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Research article

The weighted generalized Atangana-Baleanu fractional derivative in banach spaces- definition and applications

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Abstract: In this paper, we introduce the concept of the weighted generalized Atangana-Baleanu fractional derivative. We prove the existence of the stability of solutions of non-local differential equations and non-local differential inclusions, in Banach spaces, with this new fractional derivative in the presence of instantaneous and non-instantaneous impulses. We considered the case in which the lower limit of the fractional derivative was kept at the initial point and where it was changed to the impulsive points. To prove our results, we established the relationship between solutions to each of the four studied problems and those of the corresponding fractional integral equation. There has been no previous study of the weighted generalized Atangana-Baleanu fractional derivative, and so, our findings are new and interesting. The technique we used based on the properties of this new fractional differential operator and suitable fixed point theorems for single valued and set valued functions. Examples are given to illustrate the theoretical results.

Keywords: fractional differential equations; inclusions, instantaneous impulses; measure of noncompactness
Mathematics Subject Classification: 34A08, 26A33

1. Introduction

Fractional calculus has applications to numerous real-world problems in different branches of science, like physics [1], engineering and Social Sciences [2], and many other branches [3–5]. To overcome the problems arising from the presence of singular kernel in many well-known fractional integral and differential operators, Caputo and Fabrizio [6] proposed a definition based on the exponential function, and then, Atangana-Baleanu [7] generalized the Caputo- Fabrizio definition and introduced a new fractional derivative and integral with kernel based on the Mittag-Leffler function. Several studies on differential equations involving Atangana-Baleanu fractional derivative in the Caputo sense (${}^{ABC}D$) were carried out [8–10], and many applications of this fractional derivative were investigated [11–13].

Impulsive differential equations and impulsive differential inclusions are used to model sudden changes in several real life problems. Some model changes occur instantaneously using instantaneous impulse differential equations or instantaneous impulse differential inclusions. For examples of the applications of such equations in studying diseases and population growth, see [14–16]. Other changes remain active over a certain period of time. These types of changes are modeled using non-instantaneous impulse differential equations and non-instantaneous impulsive differential inclusions. In [17–19], the authors provided an extensive study of impulsive fractional differential equations and impulsive differential inclusions. See also the recent research on non-instantaneous impulsive differential inclusions [20–22]. It should be noted that there are two approaches in the literature to problems containing impulses; one by keeping the lower limit of the fractional derivative at zero and the other by changing it to the impulsive points.

In [23], the authors studied non-local impulsive differential equations and inclusions with the differential operator ${}^{ABC}D$ of order $\gamma \in (0,1)$ in spaces with infinite dimensions, and in more recent research [24], the authors investigated the existence of solutions and anti-periodic solutions for impulsive differential equations and inclusions containing ${}^{ABC}D$ of order $\alpha \in (1,2)$ in infinite dimensional Banach spaces. Many researchers introduced new concepts of fractional differential operators, which contributed to the development of fractional calculus and its increased application in more fields.

Sousa et al. [25] introduced the concept of the φ-Hilfer fractional derivative, which generalizes the φ-Caputo fractional derivative presented by Almeida [26] where $\varphi : L = [0,T] \to \mathbb{R}$ is a strictly increasing and continuously differentiable function with $\varphi'(\upsilon) \neq 0, \forall \upsilon \in L$. Vu et al. [27] introduced the φ-Atangana-Baleanu fractional derivative, and studied the existence and uniqueness of solutions for initial value problems of fuzzy differential equations involving these derivatives. Since weighted fractional derivatives can be used in the solution of several types of integral equations, differential equations and differential inclusions containing these derivatives have been studied by several authors. For example, in [20], the authors investigated the existence of solutions for a differential inclusion involving w-weighted φ-Hilfer fractional derivative. For more studies on weighted fractional differential equations and inclusions, see [28–30]. In [31], Al-Refai presented the concept of g-weighted Atangana-Baleanu fractional derivative and proved some of their properties, where $g : L \to (0,\infty)$ is continuously differentiable.

Motivated by the research mentioned above, especially [20, 27, 31], and the authors previous work [23, 24], we introduce the concept of w-weighted φ-Atangana-Baleanu fractional derivative with lower limit at $a \in [0,T]$ for some $T \in \mathbb{R}^+$, and then establish the existence of solutions for non-local differential equations and non-local differential inclusions involving this new fractional differential operator, in the presence of instantaneous and non-instantaneous impulses. We consider the case in which the lower limit of the fractional derivative is kept at the initial point and where it is changed to the impulsive points.

The key contributions of this work is as follows:

- A new concept of the fractional differential operator is introduced (Definition 1). This new differential operator generalizes both the Atangana-Baleanu [7], the φ-Atangana-Baleanu derivative introduced by HoVu, Behzad Ghanbari [27], and the g-weighted Atangana-Baleanu derivative defined by Al-Refai [31]. Some properties for the new differential operator are obtained (Lemmas 1– 3).

- Using the new differential operator, we form a list of initial value problems (Problems (2.1)–(2.4), in Section 2). The formulas of their solutions are also given (relations (4.6), (5.7), (6.3) and (7.3)).
- The relationship between solution to Problems (2.1)–(2.4) and the corresponding fractional integral equations are derived (Lemmas 4, 5, 8, 9).
- Two existence/uniqueness of solutions to Problem (2.1) are proven (Theorems 1 and 2), and three existence results of solutions to Problems (2.2)–(2.4) are proven (Theorems 3–5).
- The stability of the solution to Problem (2.1) is analyzed (Theorem 6), and in the same manner, the stability of solutions to the other considered problems can be studied.

To clarify the importance of this work and its relationship to other results, we mention the following points:

($\star$) The introduction of new fractional differential operators contributes to the development of fractional calculus and its increasing applications. The obtained results about this new fractional generalizes many other ones such as:

  - Theorems 4.3 and 5.1, in [23], if one substitutes $g(\upsilon) = 1$ and $\varphi(\upsilon) = \upsilon; \upsilon \in L$ in Theorems 1 and 5
  - Theorem 3.1 in [31], if one substitutes $\varphi(\upsilon) = \upsilon; \upsilon \in L$ in Lemma 3.
  - Theorem 2.3 in [27], if one substitutes $g(\upsilon) = 1; \upsilon \in L$ in Lemma 3
  - Problem (2.1) is studied in Theorem 3.1 in [32], Theorem 3.2 in [33], and Theorem 2.2 in [34] in the special cases: $g(\upsilon) = 1$, $\varphi(\upsilon) = \upsilon; \upsilon \in L$, $\mathfrak{R}(x) = \mathfrak{I}_0, \forall x \in PC_g(L, \Phi)$, $\Phi = \mathbb{R}$ and $I_i(x) = 0 \forall x \in \Phi$.

($\star$) The methods used in this work can generalize many of the above mentioned results when the fractional derivative in these results is replaced by the g-weighted φ-Atangana-Baleanu fractional derivative and where the right hand side represents a set-valued function, instead of function in infinite dimensional Banach spaces.

The paper is structured as follows. In Section 2, we introduce the notion of the g-weighted φ-Atangana-Baleanu fractional derivative in the Caputo sense of order γ and with lower limit at a, denoted by ${}^{ABC}D_{a,\upsilon}^{\gamma,\varphi,g}$. We also formulate the problems that will be considered. In Section 3, we present some properties of the new fractional derivative. Section 4 to Section 7 are concerned with the existence and uniqueness of solutions to these problems. The stability of such solutions are discussed in Section 8. Four examples are given in the last section for illustration.

2. Main definition and problem formulation

In this section, we introduce the definition of ${}^{ABC}D_{a,\upsilon}^{\gamma,\varphi,g}$, and present the related initial value problems. The following notations will be used in the rest of the paper.

Notation 1.

- *For each* $r \in \mathbb{N}$, *let* $\mathbb{N}_{0,r} = \{0, 1, 2, .., r\}$, $\mathbb{N}_{1,r} = \{1, 2, .., r\}$.

