Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(12): 36351–36370.
DOI: 10.3934/math.20241724
Received: 30 September 2024
Revised: 02 December 2024
Accepted: 10 December 2024
Published: 31 December 2024

Research article

On the cooling number of the generalized Petersen graphs

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Abstract: Recently, Bonato et al. (2024) introduced a new graph parameter, which is the cooling number of a graph G, denoted as $\mathrm{CL}(G)$. In contrast with burning which seeks to minimize the number of rounds to burn all vertices in a graph, cooling seeks to maximize the number of rounds to cool all vertices in the graph. Cooling number is the compelling counterpart to the well-studied burning number, offering a new perspective on dynamic processes within graphs. In this paper, we showed that the cooling number of a classic cubic graph, the generalized Petersen graphs $P(n,k)$, is $\left\lfloor \frac{n}{2k} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor + O(1)$ by the use of vertex-transitivity and combinatorial arguments. Particularly, we determined the exact values for $\mathrm{CL}(P(n,1))$ and $\mathrm{CL}(P(n,2))$.

Keywords: cooling number; generalized Petersen graphs
Mathematics Subject Classification: 05C85, 68R10, 91D30

1. Introduction

Burning number can be used to study the spread of contagion in a network, see [1,3,4]. In the past few years, burning number has been widely studied for some families of graphs, for example, circulant graphs [7], generalized Petersen graphs [19], caterpillars [9,14], grid and interval graphs [8], t-unicyclic graphs [22], hypercube [18], theta graphs [16], Q-graph [13], and spiders and path forests [5,6,15]. Mitsche, Prałat, and Roshanbin investigated the burning number of graph products in [18] and they also focused on the probabilistic aspects of the burning number in [17]. A graph G is said to be m-burnable if the burning number of G is at most m. Bonato and Lidbetter [5] and Das et al. [6] proved that every spider of order m^2 is m-burnable. A tight upper bound on the order of a spider to guarantee that it is m-burnable was then determined by Tan and Teh [20]. They have also studied the burnability of double spiders and path forests in [21].

Recently, Bonato et al. [2] introduced a new graph parameter, which is the cooling number. In contrast with burning that aims to minimize the number of rounds to burn all vertices in a graph, cooling attempts to maximize the number of rounds to cool all vertices in the graph. The cooling process on a finite simple, undirected graph G is a discrete-time process. Throughout the cooling process, vertices in G may be either uncooled or cooled.

Initially, in the first round, all vertices are uncooled. At each round $t \geq 1$, one new uncooled vertex is chosen to cool if such a node is available. We call such a chosen vertex a source. If a vertex is cooled, then it remains in that state until the end of the process. Once a vertex is cooled on round t, in round $t+1$, its uncooled neighbors become cooled. The source chosen in round $t+1$ cannot be the immediate neighbor of cooled vertex in round t. This means that distance between two consecutive sources must be of distance at least 2. If at the beginning of round $t+1$, all uncooled vertices are immediate neighbor (or neighbor) of a cooled vertex in round t, then no source is chosen in round $t+1$ and the process ends at round $t+1$. The process ends in a given round when all vertices of G are cooled.

We define the cooling number of G, written $\mathrm{CL}(G)$, to be the maximum number of rounds for the cooling process to end. Give a graph with $\mathrm{CL}(G) = c$ and let $(x_1, x_2, \dots)$ be a set of sources chosen during cooling. To completely cool G, it may have c cooling sources in a sequence $(x_1, x_2, \dots, x_c)$ or $c-1$ cooling sources, which we write, $(x_1, x_2, \dots, x_{c-1}, [x_c])$. We call $(x_1, x_2, \dots, x_c)$ or $(x_1, x_2, \dots, x_{c-1}, [x_c])$ the cooling sequence for G with $\mathrm{CL}(G) = c$. The latter implies that the whole graph may only be completely cooled in c rounds without positioning the c-th cooling source. We have that the burning number of G, $b(G) \leq \mathrm{CL}(G)$. Note that a choice of sources that burns the graphs gives an upper bound to the burning number, and a choice of sources that cools the graph gives a lower bound to the cooling number.

In [2], Bonato et al. showed the following results on the cooling number of some basic graphs. For all graphs with diameter at most 2, we have that the burning number of G, $b(G) = \mathrm{CL}(G)$.

Theorem 1.1. *[2] For a graph G on n vertices, we have that*

$$\mathrm{CL}(G) \leq \left\lceil \frac{n+1}{2} \right\rceil.$$

Let $u, v \in V(G)$, and $d(u, v)$ is the distance between u and v. Then the diameter of G, denoted by $\mathrm{diam}(G)$, is the $\max\{d(v, u) : u, v \in V(G)\}$.

Theorem 1.2. *[2] For a graph G, we have that*

$$\left\lceil \frac{\mathrm{diam}(G)+2}{2} \right\rceil \leq \mathrm{CL}(G) \leq \mathrm{diam}(G) + 1. \tag{1.1}$$

For the path P_n and cycle C_n of order n, it has been shown in [2] that

Theorem 1.3. *[2]*

$$\mathrm{CL}(P_n) = \left\lceil \frac{n+1}{2} \right\rceil = \left\lceil \frac{\mathrm{diam}(P_n)+2}{2} \right\rceil.$$

Theorem 1.4. *[2]*

$$\mathrm{CL}(C_n) = \left\lceil \frac{n+2}{3} \right\rceil.$$

For the complete caterpillar of length d and $n = 2d - 2$, denoted as CC_d, (see [2]), then

$$\mathrm{CL}(CC_d) = \left\lceil \frac{n+1}{2} \right\rceil = \mathrm{diam}(P_n) + 1.$$

Theorem 1.5. *[2] Let T be a spider with $2m$ legs, each of length r. If we have that $m < \lceil \log_2 r + 1 \rceil$, then*

$$\mathrm{CL}(T) \leq 2 \sum_{1 \leq i \leq m} \left\lfloor \frac{r+1}{2^i} \right\rfloor (1 - 1/2^m)2r.$$

They also derived the cooling number of Cartesian grids and studied the cooling number in graphs generated by the iterated local transitivity model for social networks, see [2].

In this paper, we determine completely the cooling number for the generalized Petersen graphs. The generalized Petersen graphs is a classic family of cubic graphs. The Petersen graph is not only the smallest bridgeless 3-regular graph, but it is also a remarkable configuration that serves as a counterexample to many optimistic predictions about what might be true for graphs in general [11]. Let $n \geq 3$ and $k \geq 1$ be integers such that $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. The generalized Petersen graph $P(n, k)$ is defined to be the graph on $2n$ vertices with vertex set

$$V(P(n, k)) = \{a_i, b_i \ : \ i = 0, 1, 2, \ldots, n-1\}$$

and edge set

$$E(P(n, k)) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} \ : \ i = 0, 1, 2, \ldots, n-1\},$$

where subscripts are taken modulo n. Let $D_1 = \{a_i \ : \ i = 0, 1, 2, \ldots, n-1\}$ and $D_2 = \{b_i \ : \ i = 0, 1, 2, \ldots, n-1\}$. The subgraph induced by D_1 is called the outer rim while the subgraph induced by D_2 is called the inner rim. A spoke of $P(n, k)$ is an edge of the form $a_i b_i$ for some $0 \leq i \leq n-1$.

For a graph G, for each integer $j \geq 0$, we set

$$N_j(x) = \{y \in V(G) \ : \ \text{the distance from } y \text{ to } x \text{ in } G \text{ is at most } j\},$$

and

$$N_j[x] = N_j(x) \cup \{x\}.$$

Note that if $N(x)$ is the set of vertices adjacent to x in G, then

$$N(x) = \{y \in V(G) \ : \ y \text{ is adjacent to } x \text{ in } G\} = N_1(x).$$

Furthermore, $N_0(x) = \emptyset$ and $N_0[x] = \{x\}$.

