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Research article

Global well-posedness for the 2D MHD equations with only vertical velocity damping term

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Abstract: This paper concerns two-dimensional (2D) incompressible magnetohydrodynamic (MHD) equations without magnetic diffusion with only vertical velocity damping term in the periodic domain. We prove the stability and decay rate for smooth solutions on perturbations near a background magnetic field of the system under the assumptions that the initial magnetic field satisfies the Diophantine condition.

Keywords: magnetohydrodynamic equations; global solutions; Diophantine condition
Mathematics Subject Classification: 35A05, 35Q35, 76D03

1. Introduction

1.1. Model and related studies

MHD is a discipline that studies the interaction between electric and magnetic fields in conductive fluids based on fluid mechanics. The Navier-Stokes equations in fluid mechanics and the Maxwell equations in electrodynamics contribute to the fundamental equations of the MHD system of equations. The MHD equations are also of great interest in mathematics. The following is the standard expression for the incompressible MHD equations:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \mu \Delta \mathbf{u} - \nabla P + \mathbf{B} \cdot \nabla \mathbf{B}, \\ \partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} = \eta \Delta \mathbf{B} + B \cdot \nabla \mathbf{u}, \\ \nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{B} = 0, \end{cases} \tag{1.1}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ and $P = P(\mathbf{x}, t)$ represent the velocity field, the magnetic field and the pressure, respectively. μ denotes the viscosity coefficient and η the diffusion coefficient.

In this paper, we study the 2D incompressible MHD equations without magnetic diffusion and kinetic viscosity with damping only in the vertical component of the velocity equation in the periodic domain $\mathbb{T}^2$,

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P + \nu (0, u_2)^\top = \mathbf{B} \cdot \nabla \mathbf{B}, \quad \mathbf{x} \in \mathbb{T}^2, \ t > 0, \\ \partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u}, \quad \mathbf{x} \in \mathbb{T}^2, \ t > 0, \\ \nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad \mathbf{x} \in \mathbb{T}^2, \ t > 0, \end{cases} \tag{1.2}$$

where the parameter $\nu > 0$ denotes the damping coefficient, and the 2D periodic domain is defined by $\mathbb{T}^2 = [-\pi, \pi]^2$.

In the whole plane $\mathbb{R}^2$, the global well-posedness and stability of the 2D incompressible MHD equations without magnetic diffusion with damping only in the vertical component of velocity equations remain unknown. If the magnetic field is ignored, the system reduces to Euler-like equations with an additional Riesz transform-type term, and the global well-posedness of the Euler-like equations remains an open problem in $\mathbb{R}^2$.

In recent years, important progresses have been made on the MHD equations. For the case of a viscous and resistive MHD system, the MHD system is globally well-posed in the 2D space (see [1,8]). For the case of inviscid and non-resistive MHD systems, Bardos, Sulem, and Sulem [3] proved their global well-posedness. For cases when only viscosity exists, please refer to [11, 19, 20], and only magnetic diffusion exists, please refer to [15, 21, 26]. For more results of these two dissipation cases above, see [2, 7, 25]. Ji, Lin, Wu, and Yan [10] proved the stability of the 2D MHD equations with mixed partial dissipation, and Ji and Li [9] studied the global regularity of $2\frac{1}{2}$-D MHD equations with mixed dissipation diffusion. For more results on the MHD equations with mixed partial dissipation or fractional dissipation, please see [22, 23]. Prompted by Lin and Zhang [16], there's been a lot of research on the global well-posedness of the MHD system near a constant equilibrium (see [17,28,30]).

In addition, there are some results on the 2D MHD system involving damping in the equation of the velocity. Boardman, Lin, and Wu [4] proved the stability for the MHD equations with vertical velocity damping and full magnetic diffusion. Chen, Lin, and Wu [5] improved the work to the vertical magnetic diffusion even further. Afterward, Lin and Zhang [18] proved the global well-posedness of the 2D MHD equations with horizontal magnetic diffusion and vertical damping in the velocity equation in $\mathbb{T} \times \mathbb{R}$. Besides, Lai, Wu, and Zhang [13, 14] studied the stability of a 2D MHD system with horizontal velocity damping and vertical damping in the magnetic equation.

Moreover, by using the Diophantine condition, Wu and Zhai [24] proved the global small solutions of the 3D compressible viscous non-resistive MHD system. Chen, Zhang, and Zhou [6] proved the global well-posedness of a 3D MHD system in $\mathbb{T}^3$. Compared with [6], the results of Zhai [27] reduces the regularity requirement on initial data. Zhao and Zhai [29] proved the global small solutions to the 3D MHD system with a velocity damping term. Compared to the velocity damping term in [29], in this paper, we require only its second component.

Let $\mathbf{n} \in \mathbb{R}^2$ satisfy the so-called Diophantine condition: for any $\mathbf{k} \in \mathbb{Z}^2 \backslash \{0\}$

$$|\mathbf{n} \cdot \ \mathbf{k}| \geq \frac{c}{|\mathbf{k}|^r}, \tag{1.3}$$

for some $c > 0$ and $r > 1$. Moreover, as demonstrated in [6].

The perturbation $(\mathbf{u}, \mathbf{b})$ with

$$\mathbf{b} := \mathbf{B} - \mathbf{n},$$

our Eq (1.2) becomes

$$
\begin{cases}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nu (0, u_2)^\top + \nabla P = \mathbf{b} \cdot \nabla \mathbf{b} + \mathbf{n} \cdot \nabla \mathbf{b}, \quad \mathbf{x} \in \mathbb{T}^2,\ t > 0, \\
\partial_t \mathbf{b} + \mathbf{u} \cdot \nabla \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{u} + \mathbf{n} \cdot \nabla \mathbf{u}, \quad \mathbf{x} \in \mathbb{T}^2,\ t > 0, \\
\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{b} = 0, \quad \mathbf{x} \in \mathbb{T}^2,\ t > 0, \\
\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{b}(\mathbf{x}, 0) = \mathbf{b}_0(\mathbf{x}).
\end{cases}
\tag{1.4}
$$

1.2. Main result

The main result of the paper is stated as follows.