- $\gamma \in (0,1)$, $L = [0,T]$, *where* $T \in \mathbb{R}^+$, $0 = \theta_0 \leq \theta_1 \leq ... \leq \theta_{r+1} = T$, *and* $0 = \tau_0 < b_1 \leq \tau_1 < b_2 \leq \tau_2 < ... < \tau_r < b_{r+1} = T$ *are two partitions of* L, $L_i = (\tau_i, b_{i+1}]; i \in \mathbb{N}_{0,r}$, *and* $M_i = (b_i, \tau_i]$; $i \in \mathbb{N}_{1,r}$,
- Φ *is a Banach space,* $\Im_0 \in \Phi$ *is a fixed point, and* $z : L \times \Phi \to \Phi$
- $\varphi : L \to \mathbb{R}$ *is a strictly increasing and continuously differentiable function with* $\varphi'(\upsilon) \neq 0, \forall \upsilon \in L$.
- $g : L \to (0,\infty)$ *is a continuously differentiable function and* $g^{-1}(\upsilon) = \frac{1}{g(\upsilon)}; \upsilon \in L$.
- $L_g^{q,\varphi}((0,T),\Phi), q \in [1,\infty)$ *is the Banach space of all Lebesgue measurable functions* z *such that* $zg(\varphi')^{\frac{1}{q}} \in L^q((0,T),\Phi)$, *where* $\|z\|_{L_g^{q,\varphi}((0,T),\Phi)} = (\int_0^T \|z(\upsilon)g(\upsilon)\|^q \varphi'(\upsilon)d\upsilon)^{\frac{1}{q}}$.
- $\varrho_1 = \sup_{\upsilon \in L} |g(\upsilon)|$, *and* $\varrho_2 = \sup_{\upsilon \in L} |\psi(\upsilon)|$,
- M *denote a normalizing function satisfying* $M(0) = M(1) = 1$, *and* $E_\gamma = E_{\gamma,1}$, *where* $E_{\gamma,\beta}$ *is the well known Mittag-Leffler function described by:*

$$E_{\gamma,\beta}(\mu) = \sum_{k=0}^{\infty} \frac{\mu^k}{\Gamma(\gamma k + \beta)}, \beta \in \mathbb{R},\ \mu \in \mathbb{C}.$$

We also fix the notation for the following sets

$P_b(\Phi) := \{Z \subseteq \Phi : Z \text{ is not empty, and bounded }\}$,
$P_{ck}(\Phi) = \{z \subseteq \Phi : z \text{ is not empty, convex and compact}\}$,
$P_{cc}(\Phi) := \{Z \subseteq \Phi : Z \text{ is not empty, convex and closed }\}$.
$H^1((0,T),\Phi) := \{\Im \in L^2((0,T),\Phi) : \Im^{(1)} \in L^2((0,T),\Phi)\}$,
$C(L,\Phi)$ *is the Banach space of continuous functions from* L *to* Φ.
$C_g(L,\Phi) := \{x : L \to \Phi : gx \in C(L,\Phi)\}$,
$PC_g(L,\Phi) := \{\Im : L \to \Phi : \Im g$ *is continuous except at* θ_i, $i \in \mathbb{N}_{0,r}$, $\Im(\theta_i^+)$ *and* $\Im(\theta_i^-)$ *exist with* $\Im(\theta_i) = \Im(\theta_i^-)$; $\forall i \in \mathbb{N}_{0,r}\}$,
$PC_gH^1(L,\Phi) := \{\Im : L \to \Phi, g\Im_{|(\theta_i,\theta_{i+1})} \in H^1((\theta_i,\theta_{i+1}),\Phi)$, $\Im(\theta_i^+)$ *and* $\Im(\theta_i^-)$ *exist with* $\Im(\theta_i) = \Im(\theta_i^-)$; $\forall i \in \mathbb{N}_{1,r}\}$,
$PC_g^*(L,\Phi) = \{\Im : L \to \Phi : g\Im$ *is continuous on*(τ_i, b_{i+1}), $i \in \mathbb{N}_{0,r}$, $\Im$ *is continuous on* (b_i,τ_i), $\Im(\tau_i^+) = \Im(\tau_i^-)$, $\Im(b_i^+)$ *and* $\Im(b_i^-)$ *exist with* $\Im(b_i) = \Im(b_i^-)$; $\forall i \in \mathbb{N}_{1,r}\}$.
$PC_gH^{1,*}(L,\Phi) := \{\Im : L \to \Phi : g\Im_{|(\tau_i,b_{i+1})} \in H^1((\tau_i,b_{i+1}),\Phi); \forall i \in \mathbb{N}_{0,r}$ $\Im_{|(b_i,\tau_i)} \in C((b_i,\tau_i),\Phi), \Im(\tau_i^+) = \Im(\tau_i^-), \forall i \in \mathbb{N}_{1,r}\}$.

The spaces $C_g(L,\Phi), PC_g(L,\Phi), PC_gH^1(L,\Phi), PC_g^*(L,\Phi)$, and $PC_gH^{1,*}(L,\Phi)$ are Banach spaces, where the norms function on them are given by

- $\|x\|_{C_g(L,\Phi)} := \max \|\|g(\theta)x(\theta)\| : \theta \in L\}$.
- $\|x\|_{PC(L,\Phi)} := \max \|\|g(\theta)x(\theta)\| : \theta \in L\}$.
- $\|x\|_{PC_gH^1(L,\Phi)} := \max\{\|g(\theta)x(\theta)\| : \theta \in L\}$.
- $\|\Im\|_{PC_g^*(L,\Phi))} = \max\{\max_{\substack{\upsilon \in [\tau_i,b_{i+1}] \\ i \in \mathbb{N}_{0,r}}} \|g(\upsilon)\Im(\upsilon)\|, \max_{\substack{\upsilon \in [b_i,\tau_i] \\ i \in \mathbb{N}_{1,r}}} \|\Im(\upsilon)\|\}$.
- $\|\Im\|_{PC_gH^{1,*}(L,\Phi)} = \max\{\max_{m \in \mathbb{N}_{0,r}} \|g\ \Im_{|L_m}\|_{H^1(L_m,\Phi)}, \max_{\substack{\upsilon \in [b_m,\tau_m] \\ m \in \mathbb{N}_{1,r}}} \|\Im(\upsilon)\|\}$.

where for any function $z \in H^1((0,T),\Phi)$,

$$D_\upsilon^{1,\varphi,g} z(\upsilon) := g^{-1}(\upsilon)[\frac{1}{\varphi'(\upsilon)} \frac{d}{d\upsilon} g(\upsilon)z(\upsilon)],$$

and for any $k = \mathbb{N} - \{1\}$

$$D_\upsilon^{k,\varphi,g} z(\upsilon) := D_\upsilon^{1,\varphi,g} D_\upsilon^{k-1,\varphi,g} z(\upsilon) = g^{-1}(\upsilon)[\frac{1}{\varphi'(\upsilon)} \frac{d}{d\upsilon}]^k (z(\upsilon)g(\upsilon)), k = \mathbb{N} - \{1\}.$$

We also fix the notations for Ψ, $\Re$, and $\Re_i$ to be the maps $\Psi : L \times \Phi \to P_{ck}(\Phi)$, $\Re : PC_g(L, \Phi) \to \Phi$, and $\Re_i : [b_i, \tau_i] \times \Phi \to \Phi$. For more on the definition of $H^1((0, T), \Phi)$, see [20].

In the following, we introduce the main definitions that will be used to formulate the problems studied in this work.

Definition 1. *Let $a \in (0, T)$, and $\Im : [a, T] \to \Phi$ such that $\Im \in H^1((a, T), \Phi)$.*

(1) The g-weighted φ-Atangana-Baleanu fractional integral for $\Im$ of order γ and with lower limit at a is defined by

$$^{AB}I_{a,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) := \frac{1-\gamma}{M(\gamma)}\Im(\upsilon) + \frac{\gamma}{M(\gamma)} I_{a,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon),\ \upsilon \in L,$$

where

$$I_{a,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) = \frac{1}{g(\upsilon)\Gamma(\gamma)} \int_a^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s)\Im(s)ds.$$

(2) The g-weighted φ-Atangana-Baleanu fractional derivative for $\Im$ of order γ in the Caputo sense, with lower limit at a is defined by

$$\begin{aligned} ^{ABC}D_{a,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) &:= \frac{M(\gamma)}{(1-\gamma)g(\upsilon)} \int_a^\upsilon E_\gamma(\eta_\gamma(\varphi(\upsilon) - \varphi(s))^\gamma)\varphi'(s)g(s)D_s^{1,\varphi,g}\Im(s)ds \\ &= \frac{M(\gamma)}{(1-\gamma)g(\upsilon)} \int_a^\upsilon E_\gamma(\eta_\gamma(\varphi(\upsilon) - \varphi(s))^\gamma)(g\Im)'(s)ds, \end{aligned}$$

where $\eta_\gamma = \frac{-\gamma}{1-\gamma}$

The above definition can be generalize for an order in $(n, n+1)$ foe any $n \in \mathbb{N}$ as follows.

