}a_{12}p_2\cdots p_2 = 0$, we have

$$\begin{aligned}0 &= L_n(P_n(a_{22}, a_{11}, a_{12}, p_2, \cdots, p_2))\\ &= P_n(L_n(a_{22}), a_{11}, a_{12}, p_2, \cdots, p_2) + P_n(a_{22}, L_n(a_{11}), a_{12}, p_2, \cdots, p_2)\\ &= p_1[[L_n(a_{22}), a_{11}], a_{12}]p_2 + p_1[[a_{22}, L_n(a_{11})], a_{12}]p_2\\ &= p_1[[L_n(a_{22}), a_{11}]p_2 + p_1[a_{22}, L_n(a_{11})], a_{12}]p_2.\end{aligned}$$

According to Lemma 3.1, we have

$$[L_n(a_{22}), a_{11}] + [a_{22}, L_n(a_{11})] \in \mathcal{Z}(\mathcal{T}).$$

Furthermore, we have

$$[p_1L_n(a_{22})p_1, a_{11}] \in \mathcal{Z}(A_{11}) \text{ and } [a_{22}, p_2L_n(a_{11})p_2] \in \mathcal{Z}(A_{22})$$

for all $a_{11} \in A_{11}, a_{22} \in A_{22}$. Thanks to the assumption (ii), we have

$$p_1L_n(a_{22})p_1 \in \mathcal{Z}(A_{11}) \text{ and } p_2L_n(a_{11})p_2 \in \mathcal{Z}(A_{22})$$

for all $a_{11} \in A_{11}, a_{22} \in A_{22}$. Therefore, this lemma holds.

□

Now, we define two mappings $r_{n1} : A_{11} \to \mathcal{Z}(A_{11})$ and $r_{n2} : A_{22} \to \mathcal{Z}(A_{22})$, as following

$$r_{n1}(a_{11}) = \tau^{-1}(p_2 L_n(a_{11}) p_2) + p_2 L_n(a_{11}) p_2$$

and

$$r_{n2}(a_{22}) = \tau(p_1 L_n(a_{22}) p_1) + p_1 L_n(a_{22}) p_1$$

for all $a_{11} \in A_{11}, a_{22} \in A_{22}$, respectively. It follows from Lemma 3.11 that $r_{n1} : A_{11} \to \mathcal{Z}(A_{11})$ satisfies the relation $r_{n1}(P_n(a_{11}^1, a_{11}^2, \cdots, a_{11}^n)) = 0$ for all $a_{11}^1, a_{11}^2, \cdots, a_{11}^n \in A_{11}$, with $a_{11}^1 a_{11}^2 \cdots a_{11}^n = 0$ and $r_{n2} : A_{22} \to \mathcal{Z}(A_{22})$ satisfying the relation $r_{n2}(P_n(a_{22}^1, a_{22}^2, \cdots, a_{22}^n)) = 0$ for all $a_{22}^1, a_{22}^2, \cdots, a_{22}^n \in A_{22}$ with $a_{22}^1 a_{22}^2 \cdots a_{22}^n = 0$. Now, setting

$$\begin{aligned} h_n(x) &= r_{n1} + r_{n2} \\ &= \tau^{-1}(p_2 L_n(a_{11}) p_2) + p_2 L_n(a_{11}) p_2 + \tau(p_1 L_n(a_{22}) p_1) + p_1 L_n(a_{22}) p_1 \end{aligned}$$

for all $x \in T$, which satisfies the form $x = a_{11} + a_{12} + a_{22}$, it is clear that $h_n(x) \in \mathcal{Z}(\mathcal{T})$ and $h_n(P_n(x_1, x_2, \cdots, x_n)) = 0$ for all $x_1, x_2, \cdots, x_n \in \mathcal{T}$ with $x_1 x_2 \cdots x_n = 0$.

Let's define an important mapping:

$$\Psi_n(x) = L_n(x) - h_n(x)$$

for all $x \in \mathcal{T}$.

Now, we can easily obtain the following lemmas.

Lemma 3.9. *With notations as above, for all $a_{ij} \in A_{ij}, 1 \le i \le j \le 2$, we have*

i) $\Psi_n(0) = 0$;
ii) $\Psi_n(A_{11}) \subseteq A_{11} + A_{12}, \Psi_n(A_{22}) \subseteq A_{22} + A_{12}$;
iii) $\Psi_n(A_{12}) = L_n(A_{12}); \Psi_n(p_i) \in A_{12}$.

Lemma 3.10. *With notations as above, for all $a_{ij} \in A_{ij}, 1 \le i \le j \le 2$, we have*

i) $\Psi_n(a_{11} a_{12}) = \Psi_n(a_{11}) a_{12} + a_{11} \Psi_n(b_{12})$;
ii) $\Psi_n(a_{12} a_{22}) = \Psi_n(a_{12}) a_{22} + a_{12} \Psi_n(a_{22})$.

Proof. We now only prove the conclusion (i) and the proof of conclusion (ii) can be obtained by similar methods.

(i) It follows from $a_{12} a_{11} p_1 p_2 \cdots p_2 = 0$ that

$$\begin{aligned} \Psi_n(a_{11} a_{12}) &= \Phi_n(P_n(a_{12}, a_{11}, p_1, p_2, \cdots, p_2)) \\ &= P_n(\Phi_n(a_{12}), a_{11}, p_1, p_2, \cdots, p_2) + P_n(a_{12}, \Phi_n(a_{11}), p_1, p_2, \cdots, p_2) \\ &= P_n(\Psi_n(a_{12}), a_{11}, p_1, p_2, \cdots, p_2) + P_n(a_{12}, \Psi_n(a_{11}), p_1, p_2, \cdots, p_2) \\ &= a_{11} \Psi_n(a_{12}) + a_{11} \Psi_n(a_{12}). \end{aligned}$$

By a similar method, the conclusion (ii) holds. □

Lemma 3.11. *With notations as above, for all $a_{ii}, b_{ii} \in A_{ii}, i \in \{1, 2\}$, we have $\Psi_n(a_{ii} b_{ii}) = \Psi_n(a_{ii}) b_{ii} + a_{ii} \Psi_n(b_{ii})$.*

Proof. By Lemma 3.9, for all $c_{12} \in A_{12}$, we have

$$\begin{aligned}\Psi_n(a_{11}b_{11}c_{12}) &= \Psi_n(a_{11})b_{11}c_{12} + a_{11}\Psi_n(b_{11}c_{12})\\ &= \Psi_n(a_{11})b_{11}c_{12} + a_{11}\Psi_n(b_{11})c_{12} + a_{11}b_{11}\Psi_n(c_{12})\end{aligned}$$

and

$$\Psi_n(a_{11}b_{11}c_{12}) = \Psi_n(a_{11}b_{11})c_{12} + a_{11}b_{11}\Psi_n(c_{12}).$$

Combining the two equations above, we have

$$(\Psi_n(a_{11}b_{11}) - \Psi_n(a_{11})b_{11} - a_{11}\Psi_n(b_{11}))c_{12} = 0$$

for all $a_{11}, b_{11} \in A_{11}, c_{12} \in A_{12}$. Since A_{12} is a faithful (A_{11}, A_{22})-bimodule, we have

$$\Psi_n(a_{11}b_{11}) = \Psi_n(a_{11})b_{11} + a_{11}\Psi_n(b_{11})$$

for all $a_{11}, b_{11} \in A_{11}$.

After a similar calculation process, we can get

$$\Psi_n(a_{22}b_{22}) = \Psi_n(a_{22})b_{22} + a_{22}\Psi_n(b_{22})$$

for all $a_{22}, b_{22} \in A_{22}$. □

Lemma 3.12. *With notations as above, for all $a_{ii}, b_{ii} \in A_{ii}$, we have*

i) $\Psi_n(a_{12} + b_{12}) = \Psi_n(a_{12}) + \Psi_n(a_{12})$;
ii) $\Psi_n(a_{11} + a_{12}) - \Psi_n(a_{11}) - \Psi_n(a_{12}) \in \mathcal{Z}(\mathcal{T})$ and $\Psi_n(a_{22} + a_{12}) - \Psi_n(a_{22}) - \Psi_n(a_{12}) \in \mathcal{Z}(\mathcal{T})$;
iii) $\Psi_n(a_{ii} + b_{ii}) = \Psi_n(a_{ii}) + \Psi_n(b_{ii})$.

Proof. According to the above symbols, we can prove the lemma by using the relationship between mapping Ψ_n and mapping L_n. The details are as follows:

i). It is the direct result of Lemmas 3.4 and 3.9; ii). It is the direct result of Lemma 3.3; iii). According to the above Lemma 3.5, we have

$$\Psi_n(a_{ii} + b_{ii}) = \Psi_n(a_{ii}) + \Psi_n(b_{ii}) + Z(a_{ii}, b_{ii})$$

for some $Z(a_{ii}, b_{ii}) \in \mathcal{Z}(\mathcal{T})$. Let us consider the center element $Z(a_{ii}, b_{ii})$.

On the one hand,

$$\begin{aligned}\Psi_n((a_{11} + b_{11})c_{12}) &= \Psi_n(a_{11}c_{12}) + \Psi_n(b_{11}c_{12})\\ &= \Psi_n(a_{11})c_{12} + a_{11}\Psi_n(c_{12}) + \Psi_n(b_{11})c_{12} + b_{11}\Psi_n(c_{12})\\ &= (\Psi_n(a_{11}) + \Psi_n(b_{11}))c_{12} + (a_{11} + b_{11})\Psi_n(c_{12}),\end{aligned}$$

and on the other hand, we have

$$\Psi_n((a_{11} + b_{11})c_{12}) = \Psi_n(a_{11} + b_{11})c_{12} + (a_{11} + b_{11})\Psi_n(c_{12})$$

for all $a_{11}, b_{11} \in A_{11}, c_{12} \in A_{12}$.

By observing the two equations above, we can obtain $Z(a_{11}, b_{11})c_{12} = p_1 Z(a_{11}, b_{11})p_1 c_{12} = 0$ for all $c_{12} \in A_{12}$. Since A_{12} is a faithful (A_{11}, A_{22})-bimodule, then $p_1 Z(a_{11}, b_{11})p_1 = 0$. Note that there is an algebra isomorphism $\tau : \mathcal{Z}(A_{11}) \to \mathcal{Z}(A_{22})$, and we can obtain $p_2 Z(a_{11}, b_{11})p_2 = \tau(p_1 Z(a_{11}, b_{11})p_1) = 0$ and $Z(a_{11}, b_{11})c_{12} = p_1 Z(a_{11}, b_{11})p_1 + p_2 Z(a_{11}, b_{11})p_2 = 0$. Therefore,

$$\Psi_n(a_{11} + b_{11}) = \Psi_n(a_{11}) + \Psi_n(b_{11})$$

for all $a_{11}, b_{11} \in A_{11}$.

After a similar calculation process, we can get

$$\Psi_n(a_{22} + b_{22}) = \Psi_n(a_{22}) + \Psi_n(b_{22})$$

for all $a_{22}, b_{22} \in A_{22}$. □

According to the definition of Ψ_n and Theorem 3.1, we can get the following lemma immediately.