Theorem 1.1. *For any $\gamma \geq 4r + 11$ with $r > 1$. Consider (1.4) with initial data $(\mathbf{u}_0, \mathbf{b}_0) \in H^\gamma(\mathbb{T}^2)$ with $\nabla \cdot \mathbf{u}_0 = \nabla \cdot \mathbf{b}_0 = 0$. There exists a constant ε such that, if*

$$
\begin{aligned}
&\|(\mathbf{u}_0, \mathbf{b}_0)\|_{H^\gamma} \leq \varepsilon, \\
&\int_{\mathbb{T}} \mathbf{u}_0 dx_1 = \int_{\mathbb{T}^2} \mathbf{b}_0 d\mathbf{x} = 0,
\end{aligned}
\tag{1.5}
$$

then there exists a global solution $(\mathbf{u}, \mathbf{b}) \in C([0, +\infty); H^\gamma)$ to system (1.4) satisfying

$$
\begin{aligned}
&\|\mathbf{u}(t)\|_{H^\gamma} + \|\mathbf{b}(t)\|_{H^\gamma} \leq C\varepsilon, \\
&\|\mathbf{u}(t)\|_{H^{r+5}} + \|\mathbf{b}(t)\|_{H^{r+5}} \leq C(1+t)^{-\frac{3}{2}}.
\end{aligned}
$$

Moreover, for any $t \geq 0$ and $r + 5 < \alpha < \gamma$, there holds

$$
\|\mathbf{u}(t)\|_{H^\alpha} + \|\mathbf{b}(t)\|_{H^\alpha} \leq C(1+t)^{-\frac{3(\gamma-\alpha)}{2(\gamma-r-5)}}.
$$

Remark 1.2. *If the initial data $(\mathbf{u}_0, \mathbf{b}_0)$ satisfies (1.5), for sufficiently regular solutions to the system (1.4), this property will be conserved in time,*

$$
\int_{\mathbb{T}^2} u \, d\mathbf{x} = \int_{\mathbb{T}^2} b \, d\mathbf{x} = 0,
\tag{1.6}
$$

using the two-dimensional Biot-Savart law, $u_2 = \partial_1 \Delta^{-1} \omega$, combined with the system (1.4), we can get $\frac{d}{dt} \int_{\mathbb{T}^2} b \, dx = \frac{d}{dt} \int_{\mathbb{T}^2} u \, dx = 0$.

1.3. Purpose and difficulties

According to previous studies on 2D MHD equations with a velocity damping term, it was found that the magnetic equation is either fully dissipative or partially dissipative. Our work focuses on the challenging problem of the global well-posedness of the 2D incompressible MHD equations without magnetic diffusion and kinetic viscosity with vertical velocity damping.

Due to the absence of magnetic diffusion and horizontal velocity damping, proving the theorem is challenging. Indeed, the construction of global solutions to the incompressible non-resistive MHD system with vertical velocity damping remains an open problem, even with small initial data in $\mathbb{T}^2$. This work represents a meaningful step forward. It establishes the global well-posedness and stability of solutions near background magnetic fields that satisfy a Diophantine condition.

2. PRELIMINARIES

Some useful inequalities and properties are provided in this section for assistance in demonstrating our main result.

Lemma 2.1. *If* $\mathbf{n} \in \mathbb{R}^2$ *satisfies the Diophantine condition (1.3), then it holds that*

$$\|f\|_{H^s} \le C\|\mathbf{n} \cdot \nabla f\|_{H^{s+r}}, \tag{2.1}$$

for any $s \in \mathbb{R}$, *if* $\nabla f \in H^{s+r}(\mathbb{T}^2)$ *satisfies* $\int_{\mathbb{T}^2} f\, d\mathbf{x} = 0$.

The proof of the above lemma is provided in [27].

Lemma 2.2. *Let* $s \ge 0$. *The following inequalities hold*

$$\|fg\|_{H^s} \le C(\|f\|_{L^\infty}\|g\|_{H^s} + \|f\|_{H^s}\|g\|_{L^\infty}),$$
$$\|[\Lambda^s, f \cdot \nabla]g\|_{L^2} \le C(\|\nabla f\|_{L^\infty}\|\Lambda^s g\|_{L^2} + \|\Lambda^s f\|_{L^2}\|\nabla g\|_{L^\infty}),$$

where $[a, b] = ab - ba$ *is the commutator; please refer to [12] for details.*

Lemma 2.3. *Define* $\bar{\mathbf{u}} = \frac{1}{2\pi}\int_{\mathbb{T}} \mathbf{u}(x_1, x_2)dx_1$, *let* $\mathbf{u}$ *be a smooth solution to (1.4) on* $[0, \infty) \times \mathbb{T}^2$ *satisfying* $\int_{\mathbb{T}} \mathbf{u}\, dx_1 = 0$ *, there holds*

$$\|\mathbf{u}\|_{L^2(\mathbb{T}^2)} \le C\|\partial_1 \mathbf{u}\|_{L^2(\mathbb{T}^2)}. \tag{2.2}$$

Because the integral average of $\mathbf{u}$ *equals zero in the* x_1 *direction, we have used Poincáre inequality.*

3. Proof of theorem

The proof of Theorem 1.1 is highlighted below. First, denote $\Lambda = \sqrt{-\Delta}$ and $\langle a, b\rangle$ the $L^2(\mathbb{T}^2)$ inner product of a and b. By using incompressible conditions, we can easily obtain:

$$\|\partial_1 \mathbf{u}\|_{L^2} = \|\nabla u_2\|_{L^2}. \tag{3.1}$$

The standard energy method might be used to demonstrate the local well-posedness of (1.4) given the initial data $(\mathbf{u}_0, \mathbf{b}_0) \in H^\gamma$. As a result, we can assume that there is a $T > 0$, which means that there is only one solution for $(\mathbf{u}, \mathbf{b}) \in C([0, T]; H^\gamma)$ to system (1.4).