Definition 2. *Let $a \in (0, T)$, and $\Im : [a, T] \to \Phi$ with $\Im^{(n)} \in H^1((a, T), \Phi)$.*

(1) The g-weighted φ-Atangana-Baleanu fractional integral for a function of order $\sigma \in (n, n+1); n \in \mathbb{N}$ with lower limit at a is defined by

$$^{AB}I_{a,\upsilon}^{\sigma,\varphi,g}\Im(\upsilon) := I_{a,\upsilon}^{n,\varphi,g AB}I_{a,\upsilon}^{\sigma-n,g,\varphi}\Im(\upsilon),$$

where

$$I_{a,\upsilon}^{1,\varphi,g}\Im(\upsilon) = g^{-1}(\upsilon) \int_a^\upsilon g(s)\varphi'(s)\Im(s)ds,$$

and $I_{a,\upsilon}^{n,\varphi,g} = I_{a,\upsilon}^{1,\varphi,g} I_{a,\upsilon}^{n-1,\varphi,g}$ if $n \geq 2$.

(2) The g-weighted φ-Atangana-Baleanu fractional derivative for $\Im$ of order $\sigma \in (n, n+1); n \in \mathbb{N}$ in the Caputo sense, with lower limit at a is defined by

$$^{ABC}D_{a,\upsilon}^{\sigma,\varphi,g}\Im(\upsilon) :=^{ABC} D_{a,\upsilon}^{\sigma-n,\varphi,g} D_\upsilon^{n,\varphi,g}\Im(\upsilon).$$

Remark 1.

- *If $g(\upsilon) = 1$ and $\varphi(\upsilon) = \upsilon; \upsilon \in [a, T]$, then Part 2 in Definitions 1, 2 coincide with definitions of the Atangana-Baleanu fractional derivative given by [7].*
- *If $\varphi(\upsilon) = \upsilon; \upsilon \in [a, T]$, then Part 2 in Definitions 1, 2 coincide with Definitions 2.1 and 2.2 in [31].*
- *$g(\upsilon) = 1, ; \upsilon \in [a, T]$, then Definitions 1, 2 coincide with Definitions 2.1, 2.2 in [27]*

Using the above definitions and notations, we form the following initial value problems:

P1. A fractional differential equation containing ${}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}$ in the presence of instantaneous impulses with the lower limit at the initial point 0:

$$\begin{cases} {}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) = z(\upsilon, \Im(\upsilon)), \upsilon \in (\theta_i, \theta_{i+1}), i \in \mathbb{N}_{0,r} \\ \Im(0) = \Im_0 g^{-1}(0) - g^{-1}(0)\Re(\Im), \\ \Im(\theta_i^+) = \Im(\theta_i^-) + g^{-1}(\theta_i^-)I_i(\Im(\theta_i^-)), i \in \mathbb{N}_{1,r}, \end{cases} \tag{2.1}$$

where $I_i : \Phi \to \Phi; i \in \mathbb{N}_{1,r}$.

P2. A fractional differential inclusion involving ${}^{ABC}D_{\theta_i,\upsilon}^{\gamma,\varphi,g}$ in the presence of instantaneous impulses with changing the lower limit at the impulsive points $\theta_i, i \in \mathbb{N}_{0,r}$:

$$\begin{cases} {}^{ABC}D_{\theta_i,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) \in \int_{\theta_i}^{\upsilon} \Psi(s, \Im(s))ds, \upsilon \in (\theta_i, \theta_{i+1}), i \in \mathbb{N}_{0,r}, \\ \Im(0) = \Im_0 g^{-1}(0) - g^{-1}(0)\Re(\Im), \\ \Im(\theta_i^+) = \Im(\theta_i^-) + g^{-1}(\theta_i^-)I_i(\Im(\theta_i^-)),\ i \in \mathbb{N}_{1,r}. \end{cases} \tag{2.2}$$

P3. A fractional differential fractional equation containing ${}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}$ in the presence of non-instantaneous impulses with the lower limit at the initial point 0:

$$\begin{cases} {}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) = z(\upsilon, \Im(\upsilon)), \upsilon \in \cup_{i=0}^{i=r} L_i, \\ \Im(\upsilon) = \Re_i(\upsilon, \Im(b_i^-)), \upsilon \in [b_i, \tau_i];\ i \in \mathbb{N}_{1,r}, \\ \Im(0) = \Im_0 g^{-1}(0) - g^{-1}(0)\Re(\Im). \end{cases} \tag{2.3}$$

P4. A fractional differential inclusion containing ${}^{ABC}D_{\tau_i,\upsilon}^{\gamma,\varphi,g}$ in the presence of non-instantaneous impulses with changing the lower limit at the impulsive points $\tau_i, i \in \mathbb{N}_{0,r}$:

$$\begin{cases} {}^{ABC}D_{\tau_i,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) \in \int_{\tau_i}^{\upsilon} \Psi(s, \Im(s))ds, \upsilon \in \cup_{i=0}^{i=r} L_i, \\ \Im(\upsilon) = \Re_i(\upsilon, \Im(b_i^-)), \upsilon \in [b_i, \tau_i];\ i \in \mathbb{N}_{1,r}, \\ \Im(0) = \Im_0 g^{-1}(0) - g^{-1}(0)\Re(\Im), \end{cases} \tag{2.4}$$

Note that Problems (2.1)–(2.4) are distinct. In fact, the impulses in problems (2.1) and (2.2) are instantaneous while those in problems (2.3) and (2.4) are non-instantaneous. Furthermore, the lower limit of the differential operator $g-$ weighted φ-Atangana-Baleanu fractional derivative in Problems (2.1) and (2.3) are *zero* while those in problems (2.2) and (2.4) are the impulsive points $\theta_i, i \in \mathbb{N}_{0,r}$ and $\tau_i, i \in \mathbb{N}_{0,r}$. In addition, the right hand side in both of (2.1) and (2.3) is a single-valued function, while in problems (2.2) and (2.4) is a multivalued function.

3. Basic Properties of g-weighted φ-Atangana-Baleanu fractional integral and derivative

In order to obtain some properties for ${}^{ABC}D_{a,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon)$, we present it as an infinite series. Note that

$$E_\gamma(\upsilon) = \sum_{k=0}^{\infty} \frac{\upsilon^k}{\Gamma(\gamma k + 1)},\ \gamma > 0,$$

is convergent series for all values of υ.

Lemma 1. *Let* $\Im \in H^1((0,T),\Phi)$*. For any* $\upsilon \in [0,T]$*,*

(i) $I^{1,\varphi,g}_{a,\upsilon}(D^{1,\varphi,g}_{\upsilon}\Im(\upsilon)) = g^{-1}(\upsilon)[g(\upsilon)\Im(\upsilon) - g(a)\Im(a)]$,
(ii) $D^{1,\varphi,g}_{\upsilon}(I^{1,\varphi,g}_{a,\upsilon}\Im(\upsilon)) = \Im(\upsilon)$.

Proof.

To see (i), note that

$$I^{1,\varphi,g}_{a,\upsilon}(D^{1,\varphi,g}_{\upsilon}\Im(\upsilon)) = g^{-1}(\upsilon)\int_a^{\upsilon}\frac{d}{ds}(g(s)\Im(s)]ds = g^{-1}(\upsilon)[g(\upsilon)\Im(\upsilon) - g(a)\Im(a).$$

For (ii)

$$D^{1,\varphi,g}_{\upsilon}(I^{1,\varphi,g}_{a,\upsilon}(\Im(\upsilon)) = g^{-1}(\upsilon)\frac{1}{\varphi'(\upsilon)}\frac{d}{d\upsilon}\int_a^{\upsilon} g(s)\varphi(s)\Im(s)ds = \Im(\upsilon).$$

□

Lemma 2. *Let* $\Im \in H^1((0,T),\Phi)$ *and* $a \in (0,T)$*. For any* $\upsilon \in [a,T]$*,*

$${}^{ABC}D^{\gamma,\varphi,g}_{a,\upsilon}\Im(\upsilon) = \frac{M(\gamma)}{1-\gamma}\sum_{k=0}^{\infty}\eta^k_{\gamma}(I^{\gamma k+1,g,\varphi}_{a,\upsilon}D^{1,\varphi,g}_{s}\Im(s))(\upsilon).$$

Proof. According to Definition 1, we have

$$\begin{aligned}
{}^{ABC}D^{\gamma,\varphi,g}_{a,\upsilon}\Im(\upsilon) &= \frac{M(\gamma)}{(1-\gamma)g(\upsilon)}\int_a^{\upsilon}E_{\gamma}(\eta_{\gamma}(\varphi(\upsilon)-\varphi(s))^{\gamma})\varphi'(s)g(s)D^{1,\varphi,g}_{s}\Im(s)ds\\
&= \frac{M(\gamma)}{(1-\gamma)g(\upsilon)}\int_a^{\upsilon}\sum_{k=0}^{\infty}\frac{\eta^k_{\gamma}(\varphi(\upsilon)-\varphi(s))^{k\gamma}}{\Gamma(\gamma k+1)}\varphi'(s)g(s)D^{1,\varphi,g}_{s}\Im(s)ds\\
&= \frac{M(\gamma)}{(1-\gamma)}\sum_{k=0}^{\infty}\eta^k_{\gamma}\frac{1}{\Gamma(\gamma k+1)g(\upsilon)}\int_a^{\upsilon}(\varphi(\upsilon)-\varphi(s))^{k\gamma}\varphi'(s)g(s)D^{1,\varphi,g}_{s}\Im(s)ds\\
&= \frac{M(\gamma)}{(1-\gamma)}\sum_{k=0}^{\infty}\eta^k_{\gamma}(I^{\gamma k+1,,g,\varphi}_{a,\upsilon}D^{1,\varphi,g}_{s}\Im(s))(\upsilon).
\end{aligned}$$

Since $g\Im \in C^1(L,\Phi)$, this series is convergent for all $\upsilon \in [a,T]$. □

Lemma 3. *If* $\Im \in H^1((a,T),\Phi)$ *and* $g \in C^1(L,(0,\infty))$*, then*

(i) ${}^{AB}I^{\gamma,\varphi,g}_{a,\upsilon}({}^{ABC}D^{\gamma,\varphi,g}_{a,\upsilon}\Im(\upsilon)) = \Im(\upsilon) - g^{-1}(\upsilon)g(a)\Im(a)$.