2. Main results

It is worth noting that the main difference when choosing a choice of sources to completely burn or cool a graph is such that we need to carefully choose a cooling source compared to a burning source because the cooling source cannot be a neighbor of a cooled vertex in that particular round, whereas the burning source can be a neighbor of a burned vertex in that round. In the burning number case, where we aim to minimize the number of sources, we just need to make sure a particular vertex that we want is unburned and the choice of vertex will maximize the newly burned vertices at the end of each round. However, in the cooling process, where we aim to maximize the number of sources, we need to carefully choose the cooling sources so that the chosen vertex will minimize the newly cooled vertices at the end of each round. Furthermore, to recall, two consecutive cooling sources cannot be immediate neighbors.

In general, to obtain the lower bound, we can choose a choice of cooling sources in a sequence such that the $(t+1)$-th cooling source is chosen, at best, at distance 2 from the t-th cooling source. Besides that, the $(t+1)$-th cooling source, x_{t+1} is best to be chosen with as many neighbors as possible that are contained in $N_1(x_t)$, i.e.,

$$|N_1(x_{t+1}) \cap N_1(x_t)| = \max\{|N_1(y) \cap N_1(x_t)| : \ y \ \text{ is uncooled at round } t+1 \\ \text{and it is not a neighbour of a cooled vertex}\}.$$

This will reduce the cooling spread to uncooled vertices in the $(t+1)$-th round.

2.1. Case for $k = 1$

Note that it is fairly straightforward to see that the diameter of $P(n,1)$ is $\frac{n}{2}+1$ if n is even and $\left\lceil \frac{n}{2} \right\rceil$ if n is odd.

Theorem 2.1. *If $n \geq 3$, then* $\mathrm{CL}(P(n,1)) = \left\lfloor \frac{2n+4}{5} \right\rfloor + 1$.

Proof. Let $G = P(n,1)$ with $\mathrm{CL}(G) = k$ and $(x_1, x_2, \ldots, x_{k-1}, [x_k])$, or $(x_1, x_2, \ldots, x_k)$ be a cooling sequence of G.

First, we show that $k \leq \left\lfloor \frac{2n+4}{5} \right\rfloor + 1$. Suppose $k > \frac{2n+9}{5}$. Note that G is vertex-transitive. The first cooling source x_1 can be placed at an arbitrary vertex, and we consider the spread of cooling from x_1 in k rounds. At the end of the first round, only the vertex x_1 is cooled. At the end of the second round, $N_1(x_1)$ which consists of 3 vertices are cooled. Regardless where vertex x_2 is positioned, a total of 5 vertices (which are the vertices in $N_1[x_1] \cup \{x_2\}$) are cooled at the end of the second round. After two rounds, it is observed that at the end of each i-th round for all $3 \leq i \leq k-1$, at least 5 new uncooled vertices are cooled. In fact, at the beginning of the i-th round, the cooled vertices are just the vertices contained in

$$\bigcup_{1 \leq j \leq i-1} N_{i-1-j}[x_j].$$

Recall that x_i cannot be a neighbor of a cooled vertex. Therefore,

$$x_i \notin \bigcup_{1 \leq j \leq i-1} N_{i-j}[x_j].$$

This implies that

$$N_1[x_i] \cap \left(\bigcup_{1 \leq j \leq i-1} N_{i-1-j}[x_j] \right) = \varnothing.$$

Since $i \leq k-1$, there is still another round before all the vertices are cooled. So, there are four uncooled vertices excluding x_i, where two of them are contained in the inner rim and the other two in the outer rim. Thus, at the end of the i-th round, at least 5 new uncooled vertices are cooled.

At the k-th round, at least one uncooled vertex is cooled. Following this, we have that the number of cooled vertices is at least

$$5 + 5(k-3) + 1 = 5k - 9 > 5\left(\frac{2n+9}{5}\right) - 9 = 2n,$$

a contradiction, as $|V(P(n,1))| = 2n$. Thus, $k \leq \frac{2n+9}{5}$. Since k is an integer, we have $k \leq \left\lfloor \frac{2n+9}{5} \right\rfloor = \left\lfloor \frac{2n+4}{5} \right\rfloor + 1$.

For the lower bound, we give a choice of sources $(x_1, x_2, \ldots, x_c, x_{c+1})$ or $(x_1, x_2, \ldots, x_c, [x_{c+1}])$ where $c = \left\lfloor \frac{2n+4}{5} \right\rfloor$ to cool G. Note that all vertices will be cooled at $c+1$ round and x_{c+1} may not need to be positioned in some cases. Without loss of generality, we first place x_1 at a_1 and x_2 at b_2. At the end of the second round, a_0, a_1, a_2, b_1, b_2 are cooled.

Suppose $\left\lfloor \frac{2n+4}{5} \right\rfloor$ is odd. Let $S_{2j+1} = \{a_{3j}, b_{3j}, a_{3j+1}, b_{3j+1}, a_{3j+2}, b_{3j+2}\}$ for each $1 \leq j \leq \frac{1}{2}\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 1\right)$. We set

$$x_{2j+1} = b_{3j+1};$$
$$x_{2j+2} = a_{3j+2},$$

for $1 \leq j \leq \frac{1}{2}\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 1\right) - 1$ and

$$x_{2j_0+1} = b_{3j_0+1},$$

where $j_0 = \frac{1}{2}\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 1\right)$.

At the end of the third round, a_3 and b_3 are cooled as they are adjacent to a_2 and b_2, respectively. Similarly, for each j, by the choices of x_{2j+1} and x_{2j+2}, all the vertices in S_{2j+1} are cooled. In fact, a_{3j} and b_{3j} are cooled from $a_{3j-1} \in S_{2j-3}$ and $b_{3j-1} \in S_{2j-3}$, respectively; a_{3j+1} and b_{3j+2} are cooled from b_{3j+1}.

Now, at the end of the $(2j_0+1)$-th rounds, all the following vertices are cooled:

$$a_1, a_2, a_3, \ldots, a_{3j_0};$$
$$b_1, b_2, b_3, \ldots, b_{3j_0}, b_{3j_0+1}.$$

Besides that, sourced from $x_1 = a_1$, moving toward the opposite direction, the vertices in $\{a_0, a_{n-1}, a_{n-2}, \ldots, a_{n-(c-2)}, b_0, b_{n-1}, b_{n-2}, \ldots, b_{n-(c-3)}\}$ are cooled at the end of the c-th round.

Since $\left\lfloor \frac{2n+4}{5} \right\rfloor$ is odd, $3j_0 + 1 = 1.5\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 1\right) + 1$ is an integer. We shall show that by taking $c = \left\lfloor \frac{2n+4}{5} \right\rfloor$,

$$1 \leq \left(n - (c-3) - 1.5\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 1\right) - 1\right) - 1 \leq 2. \tag{2.1}$$

In fact, the lower bound follows from

$$\left(n - (c-3) - 1.5\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 1\right) - 1\right) - 1$$

$$
\begin{aligned}
&= n - \left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 3 \right) - 1.5 \left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 1 \right) - 2 \\
&= n - 2.5 \left\lfloor \frac{2n+4}{5} \right\rfloor + 2.5 \\
&\geq n - 2.5 \left(\frac{2n+4}{5} \right) + 2.5 = 0.5,
\end{aligned}
$$

and the fact that $\left(n - (c-3) - 1.5\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 1\right) - 1\right) - 1$ is an integer, whereas the upper bound follows from

$$
\begin{aligned}
\left(n - (c-3) - 1.5\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 1\right) - 1\right) - 1 &= n - 2.5 \left\lfloor \frac{2n+4}{5} \right\rfloor + 2.5 \\
&< n - 2.5 \left(\frac{2n+4}{5} - 1 \right) + 2.5 = 3.
\end{aligned}
$$

So, we conclude that the number of uncooled vertices in the inner rim is at most 2 at the end of the $(2j_0+1)$-th round. Therefore, there are two possibilities at the end of the $(2j_0+1)$-th round,

(a) all vertices are cooled except

$$
\begin{array}{ll}
a_{3j_0+1}, & \\
 & b_{3j_0+2},
\end{array}
$$

in this case, there is exactly one vertex b_{3j_0+2} between b_{3j_0+1} and $b_{n-(c-3)}$.