We demonstrate that this local solution can be expanded to a global one using the bootstrapping argument. Theorem 1.1 is to derive a global a priori upper bound. For some $0 < \delta < 1$ to be specified later, we assume that

$$\sup_{t\in[0,T]} (\|\mathbf{u}\|_{H^\gamma} + \|\mathbf{b}\|_{H^\gamma}) \le \delta, \tag{3.2}$$

in order to initiate the bootstrapping argument. Under the assumption (1.5′),

$$\|\mathbf{u}_0\|_{H^\gamma} + \|\mathbf{b}_0\|_{H^\gamma} \le \varepsilon,$$

where $\varepsilon > 0$ is sufficiently small, our goal is to get

$$\sup_{t\in[0,T]} (\|\mathbf{u}\|_{H^\gamma} + \|\mathbf{b}\|_{H^\gamma}) \le \frac{\delta}{2}. \tag{3.3}$$

Then the intended global bound is reached via the bootstrapping argument.

3.1. Basic energy estimates

Taking the L^2-inner product of (1.4) with $(\mathbf{u}, \mathbf{b})$, integrating by parts, we can obtain the standard basic energy estimate,

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|_{L^2}^2 + \|\mathbf{b}\|_{L^2}^2) + \nu\|u_2\|_{L^2}^2 = 0, \tag{3.4}$$

where we have used

$$\begin{aligned}
&\langle \mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{u}\rangle = \langle \mathbf{u}\cdot\nabla\mathbf{b}, \mathbf{b}\rangle = 0,\\
&\langle \mathbf{b}\cdot\nabla\mathbf{b}, \mathbf{u}\rangle + \langle \mathbf{b}\cdot\nabla\mathbf{u}, \mathbf{b}\rangle = 0,\\
&\langle \mathbf{n}\cdot\nabla\mathbf{b}, \mathbf{u}\rangle + \langle \mathbf{n}\cdot\nabla\mathbf{u}, \mathbf{b}\rangle = 0,
\end{aligned}$$

due to the incompressible condition $\nabla\cdot\mathbf{u} = 0$ and $\nabla\cdot\mathbf{b} = 0$.

3.2. Higher energy estimates

To get the higher energy estimates, the purpose of this subsection is to prove the following lemma.

Lemma 3.1. *For any $\beta \in [0, \gamma]$ and $t \in [0, T]$, it holds that*

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}(t)\|_{H^\beta}^2 + \|\mathbf{b}(t)\|_{H^\beta}^2) + \nu\|u_2\|_{H^\beta}^2 \le C(\|\nabla\mathbf{u}\|_{L^\infty} + \|\nabla\mathbf{b}\|_{L^\infty})(\|\mathbf{u}(t)\|_{H^\beta}^2 + \|\mathbf{b}(t)\|_{H^\beta}^2). \tag{3.5}$$

Proof. Operating $\Lambda^k (1 \le k \le \beta)$ on (1.4) and using a commutator argument, we have

$$\begin{cases}
\partial_t\Lambda^k\mathbf{u} + \mathbf{u}\cdot\nabla\Lambda^k\mathbf{u} + \Lambda^k\nabla p + \nu\Lambda^k(0, u_2)^\top\\
\quad = \Lambda^k(\mathbf{n}\cdot\nabla\mathbf{b}) + \mathbf{b}\cdot\nabla\Lambda^k\mathbf{b} - [\Lambda^k, \mathbf{u}\cdot\nabla]\mathbf{u} + [\Lambda^k, \mathbf{b}\cdot\nabla]\mathbf{b},\\
\partial_t\Lambda^k\mathbf{b} + \mathbf{u}\cdot\nabla\Lambda^k\mathbf{b} = \Lambda^k(\mathbf{n}\cdot\nabla\mathbf{u}) + \mathbf{b}\cdot\nabla\Lambda^k\mathbf{u} - [\Lambda^k, \mathbf{u}\cdot\nabla]\mathbf{b} + [\Lambda^k, \mathbf{b}\cdot\nabla]\mathbf{u}.
\end{cases}$$

Taking L^2 inner product with $\Lambda^k\mathbf{u}, \Lambda^k\mathbf{b}$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|\Lambda^k\mathbf{u}\|_{L^2}^2 + \|\Lambda^k\mathbf{b}\|_{L^2}^2) + \nu\|\Lambda^k u_2\|_{L^2}^2 &= \Big\langle [\Lambda^k, \mathbf{b}\cdot\nabla]\mathbf{b}, \Lambda^k\mathbf{u}\Big\rangle - \Big\langle [\Lambda^k, \mathbf{u}\cdot\nabla]\mathbf{u}, \Lambda^k\mathbf{u}\Big\rangle\\
&\quad + \Big\langle [\Lambda^k, \mathbf{b}\cdot\nabla]\mathbf{u}, \Lambda^k\mathbf{b}\Big\rangle - \Big\langle [\Lambda^k, \mathbf{u}\cdot\nabla]\mathbf{b}, \Lambda^k\mathbf{b}\Big\rangle,
\end{aligned} \tag{3.6}$$