Lemma 3.13. *Let $a_{ij} \in A_{ij}, 1 \leq i \leq j \leq 2$, then $\Psi_n(a_{11} + a_{12} + a_{22}) = \Psi_n(a_{11}) + \Psi_n(a_{12}) + \Psi_n(a_{22}) + C_{a_{11}, a_{12}, a_{22}}$.*

Remark 3.1. *Lemma 3.13 enables us to establish a mapping $g_n : \mathcal{T} \to \mathcal{T}$*

$$g_n(x) = \Psi_n(x) - \Psi_n(p_1 x p_1) - \Psi_n(p_1 x p_2) - \Psi_n(p_2 x p_2)$$

for all $x \in \mathcal{T}$, then define a mapping $d_n : \mathcal{T} \to \mathcal{T}$ by

$$d_n(x) = \Psi_n(x) - g_n(x)$$

for all $x \in \mathcal{T}$. It is easy to verify that each d_n satisfies the following property:

1) $d_n(a_{ij}) = \Psi_n(a_{ij})$;
2) $d_n(a_{11} + a_{12} + a_{22}) = d_n(a_{11}) + d_n(a_{12}) + d_n(a_{22})$

for all $a_{ij} \in A_{ij}, i \leq j \in \{1, 2\}$.

Now, we are in a position to prove our main theorem.

Proof of Theorem 3.1: It follows from the definitions of d_n and g_n that

$$L_n(x) = \Phi_n(x) + h_n(x) = d_n(x) + g_n(x) + h_n(x) = d_n(x) + f_n(x),$$

where $f_n(x) = g_n(x) + h_n(x)$ is a mapping from $\mathcal{T}$ into its center $\mathcal{Z}(\mathcal{T})$ for all $x \in \mathcal{T}$.

For arbitrary $x, y \in \mathcal{T}$, elements x and y have decomposition form $x = a_{11} + a_{12} + a_{22}$ and $y = b_{11} + b_{12} + b_{22}$, where $a_{ij}, b_{ij} \in A_{ij}, 1 \leq i \leq j \leq 2$. It follows from Remark 3.1 that

$$\begin{aligned} d_n(x + y) &= d_n(a_{11} + a_{12} + a_{22} + b_{11} + b_{12} + b_{22}) \\ &= d_n(a_{11} + b_{11}) + d_n(a_{12} + b_{12}) + d_n(a_{22} + b_{22}) \\ &= d_n(a_{11}) + d_n(b_{11}) + d_n(a_{12}) + d_n(b_{12}) + d_n(a_{22}) + d_n(b_{22}) \\ &= d_n(a_{11} + a_{12} + a_{22}) + d_n(b_{11} + b_{12} + b_{22}). \end{aligned}$$

Therefore, we have

$$d_n(x + y) = d_n(x) + d_n(y).$$

Next, we prove that d_n satisfies the Leibniz formula.

First, we should prove that

$$p_1d_n(a_{11})a_{22} + a_{11}d_n(a_{22})p_2 = 0$$

for all $a_{11} \in A_{11}, a_{22} \in A_{22}$.

In fact, because of $a_{11}a_{22}p_2 \cdots p_2 = 0$ and Eqs (3.9)–(3.10), we have

$$\begin{aligned}
p_1d_n(a_{11})a_{22} + a_{11}d_n(a_{22})p_2 &= p_1(\Psi_n(a_{11}) + g_n(a_{11}))a_{22} + a_{11}(\Psi_n(a_{22}) + g_n(a_{22}))p_2 \\
&= p_1\Psi_n(a_{11})a_{22} + a_{11}\Psi_n(a_{22})p_2 \\
&= p_1L_n(a_{11})a_{22} + a_{11}L_n(a_{22})p_2 \\
&= p_1a_{11}L_n(p_2)a_{22} + a_{11}L_n(p_1)a_{22} \\
&= p_1a_{11}(p_1L_n(p_2)p_2 + p_1L_n(p_1)p_2)a_{22} \\
&= 0
\end{aligned}$$

for all $a_{11} \in A_{11}, a_{22} \in A_{22}$.

In view of Lemmas 3.8–3.11, we have

$$\begin{aligned}
d_n(xy) &= d_n(a_{11}b_{11} + a_{11}b_{12} + a_{12}b_{22} + a_{22}b_{22}) \\
&= \Psi_n(a_{11}b_{11}) + \Psi_n(a_{11}b_{12}) + \Psi_n(a_{12}b_{22}) + \Psi_n(a_{22}b_{22}) \\
&= \Psi_n(a_{11})b_{11} + a_{11}\Psi_n(b_{11}) + \Psi_n(a_{11})b_{12} + a_{11}\Psi_n(b_{12}) \\
&+ \Psi_n(a_{12})b_{22} + a_{12}\Psi_n(b_{22}) + \Psi_n(a_{22})b_{22} + a_{22}\Psi_n(b_{22}) \\
&+ \Psi_n(a_{11}b_{22}) + a_{11}\Psi_n(b_{22}) \\
&= d_n(a_{11})b_{11} + a_{11}d_n(b_{11}) + d_n(a_{11})b_{12} + a_{11}d_n(b_{12}) \\
&+ d_n(a_{12})b_{22} + a_{12}d_n(b_{22}) + d_n(a_{22})b_{22} + a_{22}d_n(b_{22}) \\
&+ d_n(a_{11})b_{22} + a_{11}d_n(b_{22}) \\
&= (d_n(a_{11}) + d_n(a_{12}) + d_n(a_{22}))(b_{11} + b_{12} + b_{22}) \\
&+ (a_{11} + a_{12} + a_{22})(d_n(b_{11}) + d_n(b_{12}) + d_n(b_{22})) \\
&= d_n(a_{11} + a_{12} + a_{22})(b_{11} + b_{12} + b_{22}) \\
&+ (a_{11} + a_{12} + a_{22})d_n(b_{11} + b_{12} + b_{22}) \\
&= d_n(x)y + xd_n(y)
\end{aligned}$$

for all $x, y \in \mathcal{T}$.

Now, we consider the properties of the mapping $f_n : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$. For arbitrary $x_1, \cdots, x_n \in \mathcal{T}$ with $x_1 \cdots x_n = 0$, note that d_n is an additive derivation of $\mathcal{T}$, and we have

$$\begin{aligned}
f_n(P_n(x_1, \cdots, x_n)) &= (L_n - d_n)(P_n(x_1, \cdots, x_n)) \\
&= L_n(P_n(x_1, \cdots, x_n)) - d_n(P_n(x_1, \cdots, x_n)) \\
&= \sum_{k=1}^{n} P_n(x_1, \cdots, (L_n - d_n)(x_k), \cdots, x_n) \\
&= \sum_{k=1}^{n} P_n(x_1, \cdots, f_n(x_k), \cdots, x_n) \\
&= 0.
\end{aligned}$$

Therefore, Theorem 3.1 holds.

4. Nonlinear Lie-type higher n-derivations: local actions

This section focuses on researching higher Lie n-derivation at zero product on triangular algebras. We will provide the more advanced version (Theorems 4.1 and 4.2) that corresponds to Theorems 3.1 and 3.2. These conclusions generalize a few previous conclusions under the same presumptions, such as [16, Theorems 2.1 and 2.2] and [17, Theorem 1 and 3].

Theorem 4.1. *Let* $\mathcal{T} = \left[\begin{smallmatrix} A_{11} & A_{12} \\ O & A_{22} \end{smallmatrix}\right]$ *be a triangular algebra over* $(n-1)$*-torsion-free commutative ring* $\mathcal{R}$*. Suppose that a sequence* $\Delta = \{\delta_m\}_{m\in\mathcal{N}}$ *of mappings* $\delta_m : \mathcal{T} \to \mathcal{T}$ *is a nonlinear mapping*

$$\delta_m(P_n(x_1, x_2, \cdots, x_n)) = \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(x_1), \delta_{i_2}(x_2), \cdots, \delta_{i_n}(x_n)) \tag{4.1}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{T}$ with $x_1x_2\cdots x_n = 0$ and $n \geq 3$. If $\pi_{A_{11}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(A_{11})$ and $\pi_{A_{22}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(A_{22})$, then every nonlinear mapping δ_m is almost additive on $\mathcal{T}$, which is

$$\delta_m(x+y) - \delta_m(x) - \delta_m(y) \in \mathcal{Z}(\mathcal{T})$$

for all $x, y \in \mathcal{T}$.

It is worth noting that the mapping δ_1 in Theorem 4.1 is equal to the mapping L_n in Theorem 3.1. Based on this, we begin to prove theorem 4.1.

Assume that a sequence $\Delta = \{\delta_m\}_{m\in\mathcal{N}}$ of nonlinear mappings $\delta_m : \mathcal{T} \to \mathcal{T}$ is a Lie-n higher derivation by local actions on triangular algebras $\mathcal{T} = \left[\begin{smallmatrix} A_{11} & A_{12} \\ O & A_{22} \end{smallmatrix}\right]$. We will use an induction method for the component index m. For $m = 1$, $\delta_1 = L_n : \mathcal{T} \to \mathcal{T}$ is a Lie n-derivation by local actions. According to Theorems 3.2 and 3.1, we obtain that a nonlinear Lie n-derivation δ_1 by local actions meet the following attributes:

$$\mathfrak{C}_1 = \begin{cases} \delta_1(0) = 0, \delta_1(a_{11}+a_{12}) - \delta_1(a_{11}) - \delta_1(a_{12}) \in \mathcal{Z}(\mathcal{T}), \delta_1(a_{22}+a_{12}) - \delta_1(a_{22}) - \delta_1(a_{12}) \in \mathcal{Z}(\mathcal{T}); \\ \delta_1(a_{12}+a'_{12}) = \delta_1(a_{12}) + \delta_1(a'_{12}); \delta_1(a_{ii}+a_{ii}) - \delta_1(a_{ii}) - \delta_1(a_{ii}) \in \mathcal{Z}(\mathcal{T}); \\ \delta_1(a_{11}+a_{12}+a_{22}) - \delta_1(a_{11}) - \delta_1(a_{12}) - \delta_1(a_{22}) \in \mathcal{Z}(\mathcal{T}) \end{cases}$$

for all $a_{ij} \in A$ with $i \leq j \in \{1, 2\}$.

We assume that the result holds for all $1 < s < m$, $m \in \mathcal{N}$, then nonlinear Lie n-derivation $\{\delta_l\}_{l=0}^{l=s}$ satisfies the following

$$\mathfrak{C}_s = \begin{cases} \delta_s(0) = 0, \delta_s(a_{11}+a_{12}) - \delta_s(a_{11}) - \delta_s(a_{12}) \in \mathcal{Z}(\mathcal{T}), \delta_s(a_{22}+a_{12}) - \delta_s(a_{22}) - \delta_s(a_{12}) \in \mathcal{Z}(\mathcal{T}); \\ \delta_s(a_{12}+a'_{12}) = \delta_s(a_{12}) + \delta_s(a'_{12}); \delta_s(a_{ii}+a_{ii}) - \delta_s(a_{ii}) - \delta_s(a_{ii}) \in \mathcal{Z}(\mathcal{T}); \\ \delta_s(a_{11}+a_{12}+a_{22}) - \delta_s(a_{11}) - \delta_s(a_{12}) - \delta_s(a_{22}) \in \mathcal{Z}(\mathcal{T}) \end{cases}$$

for all $a_{ij} \in A$ with $i \leq j \in \{1, 2\}$.

Our goal is to prove that the above conditions $\mathfrak{C}_s$ also hold for m. The process of induction can be achieved through a series of lemmas.

Lemma 4.1. *With notations as above, we have* $\delta_m(0) = 0$.

Proof. With the help of condition $\mathfrak{C}_s$, we find that

$$\delta_m(0) = \delta_m(P_n(0, 0, \cdots, 0)) = \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(0), \delta_{i_2}(0), \cdots, \delta_{i_n}(0)) = 0.$$

□

Lemma 4.2. *With notations as above, we have*

i) $\delta_m(a_{11} + a_{12}) - \delta_m(a_{11}) - \delta_m(a_{12}) \in \mathcal{Z}(\mathcal{T})$;
ii) $\delta_m(a_{22} + a_{12}) - \delta_m(a_{22}) - \delta_m(a_{12}) \in \mathcal{Z}(\mathcal{T})$

for all $a_{ij} \in A$ with $i \leq j \in \{1, 2\}$.