(b) all vertices are cooled except

$$
\begin{array}{lll}
a_{3j_0+1}, & a_{3j_0+2}, & \\
 & b_{3j_0+2}, & b_{3j_0+3},
\end{array}
$$

in this case, there are exactly two vertices b_{3j_0+2}, b_{3j_0+3} between b_{3j_0+1} and $b_{n-(c-3)}$.

In either case, all these uncooled vertices are adjacent to some cooled vertices. So, no x_{c+1} can be placed and we have the cooling sequence $(x_1, x_2, \ldots, x_c, [x_{c+1}])$. Hence, $k \geq \left\lfloor \frac{2n+4}{5} \right\rfloor + 1$.

Suppose $\left\lfloor \frac{2n+4}{5} \right\rfloor$ is even. Let $S_{2j+1} = \{a_{3j}, b_{3j}, a_{3j+1}, b_{3j+1}, a_{3j+2}, b_{3j+2}\}$ for each $1 \leq j \leq \frac{1}{2}\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 2\right)$. We set

$$
\begin{aligned}
x_{2j+1} &= b_{3j+1}; \\
x_{2j+2} &= a_{3j+2},
\end{aligned}
$$

for $1 \leq j \leq \frac{1}{2}\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 2\right)$.

At the end of the third round, a_3 and b_3 are cooled as they are adjacent to a_2 and b_2, respectively. Similarly, for each j, by the choices of x_{2j+1} and x_{2j+2}, all the vertices in S_{2j+1} are cooled. In fact, a_{3j} and b_{3j} are cooled from $a_{3j-1} \in S_{2j-3}$ and $b_{3j-1} \in S_{2j-3}$, respectively; a_{3j+1} and b_{3j+2} are cooled from b_{3j+1}.

Now, at the end of the $(2j_0+2)$-th rounds where $j_0 = \frac{1}{2}\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 2\right)$, all the following vertices are cooled:

$$
a_1, a_2, a_3, \ldots, a_{3j_0}, a_{3j_0+1}, a_{3j_0+2};
$$

$$b_1, b_2, b_3, \ldots, b_{3j_0}, b_{3j_0+1}, b_{3j_0+2}.$$

Besides that, sourced from $x_1 = a_1$, moving toward the opposite direction, the vertices in $\{a_0, a_{n-1}, a_{n-2}, \ldots, a_{n-(c-2)}, b_0, b_{n-1}, b_{n-2}, \ldots, b_{n-(c-3)}\}$ are cooled at the end of the c-th round.

Since $\left\lfloor \frac{2n+4}{5} \right\rfloor$ is even, $3j_0 + 2 = 3\left(\frac{1}{2}\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 2\right)\right) + 2 = 1.5\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 2\right) + 2$ is an integer. We shall show that by taking $c = \left\lfloor \frac{2n+4}{5} \right\rfloor$,

$$0 \leq \left(n - (c-2) - 1.5\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 2\right) - 2\right) - 1 \leq 2. \tag{2.2}$$

In fact, the lower bound follows from

$$\begin{aligned}
&\left(n - (c-2) - 1.5\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 2\right) - 2\right) - 1 \\
=&n - \left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 2\right) - 1.5\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 2\right) - 3 \\
=&n - 2.5\left\lfloor \frac{2n+4}{5} \right\rfloor + 2 \\
\geq&n - 2.5\left(\frac{2n+4}{5}\right) + 2 = 0,
\end{aligned}$$

whereas the upper bound follows from

$$\begin{aligned}
\left(n - (c-2) - 1.5\left(\left\lfloor \frac{2n+4}{5} \right\rfloor - 2\right) - 2\right) - 1 &= n - 2.5\left\lfloor \frac{2n+4}{5} \right\rfloor + 2 \\
&< n - 2.5\left(\frac{2n+4}{5} - 1\right) + 2 = 2.5.
\end{aligned}$$

So, we conclude that the number of uncooled vertices in the outer rim is at most 2 at the end of the $(2j_0 + 2)$-th round. Therefore, there are three possibilities at the end of the $(2j_0 + 2)$-th round,

(a) all vertices are cooled except

$$b_{3j_0+3},$$

in this case, there are no vertices between a_{3j_0+2} and $a_{n-(c-2)}$.

(b) all vertices are cooled except

$$\begin{array}{ll} a_{3j_0+3}, & \\ b_{3j_0+3}, & b_{3j_0+4}, \end{array}$$

in this case, there are exactly one vertex a_{3j_0+3} between a_{3j_0+2} and $a_{n-(c-2)}$.

(c) all vertices are cooled except

$$\begin{array}{lll} a_{3j_0+3}, & a_{3j_0+4}, & \\ b_{3j_0+3}, & b_{3j_0+4}, & b_{3j_0+5}, \end{array}$$

in this case, there are exactly two vertices a_{3j_0+3}, a_{3j_0+4} between a_{3j_0+2} and $a_{n-(c-2)}$.

For cases (a) and (b), all these uncooled vertices are adjacent to some cooled vertices. So, no x_{c+1} can be placed and we have the cooling sequence $(x_1, x_2, \ldots, x_c, [x_{c+1}])$. Hence, $k \geq c + 1 = \left\lfloor \frac{2n+4}{5} \right\rfloor + 1$. For case (c), we can only set $x_{c+1} = b_{3j_0+4}$ because all other uncooled vertices are adjacent to some cooled vertices. So, we have the cooling sequence $(x_1, x_2, \ldots, x_c, x_{c+1})$. Hence, $k \geq c + 1 = \left\lfloor \frac{2n+4}{5} \right\rfloor + 1$.

This completes the proof of the theorem. □

2.2. Case for $k = 2$

Proposition 2.1. *Let $3 \leq n \leq 5$. Then $CL(P(n, 2)) = 3$.*

Proof. Note that $\text{diam}(P(3, 2)) = \text{diam}(P(4, 2)) = \text{diam}(P(5, 2)) = 2$, and, hence, by Eq (1.1), $\text{CL}(P(3, 2)) \leq 3$, $\text{CL}(P(4, 2)) \leq 3$ and $\text{CL}(P(5, 2)) \leq 3$.

For $n = 3, 4, 5$ the proposition can be verified easily by having a choice of cooling sources, see Table 1. Recall that the cooling sequence of a $P(n, k)$ with c sources may only completely cooled the graph in $(c + 1)$ rounds and, hence, $\text{CL}(P(n, k)) = c + 1$.

Table 1. Cooling number of $P(n, 2)$ for $n = 3, 4, 5$.

Cooling sequence	Graph	CL($P(n, 2)$)
$(a_1, b_2, [x_3])$	$P(3, 2) = P(3, 1)$	3
(b_1, a_0, b_2)	$P(4, 2)$	3
$(a_1, b_3, [x_3])$	$P(5, 2)$	3

This completes the proof. □

For the rest of the section, we used a similar isomorphic graph of $P(n, 2)$, say $H(n, 2)$, as defined in [19] (see Figures 1 and 2).