where we have used

$$\begin{aligned}
&\Big\langle \mathbf{u}\cdot\nabla\Lambda^k\mathbf{u}, \Lambda^k\mathbf{u}\Big\rangle = \Big\langle \Lambda^k\nabla P, \Lambda^k\mathbf{u}\Big\rangle = \Big\langle \mathbf{u}\cdot\nabla\Lambda^k\mathbf{b}, \Lambda^k\mathbf{b}\Big\rangle = 0,\\
&\Big\langle \mathbf{b}\cdot\nabla\Lambda^k\mathbf{b}, \Lambda^k\mathbf{u}\Big\rangle + \Big\langle \mathbf{b}\cdot\nabla\Lambda^k\mathbf{u}, \Lambda^k\mathbf{b}\Big\rangle = 0,\\
&\Big\langle \Lambda^k(\mathbf{n}\cdot\nabla\mathbf{b}), \Lambda^k\mathbf{u}\Big\rangle + \Big\langle \Lambda^k(\mathbf{n}\cdot\nabla\mathbf{u}), \Lambda^k\mathbf{b}\Big\rangle = 0.
\end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}
&\|[\Lambda^k, \mathbf{u}\cdot\nabla]\mathbf{u}\|_{L^2} + \|[\Lambda^k, \mathbf{b}\cdot\nabla]\mathbf{b}\|_{L^2} \le C(\|\nabla\mathbf{u}\|_{L^\infty}\|\Lambda^k\mathbf{u}\|_{L^2} + \|\Lambda^k\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{L^\infty}),\\
&\|[\Lambda^k, \mathbf{u}\cdot\nabla]\mathbf{b}\|_{L^2} + \|[\Lambda^k, \mathbf{b}\cdot\nabla]\mathbf{u}\|_{L^2} \le C(\|\nabla\mathbf{u}\|_{L^\infty}\|\Lambda^k\mathbf{b}\|_{L^2} + \|\Lambda^k\mathbf{u}\|_{L^2}\|\nabla\mathbf{b}\|_{L^\infty}).
\end{aligned}$$

As a result, applying the above two estimates and the Cauchy's inequality, we can get the bound of the right-hand side of (3.6),

$$\begin{aligned}|\left\langle[\Lambda^k,\mathbf{b}\cdot\nabla]\mathbf{b},\Lambda^k\mathbf{u}\right\rangle| &\le C\|\nabla\mathbf{b}\|_{L^\infty}\|\Lambda^k\mathbf{b}\|_{L^2}\|\Lambda^k\mathbf{u}\|_{L^2}\\ &\le C\|\nabla\mathbf{b}\|_{L^\infty}(\|\Lambda^k\mathbf{u}\|_{L^2}^2+\|\Lambda^k\mathbf{b}\|_{L^2}^2),\end{aligned}\tag{3.6$'$}$$

$$\begin{aligned}|\left\langle[\Lambda^k,\mathbf{b}\cdot\nabla]\mathbf{u},\Lambda^k\mathbf{b}\right\rangle| &\le C(\|\nabla\mathbf{u}\|_{L^\infty}\|\Lambda^k\mathbf{b}\|_{L^2}+\|\Lambda^k\mathbf{u}\|_{L^2}\|\nabla\mathbf{b}\|_{L^\infty})\|\Lambda^k\mathbf{b}\|_{L^2}\\ &\le C(\|\nabla\mathbf{u}\|_{L^\infty}\|\Lambda^k\mathbf{b}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^\infty}(\|\Lambda^k\mathbf{u}\|_{L^2}^2+\|\Lambda^k\mathbf{b}\|_{L^2}^2)).\end{aligned}\tag{3.6$''$}$$

Similarly,

$$|\left\langle[\Lambda^k,\mathbf{u}\cdot\nabla]\mathbf{u},\Lambda^k\mathbf{u}\right\rangle| \le C\|\nabla\mathbf{u}\|_{L^\infty}\|\Lambda^k\mathbf{u}\|_{L^2}^2,\tag{3.6$'''$}$$

$$|\left\langle[\Lambda^k,\mathbf{u}\cdot\nabla]\mathbf{b},\Lambda^k\mathbf{b}\right\rangle| \le C(\|\nabla\mathbf{u}\|_{L^\infty}\|\Lambda^k\mathbf{b}\|_{L^2}^2+\|\nabla\mathbf{b}\|_{L^\infty}(\|\Lambda^k\mathbf{u}\|_{L^2}^2+\|\Lambda^k\mathbf{b}\|_{L^2}^2)).\tag{3.6$''''$}$$

Plugging the (3.6 ′)-(3.6 ′′′′) into (3.6) yields

$$\frac{1}{2}\frac{d}{dt}(\|\Lambda^k\mathbf{u}\|_{L^2}^2+\|\Lambda^k\mathbf{b}\|_{L^2}^2)+\nu\|\Lambda^k u_2\|_{L^2}^2 \le C(\|\nabla\mathbf{u}\|_{L^\infty}+\|\nabla\mathbf{b}\|_{L^\infty})(\|\Lambda^k\mathbf{u}\|_{L^2}^2+\|\Lambda^k\mathbf{b}\|_{L^2}^2),\tag{3.7}$$

which is the desired estimate (3.5). □

3.3. The dissipation of the magnetic field

The MHD system under consideration in this paper is not subject to magnetic diffusion. It is necessary to take advantage of the hidden dissipation caused by the background magnetic field. This subsection will show how to find the upper bound stated in the following lemma.