(ii) ${}^{ABC}D^{\gamma}_{a,\upsilon}({}^{AB}I^{\gamma}_{a,\upsilon}\Im(\upsilon)) = \Im(\upsilon) - g^{-1}(\upsilon)g(a)\Im(a)$.

Proof.

(i) Let $\upsilon \in L$ be fixed. It follows from Lemma 2 that

$${}^{AB}I^{\gamma,\varphi,g}_{a,\upsilon}({}^{ABC}D^{\gamma,\varphi,g}_{a,\upsilon}\Im(\upsilon)) = \frac{1-\gamma}{M(\gamma)}\,{}^{ABC}D^{\gamma,\varphi,g}_{a,\upsilon}\Im(\upsilon) + \frac{\gamma}{M(\gamma)}I^{\gamma,\varphi,g}_{a,\upsilon}({}^{ABC}D^{\gamma,\varphi,g}_{a,\upsilon}\Im(\upsilon))$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \eta_{\gamma}^{k} (I_{a,\upsilon}^{\gamma k+1,,\varphi,g} D_{s}^{1,\varphi,g} \Im(s))(\upsilon) + \frac{\gamma}{1-\gamma} \sum_{k=0}^{\infty} \eta_{\gamma}^{k} I_{a,\upsilon}^{\gamma,\varphi,g} I_{a,\upsilon}^{\gamma k+1,g,\varphi} D_{s}^{1,\varphi,g} \Im(s))(\upsilon) \\
&= \sum_{k=0}^{\infty} \eta_{\gamma}^{k} (I_{a,\upsilon}^{\gamma k+1,\varphi,g} D_{s}^{1,\varphi,g} \Im(s))(\upsilon) - \eta_{\gamma} \sum_{k=0}^{\infty} \eta_{\gamma}^{k} I_{a,\upsilon}^{\gamma(k+1)+1,g,\varphi} D_{s}^{1,g,\varphi} \Im(s))(\upsilon) \\
&= \sum_{k=0}^{\infty} \eta_{\gamma}^{k} (I_{a,\upsilon}^{\gamma k+1,,g,\varphi} D_{s}^{1,g,\varphi} \Im(s))(\upsilon) - \sum_{k=0}^{\infty} \eta_{\gamma}^{k+1} I_{a,\upsilon}^{\gamma(k+1)+1,g,\varphi} D_{s}^{1,g,\varphi} \Im(s))(t) \\
&= \sum_{k=0}^{\infty} \eta_{\gamma}^{k} (I_{a,t}^{\gamma k+1,,g,\varphi} D_{s}^{1,g,\varphi} \Im(s))(t) - \sum_{k=1}^{\infty} \eta_{\gamma}^{k} (I_{a,t}^{\gamma k+1,,g,\varphi} D_{s}^{1,g,\varphi} \Im(s))(t) \\
&= I_{a,t}^{1,g,\varphi} D_{s}^{1,g,\varphi} \Im(s))(t) = g^{-1}(t) \int_{a}^{t} g(s)\Im(s) ds = g^{-1}(t)(g(t)\Im(t) - g(0)\Im(0)).
\end{aligned}
$$

(ii) follows since

$$
\begin{aligned}
{}^{ABC}D_{a,\upsilon}^{\gamma,\varphi,g}({}^{ABC}I_{a,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon)) &= \frac{M(\gamma)}{(1-\gamma)} \sum_{k=0}^{\infty} \eta_{\gamma}^{k}\ I_{a,\upsilon}^{\gamma k+1,\varphi,g} (D_{s}^{1,g,\varphi}({}^{AB}I_{a,s}^{\gamma,\varphi,g}\Im(s)))(\upsilon) \\
&= \frac{M(\gamma)}{(1-\gamma)} \sum_{k=0}^{\infty} \eta_{\gamma}^{k}\ I_{a,\upsilon}^{\gamma k+1,\varphi,g} (D_{s}^{1,g,\varphi}[\frac{1-\gamma}{M(\gamma)}\Im(s) + \frac{\gamma}{M(\gamma)} I_{a,s}^{\gamma,\varphi,g}\Im(s)](\upsilon)) \\
&= \sum_{k=0}^{\infty} \eta_{\gamma}^{k}\ I_{a,\upsilon}^{\gamma k+1,\varphi,g} (D_{s}^{1,\varphi,g}\Im(s))(\upsilon) - \eta_{\gamma} \sum_{k=0}^{\infty} \eta_{\gamma}^{k}\ I_{a,\upsilon}^{\gamma k+1,\varphi,g} (D_{s}^{1,g,\varphi} I_{a,s}^{\gamma,\varphi,g}\Im(s))(\upsilon) \\
&= \sum_{k=0}^{\infty} \eta_{\gamma}^{k}\ I_{a,\upsilon}^{\gamma k,\varphi,g} I_{a,\upsilon}^{1,\varphi,g} (D_{s}^{1,\varphi,g}\Im(s))(\upsilon) - \eta_{\gamma} \sum_{k=0}^{\infty} \eta_{\gamma}^{k}\ I_{a,\upsilon}^{\gamma k,\varphi,g} I_{a,\upsilon}^{1,\varphi,g} (D_{s}^{1,\varphi,g} I_{a,s}^{\gamma,\varphi,g}\Im(s))(\upsilon) \\
&= \sum_{k=0}^{\infty} \eta_{\gamma}^{k}\ I_{a,\upsilon}^{\gamma k,\varphi,g}((g\Im)(t) - g(a)\Im(a)) - \eta_{\gamma} \sum_{k=0}^{\infty} \eta_{\gamma}^{k}\ I_{a,\upsilon}^{\gamma k,\varphi,g}[I_{a,\upsilon}^{\gamma,g,\varphi}(\Im g)(\upsilon) - I_{a,\upsilon}^{\gamma,\varphi,g}(\Im g)(a)] \\
&= \sum_{k=0}^{\infty} \eta_{\gamma}^{k}\ I_{a,\upsilon}^{\gamma k,\varphi,g}((g\Im)(\upsilon) - g(a)\Im(a)) - \eta_{\gamma} \sum_{k=0}^{\infty} \eta_{\gamma}^{k+1}\ I_{a,\upsilon}^{\gamma(k+1),\varphi,g}[(\Im g)(\upsilon) - (\Im g)(a)] \\
&= \sum_{k=0}^{\infty} \eta_{\gamma}^{k}\ I_{a,\upsilon}^{\gamma k,\varphi,g}((g\Im)(\upsilon) - g(a)\Im(a)) - \sum_{k=1}^{\infty} \eta_{\gamma}^{k}\ I_{a,\upsilon}^{\gamma k,\varphi,g}[(\Im g)(\upsilon) - (\Im g)(a)] \\
&= g^{-1}(\upsilon)((g\Im)(\upsilon) - g(a)\Im(a)) = \Im(\upsilon) - g(a)\Im(a)g^{-1}(\upsilon).
\end{aligned}
$$

□

To improve the readability of the results in the following sections, we list the following hypothesis.

Hypothesis 1.