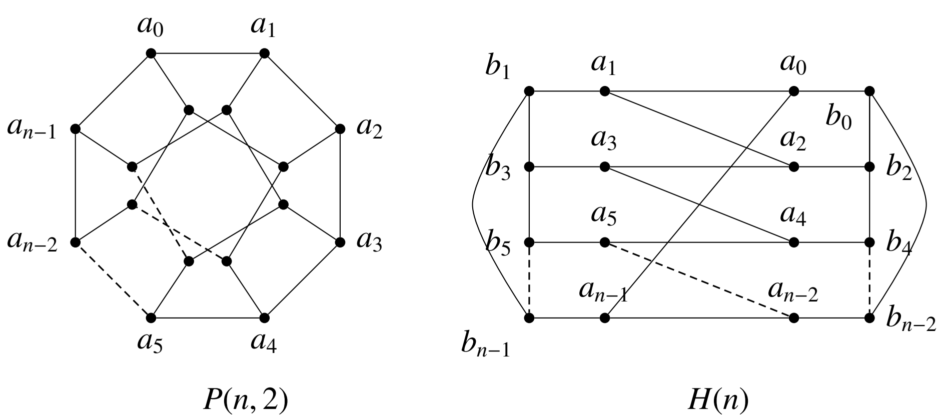

Figure 1. $H(n)$ is isomorphic to $P(n, 2)$ where n is even.

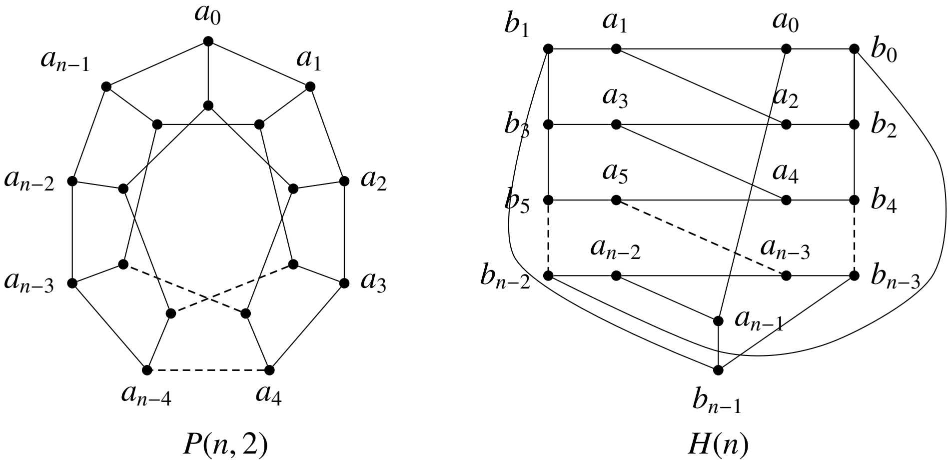

Figure 2. $H(n)$ is isomorphic to $P(n,2)$ where n is odd.

It has been shown in [10] and [12] that diameter of $P(n,2)$ is $O\left(\frac{n}{4}\right)$. Here, we show the exact value. Let $d(x,y)$ be the distance between two vertices x and y.

Lemma 2.1. *Let $n \geq 6$. Then the diameter of $P(n,2)$ is $\left\lceil \frac{n}{4} \right\rceil + 2$ if n is even and $\left\lceil \frac{n-1}{4} \right\rceil + 2$ if n is odd.*

Proof. Suppose n is even and $\frac{n}{2}$ is even. By referring to $H(n)$, if we consider only vertices of the outer rim, one of the maximum distances is $d(a_0, a_{\frac{n}{2}}) = 1 + \frac{n}{4} + 1 = \frac{n}{4} + 2$ by first going to b_0 and going to in steps of 2 to $b_{\frac{n}{2}}$ and then going to $a_{\frac{n}{2}}$. Note that $d(a_0, a_{\frac{n}{2}}) = d(a_0, a_{\frac{n}{2}-1}) = d(a_0, a_{\frac{n}{2}+1}) = \frac{n}{4} + 2$. If we consider one vertex from the outer rim, say a_0 with another vertex b_i, one of the maximum distances is $d(a_0, b_{\frac{n}{2}+1}) = 2 + \frac{n}{4}$ by first going through the path a_1, then b_1, and going to in steps of 2 in the inner rim to $b_{\frac{n}{2}+1}$. If we consider only any two vertices of the two cycles in the inner rim, one of the maximum distances is $d(b_0, b_{\frac{n}{2}-1}) = 2 + \frac{n}{4}$ by first going through path $b_0a_0a_1b_1$ and then going to in steps of 2 in the inner rim to $b_{\frac{n}{2}-1}$.

Suppose n is even and $\frac{n}{2}$ is odd. By referring to $H(n)$, if we consider only vertices of the outer rim, one of the maximum distances is $d(a_0, a_{\frac{n}{2}}) = 1 + \left\lfloor \frac{n}{4} \right\rfloor + 2 = 1 + \frac{n-2}{4} + 2 = \frac{n+2}{4} + 2 = \left\lceil \frac{n}{4} \right\rceil + 2$ by first going to a_1, then b_1, and going to in steps of 2 to $b_{\frac{n}{2}}$ and then going to $a_{\frac{n}{2}}$. If we consider one vertex from the outer rim, say a_0 with another vertex b_i, one of the maximum distances is $d(a_0, b_{\frac{n}{2}}) = 2 + \left\lfloor \frac{n}{4} \right\rfloor = \left\lceil \frac{n}{4} \right\rceil + 1$ by first going to a_1 followed by b_1 and going to in steps of 2 in the inner rim to $b_{\frac{n}{2}}$. If we consider only any two vertices of the two cycles in the inner rim, one of the maximum distances is $d(b_0, b_{\frac{n}{2}}) = 2 + \frac{n+2}{4} = \left\lceil \frac{n}{4} \right\rceil + 2$ by first going through path $b_0a_0a_1b_1$ and then going to in steps of 2 in the inner rim to $b_{\frac{n}{2}}$.

Suppose n is odd and $\frac{n-1}{2}$ is even. By referring to $H(n)$, if we consider only vertices of the outer rim, one of the maximum distances is $d(a_0, a_{\frac{n-1}{2}}) = 1 + \frac{n-1}{4} + 1 = \frac{n-1}{4} + 2$ by first going to b_0 and going to in steps of 2 to $b_{\frac{n-1}{2}}$ and then going to $a_{\frac{n-1}{2}}$. Note that $d(a_0, a_{\frac{n-1}{2}}) = d(a_0, a_{\frac{n-1}{2}-1}) = d(a_0, a_{\frac{n-1}{2}+1}) = \frac{n-1}{4} + 2$. If we consider one vertex from outer rim, say a_0 with another vertex b_i, one of the maximum distances is $d(a_0, b_{\frac{n-1}{2}}) = 1 + \frac{n-1}{4}$ by first going through the path a_0, then b_0, and going to in steps of 2 in the inner rim to $b_{\frac{n-1}{2}}$. If we consider only any two vertices of the inner rim, one of the maximum distances is $d(b_0, b_{\frac{n-1}{2}-1}) = 2 + \frac{n-1}{4}$ by first going through path $b_0a_0a_1b_1$ and then going to in steps of 2 in the inner rim to $b_{\frac{n-1}{2}-1}$.

Suppose n is odd and $\frac{n-1}{2}$ is odd. By referring to $H(n)$, if we consider only vertices of the outer rim, one of the maximum distances is $d(a_0, a_{\frac{n-1}{2}}) = 1 + \left\lceil \frac{n-1}{4} \right\rceil + 1 = \left\lceil \frac{n}{4} \right\rceil + 2$ by first going to a_1, then b_1, and going to in steps of 2 to $b_{\frac{n-1}{2}}$ and then going to $a_{\frac{n-1}{2}}$. If we consider one vertex from the outer rim, say a_0 with another vertex b_i, one of the maximum distances is $d(a_0, b_{\frac{n-1}{2}}) = 1 + \left\lceil \frac{n-1}{4} \right\rceil$ by first going to a_1 followed by b_1 and going to in steps of 2 in the inner rim to $b_{\frac{n-1}{2}}$. If we consider only any two vertices in the inner rim, one of the maximum distances is $d(b_0, b_{\frac{n-1}{2}}) = 2 + \left\lceil \frac{n-1}{4} \right\rceil$ by first going through path $b_0a_0a_1b_1$ and then going to in steps of 2 in the inner rim to $b_{\frac{n-1}{2}}$.