Proof. Here, we only prove that the conclusion i) holds and that the proof of conclusion ii) can be similarly obtained.

It is clear that $x_{12}(a_{11} + x'_{12})p_1p_2 \cdots p_2 = x_{12}a_{11}p_1p_2 \cdots p_2 = 0 = x_{12}x'_{12}p_1p_2 \cdots p_2$ for all $a_{11} \in A_{11}$ and $x_{12}, x'_{12} \in A_{12}$. On the one hand, we have

$$\begin{aligned}\delta_m(a_{11}x_{12}) &= \delta_m(P_n(x_{12}, (a_{11} + x'_{12}), p_1, p_2, \cdots, p_2)) \\ &= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(x_{12}), \delta_{i_2}(a_{11} + x'_{12}), \delta_{i_3}(p_1), \delta_{i_4}(p_2) \cdots, \delta_{i_n}(p_2)),\end{aligned}$$

and on the other hand, we have

$$\begin{aligned}\delta_n(a_{11}x_{12}) &= \delta_m(P_n(x_{12}, a_{11}, p_1, p_2, \cdots, p_2)) + \delta_m(P_n(x_{12}, x'_{12}, p_1, p_2, \cdots, p_2)) \\ &= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(x_{12}), \delta_{i_2}(a_{11}), \delta_{i_3}(p_1), \delta_{i_4}(p_2) \cdots, \delta_{i_n}(p_2)) \\ &+ \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(x_{12}), \delta_{i_2}(x'_{12}), \delta_{i_3}(p_1), \delta_{i_4}(p_2) \cdots, \delta_{i_n}(p_2)) \\ &= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(x_{12}), \delta_{i_2}(a_{11}) + \delta_{i_2}(x'_{12}), \delta_{i_3}(p_1), \delta_{i_4}(p_2) \cdots, \delta_{i_n}(p_2)).\end{aligned}$$

By observing the two equations above and inductive hypothesis $\mathfrak{C}_s$ for all $0 \leq s \leq m - 1$, we arrive at

$$\begin{aligned}0 &= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(x_{12}), \delta_{i_2}(a_{11} + x'_{12}) - (\delta_{i_2}(a_{11}) + \delta_{i_2}(x'_{12})), \delta_{i_3}(p_1), \delta_{i_4}(p_2) \cdots, \delta_{i_n}(p_2)) \\ &= \sum_{i_1+\cdots+i_n=m, i_2\neq m} P_n(\delta_{i_1}(x_{12}), \delta_{i_2}(a_{11} + x'_{12}) - (\delta_{i_2}(a_{11}) + \delta_{i_2}(x'_{12})), \delta_{i_3}(p_1), \delta_{i_4}(p_2) \cdots, \delta_{i_n}(p_2)) \\ &\quad + P_n(x_{12}, \delta_m(a_{11} + x'_{12}) - (\delta_m(a_{11}) + \delta_m(x'_{12})), p_1, p_2, \cdots, p_2) \\ &= P_n(x_{12}, \delta_m(a_{11} + x'_{12}) - (\delta_m(a_{11}) + \delta_m(x'_{12})), p_1, p_2, \cdots, p_2) \\ &= p_1(\delta_m(a_{11} + x'_{12}) - (\delta_m(a_{11}) + \delta_m(x'_{12})))x_{12} - x_{12}(\delta_m(a_{11} + x'_{12}) - (\delta_m(a_{11}) + \delta_m(x'_{12})))p_2\end{aligned}$$

for all $a_{11} \in A_{11}$ and $x_{12}, x'_{12} \in A_{12}$. It then follows from the center of algebra $\mathcal{T}$ that

$$p_1(\delta_m(a_{11} + x'_{12}) - (\delta_m(a_{11}) + \delta_m(x'_{12})))p_1 + p_2(\delta_m(a_{11} + x'_{12}) - (\delta_m(a_{11}) + \delta_m(x'_{12})))p_2 \in \mathcal{Z}(\mathcal{T})$$

for all $a_{11} \in A_{11}$ and $x_{12} \in A_{12}$.

In the following, we prove $p_1(\delta_m(a_{11} + x'_{12}) - (\delta_m(a_{11}) + \delta_m(x'_{12})))p_2 = 0$ for all $a_{11} \in A_{11}$ and $x_{12} \in A_{12}$.

With the help of $p_2(a_{11} + x_{12})p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}} = 0 = p_2 a_{11} p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}$, we have

$$\begin{aligned}\delta_m(x_{12}) &= \delta_m(P_n(p_2, x_{12} + a_{11}, p_1, \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}})) \\ &= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(p_2), \delta_{i_2}(x_{12} + a_{11}), \delta_{i_3}(p_1), \delta_{i_4}(p_2) \cdots, \delta_{i_n}(p_2))\end{aligned}$$

for all $a_{11} \in A_{11}$ and $m_{12} \in A_{12}$. On the other hand, we have

$$\begin{aligned}\delta_m(x_{12}) &= \delta_m(P_n(p_2, x_{12}, p_1, \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}})) + \delta_m(P_n(p_2, a_{11}, p_1, \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}})) \\ &= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(p_2), \delta_{i_2}(x_{12}) + \delta_{i_2}(a_{11}), \delta_{i_3}(p_1), \delta_{i_4}(p_2) \cdots, \delta_{i_n}(p_2))\end{aligned}$$

for all $a_{11} \in A_{11}$ and $x_{12} \in A_{12}$. With the help of the above two equations and inductive hypothesis $\mathfrak{C}_s$, we have

$$\begin{aligned}0 &= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(p_2), \delta_{i_2}(x_{12} + a_{11}) - (\delta_{i_2}(x_{12}) + \delta_{i_2}(a_{11})), \delta_{i_3}(p_1), \delta_{i_4}(p_2) \cdots, \delta_{i_n}(p_2)) \\ &= \sum_{i_1+\cdots+i_n=m, i_2\neq m} P_n(\delta_{i_1}(p_2), \delta_{i_2}(x_{12} + a_{11}) - (\delta_{i_2}(x_{12}) + \delta_{i_2}(a_{11})), \delta_{i_3}(p_1), \delta_{i_4}(p_2) \cdots, \delta_{i_n}(p_2)) \\ &\quad + P_n(p_2, \delta_m(x_{12} + a_{11}) - (\delta_m(x_{12}) + \delta_m(a_{11})), p_1, p_2, \cdots, p_2) \\ &= P_n(p_2, \delta_m(x_{12} + a_{11}) - (\delta_m(x_{12}) + \delta_m(a_{11})), p_1, p_2, \cdots, p_2) \\ &= p_1(\delta_m(a_{11} + x_{12}) - (\delta_m(a_{11}) + \delta_m(x_{12})))p_2\end{aligned}$$

for all $a_{11} \in A_{11}$ and $x_{12} \in A_{12}$. Therefore, we obtain that the conclusion (i) holds.

For conclusion (ii), taking into accounts the relations $x_{12}(b_{22} + x'_{12})p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}} = x_{12}x'_{12}p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}} = x_{12}b_{22}p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}} = 0$, by an analogous manner one can show that the conclusion

$$\delta_m(b_{22} + x'_{12}) - (\delta_m(b_{22}) + \delta_m(x'_{12})) \in \mathcal{Z}(\mathcal{T})$$

holds for all $b_{22} \in A_{22}$ and $x_{12},\ x'_{12} \in A_{12}$. □

Lemma 4.3. *With notations as above, we have $\delta_n(x_{12} + x'_{12}) = \delta_n(x_{12}) + \delta_n(x'_{12})$ for all $x_{12}, x'_{12} \in M_{12}$.*

Proof. Thanks to inductive hypothesis $\mathfrak{C}_s$ for all $1 \leq s < m$ and relations $(-x_{12} - p_1)(p_2 +$

$x'_{12})p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}} = 0$, we have

$$\begin{aligned}
\delta_m(x_{12} + x'_{12}) = {} & \delta_m(P_n((-x_{12} - p_1), (p_2 + x'_{12}), p_1, \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}})) \\
= & \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(-x_{12} - p_1), \delta_{i_2}(p_2 + x'_{12}), \delta_{i_3}(p_1), \underbrace{\delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-3 \text{ copies}}) \\
= & \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(-x_{12}) + \delta_{i_1}(-p_1), \delta_{i_2}(p_2) + \delta_{i_2}(x'_{12}), \delta_{i_3}(p_1), \underbrace{\delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-3 \text{ copies}}) \\
= & \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(-x_{12}), \delta_{i_2}(p_2), \delta_{i_3}(p_1), \underbrace{\delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-3 \text{ copies}}) \\
& + \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(-x_{12}), \delta_{i_2}(x'_{12}), \delta_{i_3}(p_1), \underbrace{\delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-3 \text{ copies}}) \\
& + \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(-p_1), \delta_{i_2}(p_2), \delta_{i_3}(p_1), \underbrace{\delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-3 \text{ copies}}) \\
& + \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(-p_1), \delta_{i_2}(x'_{12}), \delta_{i_3}(p_1), \underbrace{\delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-3 \text{ copies}}) \\
= & \delta_m(P_n(-x_{12}, p_2, p_1, \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}})) + \delta_m(P_n(-x_{12}, x'_{12}, p_1, \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}})) \\
& + \delta_m(P_n(-p_1, p_2, p_1, \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}})) + \delta_m(P_n(-p_1, x'_{12}, p_1, \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}})) \\
& = \delta_m(x_{12}) + \delta_m(x'_{12}).
\end{aligned}$$

That is, $\delta_m(x_{12} + x'_{12}) = \delta_m(x_{12}) + \delta_m(x'_{12})$ for all $x_{12}, x'_{12} \in A_{12}$. □

Lemma 4.4. *With notations as above, we have*

1) $\delta_m(a_{11} + a'_{11}) - \delta_m(a_{11}) - \delta_m(a'_{11}) \in \mathcal{Z}(\mathcal{T})$;
2) $\delta_m(a_{22} + a'_{22}) - \delta_m(a_{22}) - \delta_m(a'_{22}) \in \mathcal{Z}(\mathcal{T})$

for all $a_{ii}, a'_{ii} \in A_{ii}$ with $i \in \{1, 2\}$.