◇ (**HΨ**) *The map* $\Psi : L \times \Phi \to P_{ck}(\Phi)$ *satisfies*

(i) For every $x \in \Phi$, $\theta \to \Psi(\theta, x)$ *is measurable.*

(ii) For almost $\theta \in L$, $x \to \Psi(\theta, x)$ *is upper semi-continuous.*

(iii) There is a function $\tau \in L^2(J, \mathbb{R}^+)$ with for any $x \in \Phi$,

$$\sup_{y\in\Psi(\upsilon,x)} \|y\| \leq \tau(\upsilon)(1 + \|x\|),\ a.e.,\ for \upsilon \in L. \tag{3.1}$$

(iv) There is a function $\eta \in L^1(L, \mathbb{R}^+)$ such that for any bounded subset $B \subset \Phi$,

$$\varkappa(\Psi(\upsilon, B)) \leq g(\upsilon)\eta(\upsilon)\varkappa(B),\ for\ \upsilon \in L, \tag{3.2}$$

and

$$4\rho_1\rho_3\left(\frac{1-\gamma}{M(\gamma)} + \frac{6\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)}\right) < 1, \tag{3.3}$$

where $\varkappa$ is the measure of noncompactness on Φ, $\rho_1 = sup_{\upsilon\in L} g(\upsilon)$, and $\rho_3 = \|\eta\|_{L^1(L,\mathbb{R}^+)}$.

The norm of the function τ will be denoted by ρ_2, i.e., $\rho_2 = \|\tau\|_{L^1(L,\mathbb{R}^+)}$.

- (**Hz**) *The map $z : L \times \Phi \to \Phi$ is a function such that*
  - *(i) For any $\Im \in PC_g(L, \Phi)$, the function $v(\upsilon) = z(\upsilon, \Im(\upsilon)) \in PCH^1((0, T), \Phi)$ and $z(0, \Im(0)) = 0$.*
  - *(ii) For any $\upsilon \in L, x \to z(\upsilon, x)$ is uniformly continuous on bounded sets.*
  - *(iii) There is a continuous function ψ such that*

$$\|z(\upsilon, x)\| \leq \psi(\upsilon)(1 + \|x\|),\ \forall(\upsilon, x) \in L \times \Phi. \tag{3.4}$$

  - *(iv) There is a continuous function $\eta : L \to \mathbb{R}^+$ such that for any $B \in P_b(\Phi)$,*

$$\varkappa(z(\upsilon, B)) \leq \eta(\upsilon)\varkappa(B),\ for\ \upsilon \in L, \tag{3.5}$$

and

$$\kappa + 2\kappa\frac{\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)} < 1, \tag{3.6}$$

Here $\kappa = \sup_{\upsilon\in L} \eta(\upsilon)$

- (**Hz**)* *The following are satisfied*
  - *(i) For any $\Im \in PC_g(L, \Phi)$, the function $W(\upsilon) = z(\upsilon, \Im(\upsilon))$ is in $PCH^1((0, T), \Phi)$ and $W(0) = 0$.*
  - *(ii) There is $\xi_1 > 0$ such that, for any $\upsilon \in L$, and any $\Im, \varsigma \in PC_g(L, \Phi)$,*

$$\|z(\upsilon, \Im(\upsilon)) - z(\upsilon, \varsigma(\upsilon)\| \leq \xi_1\|\Im - \varsigma\|_{PC_g(L,\Phi)}.$$

- (**Hz**)** *The following hold*
  - *(i) For any $\Im \in PC^*_g(L, \Phi)$, the function $W(\upsilon) = z(\upsilon, \Im(\upsilon))$ is in $PCH^{1,*}((0, T), \Phi)$ and $W(0) = 0$*
  - *(ii) There is $\xi_1 > 0$ such that, for any $\upsilon \in L$, and any $\Im, \varsigma \in PC^*_g(L, \Phi)$,*

$$\|z(\upsilon, \Im(\upsilon)) - z(\upsilon, \varsigma(\upsilon)\| \leq \xi_1\|\Im(\upsilon) - \varsigma(\upsilon)\|_\Phi.$$

- (**H$\mathfrak{R}$**) *The function $\mathfrak{R} : PC_g(L, \Phi) \to \Phi$ is continuous, compact and there are two positive real numbers c, d such that*

$$\|\mathfrak{R}(x)| \leq c\|x\|_{PC_g(L,\Phi)} + d, \forall x \in PC_g(L, \Phi). \tag{3.7}$$

◇ $(\mathbf{H}\mathfrak{R})^*$ *There is* $\xi_2 > 0$ *such that for any* $\mathfrak{I}, \varsigma \in PC_g(L, \Phi)$,

$$\|\mathfrak{R}(\mathfrak{I}) - \mathfrak{R}(\varsigma)\| \leq \xi_2 \|\mathfrak{I} - \varsigma\|_{PC_g(L,\Phi).}$$

◇ $(\mathbf{H}\mathfrak{R})^{**}$ *The function* $\mathfrak{R} : PC_g^*(L, \Phi) \to \Phi$ *is continuous, compact and there two positive real numbers* c, d *such that*

$$\|\mathfrak{R}(x)| \leq c\|x\|_{PC_g^*(L,\Phi)} + d, \forall x \in PC_g(L, \Phi).$$

◇ $(\mathbf{HI})$ *For every* $i = 1, 2, .., r$, $I_i : \Phi \to \Phi$ *is continuous and compact on bounded subsets, and there is* $\lambda > 0$ *with*

$$\|I_i(\mathfrak{I}(\upsilon))\| \leq \lambda g(\upsilon)\|\mathfrak{I}(\upsilon)\|; \upsilon \in L. \tag{3.8}$$

◇ $(\mathbf{HI})^*$ *There is* $\xi_3 > 0$ *such that for any* $\mathfrak{I}, \varsigma \in PC_g(L, \Phi)$, *and any* $i \in \mathbb{N}_{1,r}$

$$\left\|I_i(\mathfrak{I}(\theta_i) - I_i(\varsigma(\theta_i))\right\| \leq \xi_3 \|\mathfrak{I} - \varsigma\|_{PC_g(L,\Phi)}.$$

◇ $(\mathbf{H}\mathfrak{R}_\mathbf{i})$ *There is* $\xi_4 > 0$ *such that for any* $\mathfrak{I}, \varsigma \in PC_g^*(L, \Phi)$ *and any* $i \in \mathbb{N}_{1,r}$,

$$\|\mathfrak{R}_i(\upsilon, \mathfrak{I}(b_i^-)) - \mathfrak{R}_i(\upsilon, \varsigma(b_i^-))\| \leq \xi_4 \|\mathfrak{I} - \varsigma\|_{PC_g(L,\Phi)}. \tag{3.9}$$

◇ $(\mathbf{H}\mathfrak{R}_\mathbf{i})^*$ *For every* $i \in \mathbb{N}_{1,r}$ $\mathfrak{R}_i : [b_i, \tau_i] \times \Phi \to \Phi$ *is such that, for any* $\upsilon \in [b_i, \tau_i]$, *the function* $x \to \mathfrak{R}_i(\upsilon, x)$ *is uniformly continuous and compact on bounded subsets.*

4. Existence of solutions of Problem (2.1)

In the following Lemma, we state and prove the relationship between solutions of Problem (2.1) and those of the corresponding fractional integral equation.

Lemma 4. *Let* $W : L \to \Phi$ *be continuous with* $W(0) = 0$ *and* $g \in C^1(L, (0, \infty))$.

(1) If $\mathfrak{I} \in PCH^1((0, T), \Phi)$ *is a solution for the following nonlocal impulsive g-weighted* φ*-Attangana- Baleanu:*

$$\begin{cases} {}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\mathfrak{I}(\upsilon) = W(\upsilon), \upsilon \in (\theta_i, \theta_{i+1}), i \in \mathbb{N}_{0,r}, \\ \mathfrak{I}(0) = \mathfrak{I}_0 g^{-1}(0) - g^{-1}(0)\mathfrak{R}(\mathfrak{I}), \\ \mathfrak{I}(\theta_i^+) = \mathfrak{I}(\theta_i^-) + g^{-1}(\theta_i^-)I_i(\mathfrak{I}(\theta_i^-)),\ i \in \mathbb{N}_{1,r}, \end{cases} \tag{4.1}$$

then, for any $\upsilon \in L$

$$\mathfrak{I}(\upsilon) = \begin{cases} \mathfrak{I}_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\mathfrak{R}(\mathfrak{I}) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} v(\upsilon), \upsilon \in [0, \theta_1], \\ \\ g^{-1}(\upsilon)\mathfrak{I}_0 - g^{-1}(\upsilon)\mathfrak{R}(\mathfrak{I})) + g^{-1}(\upsilon)\sum_{k=1}^{k=i} I_k(\mathfrak{I}(\theta_k^-)) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} v(\upsilon), \upsilon \in (\theta_i, \theta_{i+1}],\ i \in \mathbb{N}_{1,r}. \end{cases} \tag{4.2}$$

(2) If $W \in PCH^1((0, T), \Phi)$ *and* $\mathfrak{I}$ *is defined by (4.2), then* $\mathfrak{I} \in PCH^1((0, T), \Phi)$ *and* $\mathfrak{I}$ *is a solution for (4.1).*

Proof.