This completes the proof. □

The following corollary follows directly from Eq (1.1) and Lemma 2.1.

Corollary 2.1. *Let $n \geq 6$. Then*

$$\mathrm{CL}(P(n,2)) \leq \begin{cases} \left\lceil \frac{n-1}{4} \right\rceil + 3, & \text{if } n \text{ is odd;} \\ \left\lceil \frac{n}{4} \right\rceil + 3, & \text{if } n \text{ is even.} \end{cases}$$

Now, we are ready to prove the following theorem.

Theorem 2.2. *Let $n \geq 28$. Then*

$$\mathrm{CL}(P(n,2)) = \begin{cases} \frac{n}{4} + 3, & \text{if } n \equiv 0 \mod 4; \\ \frac{n-2}{4} + 3, & \text{if } n \equiv 2 \mod 4; \\ \frac{n-1}{4} + 3, & \text{if } n \equiv 1 \mod 4; \\ \frac{n-3}{4} + 4, & \text{if } n \equiv 3 \mod 4. \end{cases}$$

Proof. Suppose $n \equiv 0 \mod 4$, i.e., $n = 4k$ for some positive integer k. Then $\left\lceil \frac{n}{4} \right\rceil + 3 = k + 3$. By Corollary 2.1, $\mathrm{CL}(P(n,2)) \leq k + 3$. To find the lower bound, we need to determine the cooling source x_i in round i. We place $x_1 = a_0$, $x_2 = a_2$, $x_3 = a_{4k-3}$, $x_4 = a_{4k-5}$, $x_5 = a_{4k-7}$, and for $6 \leq i \leq k+2$, we place $x_i = a_{2i-5}$. Since $n \geq 28$, we have $k \geq 7$. It is not hard to see that at the end of round $k+2$, all vertices are cooled, except a_{2k} and a_{2k+1}. The vertex a_{2k} is adjacent to the cooled vertex b_{2k}, whereas a_{2k+1} is adjacent to the cooled vertex b_{2k+1}. Therefore, we have the cooling sequence $(x_1, x_2, \ldots, x_{k+2}, [x_{k+3}])$. Hence, $\mathrm{CL}(P(n,2)) \geq k + 3 = \left\lceil \frac{n}{4} \right\rceil + 3$, and the theorem holds for $n \equiv 0 \mod 4$.

Suppose $n \equiv 2 \mod 4$, i.e., $n = 4k+2$ for some positive integer k. First, we show that $\mathrm{CL}(P(n,2)) \leq k + 3$. By Corollary 2.1, $\mathrm{CL}(P(n,2)) \leq \left\lceil \frac{n}{4} \right\rceil + 3 = k + 4$. Suppose $\mathrm{CL}(P(n,2)) = k + 4$. So, it has a cooling sequence $(x_1, x_2, \ldots, x_{k+3}, [x_{k+4}])$ or $(x_1, x_2, \ldots, x_{k+3}, x_{k+4})$. This means

$$V(P(n,2)) \setminus (N_{k+2}[x_1] \cup \{x_{k+3}\}) \neq \emptyset.$$

Without loss of generality, we may assume that $x_1 = a_0$ or b_0. Suppose $x_1 = a_0$. Note that at the end of round $k+2$, all the vertices contained in $N_{k+1}[x_1]$ are cooled. Recall that x_{k+3} cannot be adjacent to a cooled vertex. Therefore, $x_{k+3} \notin N_{k+2}[x_1]$ or, equivalently,

$$x_{k+3} \in V(P(n,2)) \setminus N_{k+2}[x_1] = \{a_{2k+1}\}.$$

So, we must have $x_{k+3} = a_{2k+1}$. Now,

$$N_{k+2}[x_1] \cup \{x_{k+3}\} = V(P(n, 2)),$$

a contradiction, as $\mathrm{CL}(P(n, 2)) = k + 4$. Suppose $x_1 = b_0$. Note that at the end of round $k + 2$, all the vertices contained in $N_{k+1}[x_1]$ are cooled. As before, $x_{k+3} \notin N_{k+2}[b_0]$ or, i.e.,

$$x_{k+3} \in V(P(n, 2)) \setminus N_{k+2}[x_1] = \{b_{2k+1}\}.$$

So, we must have $x_{k+3} = b_{2k+1}$. Now,

$$N_{k+2}[x_1] \cup \{x_{k+3}\} = V(P(n, 2)),$$

again, a contradiction. Hence, $\mathrm{CL}(P(n, 2)) \leq k + 3$. To find the lower bound, we place $x_1 = a_0$, $x_2 = a_2$, $x_3 = a_{4k-1}$, $x_4 = a_{4k-3}$, $x_5 = a_{4k-5}$, and for $6 \leq i \leq k + 3$, we place $x_i = a_{2i-5}$. It is not hard to see that at the end of round $k + 2$, all vertices are cooled, except

$$\begin{gathered} a_{2k+3}, \quad a_{2k+2}, \quad a_{2k+1}, \quad a_{2k}, \quad a_{2k-1}, \\ b_{2k+1}. \end{gathered}$$

Clearly, a_{2k+3} is adjacent to the cooled vertex b_{2k+3}, a_{2k+2} is adjacent to the cooled vertex b_{2k+2}, a_{2k} is adjacent to the cooled vertex b_{2k}, a_{2k-1} is adjacent to the cooled vertex b_{2k-1}, and b_{2k+1} is adjacent to the cooled vertex b_{2k-1}. So, at the end of round $k + 3$, all vertices are cooled, and we have the cooling sequence $(x_1, x_2, \ldots, x_{k+2}, x_{k+3})$. Hence, $\mathrm{CL}(P(n, 2)) = k + 3 = \frac{n-2}{4} + 3$ and the theorem holds for $n \equiv 2 \mod 4$.

Suppose $n \equiv 1 \mod 4$, i.e., $n = 4k + 1$ for some positive integer k. Then $\left\lceil \frac{n-1}{4} \right\rceil + 3 = k + 3$. By Corollary 2.1, $\mathrm{CL}(P(n, 2)) \leq k + 3$. To find the lower bound, we need to determine the cooling source x_i in round i. We place $x_1 = a_0$, $x_2 = a_2$, $x_3 = a_{4k-2}$, $x_4 = a_{4k-4}$, $x_5 = a_{4k-6}$, and for $6 \leq i \leq k + 2$, we place $x_i = a_{2i-5}$. It is not hard to see that at the end of round $k + 2$, all vertices are cooled, except

$$a_{2k+2}, a_{2k+1}, a_{2k}.$$

The vertex a_{2k+2} is adjacent to the cooled vertex b_{2k+2}, a_{2k+1} is adjacent to the cooled vertex b_{2k+1} and a_{2k} is adjacent to the cooled vertex b_{2k}. Therefore, we have the cooling sequence $(x_1, x_2, \ldots, x_{k+2}, [x_{k+3}])$. Hence, $\mathrm{CL}(P(n, 2)) \geq k + 3 = \frac{n-1}{4} + 3$ and the theorem holds for $n \equiv 1 \mod 4$.

Suppose $n \equiv 3 \mod 4$, i.e., $n = 4k + 3$ for some positive integer k. Then $\left\lceil \frac{n-1}{4} \right\rceil + 3 = k + 4$. By Corollary 2.1, $\mathrm{CL}(P(n, 2)) \leq k + 4$. To find the lower bound, we need to determine the cooling source, x_i in round i. We place $x_1 = a_0$, $x_2 = a_2$, $x_3 = a_{4k}$, $x_4 = a_{4k-2}$, $x_5 = a_{4k-4}$, and for $6 \leq i \leq k + 3$, we place $x_i = a_{2i-5}$. It is not hard to see that at the end of round $k + 3$, all vertices are cooled, except a_{2k+2}. The vertex a_{2k+2} is adjacent to the cooled vertex b_{2k+2}. Therefore, we have the cooling sequence $(x_1, x_2, \ldots, x_{k+3}, [x_{k+4}])$. Hence, $\mathrm{CL}(P(n, 2)) \geq k+4 = \frac{n-3}{4} + 4$ and the theorem holds for $n \equiv 3 \mod 4$.

