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Research article

An introduction to the theory of OBCI-algebras

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Abstract: The purpose of this paper was to introduce the concept of "OBCI-algebras" as a partially ordered generalization of BCI-algebras. The notion of OBCI-algebras was introduced, and related properties were investigated. The notions of OBCI-subalgebras and (closed) OBCI-filters of OBCI-algebras were defined and the relationship between the OBCI-subalgebras and OBCI-filters was discussed. In addition, the direct product OBCI-algebra was discussed, and the OBCI-filter related to it was also addressed.

Keywords: OBCI-algebra; subalgebra; OBCI-subalgebra; filter; (closed) OBCI-filter
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1. Introduction

As Logic algebras, BCI-algebras, and BCK-algebras, which were first introduced by Y. Imai and K. Iséki [13] and by K. Iséki [14], respectively, in 1966, have formed an important research area in universal algebra. As is well known, these notions have two different origins, one of which is based on set theory and the other of which is based on propositional calculus. Imai and/or Iséki introduced those notions as a generalization of the concept of set-theoretic difference and of the concept of propositional implication for classical and non-classical logics [16, 18]. The logics **BCK** and **BCI** were first introduced by Lambek. The names of **BCK** and **BCI** originate from the combination of B, C, K, and I corresponding to the following implication formulas:

(B) $(\chi \to \varphi) \to ((\psi \to \chi) \to (\psi \to \varphi))$ $(prefixing)$,
(C) $(\chi \to (\varphi \to \psi)) \to (\varphi \to (\chi \to \psi))$ $(exchange, e)$,
(K) $\chi \to (\varphi \to \chi)$ $(weakening, w)$,
(I) $\chi \to \chi$ $(identity)$.

The first three together with *modus ponens* form **BCK** as the implicational fragment of the Lambek calculus with (C) and (K), and the first two and fourth together with *modus ponens* form **BCI** as the implicational fragment of the Lambek calculus with (C) [8, 19]. Note that implicational logics weaker than **BCK** and **BCI** have been introduced, see e.g., [2, 9]. *Weakly implicative logics* introduced by Cintula [1, 4], *implicational logics* by Cintula and Noguera [3–5], tonoid and partial gaggle logics by Dunn [6, 7], and implicational tonoid logics by Yang and Dunn [20, 21] are such logics with implications weaker than **BCI**. Algebraic structures for these logics can be defined by inequations but not by equations, and so those logics form order algebraizable logics (but do not form algebraizable logics). This indicates that such logics are based on partially ordered algebras, meaning they can be characterized by structures with operations and partial-order relations, though not by strictly defined algebras [8, 19]. Let $*$ and $\to$ be binary operations used as groupoid and implication in residuated lattice-ordered unital groupoids, or more generally, residuated partially ordered unital groupoids. We note also that structures for substructural logics with groupoids and implications, where the implications are weaker than **BCK** and **BCI**, require the following property:

$$x * y \leq z \text{ if and only if (iff) } y \leq x \to z \text{ (residuation).}$$

The starting point of this work is that BCK-algebras are implicational subreducts of FL_{ew}*-algebras* [8, 11, 12, 19] characterizing **BCK**, whereas BCI-algebras are not implicational subreducts of FL_e-*algebras* characterizing **BCI**. Notice that, while **BCK** is algebraizable, **BCI** is order algebraizable but not algebraizable [8, 19]. This means that the implicational subreducts of FL_e-*algebras* characterizing **BCI** can be defined as ordered algebras by inequations or equations with an underlying partial order. In other words, **BCI** requires general structures more than BCI-algebras. Note that BCI-algebras are equationally definable structures.

This series of facts gives rise to a consideration of OBCI-algebras as a generalization of BCI-algebras and as algebraic structures with implications on ordered sets for **BCI**. Therefore, we introduce the concept of OBCI-algebra, give examples to explain it, and investigate several related properties. We explore their various properties by introducing the concepts of OBCI-subalgebras and (closed) OBCI-filters. We consider the direct product OBCI-algebra and then deal with the OBCI-filter associated with it.

2. OBCI-algebras

For the development of the theory of OBCI-algebra, we will first begin by reviewing the definition of BCI-algebra which is well-known.

Definition 2.1 ([10, 15]). Let K be an abstract algebra of type (2,0) with a binary operation "$*$" and a constant "0". Then K is a *BCI-algebra* if it satisfies:

(I_1) $((x * y) * (x * z)) * (z * y) = 0$,
(I_2) $(x * (x * y)) * y = 0$,
(I_3) $x * x = 0$,
(I_4) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y, z \in K$.

We consider a binary relation "$\leq_0$" on a BCI-algebra K as follows:

$$\leq_0 := \{(x, y) \in K \times K \mid 0 = x * y\}. \tag{2.1}$$

If $(x, y) \in \leq_0$, we will write it as $x \leq_0 y$, and so $x \leq_0 y \Leftrightarrow x * y = 0$. It is well known that $(K, \leq_0)$ is a partially ordered set.

Let $(K, \leq_e)$ be a set with a binary relation $\leq_e$, and $\rightarrow$ and e are a binary operation and a constant on K, respectively. We say that $\leq_e$ is related to e (and $\rightarrow$), if $x \leq_e y$, iff $e \leq_e x \rightarrow y$, for any $x, y \in K$.

Based on this background, we introduce the notion of OBCI-algebra.