(1) Suppose that $\Im \in H^1((0,T),\Phi)$ is the solution of (4.1) and let $\upsilon \in (\theta_0,\theta_1)$, then

$$\begin{cases} {}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) = W(\upsilon), \upsilon \in (0,\theta_1], \\ \Im(0) = \Im_0 g^{-1}(0) - g^{-1}(0)\Re(\Im). \end{cases} \tag{4.3}$$

Applying ${}^{AB}I_{0,\upsilon}^{\gamma,g,\varphi}$ on both sides of (4.3), we get by Lemma (3),

$$\begin{aligned} \Im(\upsilon) &= \Im(0)g(0)g^{-1}(\upsilon) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g}W(\upsilon) \\ &= g(0)g^{-1}(\upsilon)(\Im_0 g^{-1}(0) - g^{-1}(0)\Re(\Im)) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g}W(\upsilon) \\ &= \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g}W(\upsilon); \forall \upsilon \in [0,\theta_1]. \end{aligned}$$

Let $\upsilon \in (\theta_1,\theta_2]$. The equations in (4.1) gives,

$$\begin{cases} {}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) = W(\upsilon), \upsilon \in (\theta_1,\theta_2], \\ \Im(\theta_1) = \Im(\theta_1^-) + g^{-1}(\theta_1)I_1(\Im(\theta_1^-)). \end{cases}$$

Applying ${}^{AB}I_{0,\upsilon}^{\gamma,g,\varphi}$ on both sides of ${}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) = W(\upsilon)$ and using Lemma (3), we get

$$\Im(\upsilon) = c_1 g^{-1}(\upsilon) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g}W(\upsilon); \forall \upsilon \in (\theta_1,\theta_2]. \tag{4.4}$$

To find c_1, we use the boundary condition $\Im(\theta_1) = \Im(\theta_1^-) + g^{-1}(\theta_1)I_1(\Im(\theta_1^-))$, to obtain,

$$c_1 g^{-1}(\theta_1) +^{AB} I_{0,\theta_1}^{\gamma,\varphi,g}W(\upsilon) = \Im_0 g^{-1}(\theta_1) - g^{-1}(\theta_1)\Re(\Im) +^{AB} I_{0,\theta_1}^{\gamma,\varphi,g}W(\upsilon) + g^{-1}(\theta_1)I_1(\Im(\theta_1^-)),$$

where

$$c_1 = \Im_0 - \Re(\Im) + I_1(\Im(\theta_1^-)).$$

Substitution the value of c_1 into (4.4) gives

$$\Im(\upsilon) = \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) + g^{-1}(\upsilon)I_1(\Im(\theta_1^-)) + I_{0,\upsilon}^{\gamma,\varphi,g}W(\upsilon); \forall \upsilon \in (\theta_1,\theta_2].$$

Next, let $\upsilon \in (\theta_2,\theta_3]$. Applying ${}^{AB}I_{0,\upsilon}^{\gamma,g,\varphi}$ on both sides of ${}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) = W(\upsilon)$ and using Lemma (3), to get

$$\Im(\upsilon) = c_2 g^{-1}(\upsilon) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g}(\upsilon); \forall \upsilon \in (\theta_2,\theta_3]. \tag{4.5}$$

Using the boundary condition $\Im(\theta_2^+) = \Im(\theta_2^-) + g^{-1}(\theta_2)I_2(\Im(\theta_2^-))$, we obtain

$$c_2 g^{-1}(\theta_2) +^{AB} I_{0,\theta_2}^{\gamma,\varphi,g}(\upsilon) = \Im_0 g^{-1}(\theta_2) - g^{-1}(\theta_2)\Re(\Im) + g^{-1}(\theta_2)I_1(\Im(\theta_1^-)) +^{AB} I_{0,\theta_2}^{\gamma,\varphi,g}(\upsilon) + g^{-1}(\theta_2)I_2(\Im(\theta_2^-)),$$

and therefore,

$$c_2 = \Im_0 - \Re(\Im) + I_1(\Im(\theta_1^-)) + I_2(\Im(\theta_2^-)).$$

Substitution the value of c_1 into (4.5) gives

$$\Im(\upsilon) = g^{-1}(\upsilon)\Im_0 - g^{-1}(\upsilon)\Re(\Im) + g^{-1}(\upsilon)I_1(\Im(\theta_1^-)) + g^{-1}(\upsilon)I_2(\Im(\theta_2^-)) +^{AB} I_{0,\upsilon}^{\gamma,g,\varphi}W(\upsilon); \forall \upsilon \in (\theta_2,\theta_3].$$

The same arguments leads to relation (4.2) for $i \geq 2$.

(2) Let $\Im$ given by (4.2) and $W(0) = 0$. Clearly, $\Im \in PCH^1(L, \Phi)$. If $\upsilon \in [0, \theta_1]$, then

$$\Im(\upsilon) = \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W(\upsilon).$$

By applying Lemma 3, and since $W \in PCH^1(L, \Phi)$ and $W(0) = 0$, we obtain

$$\begin{aligned} {}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) &= {}^{ABC}\ D_{0,\upsilon}^{\gamma,\varphi,g}(\Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W(\upsilon) \\ &= {}^{ABC}\ D_{0,\upsilon}^{\gamma,\varphi,g AB} I_{0,\upsilon}^{\gamma,\varphi,g} W(\upsilon) \\ &= g^{-1}(\upsilon) g(0) W(0) + W(\upsilon) = W(\upsilon), \forall \upsilon \in (0, \theta_1]. \end{aligned}$$

Moreover, $\Im(0) = \Im_0 g^{-1}(0) - g^{-1}(0)\Re(\Im)$.
Similarly, ${}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) = W(\upsilon), \upsilon \in (\theta_i, \theta_{i+1}], i \in \mathbb{N}_{1,r}$. Note that

$$\Im(\theta_i^+) = \Im(\theta_i^+) = \Im(\theta_i^-) + g^{-1}(\theta_i^-) I_i(\Im(\theta_i^-)),\ i \in \mathbb{N}_{1,r},$$

which completes the proof.

□

Remark 2. *When $g(\upsilon) = 1$ and $\varphi(\upsilon) = \upsilon, \forall \upsilon \in L$, the formula of solution function (4.3) coincides with the formula (2.8) in [23].*

Based on Lemma 4, we have the following corollary

Corollary 1. *A function $\Im \in PCH^1((0, T), \Phi)$ is a solution of Problem (2.1) if $z(0, \Im(0)) = 0$ and $\Im$ satisfies the fractional integral equation:*

$$\Im(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) + \frac{1-\gamma}{M(\gamma)} z(\upsilon, \Im(\upsilon)) \\ + \frac{\gamma}{M(\gamma)} \frac{1}{g(\upsilon)\Gamma(\gamma)} \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s, \Im(s)) ds, \upsilon \in [0, \theta_1] \\ \\ g^{-1}(\upsilon)\Im_0 - g^{-1}(\upsilon)\Re(\Im)) + g^{-1}(\upsilon) \sum_{k=1}^{k=i} I_k(\Im(\theta_k^-)) + \frac{1-\gamma}{M(\gamma)} z(\upsilon, \Im(\upsilon)) \\ + \frac{\gamma}{M(\gamma)} \frac{1}{g(\upsilon)\Gamma(\gamma)} \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s, \Im(s)) ds\ , \upsilon \in (\theta_i, \theta_{i+1}],\ i \in \mathbb{N}_{1,r}, \end{cases} \tag{4.6}$$

Theorem 1. *If $(Hz), (H\Re)$ and (HI) hold, then Problem (2.1) has a solution provided that*

$$c + \frac{1-\gamma}{M(\gamma)} \varrho_1 \varrho_2 + \lambda r + \frac{\varrho_2 \varrho_1 \varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)} < 1, \tag{4.7}$$

Proof. Let $\Upsilon : PC_g(L, \Phi) \to PC_g(L, \Phi)$ defined by:

$$\Upsilon(\Im)(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) + \frac{1-\gamma}{M(\gamma)} z(\upsilon, \Im(\upsilon)) \\ + \frac{\gamma}{M(\gamma)} \frac{1}{g(\upsilon)\Gamma(\gamma)} \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s, \Im(s)) ds, \upsilon \in [0, \theta_1] \\ \\ g^{-1}(\upsilon)\Im_0 - g^{-1}(\upsilon)\Re(\Im)) + g^{-1}(\upsilon) \sum_{k=1}^{k=i} I_k(\Im(\theta_k^-)) + \frac{1-\gamma}{M(\gamma)} z(\upsilon, \Im(\upsilon)) \\ + \frac{\gamma}{M(\gamma)} \frac{1}{g(\upsilon)\Gamma(\gamma)} \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s, \Im(s)) ds, \upsilon \in (\theta_i, \theta_{i+1}],\ i \in \mathbb{N}_{1,r}. \end{cases} \tag{4.8}$$

Note that, if $\Im$ is a fixed point for Υ, then it will be in the form of (4.6). If we define $W : L \to \Phi; W(\upsilon) = z(\upsilon, \Im(\upsilon))$, then

$$\Im(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W(\upsilon), \upsilon \in [0, \theta_1] \\ \\ g^{-1}(\upsilon)\Im_0 - g^{-1}(\upsilon)\Re(\Im)) + g^{-1}(\upsilon) \sum_{k=1}^{k=i} I_k(\Im(\theta_k^-)) \\ +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W(\upsilon), \upsilon \in (\theta_i, \theta_{i+1}],\ i \in \{1, 3, ..., r\}. \end{cases} \tag{4.9}$$

But, assumption $(Hz)(i)$ gives that $W \in PCH^1((0,T), \Phi)$ and $W(0) = 0$. Therefore, by the second part of Lemmas (4) and (4.9), $\Im$ will be a solution of Problem (2.1). We will use Schauder's fixed point, after establishing the following claims, to prove that Υ has a fixed point.