This completes the proof of the theorem. □

To remark, for $6 \leq n \leq 27$, by using similar choices of cooling sources in the proof of Theorem 2.2, it can be easily seen that $\mathrm{CL}(P(n, 2)) \geq O\left(\frac{n}{4}\right)$. Here, we omitted the choices of the cooling sources for these cases.

2.3. General case

Lemma 2.2. *Let $k \geq 3$ be a fixed positive integer and $n \geq 2k$. Then*

$$\operatorname{diam}(P(n,k)) = \begin{cases} \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } \left\lfloor \frac{n}{k} \right\rfloor \text{ is even;} \\ \left\lfloor \frac{n}{2k} \right\rfloor + 2 + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } \left\lfloor \frac{n}{k} \right\rfloor \text{ is odd.} \end{cases}$$

Proof. Note that

$$\operatorname{diam}(P(n,k)) = \max\{d(a_0, y), d(b_0, y) \ : \ y \in V(P(n,k))\},$$

where $d(x, y)$ is the distance between x and y. We consider two cases.

Case 1. Suppose $n = 2sk + r$ where $0 \leq r < k - 1$. If $y = a_j$ for some $ik \leq j \leq ik + \left\lfloor \frac{k+1}{2} \right\rfloor$ where $0 \leq i \leq s - 1$, then $d(a_0, y) = 2 + i + j - ik$, which can be seen from the path

$$a_0 \to b_0 \to b_k \to \cdots \to b_{ik} \to a_{ik} \to a_{ik+1} \to \cdots \to a_j. \tag{2.3}$$

In particular, if $i = s - 1$ and $j = (s-1)k + \left\lfloor \frac{k+1}{2} \right\rfloor$, then

$$d(a_0, y) = 2 + (s - 1) + \left\lfloor \frac{k+1}{2} \right\rfloor = \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

If $y = a_j$ for some $ik + \left\lfloor \frac{k+1}{2} \right\rfloor + 1 \leq j \leq (i+1)k$ where $0 \leq i \leq s-1$, then $d(a_0, y) = 2 + (i+1) + (i+1)k - j$, which can be seen from the path

$$a_0 \to b_0 \to b_k \to \cdots \to b_{(i+1)k} \to a_{(i+1)k} \to a_{(i+1)k-1} \to \cdots \to a_j. \tag{2.4}$$

In particular, if $i = s - 1$ and $j = (s-1)k + \left\lfloor \frac{k+1}{2} \right\rfloor + 1$, then

$$d(a_0, y) = 2 + s + k - \left\lfloor \frac{k+1}{2} \right\rfloor - 1 = 1 + s + k - \left\lfloor \frac{k+1}{2} \right\rfloor,$$

and

$$d(a_0, y) = \begin{cases} \left\lfloor \frac{n}{2k} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } k \text{ is odd;} \\ \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor & \text{if } k \text{ is even.} \end{cases}$$

If $y = a_j$ for some $sk \leq j \leq sk + \left\lfloor \frac{r}{2} \right\rfloor$, then $d(a_0, y) = 2 + s + j - sk$, which can be seen from the path

$$a_0 \to b_0 \to b_k \to \cdots \to b_{sk} \to a_{sk} \to a_{sk+1} \to \cdots \to a_j. \tag{2.5}$$

Thus, when $j = sk + \left\lfloor \frac{r}{2} \right\rfloor$, we have

$$d(a_0, y) = 2 + s + \left\lfloor \frac{r}{2} \right\rfloor \leq 2 + s + \left\lfloor \frac{k-1}{2} \right\rfloor = \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

This means if $y = a_j$ for $1 \leq j \leq sk + \left\lfloor \frac{r}{2} \right\rfloor$, then

$$d(a_0, y) \leq \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor. \tag{2.6}$$

By symmetry, if $y = a_j$ for some $sk + \left\lfloor \frac{r}{2} \right\rfloor + 1 \leq j \leq 2sk + r - 1$, then (2.6) still holds.

If $y = b_j$ for some $ik \leq j \leq ik + \left\lfloor \frac{k+1}{2} \right\rfloor + 1$ where $0 \leq i \leq s-1$, then $d(a_0, y) = j - ik + 1 + i$, which can seen from the path

$$a_0 \to a_1 \to \cdots \to a_{j-ik} \to b_{j-ik} \to b_{j-(i-1)k} \to \cdots \to b_j. \tag{2.7}$$

In particular, if $i = s-1$ and $j = (s-1)k + \left\lfloor \frac{k+1}{2} \right\rfloor + 1$, then

$$d(a_0, y) = \left\lfloor \frac{k+1}{2} \right\rfloor + 2 + (s-1) = \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

If $y = b_j$ for some $ik + \left\lfloor \frac{k+1}{2} \right\rfloor + 2 \leq j \leq (i+1)k$ where $0 \leq i \leq s-1$, then $d(a_0, y) = (i+1)k - j + 3 + (i+1)$, which can be seen from the path

$$a_0 \to b_0 \to b_k \to \cdots \to b_{(i+1)k} \to a_{(i+1)k} \to a_{(i+1)k-1} \to \cdots \to a_j \to b_j. \tag{2.8}$$

In particular, if $i = s-1$ and $j = (s-1)k + \left\lfloor \frac{k+1}{2} \right\rfloor + 2$, then

$$d(a_0, y) = k - \left\lfloor \frac{k+1}{2} \right\rfloor + 1 + s,$$

and

$$d(a_0, y) = \begin{cases} \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } k \text{ is even;} \\ \left\lfloor \frac{n}{2k} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } k \text{ is odd.} \end{cases}$$

If $y = b_j$ for some $sk \leq j \leq sk + \left\lfloor \frac{r}{2} \right\rfloor$, then $d(a_0, y) = j - sk + 1 + s$, which can be seen from the path

$$a_0 \to a_1 \to \cdots \to a_{j-sk} \to b_{j-sk} \to b_{j-(s-1)k} \to \cdots \to b_j. \tag{2.9}$$

Thus, when $j = sk + \left\lfloor \frac{r}{2} \right\rfloor$, we have

$$d(a_0, y) = \left\lfloor \frac{r}{2} \right\rfloor + 1 + s \leq \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k-1}{2} \right\rfloor = \left\lfloor \frac{n}{2k} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

This means if $y = b_j$ for $1 \leq j \leq sk + \left\lfloor \frac{r}{2} \right\rfloor$, then

$$d(a_0, y) \leq \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor. \tag{2.10}$$

By symmetry, if $y = b_j$ for some $sk + \left\lfloor \frac{r}{2} \right\rfloor + 1 \le j \le 2sk + r - 1$, then (2.10) still holds. So, we conclude that

$$\max\{d(a_0, y) \ : \ y \in V(P(n,k))\} = \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

Now, we consider $d(b_0, y)$. If $y = a_j$ for some $ik \le j \le ik + \left\lfloor \frac{k+1}{2} \right\rfloor$ where $0 \le i \le s-1$, then by considering the path in (2.3) (remove a_0), we have

$$d(b_0, y) \le \left\lfloor \frac{n}{2k} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

If $y = a_j$ for some $ik + \left\lfloor \frac{k+1}{2} \right\rfloor + 1 \le j \le (i+1)k$ where $0 \le i \le s-1$, then by considering the path in (2.4) (remove a_0), we have

$$d(b_0, y) \le \begin{cases} \left\lfloor \frac{n}{2k} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor - 1, & \text{if } k \text{ is odd;} \\ \left\lfloor \frac{n}{2k} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } k \text{ is even.} \end{cases}$$

If $y = a_j$ for some $sk \le j \le sk + \left\lfloor \frac{r}{2} \right\rfloor$, then by considering the path in (2.5) (remove a_0), we have

$$d(b_0, y) \le \left\lfloor \frac{n}{2k} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

This means if $y = a_j$ for $1 \le j \le sk + \left\lfloor \frac{r}{2} \right\rfloor$, then

$$d(b_0, y) \le \left\lfloor \frac{n}{2k} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor. \tag{2.11}$$

By symmetry, if $y = a_j$ for some $sk + \left\lfloor \frac{r}{2} \right\rfloor + 1 \le j \le 2sk + r - 1$, then (2.11) still holds.