Definition 2.2. Let $(K, \leq_e)$ be a set with a binary relation and $\rightarrow$ and e are a binary operation and a constant on K, respectively, such that $\leq_e$ is related to e (and $\rightarrow$). Then $\mathbf{K} := (K, \rightarrow, e, \leq_e)$ is called an *OBCI-algebra* if it satisfies the following conditions:

$$(\forall x, y, z \in K)((x \rightarrow y) \leq_e ((y \rightarrow z) \rightarrow (x \rightarrow z))), \tag{2.2}$$

$$(\forall x, y \in K)(x \leq_e ((x \rightarrow y) \rightarrow y)), \tag{2.3}$$

$$(\forall x \in K)(x \leq_e x), \tag{2.4}$$

$$(\forall x, y \in K)(x \leq_e y,\ y \leq_e x \ \Rightarrow\ x = y), \tag{2.5}$$

$$(\forall x, y \in K)(e \leq_e x,\ x \leq_e y \ \Rightarrow\ e \leq_e y). \tag{2.6}$$

A few special cases are noted such that, in Definition 2.2, if we use

(i) the equality "=" instead of the binary relation "$\leq_e$" related to the constant "e",
(ii) the constant "0" instead of the constant "e",
(iii) the binary operation "$*$" instead of the binary operation "$\rightarrow$" which is given by $x * y = y \rightarrow x$ for all $x, y \in K$,

then K is a BCI-algebra. This suggests that OBCI-algebras are a more generalized concept than the dual version of BCI-algebras.

Now let us look at examples that describe the OBCI-algebra.

Example 2.3. Most natural examples of OBCI-algebras are implicational subreducts of FL_e-algebras characterizing **BCI** and implicational subreducts of $InDFL_{ec}$-algebras characterizing **R** (relevance logic). On an ordered set $(K, \leq_e)$, the implicational subreduct of an FL_e-algebra requires (2.2), (2.3), and (2.4), and the implicational subreduct of an $InDFL_{ec}$-algebra further requires the following substructural condition: for all $x, y, z \in K$,

$$e \leq_e (x \rightarrow (x \rightarrow y)) \rightarrow (x \rightarrow y).$$

Example 2.4. (i) Let $K = \{1, e, \partial, 0\}$ be a set. Define a binary operation " $\rightarrow$ " on K by Table 1.

Let $\leq_e := \{(0, 0), (e, e), (\partial, \partial), (1, 1), (0, 1), (0, e), (0, \partial), (e, 1), (\partial, 1)\}$. It is clear that 1 and 0 are the greatest element and the least element of $(K, \leq_e)$, respectively, and it is straightforward to verify that $\mathbf{K} := (K, \rightarrow, e, \leq_e)$ is an OBCI-algebra.

(ii) Let $K = \{0, 1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}\}$ be a set with a binary operation " $\rightarrow$ " given by Table 2 and let $\leq_e$ be a binary relation on K related to $e = \frac{3}{4}$ given by $0 \leq_e \frac{1}{4} \leq_e \frac{1}{2} \leq_e \frac{3}{4} \leq_e 1$.

Table 1. Cayley table for the binary operation "→".

$\to$	1	e	∂	0
1	1	0	0	0
e	1	e	∂	0
∂	1	∂	e	0
0	1	1	1	1

Table 2. Cayley table for the binary operation "→".

$\to$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
1	1	0	0	0	0
$\frac{3}{4}$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	0
$\frac{1}{4}$	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	0
0	1	1	1	1	1

It is straightforward to verify that $\mathbf{K} := (K, \to, e, \leq_e)$, where $e = \frac{3}{4}$ is an OBCI-algebra.

Example 2.5. Consider a set $M := [0, 1]$, the real unit interval, and let $\leq_e$ be the total order in M where $e = \frac{1}{2}$. Define a binary operation " → " as follows:

$$x \to y = \begin{cases} \max\{\neg x, y\} & \text{if } x \leq_e y, \\ \min\{\neg x, y\} & \text{otherwise}, \end{cases}$$

for all $x, y \in M$, where $\neg x = 1 - x$. It is straightforward to verify that $\mathbf{M} := (M, \to, e, \leq_e)$ is an OBCI-algebra.

Note that $\mathbf{M} := (M, \to, e, \leq_e)$ provides the semantics for the involutive uninorm mingle logic **IUML** (see [17]).

Proposition 2.6. *Every OBCI-algebra* $\boldsymbol{K} := (K, \to, e, \leq_e)$ *satisfies:*

(i) $(\forall x, y, z \in K)\ (e \leq_e x \to y \ \Rightarrow\ e \leq_e (y \to z) \to (x \to z))$.
(ii) $(\forall x, y, z \in K)\ (e \leq_e x \to y,\ e \leq_e y \to z \ \Rightarrow\ e \leq_e x \to z)$.

Proof. (i): Let $x, y \in K$ be such that $e \leq_e x \to y$. By the combination of (2.2) and (2.6), we have that $e \leq_e (y \to z) \to (x \to z)$.

(ii): Assume that $e \leq_e x \to y$ and $e \leq_e y \to z$ for all $x, y, z \in K$. Then $e \leq_e (y \to z) \to (x \to z)$ by (i). Hence $e \leq_e x \to z$ by (2.6). □

Corollary 2.7. *If* $\boldsymbol{K} := (K, \to, e, \leq_e)$ *is an OBCI-algebra, then* $(K, \leq_e)$ *is a partially ordered set.*

Proof. It is straightforward from (2.4), (2.5), and Proposition 2.6. □

Proposition 2.8. *Every OBCI-algebra* $\boldsymbol{K} := (K, \to, e, \leq_e)$ *satisfies:*

(i) $(\forall x, y, z \in K)\ (e \leq_e (z \to (y \to x)) \to (y \to (z \to x)))$.
(ii) $(\forall x, y, z \in K)\ (z \to (y \to x) = y \to (z \to x))$.
(iii) $(\forall x, y, z \in K)\ (e \leq_e z \to (y \to x) \ \Rightarrow\ e \leq_e y \to (z \to x))$.
(iv) $(\forall x \in K)\ (e \to x = x)$.
(v) $(\forall x, y \in K)\ (((x \to y) \to y) \to y = x \to y)$.
(vi) $(\forall x \in K)\ ((x \to x) \to x = x)$.