Lemma 3.2. *For any* $\gamma \ge r+4$ *with* $r>1$. *Assume that*

$$\sup_{t\in[0,T]}(\|\mathbf{u}\|_{H^\gamma}+\|\mathbf{b}\|_{H^\gamma}) \le \delta,\tag{3.8}$$

for some $0<\delta<1$. *Then there holds that*

$$\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2-\sum_{0\le s\le r+3}\frac{d}{dt}\langle\Lambda^s\mathbf{u},\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle \le (2+C\delta)\|\mathbf{u}\|_{H^{r+4}}^2+C\delta^2\|\mathbf{b}\|_{H^3}^2.\tag{3.9}$$

Proof. Applying $\Lambda^s(0\le s\le r+3)$ *to the first equation of (1.4) and multiplying it by* $\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})$*, then taking* L^2*-inner product over* $\mathbb{T}^2$*, we obtain*

$$\begin{aligned}\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\|_{L^2}^2 &= \langle\Lambda^s\partial_t\mathbf{u},\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle+\langle\Lambda^s(\mathbf{u}\cdot\nabla\mathbf{u}),\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle\\ &\quad+\left\langle\Lambda^s(0,u_2)^\top,\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\right\rangle-\langle\Lambda^s(\mathbf{b}\cdot\nabla\mathbf{b}),\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle.\end{aligned}\tag{3.10}$$

Using Hölder's inequality and Young's inequality,

$$\langle\Lambda^s(\mathbf{u}\cdot\nabla\mathbf{u}),\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle \le C\|\Lambda^s(\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2}\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\|_{L^2}$$

$$\leq C(\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{H^s} + \|\nabla\mathbf{u}\|_{L^\infty}\|\mathbf{u}\|_{H^s})\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\|_{L^2}$$
$$\leq\varepsilon\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\|_{L^2}^2 + C\delta\|\nabla\mathbf{u}\|_{H^s}^2. \tag{3.11}$$

By (3.9), similarly,

$$\begin{aligned}\langle\Lambda^s(0,u_2)^\top,\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle &\leq C\|\Lambda^s u_2\|_{L^2}\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\|_{L^2}\\ &\leq \varepsilon\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\|_{L^2}^2 + C\|u_2\|_{H^s}^2\\ &\leq \varepsilon\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\|_{L^2}^2 + C\|\mathbf{u}\|_{H^s}^2,\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\langle\Lambda^s(\mathbf{b}\cdot\nabla\mathbf{b}),\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle \leq& C\|\Lambda^s(\mathbf{b}\cdot\nabla\mathbf{b})\|_{L^2}\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\|_{L^2}\\ \leq& C(\|\mathbf{b}\|_{L^\infty}\|\nabla\mathbf{b}\|_{H^s} + \|\nabla\mathbf{b}\|_{L^\infty}\|\mathbf{b}\|_{H^s})\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\|_{L^2}\\ \leq& C(\|\mathbf{b}\|_{H^2}\|\mathbf{b}\|_{H^{s+1}} + \|\nabla\mathbf{b}\|_{H^2}\|\mathbf{b}\|_{H^s})\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\|_{L^2}\\ \leq&\varepsilon\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\|_{L^2}^2 + C\|\mathbf{b}\|_{H^{s+1}}^2\|\mathbf{b}\|_{H^3}^2\\ \leq&\varepsilon\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\|_{L^2}^2 + C\|\mathbf{b}\|_{H^{r+4}}^2\|\mathbf{b}\|_{H^3}^2\\ \leq&\varepsilon\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\|_{L^2}^2 + C\|\mathbf{b}\|_{H^\gamma}^2\|\mathbf{b}\|_{H^3}^2\\ \leq&\varepsilon\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\|_{L^2}^2 + C\delta^2\|\mathbf{b}\|_{H^3}^2,\end{aligned} \tag{3.13}$$

where we have used assumption (3.2).

Subsequently, to control the first-time derivative term on the right-hand side of (3.10), we make use of the second equation in (1.4) and obtain

$$\begin{aligned}\langle\Lambda^s\partial_t\mathbf{u},\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle =&\frac{d}{dt}\langle\Lambda^s\mathbf{u},\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle - \langle\Lambda^s\mathbf{u},\Lambda^s(\mathbf{n}\cdot\nabla\partial_t\mathbf{b})\rangle\\ =&\frac{d}{dt}\langle\Lambda^s\mathbf{u},\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle + \langle\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u}),\Lambda^s(\mathbf{b}\cdot\nabla\mathbf{u})\rangle\\ &+\langle\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u}),\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u})\rangle - \langle\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u}),\Lambda^s(\mathbf{u}\cdot\nabla\mathbf{b})\rangle\\ =:&\frac{d}{dt}\langle\Lambda^s\mathbf{u},\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle + J_1 + J_2 + J_3.\end{aligned}$$

By Lemma 2.2 and (3.8), we get

$$\begin{aligned}|J_1| &\leq \|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u})\|_{L^2}\|\Lambda^s(\mathbf{b}\cdot\nabla\mathbf{u})\|_{L^2}\\ &\leq C\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u})\|_{L^2}(\|\mathbf{b}\|_{L^\infty}\|\nabla\mathbf{u}\|_{H^s} + \|\mathbf{b}\|_{H^s}\|\nabla\mathbf{u}\|_{L^\infty})\\ &\leq C\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u})\|_{L^2}(\|\mathbf{b}\|_{H^2}\|\nabla\mathbf{u}\|_{H^s} + \|\mathbf{b}\|_{H^\gamma}\|\nabla\mathbf{u}\|_{H^2})\\ &\leq C\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u})\|_{L^2}(\|\mathbf{b}\|_{H^\gamma}\|\nabla\mathbf{u}\|_{H^s} + \|\mathbf{b}\|_{H^\gamma}\|\nabla\mathbf{u}\|_{H^2})\\ &\leq C\delta\|\mathbf{u}\|_{H^{s+1}}(\|\nabla\mathbf{u}\|_{H^s} + \|\nabla\mathbf{u}\|_{H^2})\\ &\leq C\delta\|\mathbf{u}\|_{H^{s+1}}^2 + C\delta\|\mathbf{u}\|_{H^{s+1}}\|\mathbf{u}\|_{H^3},\\ |J_2| &\leq \|\mathbf{n}\cdot\nabla\mathbf{u}\|_{H^s}^2 \leq \|\nabla\mathbf{u}\|_{H^s}^2.\end{aligned}$$