If $y = b_j$ for some $ik \le j \le ik + \left\lfloor \frac{k+1}{2} \right\rfloor$ where $0 \le i \le s-1$, then by considering the path in (2.7) (adding $b_0 \to a_0$), we have

$$d(b_0, y) \le \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

If $y = b_j$ for some $ik + \left\lfloor \frac{k+1}{2} \right\rfloor + 1 \le j \le (i+1)k$ where $0 \le i \le s-1$, then by considering the path in (2.8) (remove a_0), we have

$$d(b_0, y) \le \begin{cases} \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } k \text{ is even;} \\ \left\lfloor \frac{n}{2k} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } k \text{ is odd.} \end{cases}$$

If $y = b_j$ for some $sk \le j \le sk + \left\lfloor \frac{r}{2} \right\rfloor$, then by considering the path in (2.9) (adding $b_0 \to a_0$), we have

$$d(b_0, y) \le \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

This means if $y = b_j$ for $1 \le j \le sk + \left\lfloor \frac{r}{2} \right\rfloor$, then

$$d(b_0, y) \le \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor. \tag{2.12}$$

By symmetry, if $y = b_j$ for some $sk + \left\lfloor \frac{r}{2} \right\rfloor + 1 \le j \le 2sk + r - 1$, then (2.12) still holds. So, we conclude that

$$\max\{d(b_0, y) \ : \ y \in V(P(n,k))\} = \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

Hence,

$$\mathrm{diam}(P(n,k)) = \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

Case 2. Suppose $n = (2s+1)k + r$ where $0 \le r < k - 1$. We consider $d(a_0, y)$. If $y = a_j$ for some $ik \le j \le ik + \left\lfloor \frac{k+1}{2} \right\rfloor$ where $0 \le i \le s$, then $d(a_0, y) = 2 + i + j - ik$, which can be seen from the path in (2.3). In particular, if $i = s$ and $j = sk + \left\lfloor \frac{k+1}{2} \right\rfloor$, then

$$d(a_0, y) = 2 + s + \left\lfloor \frac{k+1}{2} \right\rfloor = \left\lfloor \frac{n}{2k} \right\rfloor + 2 + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

If $y = a_j$ for some $ik + \left\lfloor \frac{k+1}{2} \right\rfloor + 1 \le j \le (i+1)k$ where $0 \le i \le s$, then $d(a_0, y) = 2 + (i+1) + (i+1)k - j$, which can be seen from the path in (2.4). In particular, if $i = s$ and $j = sk + \left\lfloor \frac{k+1}{2} \right\rfloor + 1$, then

$$d(a_0, y) = 3 + s + k - \left\lfloor \frac{k+1}{2} \right\rfloor - 1 = 2 + s + k - \left\lfloor \frac{k+1}{2} \right\rfloor,$$

and

$$d(a_0, y) = \begin{cases} \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } k \text{ is odd;} \\ \\ \left\lfloor \frac{n}{2k} \right\rfloor + 2 + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } k \text{ is even.} \end{cases}$$

This means if $y = a_j$ for $1 \le j \le (s+1)k$, then

$$d(a_0, y) \le \left\lfloor \frac{n}{2k} \right\rfloor + 2 + \left\lfloor \frac{k+1}{2} \right\rfloor. \tag{2.13}$$

By symmetry, if $y = a_j$ for some $sk + r - 1 \le j \le (2s+1)k + r - 1$, then (2.13) still holds.

If $y = b_j$ for some $ik \le j \le ik + \left\lfloor \frac{k+1}{2} \right\rfloor + 1$ where $0 \le i \le s$, then $d(a_0, y) = j - ik + 1 + i$, which can be seen from the path in (2.7). In particular, if $i = s$ and $j = sk + \left\lfloor \frac{k+1}{2} \right\rfloor + 1$, then

$$d(a_0, y) = \left\lfloor \frac{k+1}{2} \right\rfloor + 2 + s = \left\lfloor \frac{n}{2k} \right\rfloor + 2 + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

If $y = b_j$ for some $ik + \left\lfloor \frac{k+1}{2} \right\rfloor + 2 \le j \le (i+1)k$ where $0 \le i \le s$, then $d(a_0, y) = (i+1)k - j + 3 + (i+1)$, which can be seen from the path in (2.8). In particular, if $i = s$ and $j = sk + \left\lfloor \frac{k+1}{2} \right\rfloor + 2$, then

$$d(a_0, y) = k - \left\lfloor \frac{k+1}{2} \right\rfloor + 2 + s,$$

and

$$d(a_0, y) = \begin{cases} \left\lfloor \frac{n}{2k} \right\rfloor + 2 + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } k \text{ is even;} \\ \\ \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } k \text{ is odd.} \end{cases}$$

This means if $y = b_j$ for $1 \leq j \leq (s+1)k$, then

$$d(a_0, y) \leq \left\lfloor \frac{n}{2k} \right\rfloor + 2 + \left\lfloor \frac{k+1}{2} \right\rfloor. \tag{2.14}$$

By symmetry, if $y = b_j$ for some $sk + r - 1 \leq j \leq (2s+1)k + r - 1$, then (2.14) still holds. Thus,

$$\max\{d(a_0, y) \ : \ y \in V(P(n,k))\} = \left\lfloor \frac{n}{2k} \right\rfloor + 2 + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

Now, we consider $d(b_0, y)$. If $y = a_j$ for some $ik \leq j \leq ik + \left\lfloor \frac{k+1}{2} \right\rfloor$ where $0 \leq i \leq s$, then by considering the path in (2.3) (remove a_0), we have

$$d(b_0, y) \leq \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

If $y = a_j$ for some $ik + \left\lfloor \frac{k+1}{2} \right\rfloor + 1 \leq j \leq (i+1)k$ where $0 \leq i \leq s$, then by considering the path in (2.4) (remove a_0), we have

$$d(b_0, y) \leq \begin{cases} \left\lfloor \frac{n}{2k} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } k \text{ is odd;} \\ \\ \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor & \text{if } k \text{ is even.} \end{cases}$$

This means if $y = a_j$ for $1 \leq j \leq (s+1)k$, then

$$d(b_0, y) \leq \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor. \tag{2.15}$$

By symmetry, if $y = a_j$ for some $sk + r - 1 \leq j \leq (2s+1)k + r - 1$, then (2.15) still holds.