Proof. (i) Let $x, y, z \in K$. The condition (2.3) induces $z \leq_e (z \to x) \to x$. It follows from (2.3) and Proposition 2.6(i) that

$$y \to (z \to x) \leq_e ((z \to x) \to x) \to (y \to x) \leq_e z \to (y \to x).$$

Since $\leq_e$ is transitive (see Corollary 2.7), we have $y \to (z \to x) \leq_e z \to (y \to x)$ and so

$$e \leq_e (y \to (z \to x)) \to (z \to (y \to x)).$$

(ii) The result (i) is true for all $x, y, z \in K$, and so

$$e \leq_e (y \to (z \to x)) \to (z \to (y \to x))$$

for all $x, y, z \in K$. Thus $y \to (z \to x) = z \to (y \to x)$ by (2.5).

(iii) It is straightforward by (ii).

(iv) Using (2.4), we have $e \leq_e (e \to x) \to (e \to x)$ for all $x \in K$, and so $e \leq_e e \to ((e \to x) \to x)$ by (iii). It follows that $e \leq_e (e \to x) \to x$. The combination of (2.4) and (ii) induces $e \leq_e e \to (x \to x) = x \to (e \to x)$ for all $x \in K$. Consequently, $e \to x = x$ for all $x \in K$ by (2.5).

(v) The combination of (2.3) and Proposition 2.6(i) derives

$$e \leq_e (((x \to y) \to y) \to y) \to (x \to y)$$

for all $x, y \in K$. On the other hand, since

$$e \leq_e ((x \to y) \to y) \to ((x \to y) \to y)$$

by (2.4), we have $e \leq_e (x \to y) \to (((x \to y) \to y) \to y)$ by (iii). It follows from (2.5) that $((x \to y) \to y) \to y = x \to y$ for all $x, y \in K$.

(vi) Using (2.4) and (ii), we have

$$e \leq_e ((x \to x) \to x) \to ((x \to x) \to x) = (x \to x) \to (((x \to x) \to x) \to x).$$

Since $e \leq_e x \to x$ by (2.4), it follows from (2.6) that $e \leq_e ((x \to x) \to x) \to x$. On the other hand, we have $e \leq_e x \to ((x \to x) \to x)$ by (2.3). Therefore $(x \to x) \to x = x$ for all $x \in K$ by (2.5). □

The combination of (2.2) and Proposition 2.8(iii) induces the following corollary.

Corollary 2.9. *Every OBCI-algebra* $\boldsymbol{K} := (K, \to, e, \leq_e)$ *satisfies:*

$$(\forall x, y, z \in K)(e \leq_e (y \to z) \to ((x \to y) \to (x \to z))). \tag{2.7}$$

Proposition 2.10. *Every OBCI-algebra* $\mathbf{K} := (K, \rightarrow, e, \leq_e)$ *satisfies:*

$$(\forall x, y, z \in K)(e \leq_e x \rightarrow y \ \Rightarrow \ e \leq_e (z \rightarrow x) \rightarrow (z \rightarrow y)). \tag{2.8}$$

Proof. Let $x, y, z \in K$ be such that $e \leq_e x \rightarrow y$. If we replace x, y, and z with z, x, and y, respectively, in Corollary 2.9 and use (2.6), then $e \leq_e (z \rightarrow x) \rightarrow (z \rightarrow y)$. □

Let $\mathbf{K} := (K, \rightarrow_K, e_K, \leq_K)$ and $\mathbf{M} := (M, \rightarrow_M, e_M, \leq_M)$ be OBCI-algebras. Consider a binary operation " $\Rightarrow$ ", a constant "$\mathbf{e}$", and a binary relation " $\ll$ " in the Cartesian product $K \times M$ defined as follows:

$$\begin{aligned}
&(x, a) \Rightarrow (y, b) = (x \rightarrow_K y, a \rightarrow_M b),\\
&\mathbf{e} = (e_K, e_M),\\
&(x, a) \ll (y, b) \ \Leftrightarrow \ x \leq_K y, \ a \leq_M b)
\end{aligned}$$

for all $(x, a), (y, b) \in K \times M$. It is easy to check that $\mathbf{K} \times \mathbf{M} := (K \times M, \Rightarrow, \mathbf{e}, \ll)$ is an OBCI-algebra, and it is called a *direct product OBCI-algebra*.

Example 2.11. Let $K := \{e_K, 1, x, y\}$ and $M = \{e_M, a, b\}$ be sets with binary operations " $\rightarrow_K$ " and " $\rightarrow_M$ " given by Tables 3 and 4, respectively.

Table 3. Cayley table for the binary operation "$\rightarrow_K$".

$\rightarrow_K$	e_K	1	x	y
e_K	e_K	1	x	y
1	e_K	e_K	x	y
x	e_K	1	e_K	y
y	y	y	y	e_K

Table 4. Cayley table for the binary operation "$\rightarrow_M$".