Proof. We only prove the statement 1). The statement 2) can be proved in a similar way. Because of relations $x_{12}p_1(a_{11} + a'_{11}) \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}} = x_{12}p_1a_{11} \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}} = 0 = x_{12}p_1a'_{11} \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}}$, we arrive at

$$\begin{aligned}
\delta_m((a_{11} + a'_{11})x_{12}) = {} & \delta_m(P_n(x_{12}, p_1, (a_{11} + a'_{11}), \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}})) \\
= & \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(x_{12}), \delta_{i_2}(p_1), \delta_{i_3}(a_{11} + a'_{11}), \underbrace{\delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-3 \text{ copies}}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\delta_m((a_{11}+a'_{11})x_{12}) &= \delta_m(P_n(x_{12},p_1,(a_{11}),\underbrace{p_2,\cdots,p_2}_{n-3\text{ copies}})) + \delta_m(P_n(x_{12},p_1,(a'_{11}),\underbrace{p_2,\cdots,p_2}_{n-3\text{ copies}}))\\ &= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(x_{12}),\delta_{i_2}(p_1),\delta_{i_3}(a_{11}),\underbrace{\delta_{i_4}(p_2),\cdots,\delta_{i_n}(p_2)}_{n-3\text{ copies}})\\ &+ \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(x_{12}),\delta_{i_2}(p_1),\delta_{i_3}(a'_{11}),\underbrace{\delta_{i_4}(p_2),\cdots,\delta_{i_n}(p_2)}_{n-3\text{ copies}})\\ &= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(x_{12}),\delta_{i_2}(p_1),\delta_{i_3}(a_{11})+\delta_{i_3}(a'_{11}),\underbrace{\delta_{i_4}(p_2),\cdots,\delta_{i_n}(p_2)}_{n-3\text{ copies}})\end{aligned}$$

for all $a_{11} \in A_{11}, x_{12} \in A_{12}$. On comparing the above two relations together with inductive hypothesis $\mathfrak{C}_s$ for all $1 \leq s < m$, we see that

$$\begin{aligned}0 &= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(x_{12}),\delta_{i_2}(p_1),\delta_{i_3}(a_{11}+a'_{11})-(\delta_{i_3}(a_{11})+\delta_{i_3}(a'_{11})),\underbrace{\delta_{i_4}(p_2),\cdots,\delta_{i_n}(p_2)}_{n-3\text{ copies}})\\ &= \sum_{i_1+\cdots+i_n=m, i_3\neq m} P_n(\delta_{i_1}(x_{12}),\delta_{i_2}(p_1),\delta_{i_3}(a_{11}+a'_{11})-(\delta_{i_3}(a_{11})+\delta_{i_3}(a'_{11})),\underbrace{\delta_{i_4}(p_2),\cdots,\delta_{i_n}(p_2)}_{n-3\text{ copies}})\\ &+ P_n(x_{12},p_1,\delta_m(a_{11}+a'_{11})-(\delta_m(a_{11})+\delta_m(a'_{11})),\underbrace{p_2,\cdots,p_2}_{n-3\text{ copies}})\\ &= P_n(x_{12},p_1,\delta_m(a_{11}+a'_{11})-(\delta_m(a_{11})+\delta_m(a'_{11})),\underbrace{p_2,\cdots,p_2}_{n-3\text{ copies}}),\end{aligned}$$

which is

$$p_1(\delta_m(a_{11}+a'_{11})-(\delta_m(a_{11})+\delta_m(a'_{11})))x_{12} = x_{12}(\delta_m(a_{11}+a'_{11})-(\delta_m(a_{11})+\delta_m(a'_{11})))p_2. \tag{4.4}$$

It follows from the center of triangular algebra $\mathcal{T}$ and the above equation that

$$p_1(\delta_m(a_{11}+a'_{11})-(\delta_m(a_{11})+\delta_m(a'_{11})))p_1 \oplus p_2(\delta_m(a_{11}+a'_{11})-(\delta_m(a_{11})+\delta_n(a'_{11})))p_2 \in \mathcal{Z}(\mathcal{T}). \tag{4.5}$$

In the following, we prove

$$p_1(\delta_m(a_{11}+a'_{11})-(\delta_m(a_{11})+\delta_m(a'_{11})))p_2 = 0$$

for all $a_{11}, a'_{11} \in A_{11}$.

Benefitting from $(a_{11}+a'_{11})\underbrace{p_2\cdots p_2}_{n-1\text{ copies}} = a_{11}\underbrace{p_2\cdots p_2}_{n-1\text{ copies}} = a'_{11}\underbrace{p_2\cdots p_2}_{n-1\text{ copies}} = 0$, we have

$$\begin{aligned}0 &= \delta_m(P_n((a_{11}+a'_{11}),p_2,\cdots,p_2))\\ &= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(a_{11}+a'_{11}),\delta_{i_2}(p_2),\underbrace{\delta_{i_3}(p_2),\cdots,\delta_{i_n}(p_2)}_{n-2\text{ copies}})\end{aligned}$$

and

$$\begin{aligned}
0 &= \delta_m(P_n(a_{11}, p_2, \cdots, p_2)) + \delta_m(P_n(a'_{11}, p_2, \cdots, p_2)) \\
&= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(a_{11}), \delta_{i_2}(p_2), \underbrace{\delta_{i_3}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-2 \text{ copies}}) \\
&+ \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(a'_{11}), \delta_{i_2}(p_2), \underbrace{\delta_{i_3}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-2 \text{ copies}}) \\
&= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(a_{11}) + \delta_{i_1}(a'_{11}), \delta_{i_2}(p_2), \underbrace{\delta_{i_3}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-2 \text{ copies}}).
\end{aligned}$$

By combining the above two equations with inductive hypothesis $\mathfrak{C}_s$ for all $1 \leq s < m$, we can get

$$\begin{aligned}
0 &= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(a_{11} + a'_{11}) - (\delta_{i_1}(a'_{11}) + \delta_{i_1}(a'_{11})), \delta_{i_2}(p_2), \underbrace{\delta_{i_3}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-2 \text{ copies}}) \\
&= \sum_{i_1+\cdots+i_n=m, i_1 \neq m} P_n(\delta_{i_1}(\delta_{i_1}(a_{11} + a'_{11}) - (\delta_{i_1}(a'_{11}) + \delta_{i_1}(a'_{11})), \delta_{i_2}(p_2), \underbrace{\delta_{i_3}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-2 \text{ copies}}) \\
&+ P_n(\delta_m(a_{11} + a'_{11}) - (\delta_m(a'_{11}) + \delta_m(a'_{11})), p_2, \underbrace{p_2, \cdots, p_2}_{n-2 \text{ copies}}) \\
&= P_n(\delta_m(a_{11} + a'_{11}) - (\delta_m(a'_{11}) + \delta_m(a'_{11})), p_2, \underbrace{p_2, \cdots, p_2}_{n-2 \text{ copies}}),
\end{aligned}$$

which is

$$p_1(\delta_m(a_{11} + a'_{11}) - (\delta_m(a_{11}) + \delta_m(a'_{11})))p_2 = 0. \tag{4.6}$$

Combining (4.5) and (4.6), this claim holds. □

Lemma 4.5. *With notations as above, we have* $\delta_m(a_{11} + x_{12} + b_{22}) - \delta_m(a_{11}) - \delta_m(x_{12}) - \delta_m(b_{22}) \in \mathcal{Z}(\mathcal{T})$ *for all* $a_{11} \in A_{11}, x_{12} \in A_{12}, a_{22} \in A_{22}$.

Proof. For arbitrary $a_{11} \in A_{11}, x_{12} \in A_{12}, a_{22} \in A_{22}$, in view of $(a_{11} + x_{12} + b_{22})x'_{12}p_1p_2 \cdots p_2 = 0$, we have

$$\begin{aligned}
\delta_m(x_{12'}b_{22} - a_{11}x'_{12}) &= \delta_m(P_n(a_{11} + x_{12} + b_{22}, x'_{12}, p_1, p_2, \cdots, p_2)) \\
&= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(a_{11} + x_{12} + b_{22}), \delta_{i_2}(x'_{12}), \delta_{i_3}(p_1), \underbrace{\delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-3 \text{ copies}})
\end{aligned}$$

and

$$\begin{aligned}
&\delta_m(x'_{12}b_{22} - a_{11}x'_{12}) = \delta_m(x'_{12}b_{22}) + \delta_m(-a_{11}x'_{12}) \\
&= \delta_m(P_n(a_{11}, x'_{12}, p_1, p_2, \cdots, p_2)) + \delta_m(P_n(x_{12}, x'_{12}, p_1, p_2, \cdots, p_2)) \\
&+ \delta_m(P_n(b_{22}, x'_{12}, p_1, p_2, \cdots, p_2)) \\
&= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(a_{11}) + \delta_{i_1}(x_{12}) + \delta_{i_1}(b_{22}), \delta_{i_2}(x'_{12}), \delta_{i_3}(p_1), \underbrace{\delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-3 \text{ copies}}).
\end{aligned}$$

Let us set $W_i = \delta_i(a_{11} + x_{12} + b_{22}) - (\delta_i(a_{11}) + \delta_i(x_{12}) + \delta_i(b_{22}))$. Taking into accounts the above two equations and using inductive hypothesis $\mathfrak{C}_s$ for all $1 \leq s \leq n$, we have

$$\begin{aligned}
0 &= \sum_{i_1+\cdots+i_n=m} P_n(W_{i_1}, \delta_{i_2}(x'_{12}), \delta_{i_3}(p_1), \underbrace{\delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-3 \text{ copies}}) \\
&= \sum_{i_1+\cdots+i_n=m, i_1\neq m} P_n(W_{i_1}, \delta_{i_2}(x'_{12}), \delta_{i_3}(p_1), \underbrace{\delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-3 \text{ copies}}) \\
&+ P_n(W_m, x'_{12}, p_1, \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}}) \\
&= P_n(W_m, x'_{12}, p_1, \underbrace{p_2, \cdots, p_2}_{n-3 \text{ copies}}),
\end{aligned}$$

which is $p_1 W_m x'_{12} - x'_{12} W_m p_2 = 0$, i.e.,

$$p_1 W_m p_1 \oplus p_2 W_m p_2 \in \mathcal{Z}(\mathcal{T}). \tag{4.7}$$

In the following part, we prove $p_1 W_m p_2 = 0$. It is clear that $(a_{11} + x_{12} + b_{22})(-p_1)\underbrace{p_2 \cdots p_2}_{n-2 \text{ copies}} = 0$, then

$$\begin{aligned}
\delta_m(x_{12}) &= \delta_m(P_n(a_{11} + x_{12} + b_{22}, -p_1, \underbrace{p_2, \cdots, p_2}_{n-2 \text{ copies}}) \\
&= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(a_{11} + x_{12} + b_{22}), \delta_{i_2}(-p_1), \underbrace{\delta_{i_3}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-2 \text{ copies}})
\end{aligned}$$

and

$$\begin{aligned}
\delta_n(x_{12}) &= \delta_m(P_n(a_{11}, -p_1, \underbrace{p_2, \cdots, p_2}_{n-2 \text{ copies}}) + \delta_m(P_n(x_{12}, -p_1, \underbrace{p_2, \cdots, p_2}_{n-2 \text{ copies}}) \\
&+ \delta_m(P_n(b_{22}, -p_1, \underbrace{p_2, \cdots, p_2}_{n-2 \text{ copies}}) \\
&= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(a_{11}) + \delta_{i_1}(x_{12}) + \delta_{i_1}(b_{22}), \delta_{i_2}(-p_1), \underbrace{\delta_{i_3}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-2 \text{ copies}}).
\end{aligned}$$

According to the above two equations and inductive hypothesis $\mathfrak{C}_s$ for all $1 \leq s \leq m$, we can get

$$\begin{aligned}
0 &= \sum_{i_1+\cdots+i_n=m} P_n(W_{i_1}, \delta_{i_2}(-p_1), \underbrace{\delta_{i_3}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-2 \text{ copies}}) \\
&= \sum_{i_1+\cdots+i_n=m, i_1\neq m} P_n(W_{i_1}, \delta_{i_2}(-p_1), \underbrace{\delta_{i_3}(p_2), \cdots, \delta_{i_n}(p_2)}_{n-2 \text{ copies}}) + P_n(W_m, -p_1, \underbrace{p_2, \cdots, p_2}_{n-2 \text{ copies}}) \\
&= P_n(W_m, -p_1, \underbrace{p_2, \cdots, p_2}_{n-2 \text{ copies}}),
\end{aligned}$$

which is

$$p_1 W_m p_2 = 0. \tag{4.8}$$

It follows from relations (4.7)–(4.8) that this lemma holds. □

Next, we give the proof of this theorem. For arbitrary $x = a_{11} + x_{12} + b_{22}$ and $y = a'_{11} + x'_{12} + b'_{22}$, we have

$$\begin{aligned}\delta_m(x+y) &= \delta_m(a_{11} + a'_{11} + x_{12} + x'_{12} + b_{22} + b'_{22})\\ &= \delta_m(a_{11} + a'_{11}) + \delta_m(x_{12} + x'_{12}) + \delta_m(b_{22} + b'_{22}) + Z_1\\ &= \delta_m(a_{11}) + \delta_m(a'_{11}) + \delta_m(x_{12}) + \delta_m(x'_{12}) + \delta_m(b_{22}) + \delta_m(b'_{22}) + Z_1 + Z_2 + Z_3\\ &= \delta_m(x) + \delta_m(y) + Z_1 + Z_2 + Z_3 + Z_4 + Z_5,\end{aligned}$$

which implies that $\delta_m(x+y) - \delta_m(x) - \delta_m(y) \in \mathcal{Z}(\mathcal{T})$.