Likewise,

$$|J_3| \leq \|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u})\|_{L^2}\|\Lambda^s(\mathbf{u}\cdot\nabla\mathbf{b})\|_{L^2}$$

$$
\begin{aligned}
&\leq C\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u})\|_{L^2}(\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{b}\|_{H^s}+\|\mathbf{u}\|_{H^s}\|\nabla\mathbf{b}\|_{L^\infty})\\
&\leq C\|\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{u})\|_{L^2}(\|\mathbf{u}\|_{H^2}\|\mathbf{b}\|_{H^\gamma}+\|\mathbf{u}\|_{H^s}\|\mathbf{b}\|_{H^\gamma})\\
&\leq C\delta\|\mathbf{u}\|_{H^{s+1}}(\|\mathbf{u}\|_{H^s}+\|\mathbf{u}\|_{H^2})\\
&\leq C\delta\|\mathbf{u}\|_{H^{s+1}}^2+C\delta\|\mathbf{u}\|_{H^{s+1}}\|\mathbf{u}\|_{H^3}.
\end{aligned}
$$

This shows that

$$
\langle\Lambda^s\partial_t\mathbf{u},\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle\leq\frac{d}{dt}\langle\Lambda^s\mathbf{u},\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle+(1+C\delta)\|\mathbf{u}\|_{H^{s+1}}^2+C\delta\|\mathbf{u}\|_{H^{s+1}}\|\mathbf{u}\|_{H^3}. \tag{3.14}
$$

For any $0\leq s\leq r+3$ *and* $r>1$*, summing up (3.10)–(3.14) and taking* δ *small enough, we get the result of Lemma 3.2.* □

3.4. Proof of Theorem 1.1

There are no dissipative terms in equations **u** and **b** in (1.4). However, in order to demonstrate the intended stability results, we do require these stabilizing effects. We solve this difficulty by combining Lemma 2.3 and $\mathbf{n}\cdot\nabla\mathbf{b}$.

First, taking $\beta=r+5$ in Lemma 3.1,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}(t)\|_{H^{r+5}}^2+\|\mathbf{b}(t)\|_{H^{r+5}}^2)+\nu\|u_2\|_{H^{r+5}}^2\\
&\leq C(\|\nabla\mathbf{u}\|_{L^\infty}+\|\nabla\mathbf{b}\|_{L^\infty})(\|\mathbf{u}(t)\|_{H^{r+5}}^2+\|\mathbf{b}(t)\|_{H^{r+5}}^2).
\end{aligned} \tag{3.15}
$$

Multiplying (3.15) by a suitable large constant λ and then adding it to (3.9), we have

$$
\begin{aligned}
&\frac{d}{dt}\left\{\lambda(\|\mathbf{u}(t)\|_{H^{r+5}}^2+\|\mathbf{b}(t)\|_{H^{r+5}}^2)-\sum_{0\leq s\leq r+3}\langle\Lambda^s\mathbf{u},\Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle\right\}\\
&+\lambda\|u_2\|_{H^{r+5}}^2+\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2\\
&\leq C\lambda\|\nabla\mathbf{u}\|_{L^\infty}\|\mathbf{u}\|_{H^{r+5}}^2+C\lambda\|\nabla\mathbf{u}\|_{L^\infty}\|\mathbf{b}\|_{H^{r+5}}^2+C\lambda\|\nabla\mathbf{b}\|_{L^\infty}\|\mathbf{u}\|_{H^{r+5}}^2\\
&+C\lambda\|\nabla\mathbf{b}\|_{L^\infty}\|\mathbf{b}\|_{H^{r+5}}^2+(2+C\delta)\|\mathbf{u}\|_{H^{r+4}}^2+C\delta^2\|\mathbf{b}\|_{H^3}^2,
\end{aligned} \tag{3.16}
$$

where $\lambda>1$ is a constant to be determined later.

The terms on the right-hand side of (3.16) can be bounded. Next, for any $\gamma\geq 2r+7$, by (3.1), Lemma 2.1, 2.3, and the interpolation inequality, we have

$$
\begin{aligned}
&\|\mathbf{b}\|_{H^3}^2\leq C\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2,\\
&\|\mathbf{b}\|_{H^{r+5}}^2\leq C\|\mathbf{b}\|_{H^3}\|\mathbf{b}\|_{H^\gamma}\leq C\delta\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}},\\
&\|\mathbf{u}\|_{H^{r+4}}^2\leq\|\partial_1\mathbf{u}\|_{H^{r+4}}^2\leq\|u_2\|_{H^{r+5}}^2,\\
&\|\mathbf{u}\|_{H^{r+5}}^2\leq C\|\mathbf{u}\|_{H^3}\|\mathbf{u}\|_{H^\gamma}\leq C\delta\|\mathbf{u}\|_{H^3}\leq C\delta\|\mathbf{u}\|_{H^{r+4}},
\end{aligned} \tag{3.17}
$$

which gives

$$
\begin{aligned}
&\|\mathbf{b}\|_{H^{r+5}}^4\leq C\delta^2\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2,\\
&\|\mathbf{u}\|_{H^{r+5}}^4\leq C\delta^2\|\mathbf{u}\|_{H^{r+4}}^2.
\end{aligned} \tag{3.18}
$$