$\rightarrow_M$	e_M	a	b
e_M	e_M	a	b
a	b	e_M	a
b	a	b	e_M

Let

$$\leq_K := \{(e_K, e_K), (1, 1), (x, x), (y, y), (1, e_K), (x, e_K)\}$$

and $\leq_M := \{(e_M, e_M), (a, a), (b, b)\}$. Then $\mathbf{K} := (K, \rightarrow_K, e_K, \leq_K)$ and $\mathbf{M} := (M, \rightarrow_M, e_M, \leq_M)$ are OBCI-algebras, and $\mathbf{K} \times \mathbf{M} := (K \times M, \Rightarrow, \mathbf{e}, \ll)$ is the direct product OBCI-algebra of $\mathbf{K}$ and $\mathbf{M}$ in which

$$\begin{aligned}
K \times M = \{&(e_K, e_M), (e_K, a), (e_K, b), (1, e_M), (1, a), (1, b),\\
&(x, e_M), (x, a), (x, b), (y, e_M), (y, a), (y, b)\},
\end{aligned}$$

and the operation " ⇒ " is given by Table 5.

Table 5. Cayley table for the operation " ⇒ ".

$\Rightarrow$	(e_K, e_M)	(e_K, a)	(e_K, b)	$(1, e_M)$	$(1, a)$	$(1, b)$
(e_K, e_M)	(e_K, e_M)	(e_K, a)	(e_K, b)	$(1, e_M)$	$(1, a)$	$(1, b)$
(e_K, a)	(e_K, b)	(e_K, e_M)	(e_K, a)	$(1, b)$	$(1, e_M)$	$(1, a)$
(e_K, b)	(e_K, a)	(e_K, b)	(e_K, e_M)	$(1, a)$	$(1, b)$	$(1, e_M)$
$(1, e_M)$	(e_K, e_M)	(e_K, a)	(e_K, b)	(e_K, e_M)	(e_M, a)	(e_K, b)
$(1, a)$	(e_K, b)	(e_K, e_M)	(e_K, a)	(e_K, b)	(e_K, e_M)	(e_K, a)
$(1, b)$	(e_K, a)	(e_K, b)	(e_K, e_M)	(e_K, a)	(e_K, b)	(e_K, e_M)

$\Rightarrow$	(x, e_M)	(x, a)	(x, b)	(y, e_M)	(y, a)	(y, b)
(e_K, e_M)	(x, e_M)	(x, a)	(x, b)	(y, e_M)	(y, a)	(y, b)
(e_K, a)	(x, b)	(x, e_M)	(x, a)	(y, b)	(y, e_M)	(y, a)
(e_K, b)	(x, a)	(x, b)	(x, e_M)	(y, a)	(y, b)	(y, e_M)
$(1, e_M)$	(x, e_M)	(x, a)	(x, b)	(y, e_M)	(y, a)	(y, b)
$(1, a)$	(x, b)	(x, e_M)	(x, a)	(y, b)	(y, e_M)	(y, a)
$(1, b)$	(x, a)	(x, b)	(x, e_M)	(y, a)	(y, b)	(y, e_M)

$\Rightarrow$	(e_K, e_M)	(e_K, a)	(e_K, b)	$(1, e_M)$	$(1, a)$	$(1, b)$
(x, e_M)	(e_K, e_M)	(e_K, a)	(e_K, b)	$(1, e_M)$	$(1, a)$	$(1, b)$
(x, a)	(e_K, b)	(e_K, e_M)	(e_K, a)	$(1, b)$	$(1, e_M)$	$(1, a)$
(x, b)	(e_K, a)	(e_K, b)	(e_K, e_M)	$(1, a)$	$(1, b)$	$(1, e_M)$
(y, e_M)	(y, e_M)	(y, a)	(y, b)	(y, e_M)	(y, a)	(y, b)
(y, a)	(y, b)	(y, e_M)	(y, a)	(y, b)	(y, e_M)	(y, a)
(y, b)	(y, a)	(y, b)	(y, e_M)	(y, a)	(y, b)	(y, e_M)

$\Rightarrow$	(x, e_M)	(x, a)	(x, b)	(y, e_M)	(y, a)	(y, b)
(x, e_M)	(e_K, e_M)	(e_K, a)	(e_K, b)	(y, e_M)	(y, a)	(y, b)
(x, a)	(e_K, b)	(e_K, e_M)	(e_K, a)	(y, b)	(y, e_M)	(y, a)
(x, b)	(e_K, a)	(e_K, b)	(e_K, e_M)	(y, a)	(y, b)	(y, e_M)
(y, e_M)	(y, e_M)	(y, a)	(y, b)	(e_K, e_M)	(e_K, a)	(e_K, b)
(y, a)	(y, b)	(y, e_M)	(y, a)	(e_K, b)	(e_K, e_M)	(e_K, a)
(y, b)	(y, a)	(y, b)	(y, e_M)	(e_K, a)	(e_K, b)	(e_K, e_M)

3. OBCI-subalgebras and OBCI-filters

In what follows, let $\mathbf{K} := (K, \rightarrow, e, \leq_e)$ denote an OBCI-algebra unless otherwise specified.

Definition 3.1. A non-empty subset A of K is called

- a *subalgebra* of $\mathbf{K}$ if it satisfies:

$$(\forall x, y \in K)(x, y \in A \ \Rightarrow \ x \to y \in A), \tag{3.1}$$

- an *OBCI-subalgebra* of $\mathbf{K}$ if it satisfies:

$$(\forall x, y \in K)(x, y \in A,\ e \leq_e x,\ e \leq_e y \ \Rightarrow \ x \to y \in A). \tag{3.2}$$

Example 3.2. (i) Consider the OBCI-algebra $\mathbf{K}$ in Example 2.4(i). Then the set $A := \{1, e, 0\}$ is a subalgebra of $\mathbf{K}$. Consider the OBCI-algebra $\mathbf{K}$ in Example 2.4(ii). Then the set $B := \{1, \frac{3}{4}, 0\}$ is a subalgebra of $\mathbf{K}$.