Claim 1. *There is a natural number k_0 such that $\Upsilon(\Delta_{k_0}) \subseteq \Delta_{k_0}$, where* $\Delta_{k_0} = \{u \in PC_g(L, \Phi) : \|gu\|_{PC_g(L,\Phi)} \leq k_0\}$.

Pf: If this is not true, then for every natural number n there is $\Im_n$ with $\|\Im_n\|_{PC_g(L,\Phi)} \leq n$, but $\|\Upsilon(\Im_n)\|_{PC_g(L,\Phi)} > n$. Let $\upsilon \in [0, \theta_1]$. Using (3.4), (3.7) and (4.9), we obtain

$$\begin{aligned} \|g(\upsilon)\Upsilon(\Im_n)(\upsilon)\| \leq \|\Im_0\| + \ c\,n + d + \frac{1-\gamma}{M(\gamma)}\varrho_1\varrho_2(1+n) \\ + \frac{\gamma\varrho_2\varrho_1(1+n)}{M(\gamma)\Gamma(\gamma)} \int_0^{\upsilon} (\varphi(\upsilon) - \varphi(s))^{\gamma-1}\varphi'(s)ds \\ \leq \|\Im_0\| + \ cn + d + \frac{1-\gamma}{M(\gamma)}\varrho_1\varrho_2(1+n) + \frac{\varrho_2\varrho_1(1+n)\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}. \end{aligned} \tag{4.10}$$

Let $\upsilon \in (\theta_i, \theta_{i+1}],\ i \in \mathbb{N}_{1,r}$. It follows from (3.4), (3.7), (3.8) and (4.8) that

$$\begin{aligned} \|g(\upsilon)\Upsilon(\Im_n)(\upsilon)\| \leq \|\Im_0\| + \ cn + d + \frac{1-\gamma}{M(\gamma)}\varrho_1\varrho_2(1+n) \\ + \lambda rn + \frac{\varrho_2\varrho_1(1+n)\varphi(T)^{\gamma}}{M(\gamma)g(\upsilon)\Gamma(\gamma)}. \end{aligned} \tag{4.11}$$

From (4.10) and (4.11), we get

$$n < \|\Upsilon(x)\|_{PC_g(L,\Phi)} \leq \|\Im_0\| + cn + d + \frac{1-\gamma}{M(\gamma)}\varrho_1\varrho_2(1+n) + \lambda rn + \frac{\varrho_2\varrho_1(1+n)\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}. \tag{4.12}$$

By dividing both sides of (4.12) by n and then, taking the limit when $n \to \infty$, we get

$$1 < c + \frac{1-\gamma}{M(\gamma)}\varrho_1\varrho_2 + \lambda r + \frac{\varrho_2\varrho_1\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}, \tag{4.13}$$

which contradicts (4.7). ¶

Claim 2. $\Upsilon : \Delta_{k_0} \to \Delta_{k_0}$ *is continuous.*

Pf: Suppose that $\Im_m \in B_{k_0}, \Im_m \to \Im$. By definition of Υ,

$$\Upsilon(\Im_m)(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im_m) + \frac{1-\gamma}{M(\gamma)} z(\upsilon, \Im_m(\upsilon)) \\ +\frac{\gamma}{M(\gamma)} \frac{1}{g(\upsilon)\Gamma(\gamma)} \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s, \Im_m(s)) ds, \upsilon \in [0, \theta_1] \\ \\ g^{-1}(\upsilon)\Im_0 - g^{-1}(\upsilon)\Re(\Im_m)) + g^{-1}(\upsilon) \sum_{k=1}^{k=i} I_k(\Im_m(\theta_k^-)) + \frac{1-\gamma}{M(\gamma)} z(\upsilon, \Im_m(\upsilon)) \\ +\frac{\gamma}{M(\gamma)} \frac{1}{g(\upsilon)\Gamma(\gamma)} \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s, \Im_m(s)) ds, \upsilon \in (\theta_i, \theta_{i+1}], \ i \in \mathbb{N}_{1,r}. \end{cases} \tag{4.14}$$

By the continuity of z, $\Re$ and $I_k, k \in \mathbb{N}_{1,r}$, we obtain from Lebesgue dominated convergence theorem that $\lim_{m\to\infty} \Upsilon(\Im_m) = \Upsilon(\Im)$ in $PC_g(L, \Phi)$.¶

Claim 3. *The sets* $\Delta_1|_{\overline{L_i}}$, *for any* $i \in \mathbb{N}_{0,r}$, *where* $\Delta_1 = \Upsilon(\Delta_{k_0})$ *and*

$$\Delta_1|_{\overline{L_i}} = \{\Im^* \in (\overline{L_i}, \Phi) : \Im^*(\upsilon) = g(\upsilon)\Im(\upsilon), \upsilon \in (\upsilon_i, \upsilon_{i+1}], \Im^*(\upsilon_i) = \lim_{\upsilon \to \upsilon_i^+} g(\upsilon)\Im(\upsilon), \Im \in \Delta_1\}.$$

are equicontinuous.

Pf: Let $\Im = \Upsilon(W);\ W \in \Delta_{k_0}$. Then

$$\Im(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(W) + \frac{1-\gamma}{M(\gamma)} z(\upsilon, W(\upsilon)) \\ +\frac{\gamma}{M(\gamma)} \frac{1}{g(\upsilon)\Gamma(\gamma)} \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s, \Im(s)) ds, \upsilon \in [0, \theta_1] \\ \\ g^{-1}(\upsilon)\Im_0 - g^{-1}(\upsilon)\Re(\Im)) + g^{-1}(\upsilon) \sum_{k=1}^{k=i} I_k(\Im(\theta_k^-)) + \frac{1-\gamma}{M(\gamma)} z(\upsilon, \Im(\upsilon)) \\ +\frac{\gamma}{M(\gamma)} \frac{1}{g(\upsilon)\Gamma(\gamma)} \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s, \Im(s)) ds, \upsilon \in (\theta_i, \theta_{i+1}], \ i \in \mathbb{N}_{1,r}. \end{cases}$$

and so,

$$\Im^*(\upsilon) = \begin{cases} \Im_0 - \Re(W) + \frac{1-\gamma}{M(\gamma)} g(\upsilon) z(\upsilon, W(\upsilon)) \\ +\frac{\gamma}{M(\gamma)\Gamma(\gamma)} \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s, \Im(s)) ds, \upsilon \in [0, \theta_1] \\ \\ \Im_0 - \Re(\Im)) + \sum_{k=1}^{k=i} I_k(\Im(\theta_k^-)) + \frac{1-\gamma}{M(\gamma)} g(\upsilon) z(\upsilon, \Im(\upsilon)) \\ +\frac{\gamma}{M(\gamma)\Gamma(\gamma)} \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s, \Im(s)) ds, \upsilon \in (\theta_i, \theta_{i+1}], \ i \in \mathbb{N}_{1,r}. \end{cases} \tag{4.15}$$