By Young's inequality, (2.2), (3.1), (3.17), and (3.18), we get

$$\begin{aligned}
C\lambda\|\nabla\mathbf{u}\|_{L^\infty}\|\mathbf{u}\|_{H^{r+5}}^2 \le& C\lambda\|\nabla\mathbf{u}\|_{H^2}\|\mathbf{u}\|_{H^{r+5}}^2\\
\le& \lambda(\varepsilon\|\mathbf{u}\|_{H^3}^2 + C\|\mathbf{u}\|_{H^{r+5}}^4)\\
\le& \lambda\varepsilon\|\mathbf{u}\|_{H^{r+4}}^2 + C\lambda\delta^2\|\mathbf{u}\|_{H^{r+4}}^2\\
\le& \lambda\varepsilon\|\partial_1\mathbf{u}\|_{H^{r+4}}^2 + C\lambda\delta^2\|\partial_1\mathbf{u}\|_{H^{r+4}}^2\\
\le& \lambda\varepsilon\|\nabla u_2\|_{H^{r+4}}^2 + C\lambda\delta^2\|\nabla u_2\|_{H^{r+4}}^2\\
\le& \lambda\varepsilon\|u_2\|_{H^{r+5}}^2 + C\lambda\delta^2\|u_2\|_{H^{r+5}}^2,\\
C\lambda\|\nabla\mathbf{u}\|_{L^\infty}\|\mathbf{b}\|_{H^{r+5}}^2 \le& C\lambda\|\nabla\mathbf{u}\|_{H^2}\|\mathbf{b}\|_{H^{r+5}}^2\\
\le& \lambda\varepsilon\|\mathbf{u}\|_{H^{r+4}}^2 + C\|\mathbf{b}\|_{H^{r+5}}^4\\
\le& \lambda\varepsilon\|\partial_1\mathbf{u}\|_{H^{r+4}}^2 + C\delta^2\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2\\
\le& \lambda\varepsilon\|u_2\|_{H^{r+5}}^2 + C\delta^2\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2.
\end{aligned} \tag{3.19}$$

Similarly,

$$\begin{aligned}
C\lambda\|\nabla\mathbf{b}\|_{L^\infty}\|\mathbf{u}\|_{H^{r+5}}^2 \le& C\lambda\|\mathbf{b}\|_{H^3}\|\mathbf{u}\|_{H^{r+5}}^2\\
\le& \lambda\varepsilon\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2 + C\lambda\|\mathbf{u}\|_{H^{r+5}}^4\\
\le& \lambda\varepsilon\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2 + C\lambda\delta^2\|u_2\|_{H^{r+5}}^2,\\
C\lambda\|\nabla\mathbf{b}\|_{L^\infty}\|\mathbf{b}\|_{H^{r+5}}^2 \le& C\lambda\|\mathbf{b}\|_{H^3}\|\mathbf{b}\|_{H^{r+5}}^2\\
\le& C\lambda\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}(C\delta\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}})\\
\le& C\lambda\delta\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2.
\end{aligned} \tag{3.20}$$

Hence, inserting (3.17)–(3.20) in (3.16), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{\lambda(\|\mathbf{u}(t)\|_{H^{r+5}}^2 + \|\mathbf{b}(t)\|_{H^{r+5}}^2) - \sum_{0\le s\le r+3}\langle\Lambda^s\mathbf{u}, \Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle\right\}\\
&\quad + \lambda\|u_2\|_{H^{r+5}}^2 + \|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2\\
&\le (C\lambda\delta^2 + 2 + C\delta + 2\lambda\varepsilon)\|u_2\|_{H^{r+5}}^2 + (C\lambda\delta + C\delta^2 + \lambda\varepsilon)\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2.
\end{aligned} \tag{3.21}$$

Taking $\delta, \varepsilon > 0$ small enough, we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\lambda(\|\mathbf{u}(t)\|_{H^{r+5}}^2 + \|\mathbf{b}(t)\|_{H^{r+5}}^2) - \sum_{0\le s\le r+3}\langle\Lambda^s\mathbf{u}, \Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle\right\}\\
&\quad + \frac{\lambda}{2}\|u_2\|_{H^{r+5}}^2 + \frac{1}{2}\|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2\\
&\le 0.
\end{aligned} \tag{3.22}$$

Define

$$\begin{aligned}
E(t) &= \lambda(\|\mathbf{u}(t)\|_{H^{r+5}}^2 + \|\mathbf{b}(t)\|_{H^{r+5}}^2) - \sum_{0\le s\le r+3}\langle\Lambda^s\mathbf{u}, \Lambda^s(\mathbf{n}\cdot\nabla\mathbf{b})\rangle,\\
D(t) &= \lambda\|u_2\|_{H^{r+5}}^2 + \|\mathbf{n}\cdot\nabla\mathbf{b}\|_{H^{r+3}}^2,
\end{aligned}$$

then (3.22) becomes

$$\frac{d}{dt}E(t) + \frac{1}{2}D(t) \leq 0. \tag{3.23}$$

We take $\lambda > 1$ so that

$$E(t) \geq (\|\mathbf{u}(t)\|_{H^{r+5}}^2 + \|\mathbf{b}(t)\|_{H^{r+5}}^2).$$

For any $\gamma \geq 4r + 11$, we use interpolation inequality

$$\|\mathbf{b}\|_{H^{r+5}}^2 \leq \|\mathbf{b}\|_{H^3}^{\frac{3}{2}} \|\mathbf{b}\|_{H^\gamma}^{\frac{1}{2}} \leq C\delta^{\frac{1}{2}} \|\mathbf{n} \cdot \nabla \mathbf{b}\|_{H^{r+3}}^{\frac{3}{2}},$$