(ii) Consider the OBCI-algebra $\mathbf{K}$ in Example 2.4(i). Then the set $A := \{e, 0\}$ is an OBCI-subalgebra of $\mathbf{K}$. Consider the OBCI-algebra $\mathbf{K}$ in Example 2.4(ii). Then the set $B := \{\frac{3}{4}, 0\}$ is an OBCI-subalgebra of $\mathbf{K}$.

It is clear that every subalgebra is an OBCI-subalgebra. However, its converse does not hold as seen in the following example.

Example 3.3. Consider the OBCI-algebras and their OBCI-subalgebras in Example 3.2(ii). Each OBCI-subalgebra is not a subalgebra of the corresponding OBCI-algebra since $0 \to e = 1$ and $0 \to \frac{3}{4} = 1$, respectively.

Theorem 3.4. *Let A be an OBCI-subalgebra of $\mathbf{K} := (K, \to, e, \leq_e)$ such that $e \leq_e x$ for all $x \in A$. Then A is a subalgebra of $\mathbf{K}$.*

Proof. This is straightforward. □

Definition 3.5. A subset F of K is called

- a *filter* of $\mathbf{K}$ if it satisfies:

$$e \in F, \tag{3.3}$$

$$(\forall x, y \in K)(x \to y \in F,\ x \in F \ \Rightarrow \ y \in F), \tag{3.4}$$

- an *OBCI-filter* of $\mathbf{K}$ if it satisfies (3.3) and

$$(\forall x, y \in K)(x \in F,\ e \leq_e x \to y \ \Rightarrow \ y \in F). \tag{3.5}$$

It is clear that $\{e\}$ and K are OBCI-filters of $\mathbf{K}$.

Example 3.6. Consider the OBCI-algebra $\mathbf{K}$ in Example 2.4(i). Then the set $F := \{1, e\}$ is a filter and an OBCI-filter of $\mathbf{K}$.

Example 3.7. Consider the OBCI-algebra $\mathbf{K}$ in Example 2.4(ii). Then the set $F := \{1, \frac{3}{4}, \frac{1}{2}\}$ is an OBCI-filter of $\mathbf{K}$, but it is not a filter of $\mathbf{K}$ since $\frac{1}{2} \to \frac{1}{4} = \frac{1}{2} \in F$ but $\frac{1}{4} \notin F$. The set $G := \{1, \frac{3}{4}\}$ is a filter and an OBCI-filter of $\mathbf{K}$.

Example 3.8. In Example 2.5, the set $F := [\frac{1}{2}, 1]$ is an OBCI-filter of $\mathbf{K}$. But it is not a filter of $\mathbf{K}$ since $\frac{1}{2} \to \frac{3}{8} = \frac{1}{2} \in F$ but $\frac{3}{8} \notin F$.

In Examples 3.7 and 3.8, we can see that any OBCI-filter may not be a filter. In Example 2.4(i), we know that $F := \{e, \partial\}$ is a filter of $\mathbf{K}$, but it is not an OBCI-filter of $\mathbf{K}$ because $e \leq_e e \rightarrow 1 = 1$. Thus, we observe that a filter and OBCI-filter are mutually independent concepts.

We provide a condition for filters to be OBCI-filters and vice versa.

Theorem 3.9. (i) *If a filter F of $\boldsymbol{K} := (K, \rightarrow, e, \leq_e)$ satisfies*

$$(\forall x \in K)(e \leq_e x \;\Rightarrow\; x \in F), \tag{3.6}$$

then it is an OBCI-filter of $\boldsymbol{K}$.

(ii) *If an OBCI-filter F of $\boldsymbol{K} := (K, \rightarrow, e, \leq_e)$ satisfies*

$$(\forall x \in K)(x \in F \;\Rightarrow\; e \leq_e x), \tag{3.7}$$

then it is a filter of $\boldsymbol{K}$.

Proof. This is straightforward. □

Given a subset F of K, consider the following assertion:

$$(\forall x, y \in K)(x \rightarrow y \in F \;\Rightarrow\; x \leq_e y). \tag{3.8}$$

The following example provides an OBCI-filter F of $\mathbf{K}$ that does not satisfy the condition (3.8).

Example 3.10. Consider the OBCI-algebra $\mathbf{K}$ in Example 2.4(ii) and take the OBCI-filter $F := \{1, \frac{3}{4}, \frac{1}{2}\}$ of $\mathbf{K}$. We can see that it does not satisfy the condition (3.8) since $\frac{1}{2} \rightarrow \frac{1}{4} = \frac{1}{2} \in F$ but $\frac{1}{2} \not\leq_e \frac{1}{4}$.

Theorem 3.11. *Let F be an OBCI-filter of $\boldsymbol{K} := (K, \rightarrow, e, \leq_e)$ satisfying* (3.8)*. Then F is a filter of $\boldsymbol{K}$.*

Proof. Let $x, y \in K$ be such that $x \rightarrow y \in F$ and $x \in F$. Then $x \leq_e y$ by (3.8), and so $e \leq_e x \rightarrow y$. Hence $y \in F$, and therefore F is a filter of $\mathbf{K}$. □

Proposition 3.12. *Every OBCI-filter is an up-set, that is, every OBCI-filter F of $\boldsymbol{K} := (K, \rightarrow, e, \leq_e)$ satisfies:*

$$(\forall x, y \in K)(x \leq_e y,\ x \in F \;\Rightarrow\; y \in F). \tag{3.9}$$

Proof. Let F be an OBCI-filter of $\mathbf{K}$. Let $x, y \in K$ be such that $x \leq_e y$ and $x \in F$. Then $e \leq_e x \rightarrow y$, and so $y \in F$ by (3.5). □

Corollary 3.13. *Every OBCI-filter F of $\boldsymbol{K} := (K, \rightarrow, e, \leq_e)$ satisfies* (3.6).