If $y = b_j$ for some $ik \leq j \leq ik + \left\lfloor \frac{k+1}{2} \right\rfloor$ where $0 \leq i \leq s$, then by considering the path in (2.7) (adding $b_0 \to a_0$), we have

$$d(b_0, y) \leq \left\lfloor \frac{n}{2k} \right\rfloor + 2 + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

If $y = b_j$ for some $ik + \left\lfloor \frac{k+1}{2} \right\rfloor + 1 \leq j \leq (i+1)k$ where $0 \leq i \leq s$, then by considering the path in (2.8) (remove a_0), we have

$$d(b_0, y) \leq \begin{cases} \left\lfloor \frac{n}{2k} \right\rfloor + 2 + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } k \text{ is even;} \\ \\ \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } k \text{ is odd.} \end{cases}$$

This means if $y = b_j$ for $1 \leq j \leq (s+1)k$, then

$$d(b_0, y) \leq \left\lfloor \frac{n}{2k} \right\rfloor + 2 + \left\lfloor \frac{k+1}{2} \right\rfloor. \tag{2.16}$$

By symmetry, if $y = b_j$ for some $sk + r - 1 \leq j \leq (2s + 1)k + r - 1$, then (2.16) still holds. Hence,

$$\max\{d(b_0, y) \ : \ y \in V(P(n, k))\} = \left\lfloor \frac{n}{2k} \right\rfloor + 2 + \left\lfloor \frac{k+1}{2} \right\rfloor,$$

and

$$\operatorname{diam}(P(n, k)) = \left\lfloor \frac{n}{2k} \right\rfloor + 2 + \left\lfloor \frac{k+1}{2} \right\rfloor.$$

This completes the proof of the lemma. □

Theorem 2.3. *Let $k \geq 3$ be a fixed positive integer and $n \geq 2k$. Then*

$$\left\lfloor \frac{n}{2k} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor \leq \mathrm{CL}(P(n, k)) \leq \begin{cases} \left\lfloor \frac{n}{2k} \right\rfloor + 2 + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } \left\lfloor \frac{n}{k} \right\rfloor \text{ is even;} \\ \left\lfloor \frac{n}{2k} \right\rfloor + 3 + \left\lfloor \frac{k+1}{2} \right\rfloor, & \text{if } \left\lfloor \frac{n}{k} \right\rfloor \text{ is odd.} \end{cases}$$

Proof. Let $c = \left\lfloor \frac{n}{2k} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor$. In order to show the lower bound, instead of providing a cooling sequence $(x_1, x_2, \ldots, x_c)$ or $(x_1, x_2, \ldots, x_{c-1}, [x_c])$ that can completely cool the whole graph, we provide choices of cooling sources in the cooling sequence so that each assigned cooling source is of $d(x_j, x_i) \geq j - i + 1$ for $i = 1, 2, \ldots, j - 1$. Let $n = sk + r$ for some $0 \leq r \leq k - 1$. To ease the explanation, we draw $P(n, k)$ as follows:

(i) Place the vertices $a_0, a_1, a_2, \ldots, a_{k-1}$ horizontally from right to left.

(ii) Then, place the vertices $a_k, a_{k+1}, \ldots, a_{sk-1}$ such that $a_{(j+1)k+i}$ is positioned right below a_{jk+i} for each $i = 0, 1, 2, \ldots, k - 1$ and $j = 0, 1, \ldots, s - 1$. Place a_{sk+i} below $a_{(s-1)k+i}$ for $i = 0, 1, 2, \ldots r - 1$. Add edges to form the outer rim cycle $C_n = a_0 a_1 a_2 \ldots a_{n-1}$.

(iii) Add the vertex b_j beside a_j for all $j = 0, 1, 2, \ldots, n - 1$ and then draw the corresponding spoke $a_j b_j$ without any crossing.

(iv) Add the edges of the inner rim induced by the vertices $\{b_0, b_1, b_2, \ldots, b_{n-1}\}$.

Here, we do the following to position each cooling source. Let

(i) $x_1 = b_{\lfloor \frac{n}{2k} \rfloor k + \lfloor \frac{k-1}{2} \rfloor}$;

(ii) $x_i = a_{\lfloor \frac{n}{2k} \rfloor k + \lfloor \frac{k-1}{2} \rfloor - (i-1)k}$ for each $i = 2, 3, \ldots, \left\lfloor \frac{n}{2k} \right\rfloor + 1$;

(iii) $x_{\lfloor \frac{n}{2k} \rfloor + 1 + j} = b_{\lfloor \frac{k-1}{2} \rfloor} + j$ for each $j = 1, 2, \ldots, \left\lfloor \frac{k-1}{2} \right\rfloor$.

Hence, $\mathrm{CL}(P(n, k)) \geq \left\lfloor \frac{n}{2k} \right\rfloor + 1 + \left\lfloor \frac{k-1}{2} \right\rfloor = \left\lfloor \frac{n}{2k} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor$.

Figure 3 depicts the case for $P(52, 7)$ such that $\mathrm{CL}(P(52, 7)) > 7$.

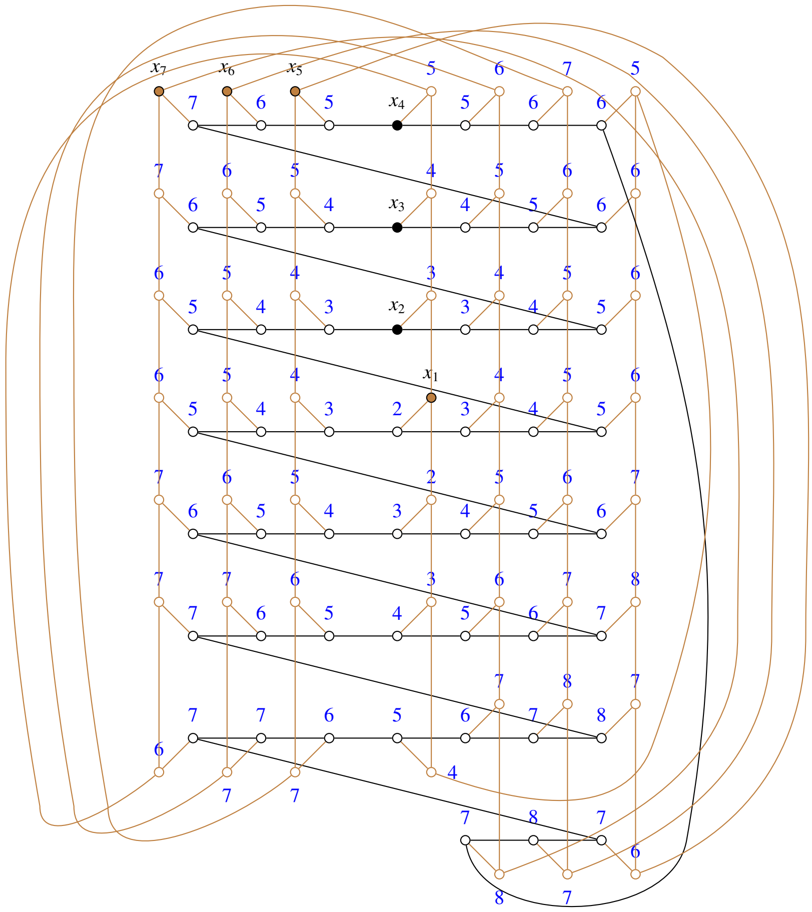

Figure 3. Cooling in $P(52, 7)$. The inner rim is induced by the black vertices while the outer rim is induced by the brown vertices. Black labels indicate the vertices of the cooling sequence in increasing order. Blue labels indicate the round that the corresponding vertex was cooled.

The upper bound follows from Eq (1.1) and Lemma 2.2. □

The following corollary is an immediate consequence of Corollary 2.1 and Theorems 2.2 and 2.3.

Corollary 2.2. *Let $k \geq 2$ be a fixed positive integer and $n \geq 2k$. Then*

$$\mathrm{CL}(P(n, k)) = \left\lfloor \frac{n}{2k} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor + O(1).$$

3. Concluding remark

The cooling number, a relatively new graph parameter, aims to maximize the number of rounds required to cool all vertices in a graph. It is the compelling counterpart to the extensively researched burning number, offering a new perspective on dynamic processes within graphs. We presented the exact results for the cooling numbers of $P(n, 1)$ and $P(n, 2)$, while providing an asymptotic formula for $P(n, k)$ for general k.

While this paper focuses on generalized Petersen graphs, it could benefit a discussion on how the results might generalize to other families of graphs or how the use of vertex-transitivity and combinatorial arguments to derive the cooling sequences could be adapted. This would broaden the impact of the results and suggest potential avenues for future research.

Author contributions

Kai An Sim: Conceptualization, investigation, methodology, funding acquisition, writing-original draft, writing-review & editing; Kok Bin Wong: Conceptualization, investigation, methodology, writing-original draft, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The author(s) declare(s) that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This project is supported by Publication Support Scheme by Sunway University, Malaysia.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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