which implies

$$\begin{aligned}
E(t) &\leq C(\|\mathbf{u}\|_{H^{r+5}}^2 + \|\mathbf{b}\|_{H^{r+5}}^2) \\
&\leq C\|\mathbf{u}\|_{H^3}^{\frac{3}{2}} \|\mathbf{u}\|_{H^\gamma}^{\frac{1}{2}} + \|\mathbf{b}\|_{H^3}^{\frac{3}{2}} \|\mathbf{b}\|_{H^\gamma}^{\frac{1}{2}} \\
&\leq C\delta^{\frac{1}{2}} \|\mathbf{n} \cdot \nabla \mathbf{u}\|_{H^{r+3}}^{\frac{3}{2}} + C\delta^{\frac{1}{2}} \|\mathbf{n} \cdot \nabla \mathbf{b}\|_{H^{r+3}}^{\frac{3}{2}} \\
&\leq C\delta^{\frac{1}{2}} \|\mathbf{u}\|_{H^{r+4}}^{\frac{3}{2}} + C\delta^{\frac{1}{2}} \|\mathbf{n} \cdot \nabla \mathbf{b}\|_{H^{r+3}}^{\frac{3}{2}} \\
&\leq C\delta^{\frac{1}{2}} \|\partial_1 \mathbf{u}\|_{H^{r+4}}^{\frac{3}{2}} + C\delta^{\frac{1}{2}} \|\mathbf{n} \cdot \nabla \mathbf{b}\|_{H^{r+3}}^{\frac{3}{2}} \\
&\leq C\delta^{\frac{1}{2}} \|u_2\|_{H^{r+5}}^{\frac{3}{2}} + C\delta^{\frac{1}{2}} \|\mathbf{n} \cdot \nabla \mathbf{b}\|_{H^{r+3}}^{\frac{3}{2}} \\
&\leq C(D(t))^{\frac{3}{4}},
\end{aligned}$$

i.e., $D(t) \geq C(E(t))^{\frac{4}{3}}$ for some $C > 0$. Thus, inserting this inequality in (3.23), we get

$$\frac{d}{dt}E(t) + c(E(t))^{\frac{4}{3}} \leq 0.$$

Integrating this inequality gives

$$E(t) \leq C(1 + t)^{-3}. \tag{3.24}$$

Thus, by (3.24), we can demonstrate the decay rate

$$\|\mathbf{u}(t)\|_{H^{r+5}} + \|\mathbf{b}(t)\|_{H^{r+5}} \leq C(1 + t)^{-\frac{3}{2}},$$

for any $t \in [0, +\infty)$.

Next, for any $\alpha > r + 5$, choosing $\gamma > \alpha$, we have the following interpolation inequality

$$\|f(t)\|_{H^\alpha} \leq \|f(t)\|_{H^{r+5}}^{\frac{\gamma-\alpha}{\gamma-r-5}} \|f(t)\|_{H^\gamma}^{\frac{\alpha-r-5}{\gamma-r-5}}. \tag{3.25}$$

In the end, in order to get the stability of $\mathbf{u}$ and $\mathbf{b}$ in the desired Sobolev space, we need to sacrifice some decay of $\mathbf{u}$ and $\mathbf{b}$ in much higher space. Inserting (3.25) in (3.24), we obtain

$$\|\mathbf{u}(t)\|_{H^\alpha} + \|\mathbf{b}(t)\|_{H^\alpha} \leq C(1 + t)^{-\frac{3(\gamma-\alpha)}{2(\gamma-r-5)}}.$$

Then using Sobolev's inequalities and Lemma 3.1 with $\beta = \gamma$, for any $\gamma \geq 4r + 11$, we obtain

$$\frac{d}{dt}(\|\mathbf{u}(t)\|_{H^\gamma}^2 + \|\mathbf{b}(t)\|_{H^\gamma}^2) \leq C(\|\mathbf{u}\|_{H^3} + \|\mathbf{b}\|_{H^3})(\|\mathbf{u}(t)\|_{H^\gamma}^2 + \|\mathbf{b}(t)\|_{H^\gamma}^2).$$

Obviously, (3.24) implies the decay upper bound

$$\int_0^t (\|\mathbf{u}(\tau)\|_{H^3} + \|\mathbf{b}(\tau)\|_{H^3})\, d\tau \leq C.$$

By Gronwall inequality, we have

$$\begin{aligned}
\|\mathbf{u}\|_{H^\gamma}^2 + \|\mathbf{b}\|_{H^\gamma}^2 \leq & C(\|\mathbf{u}_0\|_{H^\gamma}^2 + \|\mathbf{b}_0\|_{H^\gamma}^2) \exp(\int_0^t (\|\mathbf{u}(\tau)\|_{H^3} + \|\mathbf{b}(\tau)\|_{H^3})\, d\tau) \\
\leq & C(\|\mathbf{u}_0\|_{H^\gamma}^2 + \|\mathbf{b}_0\|_{H^\gamma}^2) \\
\leq & C\varepsilon^2.
\end{aligned}$$

Taking ε small enough so that $\sqrt{C}\varepsilon \leq \frac{\delta}{2}$, we have

$$\|\mathbf{u}\|_{H^\gamma} + \|\mathbf{b}\|_{H^\gamma} \leq \frac{\delta}{2}.$$

Consequently, the bootstrapping argument suggests that the local solution can eventually be expanded to a global one.

This completes the proof of Theorem 1.1. □

4. Conclusions

Based on the above proof process of the theorem, we can obtain that there has a global solution $(\mathbf{u}, \mathbf{b}) \in C([0, +\infty); H^\gamma)$ to system (1.4) satisfying

$$\begin{aligned}
&\|\mathbf{u}(t)\|_{H^\gamma} + \|\mathbf{b}(t)\|_{H^\gamma} \leq C\varepsilon, \\
&\|\mathbf{u}(t)\|_{H^{r+5}} + \|\mathbf{b}(t)\|_{H^{r+5}} \leq C(1 + t)^{-\frac{3}{2}}.
\end{aligned}$$

Moreover, for any $t \geq 0$ and $r + 5 < \alpha < \gamma$, there holds

$$\|\mathbf{u}(t)\|_{H^\alpha} + \|\mathbf{b}(t)\|_{H^\alpha} \leq C(1 + t)^{-\frac{3(\gamma-\alpha)}{2(\gamma-r-5)}}.$$

Author contributions

Long: Study conception, design, methodology and write the first draft of the manuscript. Ye: Material preparation, data collection and analysis. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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