Proposition 3.14. *Every OBCI-filter F of $\boldsymbol{K} := (K, \rightarrow, e, \leq_e)$ satisfies:*

$$(\forall x \in K)(x \in F \;\Rightarrow\; (x \rightarrow e) \rightarrow e \in F). \tag{3.10}$$

Proof. Let $x \in F$. Using (2.4) and Proposition (2.8)(ii), we have

$$e \leq_e (x \rightarrow e) \rightarrow (x \rightarrow e) = x \rightarrow ((x \rightarrow e) \rightarrow e).$$

Hence $(x \rightarrow e) \rightarrow e \in F$. □

The example below shows that any filter F of $\mathbf{K}$ does not satisfy (3.9) and (3.6).

Example 3.15. Let $\mathbf{K}$ be an OBCI-algebra in Example 2.4(i). Then $F := \{e, \partial\}$ is a filter of $\mathbf{K}$ that does not satisfy (3.9) and (3.6).

In the following examples, we know that any OBCI-filter is neither a subalgebra nor an OBCI-subalgebra.

Example 3.16. Let $K = \{1, e, \partial, 0\}$ be a set, where 1 and 0 are the greatest element and the least element of K, respectively. Define a binary operation " $\rightarrow$ " on K by Table 6.

Table 6. Cayley table for the binary operation "$\rightarrow$".

$\rightarrow$	1	e	∂	0
1	1	0	0	0
e	1	e	∂	0
∂	1	0	e	0
0	1	1	1	1

Let $\leq_e := \{(0,0), (e,e), (\partial,\partial), (1,1), (0,1), (0,e), (0,\partial), (e,1), (\partial,1)\}$. Then it is straightforward to verify that $\mathbf{K} := (K, \rightarrow, e, \leq_e)$ is an OBCI-algebra. The filter $F := \{e, \partial\}$ in $\mathbf{K}$ is not a subalgebra of $\mathbf{K}$ since $\partial \rightarrow e = 0 \notin F$.

Example 3.17. The OBCI-filter $F := \{1, \frac{3}{4}, \frac{1}{2}\}$ in Example 3.7 is not an OBCI-subalgebra of $\mathbf{K}$ since $e = \frac{3}{4} \leq_e 1$ and $e = \frac{3}{4} \leq_e \frac{3}{4}$, but $1 \rightarrow \frac{3}{4} = 0 \notin F$.

Definition 3.18. An OBCI-filter (resp., filter) F of $\mathbf{K} := (K, \rightarrow, e, \leq_e)$ is said to be *closed* if it is an OBCI-subalgebra (resp., subalgebra) of $\mathbf{K}$.

Example 3.19. Let $K = \{e, a, b, c, d\}$ be a set with a binary operation " $\rightarrow$ " given by Table 7.

Table 7. Cayley table for the binary operation "$\rightarrow$".

$\rightarrow$	e	a	b	c	d
e	e	a	b	c	d
a	e	e	b	c	c
b	e	a	e	c	d
c	c	d	c	e	a
d	c	c	c	e	e

Let $\leq_e := \{(e,e), (a,a), (b,b), (c,c), (d,d), (a,e), (b,e), (d,c)\}$. Then $\mathbf{K} := (K, \rightarrow, e, \leq_e)$ is an OBCI-algebra, and $F_1 := \{e, a\}$ and $F_2 := \{e, b, c\}$ are closed OBCI-filters of $\mathbf{K}$.

Proposition 3.20. *Every closed OBCI-filter F of $\mathbf{K} := (K, \rightarrow, e, \leq_e)$ satisfies:*

$$(\forall x \in K)(e \leq_e x \ \Rightarrow \ x \rightarrow e \in F). \tag{3.11}$$

Proof. Using Definition 3.18, we obtain (3.11). □

In Example 3.19, we know that $b \to c = c \in F_2$. But $(b, c) \notin \leq_e$, i.e., $b \leq_e c$ does not hold. This shows that any (closed) filter F of $\mathbf{K}$ does not satisfy (3.8). Also, OBCI-filter F of $\mathbf{K}$ does not satisfy (3.8). In fact, the OBCI-filter $F := \{1, \frac{3}{4}, \frac{1}{2}\}$ in Example 3.7 does not satisfy (3.8) since $\frac{3}{4} \to \frac{1}{2} = \frac{1}{2} \in F$ but $\frac{3}{4} \not\leq_e \frac{1}{2}$.

We provide conditions for a filter and an OBCI-filter to be closed.

Theorem 3.21. *If a filter F of $\boldsymbol{K} := (K, \to, e, \leq_e)$ satisfies* (3.6) *and the following condition:*

$$(\forall x \in K)(x \in F \;\Rightarrow\; x \to e \in F), \tag{3.12}$$

then it is closed.