Based on the additive of δ_m on $\mathcal{T}$, we give the main result in this section reading as follows.

Theorem 4.2. *Let $\mathcal{T} = \left[\begin{smallmatrix} A_{11} & A_{12} \\ O & A_{22}\end{smallmatrix}\right]$ be a triangular algebra satisfying*

i) $\pi_{A_{11}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A}_{11})$ and $\pi_{A_{22}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A}_{22})$;
ii) For any $a_{11} \in A_{11}$, if $[a_{11}, A_{11}] \in \mathcal{Z}(\mathcal{A}_{11})$, then $a_{11} \in \mathcal{Z}(\mathcal{A}_{11})$, or for any $a_{22} \in A_{22}$, if $[a_{22}, A_{22}] \in \mathcal{Z}(\mathcal{A}_{22})$, then $a_{22} \in \mathcal{Z}(\mathcal{A}_{22})$.

Suppose that a sequence $\Delta = \{\delta_m\}_{m\in\mathcal{N}}$ of mappings $\delta_m : \mathcal{T} \to \mathcal{T}$ is a nonlinear map satisfying

$$\delta_m(P_n(x_1, x_2, \cdots, x_n)) = \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(x_1), \delta_{i_2}(x_2), \cdots, \delta_{i_n}(x_n))$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{T}$ with $x_1x_2\cdots x_n = 0$ and $n \geq 3$. For every $m \in \mathcal{N}$,

$$\delta_m(x) = \chi_m(x) + f_m(x)$$

for all $x \in \mathcal{T}$, where a sequence $\Upsilon = \{\chi_m\}_{m\in\mathcal{N}}$ of additive mapping $\chi_m : \mathcal{T} \to \mathcal{T}$ is a higher derivation, and $f_m : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ is a nonlinear mapping such that $f_m(P_n(x_1, x_2, \cdots, x_n)) = 0$ for any $x_1, x_2, \cdots, x_n \in \mathcal{T}$ with $x_1x_2\cdots x_n = 0$.

In the process of proof, we will use mathematical induction for index m. For $m = 1$, $\delta_1 = L_n$ is a Lie-n derivation on $\mathcal{T}$ by local action at zero. By Theorem 3.2, it follows from Theorem 3.2 that there exists an additive derivation d_1 and a nonlinear center mapping f_1, satisfying $f_1(P_n(x_1, x_2, \cdots, x_n)) = 0$ for any $x_1, x_2, \cdots, x_n \in \mathcal{T}$ with $x_1x_2\cdots x_n = 0$ and $n \geq 3$, such that $\delta_1(x) = d_1(x) + f_1(x)$ for all $x \in \mathcal{T}$. Furthermore, δ_1 and d_1 satisfy the following properties

$$\mathfrak{F}_1 = \begin{cases} \delta_1(0) = 0, \delta_1(A_{11}) \subseteq A_{11} + A_{12} + \mathcal{Z}(\mathcal{T}), \delta_1(A_{22}) \subseteq A_{22} + A_{12} + \mathcal{Z}(\mathcal{T});\\ \delta_1(A_{12}) \subseteq A_{12}, \delta_1(p_i) \subseteq A_{12} + \mathcal{Z}(\mathcal{T});\\ d_1(A_{ii}) \subseteq A_{ii} + A_{ij}, d_1(A_{12}) \subseteq A_{12}; f_1(P_n(x_1, x_2, \cdots, x_n)) = 0\end{cases}$$

for $i \leq j \in \{1, 2\}$ and for any $x_1, \cdots, x_n \in \mathcal{T}$ with $x_1\cdots x_n = 0$.

We assume that the result holds for s for all $1 < s < m$, $m \in \mathcal{N}$, then there exists an additive derivation d_s and a nonlinear center mapping f_s, satisfying $f_s(P_n(x_1, x_2, \cdots, x_n)) = 0$ for any $x_1, x_2, \cdots, x_n \in \mathcal{T}$ with $x_1x_2\cdots x_n = 0$, such that $\delta_s(x) = d_s(x) + f_s(x)$ for all $x \in \mathcal{T}$. Moreover, δ_s and d_s satisfy the following properties

$$\mathfrak{F}_s = \begin{cases} \delta_s(0) = 0, \delta_1(A_{11}) \subseteq A_{11} + A_{12} + \mathcal{Z}(\mathcal{T}), \delta_1(A_{22}) \subseteq A_{22} + A_{12} + \mathcal{Z}(\mathcal{T});\\ \delta_s(A_{12}) \subseteq A_{12}, \delta_s(p_i) \subseteq A_{12} + \mathcal{Z}(\mathcal{T});\\ d_s(A_{ii}) \subseteq A_{ii} + A_{ij}, d_s(A_{12}) \subseteq A_{12}; f_s(P_n(x_1, x_2, \cdots, x_n)) = 0\end{cases}$$

for $i \leq j \in \{1, 2\}$ and for any $x_1, x_2, \cdots, x_n \in \mathcal{T}$ with $x_1x_2\cdots x_n = 0$.

The induction process can be realized through a series of lemmas.

Lemma 4.6. *With notations as above, we have*

1) $\delta_m(A_{12}) \subseteq A_{12}$;
2) $p_1\delta_m(p_1)p_1 \oplus p_2\delta_m(p_1)p_2 \in \mathcal{Z}(\mathcal{T})$ and $p_1\delta_m(p_2)p_1 \oplus p_2\delta_m(p_2)p_2 \in \mathcal{Z}(\mathcal{T})$;
3) $\delta_m(p_1) \in M_{12} + \mathcal{Z}(\mathcal{T})$.

Proof. Because of $x_{12}p_1p_1\underbrace{p_2\cdots p_2}_{n-3 \text{ copies}} = 0$ for $x_{12} \in A_{12}$, with the help of condition $\mathfrak{F}_s$ for all $1 < s < m$, we have

$$\begin{aligned}
\delta_m(x_{12}) &= \delta_m(P_n(x_{12}, p_1, p_1, p_2, \cdots, p_2)) \\
&= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(x_{12}), \delta_{i_2}(p_1), \delta_{i_3}(p_1), \delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)) \\
&= \sum_{i_1+\cdots+i_n=m, 0<i_1,\cdots,i_n<m} P_n(\delta_{i_1}(x_{12}), \delta_{i_2}(p_1), \delta_{i_3}(p_1), \delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)) \\
&+ P_n(\delta_m(x_{12}), p_1, p_1, p_2, \cdots, p_2) + P_n(x_{12}, \delta_m(p_1), p_1, p_2, \cdots, p_2) \\
&+ P_n(x_{12}, p_1, \delta_m(p_1), p_2, \cdots, p_2) + \sum_{s=4}^{n} P_n(x_{12}, p_1, p_1, p_2, \cdots, p_2, \underbrace{\delta_m(p_2)}_{s-th\ component}, \cdots, p_2) \\
&= P_n(\delta_m(x_{12}), p_1, p_1, p_2, \cdots, p_2) + P_n(x_{12}, \delta_m(p_1), p_1, p_2, \cdots, p_2) \\
&+ P_n(x_{12}, p_1, \delta_m(p_1), p_2, \cdots, p_2) \\
&= p_1\delta_m(x_{12})p_2 + p_1\delta_m(p_1)x_{12} + \delta_m(p_1)x_{12} - 2x_{12}\delta_m(p_1),
\end{aligned}$$

then we can obtain that $\delta_m(x_{12}) \in A_{12}$. Multiplying by p_1 on the left side and p_2 on the right side of the above equation, we can obtain that $p_1\delta_m(p_1)x_{12} = x_{12}\delta_m(p_1)p_2$ for all $x_{12} \in A_{12}$. It follows from definition of center that

$$p_1\delta_m(p_1)p_1 \oplus p_2\delta_m(p_1)p_2 \in \mathcal{Z}(\mathcal{T}).$$

Because of $p_2x_{12}p_1p_2\cdots p_2 = 0$, we adopt the same discussion as relations

$$\begin{aligned}
\delta_m(x_{12}) &= \delta_m(P_n(p_2, x_{12}, p_1, p_2 \cdots, p_2)) \\
&= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(p_2), \delta_{i_2}(x_{12}), \delta_{i_3}(p_1), \delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)) \\
&= \sum_{i_1+\cdots+i_n=m, 0<i_1,\cdots,i_n<m} P_n(\delta_{i_1}(p_2), \delta_{i_2}(x_{12}), \delta_{i_3}(p_1), \delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)) \\
&+ P_n(\delta_m(p_2), x_{12}, p_1, p_2, \cdots, p_2) + P_n(p_2, \delta_m(x_{12}), p_1, p_2, \cdots, p_2) \\
&+ P_n(p_2, x_{12}, \delta_m(p_1), p_2, \cdots, p_2) + \sum_{s=4}^{n} P_n(p_2, x_{12}, p_1, p_2, \cdots, p_2, \underbrace{\delta_m(p_2)}_{s-th\ component}, \cdots, p_2) \\
&= P_n(\delta_m(p_2), x_{12}, p_1, p_2, \cdots, p_2) + P_n(p_2, \delta_m(x_{12}), p_1, p_2, \cdots, p_2) \\
&+ P_n(p_2, x_{12}, \delta_m(p_1), p_2, \cdots, p_2) \\
&= -p_1\delta_m(p_2)x_{12} + x_{12}\delta_m(p_2)p_2 + p_1\delta_m(x_{12})p_2 - x_{12}\delta_m(p_1)p_2 + \delta_m(p_1)x_{12} \\
&= -p_1\delta_m(p_2)x_{12} + x_{12}\delta_m(p_2)p_2 + p_1\delta_m(x_{12})p_2
\end{aligned}$$

Multiplying by p_1 on the left side and p_2 on the right side of the above equation, we can obtain that $p_1\delta_m(p_2)x_{12} = x_{12}\delta_m(p_2)p_2$ for all $x_{12} \in A_{12}$. It follows from definition of center and we can prove that $p_1\delta_m(p_2)p_1 \oplus p_2\delta_m(p_2)p_2 \in \mathcal{Z}(\mathcal{T})$ holds. □

Lemma 4.7. *With notations as above, we have*

1) $\delta_m(a_{11}) \in A_{11} + A_{12} + \mathcal{Z}(\mathcal{T})$, where $p_2\delta_m(a_{11})p_2 \in \mathcal{Z}(A_{11})$;
2) $\delta_m(a_{22}) \in A_{22} + A_{12} + \mathcal{Z}(\mathcal{T})$, where $p_1\delta_m(a_{22})p_1 \in \mathcal{Z}(A_{22})$

for all $a_{ii} \in A_{ii}$ with $i \in \{1,2\}$.

Proof. In fact, it is clear that $a_{22}a_{11}a_{12}\underbrace{p_2\cdots p_2}_{n-3 \text{ copies}} = 0$ for all $a_{ij} \in A_{ij}$ and for all $i,j \in \{1,2\}$, then according to inductive hypothesis $\mathfrak{F}_s$, we have

$$\begin{aligned}0 =&\delta_m(P_n(a_{22}, a_{11}, a_{12}, p_2, \cdots, p_2))\\ =&\sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(a_{22}), \delta_{i_2}(a_{11}), \delta_{i_3}(a_{12}), \delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2))\\ =&\sum_{i_1+\cdots+i_n=m, i_1,\cdots,i_n<m} P_n(\delta_{i_1}(a_{22}), \delta_{i_2}(a_{11}), \delta_{i_3}(a_{12}), \delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2))\\ &+ P_n(\delta_m(a_{22}), a_{11}, a_{12}, p_2, \cdots, p_2) + P_n(a_{22}, \delta_m(a_{11}), a_{12}, p_2, \cdots, p_2)\\ =&P_n(\delta_m(a_{22}), a_{11}, a_{12}, p_2, \cdots, p_2) + P_n(a_{22}, \delta_m(a_{11}), a_{12}, p_2, \cdots, p_2)\\ =&P_n(p_1\delta_m(a_{22})p_1, a_{11}, a_{12}, p_2, \cdots, p_2) + P_n(a_{22}, p_2\delta_m(a_{11})p_2, a_{12}, p_2, \cdots, p_2)\\ =&P_{n-1}([p_1\delta_m(a_{22})p_1, a_{11}] + [a_{22}, p_2\delta_m(a_{11})p_2], a_{12}, p_2, \cdots, p_2)\end{aligned}$$

for all $a_{ij} \in A_{ij}$ and for all $i \leq j \in \{1,2\}$. In light of Lemma 3.1, we obtain

$$[p_1\delta_m(a_{22})p_1, a_{11}] \oplus [a_{22}, p_2\delta_m(a_{11})p_2] \in \mathcal{Z}(\mathcal{T}) \tag{4.9}$$

for all $a_{ii} \in A_{ii}$ and or all $i \in \{1,2\}$. With the help of characterization of algebraic center, we have

$$[p_1\delta_m(a_{22})p_1, a_{11}] \in \mathcal{Z}(A_{11}) \text{ and } [a_{22}, p_2\delta_m(a_{11})p_2] \in \mathcal{Z}(A_{22}),$$

then

$$p_1\delta_m(a_{22})p_1 \in \mathcal{Z}(A_{11}) \text{ and } p_2\delta_m(a_{11})p_2 \in \mathcal{Z}(A_{22}) \tag{4.10}$$

for all $a_{ii} \in A_{ii}$ and for all $i \in \{1,2\}$. Further based on theorem hypothesis (ii) and above Eqs (4.9) and (4.10), we arrive at

$$\begin{aligned}\delta_m(a_{11}) = &\, p_1\delta_m(a_{11})p_1 - \tau^{-1}(p_2\delta_m(a_{11})p_2) + p_1\delta_m(a_{11})p_2\\ &+ \tau^{-1}(p_2\delta_m(a_{11})p_2) + p_2\delta_m(a_{11})p_2 \in A_{11} + A_{12} + \mathcal{Z}(\mathcal{T})\end{aligned}$$

and

$$\begin{aligned}\delta_m(a_{22}) = &\, p_2\delta_m(a_{22})p_2 - \tau(p_1\delta_m(a_{22})p_1) + p_1\delta_m(a_{22})p_2\\ &+ p_1\delta_m(a_{22})p_1 + \tau(p_1\delta_m(a_{22})p_1) \in A_{22} + A_{12} + \mathcal{Z}(\mathcal{T})\end{aligned}$$

for all $a_{ij} \in A_{ij}$ with $i \leq j \in \{1,2\}$. We can conclude that this claim can be established. □

Now, we define mapping $f_{m1}(a_{11}) = \tau^{-1}(p_2\delta_m(a_{11})p_2) + p_2\delta_m(a_{11})p_2$ and $f_{m2}(a_{22}) = p_1\delta_m(a_{22})p_1 + \tau(p_1\delta_m(a_{22})p_1)$ for all $a_{11} \in A_{11}$ and $a_{22} \in A_{22}$. It follows from Lemma 4.7 that $f_{m1} : A_{11} \to \mathcal{Z}(A_{11})$ such that $f_{n1}(P_n(a_{11}^1, \cdots, a_{11}^n)) = 0$ for all $a_{11}^1, \cdots, a_{11}^n \in A_{11}$ with $a_{11}^1 a_{11}^2 \cdots a_{11}^n = 0$ and $f_{m2} : A_{22} \to \mathcal{Z}(A_{22})$, such that $f_{m2}(P_n(a_{22}^1, \cdots, a_{22}^n)) = 0$ for all $a_{22}^1, \cdots, a_{22}^n \in A_{22}$ with $a_{22}^1 a_{22}^2 \cdots a_{22}^n = 0$. Now, set

$$\begin{aligned} f_m(x) = f_{m1}(a_{11}) + f_{m2}(a_{22}) = \tau^{-1}(p_2\delta_m(a_{11})p_2) + p_2\delta_m(a_{11})p_2 \\ + p_1\delta_m(a_{22})p_1 + \tau(p_1\delta_m(a_{22})p_1) \end{aligned} \tag{4.11}$$

for all $x = a_{11} + a_{12} + a_{22} \in \mathcal{T}$. It is clear that $f_m(x) \in \mathcal{Z}(\mathcal{T})$ and $f_m(P_n(x_1, x_2, \cdots, x_n)) = 0$ with $x_1x_2 \cdots x_n = 0$ for all $x_1, x_2, \cdots, x_n \in \mathcal{T}$. Define a new mapping

$$\varpi_m(x) = \delta_m(x) - f_m(x) \tag{4.12}$$

for all $x \in \mathcal{T}$.

Taking into account Lemmas 4.6 and 4.7 together with (4.11) and (4.12), we can easily get the following Lemma 4.8.

Lemma 4.8. *With notations as above, we have*

1) $\varpi_m(0) = 0$, $\varpi_m(a_{12}) = \delta_m(a_{12}) \in A_{12}$, $\varpi_m(p_i) \in A_{12}$;
2) $\varpi_m(a_{11}) \in A_{11} + A_{12}$ and $\varpi_m(a_{22}) \in A_{22} + A_{12}$,

for all $a_{ij} \in A_{ij}$ with $i \leq j \in \{1, 2\}$.

Lemma 4.9. *With notations as above, we have*

1) $\varpi_m(a_{11}a_{12}) = a_{11}\varpi_m(a_{12}) + \varpi_m(a_{11})a_{12} + \sum_{i+j=m,0<i,j<m} d_i(a_{11})d_j(a_{12})$;
2) $\varpi_m(a_{12}a_{22}) = a_{12}\varpi_m(a_{22}) + \varpi_m(a_{12})a_{22} + \sum_{i+j=n,0<i,j<m} d_i(a_{12})d_j(a_{22})$

for all $a_{ij} \in A_{ij}$ with $i \leq j \in \{1, 2\}$.

Proof. Now, we only prove the conclusion 1), and conclusion 2) can be proved by similar methods. It follows from $a_{12}a_{11}p_1 \underbrace{p_2 \cdots p_2}_{n-3 \text{ copies}} = 0$ and the induction hypothesis $\mathfrak{F}_s$ for all $1 \leq s \leq m-1$ that

$$\begin{aligned} \varpi_m(a_{11}a_{12}) &= \delta_m(a_{11}a_{12}) = \delta_m(P_n(a_{12}, a_{11}, p_1, p_2, \cdots, p_2)) \\ &= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(a_{12}), \delta_{i_2}(a_{11}), \delta_{i_3}(p_1), \delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)) \\ &= \sum_{i_1+\cdots+i_n=m, i_1,\cdots,i_n<m} P_n(\delta_{i_1}(a_{12}), \delta_{i_2}(a_{11}), \delta_{i_3}(p_1), \delta_{i_4}(p_2), \cdots, \delta_{i_n}(p_2)) \\ &+ P_n(\delta_m(a_{12}), a_{11}, p_1, p_2, \cdots, p_2) + P_n(a_{12}, \delta_m(a_{11}), p_1, p_2, \cdots, p_2) \\ &+ P_n(a_{12}, a_{11}, \delta_m(p_1), p_2, \cdots, p_2) \\ &= \sum_{i_1+\cdots+i_n=m, i_1,\cdots,i_n<m} P_n(d_{i_1}(a_{12}), d_{i_2}(a_{11}), d_{i_3}(p_1), d_{i_4}(p_2), \cdots, d_{i_n}(p_2)) \\ &+ a_{11}\varpi_n(a_{12}) + \varpi_n(a_{11})a_{12} \\ &= \sum_{i_1+i_2=m, 0<i_1,i_2<m} P_n(d_{i_1}(a_{12}), d_{i_2}(a_{11}), p_1, p_2, \cdots, p_2) + a_{11}\varpi_n(a_{12}) + \varpi_n(a_{11})a_{12} \\ &= a_{11}\varpi_n(a_{12}) + \varpi_n(a_{11})a_{12} + \sum_{i_1+i_2=m, 0<i_1,i_2<m} d_{i_2}(a_{11})d_{i_1}(a_{12}) \end{aligned}$$

for all $a_{st} \in A_{st}$ with $s \leq t \in \{1,2\}$.

Adopt the same discussion as relations $\varpi_m(a_{12}a_{22}) = \delta_m(a_{12}a_{22}) = \delta_m(P_n(a_{22}, a_{12}, p_1, p_2, \cdots, p_2))$ with $a_{22}a_{12}p_1p_2\cdots p_2 = 0$, and we can prove

$$\varpi_n(a_{12}a_{22}) = a_{12}\varpi_n(a_{22}) + \varpi_n(a_{12})a_{22} + \sum_{i+j=n,0<i,j<n} d_i(a_{12})d_j(a_{22})$$

for all $a_{st} \in A_{st}$ with $s \leq t \in \{1,2\}$. □

Lemma 4.10. *With notations as above, we have*

1) $\varpi_m(a_{11}a'_{11}) = \varpi_m(a_{11})a'_{11} + a_{11}\varpi_m(a'_{11})p_2 + \sum_{i+j=m,0<i,j<m} d_i(a_{11})d_j(a'_{11})$;
2) $\varpi_m(b_{22}b'_{22}) = \varpi_m(b_{22})b'_{22} + b_{22}\varpi_m(b'_{22})p_2 + \sum_{i+j=n,0<i,j<m} d_i(b_{22})d_j(b'_{22})$

for all $a_{ii}, a'_{ii} \in A_{ii}$ with $i \in \{1,2\}$.