Proof. Assume that F is a filter of $\mathbf{K}$ satisfying (3.6) and (3.12). It suffices to show that F is a subalgebra of $\mathbf{K}$. Let $x, y \in F$. Then $x \to e \in F$ and $y \to e \in F$ by (3.12). Also, we obtain $e \to x = x \in F$ and $e \to y = y \in F$ by Proposition 2.8(iv). Since

$$e \leq_e (x \to e) \to ((e \to y) \to (x \to y))$$

by (2.2), it follows from (3.6) that $(x \to e) \to ((e \to y) \to (x \to y)) \in F$. Hence $x \to y \in F$ by (3.4), and therefore F is a subalgebra of $\mathbf{K}$, and the proof is complete. □

Theorem 3.22. *If an OBCI-filter F of $\boldsymbol{K} := (K, \to, e, \leq_e)$ satisfies* (3.11) *and* (3.8)*, then it is closed.*

Proof. Let F be an OBCI-filter of $\mathbf{K}$ that satisfies (3.11) and (3.8). It is sufficient to show that F is an OBCI-subalgebra of $\mathbf{K}$. Let $x, y \in F$ be such that $e \leq_e x$ and $e \leq_e y$. Then $x \to e \in F$ and $y \to e \in F$ by (3.11). Moreover, we obtain $e \to x = x \in F$ and $e \to y = y \in F$ by Proposition 2.8(iv). Since $e \leq_e (x \to e) \to ((e \to y) \to (x \to y))$ by (2.2), we get

$$(x \to e) \leq_e (e \to y) \to (x \to y).$$

Since $x \to e \in F$, it follows from Proposition 3.12 that $(e \to y) \to (x \to y) \in F$. Hence $e \to y \leq_e x \to y$ by (3.8), and so $x \to y \in F$ by Proposition 3.12. Therefore F is an OBCI-subalgebra of $\mathbf{K}$, and the proof is complete. □

Proposition 3.23. *Every OBCI-filter F of $\boldsymbol{K} := (K, \to, e, \leq_e)$ satisfies:*

$$(\forall x, y, z \in K)(x \in F,\ e = x \to (y \to z) \;\Rightarrow\; y \to z \in F). \tag{3.13}$$

$$(\forall x, y \in K)\left(\begin{array}{l} \left\{ \begin{array}{l} e \leq_e (x \to y) \to (x \to y) \\ x \in F,\ x \to ((x \to y) \to y) \in F \end{array} \right\} \\ \Rightarrow (x \to y) \to y \in F \end{array} \right). \tag{3.14}$$

Proof. The result (3.13) is straightforward. Let $x, y \in K$ be such that $x \in F$, $x \to ((x \to y) \to y) \in F$ and $e \leq_e (x \to y) \to (x \to y)$. Then $e \leq_e x \to ((x \to y) \to y)$ by Proposition 2.8(iii). It follows from (3.5) that $(x \to y) \to y \in F$. □

Theorem 3.24. *The intersection of the OBCI-filters of $\boldsymbol{K} := (K, \to, e, \leq_e)$ is an OBCI-filter of $\boldsymbol{K}$.*

Table 8. Cayley table for the binary operation "→".

$\rightarrow$	e	1	x	y
e	e	1	x	y
1	1	e	y	x
x	x	y	e	1
y	y	x	1	e

Proof. This is straightforward. □

The following example shows that the union of the filters of **K** may not be a filter of **K**.

Example 3.25. Let $K := \{e, 1, x, y\}$ be a set with binary operations " → " given by Table 8. Let $\leq_e := \{(e, e), (1, 1), (x, x), (y, y)\}$. Then $\mathbf{K} := (K, \rightarrow, e, \leq_e)$ is an OBCI-algebra. It is straightforward to verify that $F_1 := \{e, 1\}$ and $F_2 := \{e, y\}$ are filters of **K**. But $F_1 \cup F_2 = \{e, 1, y\}$ is not a filter of **K** since $y \in F_1 \cup F_2$ and $y \rightarrow x = 1 \in F_1 \cup F_2$, but $x \notin F_1 \cup F_2$.

It is interesting to note that the union of the OBCI-filters of **K** forms an OBCI-filter of **K** as the following theorem shows.

Theorem 3.26. *The union of the OBCI-filters of* $\boldsymbol{K} := (K, \rightarrow, e, \leq_e)$ *is an OBCI-filter of* $\boldsymbol{K}$*.*

Proof. Let F_1 and F_2 be OBCI-filters of **K**. It is clear that $e \in F_1 \cup F_2$. Let $x, y \in K$ be such that $e \leq_e x \rightarrow y$ and $x \in F_1 \cup F_2$. Then $x \in F_1$ or $x \in F_2$. If $x \in F_1$, then $y \in F_1$ by (3.5). Similarly if $x \in F_2$, then $y \in F_2$ by (3.5). Hence $y \in F_1 \cup F_2$. Therefore $F_1 \cup F_2$ is an OBCI-filters of **K**. □

We then deal with OBCI-filters of direct product OBCI-algebras.

Theorem 3.27. *If F_K and F_M are OBCI-filters of OBCI-algebras* $\boldsymbol{K} := (K, \rightarrow, e, \leq_e)$ *and* $\boldsymbol{M} := (M, \rightarrow, e, \leq_e)$*, respectively, then $F_K \times F_M$ is an OBCI-filter of the direct product OBCI-algebra* $\boldsymbol{K} \times \boldsymbol{M} := (K \times M, \Rightarrow, \mathbf{e}, \ll)$*.*

Proof. This is straightforward. □

Theorem 3.28. *Every OBCI-filter F of* $\boldsymbol{K} \times \boldsymbol{M} := (K \times M, \Rightarrow, \mathbf{e}, \ll)$ *can be expressed as $F \subseteq F_K \times F_M$ for OBCI-filters F_K and F_M of OBCI-algebras* $\boldsymbol{K} := (K, \rightarrow, e, \leq_e)$ *and* $\boldsymbol{M} := (M, \rightarrow, e, \leq_e)$*, respectively.*

Proof. This is straightforward. □

The following example illustrates Theorem 3.27.