Proof. For conclusion 1), arbitrary $a_{11}, a'_{11} \in A_{11}$ and $a_{12} \in A_{12}$ and by conclusion 1) in Lemma 4.9, we have

$$\begin{aligned}
\varpi_m(a_{11}a'_{11}a_{12}) &= a_{11}a'_{11}\varpi_m(a_{12}) + \varpi_m(a_{11}a'_{11})a_{12} \\
&+ \sum_{i+j=m,0<i,j<m} d_i(a_{11}a'_{11})d_j(a_{12}) \\
&= a_{11}a'_{11}\varpi_m(a_{12}) + \varpi_m(a_{11}a'_{11})a_{12} \\
&+ \sum_{i+j=m,0<i,j<m} \Big(\sum_{i_1+i_2=i,0<i<m} d_{i_1}(a_{11})d_{i_2}(a'_{11})\Big)d_j(a_{12}) \\
&= a_{11}a'_{11}\varpi_m(a_{12}) + \varpi_m(a_{11}a'_{11})a_{12} \\
&+ \sum_{i_1+i_2+j=m,0<i_1,i_2,j<m} d_{i_1}(a_{11})d_{i_2}(a'_{11})d_j(a_{12})
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
\varpi_m(a_{11}a'_{11}a_{12}) &= a_{11}\varpi_m(a'_{11}a_{12}) + \varpi_m(a_{11})a'_{11}a_{12} \\
&+ \sum_{i+j=m,0<i,j<m} d_i(a_{11})d_j(a'_{11}a_{12}) \\
&= a_{11}a'_{11}\varpi_m(a_{12}) + a_{11}\varpi_m(a'_{11})a_{12} + \varpi_m(a_{11})a'_{11}a_{12} \\
&+ \sum_{i+j=n,0<i,j<m} a_{11}d_i(a'_{11})d_j(a_{12}) + \sum_{i+j=m,0<i,j<m} d_i(a_{11})d_j(a'_{11}a_{12}) \\
&= a_{11}a'_{11}\varpi_m(a_{12}) + a_{11}\varpi_m(a'_{11})a_{12} + \varpi_m(a_{11})a'_{11}a_{12} \\
&+ \sum_{i+j=m,0<i,j<m} a_{11}d_i(a'_{11})d_j(a_{12}) + \sum_{i+j=m,0<i,j<m} d_i(a_{11})\Big(\sum_{j_1+j_2=j,0<j<m} d_{j_1}(a'_{11})d_{j_2}(a_{12})\Big) \\
&= a_{11}a'_{11}\varpi_m(a_{12}) + a_{11}\varpi_m(a'_{11})a_{12} + \varpi_m(a_{11})a'_{11}a_{12} \\
&+ \sum_{i+j=m,0<i,j<m} a_{11}d_i(a'_{11})d_j(a_{12}) + \sum_{i+j_1+j_2=m,0<i,j_1,j_2<m} d_i(a_{11})d_{j_1}(a'_{11})d_{j_2}(a_{12}) \\
&= a_{11}a'_{11}\varpi_m(a_{12}) + a_{11}\varpi_m(a'_{11})a_{12} + \varpi_m(a_{11})a'_{11}a_{12} \\
&+ \sum_{i+j=m,0<i,j<m} d_i(a_{11})d_j(a'_{11})a_{12} + \sum_{i+j_1+j_2=m,0<i,j_1,j_2<m} d_i(a_{11})d_{j_1}(a'_{11})d_{j_2}(a_{12})
\end{aligned} \tag{4.13}$$

for all $a_{tt}, a'_{tt} \in A_{tt}$ with $t \in \{1, 2\}$.

Combining (4.12) with (4.13) leads to

$$\varpi_m(a_{11}a'_{11})a_{12} = (\varpi_m(a_{11})a'_{11} + a_{11}\varpi_m(a'_{11}) + \sum_{i+j=m,0<i,j<m} d_i(a_{11})d_j(a'_{11}))a_{12}$$

for all $a_{tt}, a'_{tt} \in A_{tt}$ with $t \in \{1, 2\}$.

Since $\varpi_m(A_{11}) \subseteq A_{11} + A_{12}$ and A_{12} are faithful as a left A_{11}-module, the above relation implies that

$$\varpi_m(a_{11}a'_{11})p_1 = \{\varpi_m(a_{11})a'_{11} + a_{11}\varpi_m(a'_{11}) + \sum_{i+j=m,0<i,j<m} d_i(a_{11})d_j(a'_{11})\}p_1 \tag{4.14}$$

for all $a_{11}, a'_{11} \in A_{11}$.

On the other hand, by $a_{11}\underbrace{p_2 \cdots p_2}_{n-1 \text{ copies}} = 0$ for all $a_{11} \in A_{11}$, we arrive at

$$\begin{aligned}
0 &= \delta_m(P_n(a_{11}, \underbrace{p_2, \cdots, p_2}_{n-1 \text{ copies}})) \\
&= P_n(\delta_m(a_{11}), \underbrace{p_2, \cdots, p_2}_{n-1 \text{ copies}}) + P_n(a_{11}, \delta_m(p_2), \underbrace{p_2, \cdots, p_2}_{n-2 \text{ copies}}) \\
&+ \sum_{i_1+\cdots+i_n=m, i_1,\cdots,i_n<m} P_n(\delta_{i_1}(a_{11}), \delta_{i_2}(p_2), \cdots, \delta_{i_n}(p_2)) \\
&= P_n(\varpi_m(a_{11}), \underbrace{p_2, \cdots, p_2}_{n-1 \text{ copies}}) + P_n(a_{11}, \varpi_m(p_2), \underbrace{p_2, \cdots, p_2}_{n-2 \text{ copies}}) \\
&+ \sum_{i_1+\cdots+i_n=m, i_1,\cdots,i_n<m} P_n(d_{i_1}(a_{11}), d_{i_2}(p_2), \cdots, d_{i_n}(p_2))
\end{aligned}$$

for all $a_{11}, a'_{11} \in A_{11}$.

Since $\varpi_n(A_{11}) \subseteq A_{11} + A_{12}, \varpi_n(p_2) \in A_{12}$ and $d_i(p_2) \in A_{12}$, the above equation implies that

$$0 = \varpi_m(a_{11})p_2 + a_{11}\varpi_m(p_2) + \sum_{i+j=m,0\leq i,j<m} d_i(a_{11})d_j(p_2)$$

for all $a_{11}, a'_{11} \in A_{11}$.

On substituting a_{11} by $a_{11}a'_{11}$ in above equation, we get

$$\begin{aligned}
0 &= \varpi_m(a_{11}a'_{11})p_2 + a_{11}a'_{11}\varpi_m(p_2) + \sum_{i+j=m,0\leq i,j<m} d_i(a_{11}a'_{11})d_j(p_2) \\
&= \varpi_m(a_{11}a'_{11})p_2 + a_{11}a'_{11}\varpi_m(p_2) \\
&+ \sum_{i+j=m,0\leq i,j<m} (\sum_{i_1+i_2=i,0\leq i_1,i_2<m} d_{i_1}(a_{11})d_{i_2}(a'_{11}))d_j(p_2) \\
&= \varpi_m(a_{11}a'_{11})p_2 + a_{11}a'_{11}\varpi_m(p_2) \\
&+ \sum_{i_1+i_2+j=m,0\leq i_1,i_2,j<m} d_{i_1}(a_{11})d_{i_2}(a'_{11})d_j(p_2)
\end{aligned}$$

for all $a_{11}, a'_{11} \in A_{11}$. Therefore, we have

$$p_1(\varpi_m(a_{11}a'_{11})p_2 + a_{11}a'_{11}\varpi_m(p_2) + \sum_{i_1+i_2+j=m, 0\le i_1,i_2,j<m} d_{i_1}(a_{11})d_{i_2}(a'_{11})d_j(p_2))p_2 = 0. \tag{4.15}$$

Again, note that $a'_{11}p_2\cdots p_2 = 0$ for all $a'_{11} \in A_{11}$, and we have

$$\begin{aligned} 0 = \delta_m(P_n(a'_{11}, p_2, \cdots, p_2)) \\ &= \sum_{i_1+\cdots+i_n=m} P_n(\delta_{i_1}(a'_{11}), \delta_{i_2}(p_2), \cdots, \delta_{i_n}(p_2)) \\ &= P_n(\varpi_m(a'_{11}), p_2, \cdots, p_2)] + P_n(a'_{11}, \varpi_m(p_2), p_2, \cdots, p_2] \\ &+ \sum_{i_1+\cdots+i_n=m, i_1,\cdots,i_n<m} P_n(d_{i_1}(a'_{11}), d_{i_2}(p_2), \cdots, d_{i_n}(p_2)). \end{aligned}$$

This gives us

$$0 = \varpi_m(a'_{11})p_2 + a'_{11}\varpi_m(p_2) + \sum_{i+j=m, 0\le i,j<m} d_i(a'_{11})d_j(p_2). \tag{4.16}$$

Now, left multiplying a_{11} in (4.16) and combining it with (4.15) gives us

$$\varpi_m(a_{11}a'_{11})p_2 + \sum_{i+j+k=m, 0\le i, 0<j} d_i(a_{11})d_j(a'_{11})d_k(p_2) = a_{11}\varpi_m(a'_{11})p_2.$$

This implies that

$$\varpi_m(a_{11}a'_{11})p_2 + \sum_{i=1}^{m} d_i(a_{11}) \sum_{j+k=m-i, 0\le i} d_j(a'_{11})d_k(p_2) = a_{11}\varpi_m(a'_{11})p_2.$$

Now, using the condition $\mathfrak{F}_s$, we find that

$$\varpi_m(a_{11}a'_{11})p_2 - \sum_{i=1}^{m-1} d_i(a_{11})d_{m-i}(a'_{11})p_2 = a_{11}\varpi_m(a'_{11})p_2,$$

which gives

$$\varpi_m(a_{11}a'_{11})p_2 = a_{11}\varpi_m(a'_{11})p_2 + \sum_{i=1}^{m-1} d_i(a_{11})d_{m-i}(a'_{11})p_2.$$

Hence,

$$\varpi_m(a_{11}a'_{11})p_2 = \{\varpi_m(a_{11})a'_{11} + a_{11}\varpi_m(a'_{11})p_2 + \sum_{i+j=m, 0<i,j<m} d_i(a_{11})d_{m-i}(a'_{11})\}p_2. \tag{4.17}$$

Now, adding (4.14) and (4.17), we have

$$\varpi_m(a_{11}a'_{11}) = \varpi_m(a_{11})a'_{11} + a_{11}\varpi_m(a'_{11})p_2 + \sum_{i+j=m, 0<i,j<m} d_i(a_{11})d_j(a'_{11}).$$

Adopting the same discussion, we have

$$\varpi_m(b_{22}b'_{22}) = \varpi_m(b_{22})b'_{22} + b_{22}\varpi_m(b'_{22})p_2 + \sum_{i+j=m, 0<i,j<m} d_i(b_{22})d_j(b'_{22})$$

for all $b_{22}, b'_{22} \in A_{22}$. □

Remark 4.1. *Now, we establish a mapping* $g_m : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ *by*

$$g_m(x) = \varpi_m(x) - \varpi_m(p_1xp_1) - \varpi_m(p_1xp_2) - \varpi_m(p_2xp_2)$$

and $g_m(P_n(x_1, \cdots, x_n)) = 0$ with $x_1 \cdots x_n = 0$ for all $x_1, \cdots, x_n \in \mathcal{T}$, then define a mapping $\chi_m(x) = \varpi_m(x) - g_m(x)$ for all $x \in \mathcal{T}$. It is easy to verify that

$$\chi_m(a_{11} + a_{12} + a_{22}) = \chi_m(a_{11}) + \chi_m(a_{12}) + \chi_m(a_{22}).$$

From the definition of χ_m and g_m, we find that

$$\varphi_m(x) = \varpi_m(x) + f_m(x) = \chi_m(x) + g_m(x) + f_m(x) = \chi_m(x) + h_m(x),$$

where $h_m(x) = g_m(x) + f_m(x)$ for all $x \in \mathcal{T}$.