Example 3.29. Consider the OBCI-algebras **K** with $e = \frac{3}{4}$ and **M** with $e = \frac{1}{2}$ in Example 2.4(ii) and Example 2.5, respectively. If we take OBCI-filters $F_K := \{1, \frac{3}{4}, \frac{1}{2}\}$ and $F_M := [\frac{1}{2}, 1]$ of **K** and **M**, respectively, then

$$F_K \times F_M = \{(x, y) \in K \times M \mid x \in F_K,\ y \in F_M\}$$

is an OBCI-filter of $\mathbf{K} \times \mathbf{M} := (K \times M, \Rightarrow, \mathbf{e}, \ll)$ where $\mathbf{e} = (\frac{3}{4}, \frac{1}{2})$.

Theorem 3.30. *Given subsets F_K and F_M of OBCI-algebras* $\boldsymbol{K} := (K, \rightarrow_K, e_K, \leq_K)$ *and* $\boldsymbol{M} := (M, \rightarrow_M, e_M, \leq_M)$*, respectively, we define two sets:*

$$\mathcal{OF}^{e_M} := \{(x,a) \in K \times M \mid x \in F_K,\ (\exists b \in F_M)(e_M \leq_M b \rightarrow_M a)\} \text{ and}$$
$$\mathcal{OF}^{e_K} := \{(y,b) \in K \times M \mid (\exists x \in F_K)(e_K \leq_K x \rightarrow_K y),\ b \in F_M\}.$$

If F_K and F_M are OBCI-filters of $\boldsymbol{K}$ and $\boldsymbol{M}$, respectively, then $\mathcal{OF}^{e_M}$ and $\mathcal{OF}^{e_K}$ are OBCI-filters of $\boldsymbol{K} \times \boldsymbol{M} := (K \times M, \Rightarrow, \mathbf{e}, \ll)$.

Proof. Let F_K and F_M be OBCI-filters of $\mathbf{K}$ and $\mathbf{M}$, respectively. It is clear that $\mathbf{e} = (e_K, e_M) \in \mathcal{OF}^{e_M}$. Let $(x,a), (y,b) \in K \times M$ be such that $(x,a) \in \mathcal{OF}^{e_M}$ and $\mathbf{e} \ll (x,a) \Rightarrow (y,b)$. Then $\mathbf{e} \ll (x \rightarrow_K y, a \rightarrow_M b)$, and so $e_K \leq_K x \rightarrow_K y$ and $e_M \leq_M a \rightarrow_M b$. Since $(x,a) \in \mathcal{OF}^{e_M}$, we know that $x \in F_K$ and there exists $c \in F_M$ such that $e_M \leq_M c \rightarrow_M a$. Hence $y \in F_K$ by (3.5) and $e_M \leq_M (a \rightarrow_M b) \rightarrow_M (c \rightarrow_M b)$ by (2.2) and (2.6). Again it follows from $e_M \leq_M a \rightarrow_M b$ and (2.6) that $e_M \leq_M c \rightarrow_M b$. Therefore $\mathcal{OF}^{e_M}$ is an OBCI-filter of $\mathbf{K} \times \mathbf{M} := (K \times M, \Rightarrow, \mathbf{e}, \ll)$. In a similar way, we can verify that $\mathcal{OF}^{e_K}$ is an OBCI-filter of $\mathbf{K} \times \mathbf{M}$. □

4. Conclusions

BCK-algebras were developed by Iséki and S. Tannaka to generalize the set difference in set theory, and by Y. Imai and Iséki as the algebras of certain propositional calculi. BCI-algebras introduced by K. Iséki are a generalized version of BCK-algebras. It turns out that Abelian groups are a special case of BCI-algebras. As a type of algebraic structure, the BCI-algebra has theoretical significance in areas such as logic, especially in the study of non-classical logics and the algebraic structures underlying them. They provide a framework for generalizing notions of difference and implications beyond classical Boolean structures. The generalization of BCK/BCI-algebras has developed several forms of algebra, for example, d-algebras, BCC-algebras, BCH-algebras, BH-algebras, BZ-algebras, near-BCK-algebras, pre-BCK-algebras, B-algebras, BE-algebras, BF-algebras, BG-algebras, BI-algebras, BM-algebras, BO-algebras, C-algebras, CI-algebras, Q-algebras, QS-algebras, etc. This series of generalizations is achieved through processes such as subtracting too complex things from the axioms/conditions of BCK/BCI-algebras or replacing them with other axioms.

In this article, we have attempted to generalize BCI-algebras in a different direction from conventional generalization methods that try to subtract complex conditions or replace them with other conditions. We have introduced the concept of OBCI-algebras, and investigated several related properties. We have introduced subalgebras, OBCI-subalgebras, (closed) filters, and (closed) OBCI-filters as partial structures of OBCI-algebras and explored their relationships further. We have provided conditions for filters to be OBCI-filters and vice versa. We also have provided conditions for filters and OBCI-filters to be closed. We have formed the direct product OBCI-algebra and discussed OBCI-filters on it.

The ideas and results obtained in this paper will contribute to future research on topics such as various types of OBCI-filters, applications of various types of fuzzy set theory, the study of soft and rough set theory, derivations, coding theory, and decision-making theory related to OBCI-algebras.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

Eunsuk Yang: Conceptualization, Funding acquisition, Resources, Validation; Eun Hwan Roh: Conceptualization, Resources, Validation; Young Bae Jun: Conceptualization, Writing-original draft, Validation, Writing – review & editing. All authors have read and agreed to the published version of the manuscript. The three authors contributed equally.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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