Lemma 4.11. *With notations as above, we obtain that* $\{\chi_i\}_{i=0}^{i=m}$ *is an additive higher derivation on triangular algebras* $\mathcal{T}$*.*

Proof. Suppose that $x, y \in \mathcal{T}$, such that $x = a_{11} + a_{12} + a_{22}$ and $y = a'_{11} + a'_{12} + a'_{22}$, where $a_{ij}, a'_{ij} \in A_{ij}$ with $i \leqslant j \in \{1, 2\}$, then

$$\begin{aligned}
\chi_m(x + y) &= \chi_m((a_{11} + a_{12} + a_{22}) + (a'_{11} + a'_{12} + a'_{22}))\\
&= \chi_m((a_{11} + a'_{11}) + (a_{12} + a'_{12}) + (a_{22} + a'_{22}))\\
&= \chi_m(a_{11} + a'_{11}) + \chi_m(a_{12} + a'_{12}) + \chi_m(a_{22} + a'_{22})\\
&= \chi_m(a_{11}) + \chi_m(a'_{11}) + \varpi_m(a_{12}) + \chi_m(a'_{12}) + \chi_m(a_{22}) + \chi_m(a'_{22})\\
&= \chi_m(a_{11} + a_{12} + a_{22}) + \chi_m(a'_{11} + a'_{12} + a'_{22}))\\
&= \chi_m(x) + \chi_m(y).
\end{aligned}$$

By Lemmas 4.8 and 4.10, we have

$$\begin{aligned}
\chi_m(xy) &= \chi_m((a_{11} + a_{12} + a_{22})(a'_{11} + a'_{12} + a'_{22}))\\
&= \chi_m(a_{11}a'_{11} + a_{11}a'_{12} + a_{12}a'_{22} + a_{22}a'_{22})\\
&= \varpi_m(a_{11})a'_{11} + a_{11}\varpi_m(a'_{11}) + \sum_{i+j=m,0<i<m} d_i(a_{11})d_j(a'_{11})\\
&\quad + \varpi_m(a_{11})a'_{12} + a_{11}\varpi_m(a'_{12}) + \sum_{i+j=m,0<i<m} d_i(a_{11})d_j(a'_{12})\\
&\quad + \varpi_m(a_{12})a'_{22} + a_{12}\varpi_m(a'_{22}) + \sum_{i+j=m,0<i<m} d_i(a_{12})d_j(a'_{22})\\
&\quad + \varpi_m(a_{22})a'_{22} + a_{22}\varpi_m(a'_{22}) + \sum_{i+j=m,0<i<m} d_i(a_{22})d_j(a'_{22}).
\end{aligned} \tag{4.18}$$

On the other hand, we have

$$
\begin{aligned}
&\chi_m(x)y + x\chi_m(y) + \sum_{i+j=m,0<i<m} \chi_i(x)\chi_j(y)\\
&= \chi_m(a_{11} + a_{12} + a_{22})y + x\chi_m(a'_{11} + a'_{12} + a'_{22}) + \sum_{i+j=m,0<i<m} \chi_i(x)\chi_j(y)\\
&= (\varpi_m(a_{11}) + \varpi_m(a_{12}) + \varpi_m(a_{22}))y + \sum_{i+j=m,0<i<m} d_i(a_{11})d_j(a'_{11})\\
&+ x(\varpi_m(a'_{11}) + \varpi_m(a'_{12}) + \varpi_m(a'_{22})) + \sum_{i+j=m,0<i<m} d_i(a_{11})d_j(a'_{12}) + \sum_{i+j=m,0<i<m} d_i(a_{11})d_j(a'_{22})\\
&+ \sum_{i+j=m,0<i<m} d_i(a_{12})d_j(a'_{11}) + \sum_{i+j=m,0<i<m} d_i(a_{12})d_j(a'_{12}) + \sum_{i+j=m,0<i<m} d_i(a_{12})d_j(a'_{22})\\
&+ \sum_{i+j=m,0<i<m} d_i(a_{22})d_j(a'_{11}) + \sum_{i+j=m,0<i<m} d_i(a_{22})d_j(a'_{12}) + \sum_{i+j=m,0<p<m} d_i(a_{22})d_j(a'_{22}).
\end{aligned}
$$

Taking into account the induction hypothesis $\mathfrak{F}_s$, Lemmas 4.9 and 4.10, we calculate that

$$
\begin{aligned}
&\chi_m(x)y + x\chi_m(y) + \sum_{i+j=m,0<i<m} d_i(x)d_j(y)\\
&= \varpi_m(a_{11})a'_{11} + \varpi_m(a_{11})a'_{12} + \varpi_m(a_{12})a'_{22} + \varpi_m(a_{22})a'_{22}\\
&+ a_{11}\varpi_m(a'_{11}) + a_{11}\varpi_m(a'_{12}) + a_{12}\varpi_m(a'_{22}) + a_{22}\varpi_m(a'_{22})\\
&+ \sum_{i+j=m,0<i<m} d_i(a_{11})d_j(a'_{11}) + \sum_{i+j=m,0<i<m} d_i(a_{11})d_j(a'_{12})\\
&+ \sum_{i+j=m,0<i<m} d_i(a_{12})d_j(a'_{22}) + \sum_{i+j=m,0<i<m} d_i(a_{22})d_j(a'_{22}).
\end{aligned} \tag{4.19}
$$

Combining (4.18) and (4.19), we get

$$\chi_m(xy) = \chi_m(x)y + x\chi_m(y) + \sum_{i+j=m,0<i<m} \chi_i(x)\chi_j(y)$$

for all $x, y \in \mathcal{T}$. This shows that each χ_m satisfies the Leibniz formula of higher order on $\mathcal{T}$. □

Finally, we need to prove that each h_m vanishes $P_n(x_1, \cdots, x_n)$ with $x_1 \cdots x_n = 0$ for all $x_1, \cdots, x_n \in \mathcal{T}$. Note that h_m maps into $\mathcal{Z}(\mathcal{T})$, $\{\chi_i\}_{i=0}^m$ as an additive higher derivation of $\mathcal{T}$. It follows from inductive hypothesis $\mathfrak{F}_s$ that

$$h_m(P_n(x_1, \cdots, x_n)) = \delta_n(P_n(x_1, \cdots, x_n)) - \chi_n(P_n(x_1, \cdots, x_n)) = 0$$

with $x_1 \cdots x_n = 0$ for all $x_1, \cdots, x_n \in \mathcal{T}$. We lastly complete the proof of the main theorem.

In particular, we have the following corollaries.

When $n = 3$, we have the following corollary.

Corollary 4.1. [17, Theorem 3.3] *Let* $\mathcal{T} = \left[\begin{smallmatrix} A_{11} & A_{12} \\ O & A_{22} \end{smallmatrix}\right]$ *be a triangular algebra satisfying*

i) $\pi_{A_{11}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A}_{11})$ and $\pi_{A_{22}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A}_{22})$.

ii) For any $a_{11} \in A_{11}$, if $[a_{11}, A_{11}] \in \mathcal{Z}(\mathcal{A})_{11}$, then $a_{11} \in \mathcal{Z}(\mathcal{A})$, or for any $a_{22} \in A_{22}$, if $[a_{22}, A_{22}] \in \mathcal{Z}(\mathcal{A}_{22})$, then $a_{22} \in \mathcal{Z}(\mathcal{A}_{22})$.

Suppose that a sequence $\Delta = \{\delta_m\}_{m\in\mathcal{N}}$ of mappings $\delta_m : \mathcal{T} \to \mathcal{T}$ is a nonlinear map satisfying

$$\delta_m([[x, y], z]) = \sum_{i+j+k=m} [[\delta_i(x), \delta_j(y)], \delta_k(z)]$$

for all $x, y, z \in \mathcal{T}$ with $xyz = 0$. For every $m \in \mathcal{N}$,

$$\delta_m(x) = \chi_m(x) + h_m(x)$$

for all $x \in \mathcal{T}$, where a sequence $\Upsilon = \{\chi_m\}_{m\in\mathcal{N}}$ of additive mapping $\chi_m : \mathcal{T} \to \mathcal{T}$ is a higher derivation and $h_m : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ is a nonlinear mapping, such that $h_m([[x, y], z]) = 0$ for any $x, y, z \in \mathcal{T}$ with $xyz = 0$.

When $n = 3$ and $m = 1$, we have the following corollary.

Corollary 4.2. [16, Theorem 2.2] *Let* $\mathcal{T} = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}$ *be a triangular algebra satisfying*

i) $\pi_{A_{11}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A}_{11})$ and $\pi_{A_{22}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A}_{22})$.

ii) For any $a_{11} \in A_{11}$, if $[a_{11}, A_{11}] \in \mathcal{Z}(\mathcal{A}_{11})$, then $a_{11} \in \mathcal{Z}(\mathcal{A})$, or for any $a_{22} \in A_{22}$, if $[a_{22}, A_{22}] \in \mathcal{Z}(\mathcal{A}_{22})$ then $a_{22} \in \mathcal{Z}(\mathcal{A}_{22})$.

Suppose $\delta_1 : \mathcal{T} \to \mathcal{T}$ is a nonlinear map satisfying

$$\delta_1([[x, y], z]) = \sum_{i+j+k=1} [[\delta_i(x), \delta_j(y)], \delta_k(z)]$$

for all $x, y, z \in \mathcal{T}$ with $xyz = 0$, then there exists an additive derivation ϖ_1 of $\mathcal{T}$ and a nonlinear map $h_1 : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$, such that

$$\delta_1(x) = \chi_1(x) + h_1(x)$$

for all $x \in \mathcal{T}$, where $\tau_1([[x, y], z])$ for any $x, y, z \in \mathcal{T}$ with $xyz = 0$.

5. Applications

In this section, we apply Theorem 4.2 to certain classes of triangular algebras that satisfy the hypotheses of Theorem 4.2. Some standard examples of triangular rings satisfying the hypotheses of Theorem 4.1 are: Upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras (see [24, 4. More applications and further topics] for details).

According to the Theorem 4.2, we get the following corollaries.

Corollary 5.1. *Let* $\mathcal{R}$ *be a* 2*-torsion free commutative ring with identity and* $\mathcal{T}_k(\mathcal{R})(k \geq 2)$ *be the algebra of all* $k \times n$ *upper triangular matrices over* $\mathcal{R}$*. Let* $\{L_m\}_{m\in\mathcal{N}}$ *be a family of nonlinear mapping* $L_m : \mathcal{T}_k(\mathcal{R}) \longrightarrow \mathcal{T}_k(\mathcal{R})$ *satisfying Eq* (1.3)*, then every Lie-m higher derivation* $L_m : \mathcal{T}_k(\mathcal{R}) \longrightarrow \mathcal{T}_k(\mathcal{R})$ *be of standard form.*

Corollary 5.2. *Let* $\mathcal{R}$ *be a* 2*-torsion free commutative ring with identity and* $\mathcal{T}_s^{\bar{k}}(\mathcal{R})(s \geq 3)$ *be a block upper triangular matrix ring with over* $\mathcal{T}_s^{\bar{k}}(\mathcal{R}) \neq M_s(R)$*. Let* $\{L_m\}_{m\in\mathcal{N}}$ *be a family of nonlinear mapping* $L_m : \mathcal{T}_s^{\bar{k}}(\mathcal{R}) \longrightarrow \mathcal{T}_s^{\bar{k}}(\mathcal{R})$ *satisfying Eq* (1.3)*, then every Lie-m higher derivation* $L_m : \mathcal{T}_s^{\bar{k}}(\mathcal{R}) \longrightarrow \mathcal{T}_s^{\bar{k}}(\mathcal{R})$ *be of standard form.*

Corollary 5.3. *Let* H *be a Hilbert space,* $\mathcal{N}$ *be a nest of* H *and* Alg($\mathcal{N}$) *be the nest algebra associated with* $\mathcal{N}$. *Let* $\{L_m\}_{m\in\mathcal{N}}$ *be a family of nonlinear mapping* $L_m : \text{Alg}(\mathcal{N}) \longrightarrow \text{Alg}(\mathcal{N})$ *satisfying Eq* (1.3), *then every Lie-m higher derivation* $L_m : \text{Alg}(\mathcal{N}) \longrightarrow \text{Alg}(\mathcal{N})$ *be of standard form.*

6. Conclusions

The purpose of this article was to prove that every nonlinear Lie-n higher derivation by local actions on the triangular algebras is of a standard form. As an application, we gave a characterization of Lie-*n* higher derivations by local actions on upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras, respectively.

Use of AI tools declaration

The author declares she has not used Artificial Intelligence(AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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