Case 1. $i = 0$. Let $\upsilon, \upsilon + \delta$ be two points in $\overline{L_0} = [0, \theta_1]$. From (4.15) and the uniform continuity of z on bounded sets, we obtain

$$\begin{aligned} &\lim_{\delta\to 0} \|\Im^*(\upsilon + \delta) - \Im^*(\upsilon)\| \\ &\le \frac{1-\gamma}{M(\gamma)} \lim_{\delta\to 0} [g(\upsilon + \delta) z(\upsilon + \delta, W(\upsilon + \delta)) - g(\upsilon) z(\upsilon, W(\upsilon))] \\ &+ \lim_{\delta\to 0} \frac{\gamma}{M(\gamma)\Gamma(\gamma)} \| \int_0^{\upsilon+\delta} (\varphi(\upsilon + \delta) - \varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s, \Im(s)) ds \\ &- \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s, \Im(s))\| \\ &= \frac{\gamma}{M(\gamma)\Gamma(\gamma)} \lim_{\delta\to 0} \int_0^\upsilon |((\varphi(\upsilon + \delta) - \varphi(s))^{\gamma-1} \varphi'(s) \end{aligned}$$

$$- (\varphi(\upsilon) - \varphi(s))^{\gamma-1})\varphi'(s)| \, \|g(s)z(s, \Im(s))\|ds$$
$$+ \frac{\gamma}{M(\gamma)\Gamma(\gamma)} \lim_{\delta\to 0} \int_{\upsilon}^{\upsilon+\delta} |((\varphi(\upsilon+\delta) - \varphi(s))^{\gamma-1}\varphi'(s)| \, \|g(s)z(s, \Im(s))\|ds$$
$$\leq \frac{\varrho_1\varrho_2\gamma(1+k_0)}{M(\gamma)\Gamma(\gamma)} \lim_{\delta\to 0} \int_0^{\upsilon} |((\varphi(\upsilon+\delta) - \varphi(s))^{\gamma-1}\varphi'(s) - (\varphi(\upsilon) - \varphi(s))^{\gamma-1})\varphi'(s)ds|$$
$$+ \frac{\varrho_1\varrho_2\gamma(1+k_0)}{M(\gamma)\Gamma(\gamma)} \lim_{\delta\to 0} \int_{\upsilon}^{\upsilon+\delta} |((\varphi(\upsilon+\delta) - \varphi(s))^{\gamma-1}\varphi'(s)|ds = 0,$$

independently of $\Im$.

Case 2. $i \in \mathbb{N}_{1,r}$. Let $\upsilon, \upsilon+\delta$ be two points in $(\theta_i, \theta_{i+1}]$. Using the same arguments as in Case 1, we have,

$$\lim_{\delta\to 0} \|\Im^*(\upsilon+\delta) - \Im^*(\upsilon)\| \leq \frac{1-\gamma}{M(\gamma)} \lim_{\delta\to 0}[g(\upsilon+\delta)z(\upsilon+\delta, W(\upsilon+\delta)) - g(\upsilon)z(\upsilon, W(\upsilon))]$$
$$+ \lim_{\delta\to 0}[g^{-1}(\upsilon+\delta) - g^{-1}(\upsilon+\delta)] \sum_{k=1}^{k=i} I_k(\Im(\theta_k^-))$$
$$+ \lim_{\delta\to 0} \frac{\gamma}{M(\gamma)\Gamma(\gamma)} \| \int_0^{\upsilon+\delta} (\varphi(\upsilon+\delta) - \varphi(s))^{\gamma-1} g(s)\varphi'(s)z(s, \Im(s))ds$$
$$- \int_0^{\upsilon} (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s)z(s, \Im(s))\| = 0,$$

independently of $\Im$, and the claim is proved. ¶

Claim 4. *The set* $\Delta = \cap_{k=1}\Delta_k$ *is compact, where* $\Delta_0 = \Delta_{k_0}$ *and* $\Delta_k = \Upsilon(\Delta_{k-1}), k \geq 1$.

Pf: From Claim 1 and Claim 2, $\Delta_k; k \geq 1$ is a non-empty, convex, bounded, and closed set; thus, Δ is bounded and closed. Moreover, $\Delta_2 = \Upsilon(\Delta_1) = \Upsilon(\Upsilon(\Delta_{k_0})) \subseteq \Upsilon(\Delta_{k_0}) = \Delta_1$. So, by induction, (Δ_k) is a non-increasing sequence. We show that Δ is relatively compact, and hence it is compact. By the Generalized Cantor's intersection property [35], it is enough to prove that

$$\lim_{n\to\infty} \varkappa_{PC_g}(\Delta_n) = 0, \tag{4.16}$$

where $\varkappa_{PC_g}$ is the Hausdorff measure of noncompactness in $PC_g(L, \Phi)$ which is defined by:
$\chi_{PC_g(L,\Phi))} : P_b(PC_g(L, \Phi)) \to [0, \infty)$,

$$\chi_{PC_g(L,\Phi)}(D) := \max_{i\in\mathbb{N}_{0,r}} \chi_i(D|_{[\theta_i,\theta_{i+1}]}), \tag{4.17}$$

where

$$D|_{[\theta_i,\theta_{i+1}]} := \{h^* \in C([\theta_i, \theta_{i+1}], \Phi) : h^*(\varrho) = g(\varrho)h(\varrho), \varrho \in (\theta_i, \theta_{i+1}],$$
$$h^*(\theta_i) = \lim_{\varrho\to\theta_i^+} h^*(\varrho), h \in D\}. \tag{4.18}$$

and χ_i is the Hausdorff measure of noncompactness on $C([\theta_i, \theta_{i+1}], \Phi)$. To prove (4.16), let $\epsilon > 0$, and $n \geq 1$ be fixed. Then, (see [36]) there is a sequence $(\Im_k)$ in Δ_n such that

$$\chi_{PC_g(L,\Phi)}(\Delta_n) \leq 2\chi_{PC_g(L,\Phi)}\{\Im_m : m \geq 1\} + \epsilon. \tag{4.19}$$

Set $\Pi = \{\Im_m : m \geq 1\}$. It follows from (4.17) and (4.19) that

$$\chi_{PC_g(L,\Phi)}(\Delta_n) \leq 2 \max_{i\in\mathbb{N}_{0,r}} \chi_i(\Pi|_{|[\theta_i,\theta_{i+1}]}) + \epsilon, \tag{4.20}$$

but, from Claim 3, the sets $\Pi|_{|[\theta_i,\theta_{i+1}]}; i \in \mathbb{N}_{0,r}$ are equicontinuous, and consequently, inequality (4.20) becomes

$$\begin{aligned}\chi_{PC_g(L,\Phi)}(\Delta_n) &\leq 2 \max_{i\mathbb{N}_{0,r}} \max_{\upsilon\in[\theta_i,\theta_{i+1}]} \chi\{g(\upsilon)\Im_m(\upsilon) : m \geq 1\} + \epsilon \\ &\leq 2 \max_{\upsilon\in L} \chi\{g(\upsilon)\Im_m(\upsilon) : m \geq 1\} + \epsilon. \end{aligned}\tag{4.21}$$

To evaluate the quantity $\chi\{g(\upsilon)\Im_m(\upsilon) : m \geq 1\}; \upsilon \in L$, we note that, since $\Im_m \in \Delta_n = \Upsilon(\Delta_{n-1})$, there is $W_m \in \Delta_{n-1}$ with $\Im_m = \Upsilon(W_m)$, and hence, for any $m \geq 1$,

$$g(\upsilon)\Im_m(\upsilon) = \begin{cases} \Im_0 - \Re(W_m) + \frac{1-\gamma}{M(\gamma)} g(\upsilon)z(\upsilon, W_m(\upsilon)) \\ + \frac{\gamma}{M(\gamma)\Gamma(\gamma)} \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s)z(s, W_m(s))ds, \upsilon \in [0, \theta_1] \\ \\ \Im_0 - \Re(W_m) + \sum_{k=1}^{k=i} I_k(W_m(\theta_k^-)) + \frac{1-\gamma}{M(\gamma)} g(\upsilon)z(\upsilon, W_m(\upsilon)) \\ + \frac{\gamma}{M(\gamma)\Gamma(\gamma)} \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s)z(s, W_m(s))ds, \upsilon \in (\theta_i, \theta_{i+1}],\ i \in \mathbb{N}_{1,r}. \end{cases}\tag{4.22}$$

In view of (3.5), for $\upsilon \in L$,

$$\chi\{g(\upsilon)z(\upsilon, W_m(\upsilon)) : m \geq 1\} \leq \eta(\upsilon)\chi\{g(\upsilon)W_m(\upsilon) : m \geq 1 \leq \kappa\chi_{PC_g(L,\Phi)}(\Delta_{n-1}), \tag{4.23}$$

where $\kappa = \sup_{\upsilon\in L} \eta(\upsilon)$. However,

$$\Lambda(\upsilon) = \chi\{\int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s)z(s, W_m(s))ds : k \geq 1\}.$$

From the properties of χ and (4.23), it follows that for $\upsilon \in (\theta_i, \theta_{i+1}],\ i \in \mathbb{N}_{0,r}$,

$$\begin{aligned}\Lambda(\upsilon) &\leq 2 \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1}\varphi'(s)\chi\{g(s)z(s, W_m(s)) : m \geq 1\}ds \\ &\leq 2\kappa\chi_{PC_g(L,\Phi)}(\Delta_{n-1}) \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1}\varphi'(s)ds \\ &\leq 2\kappa\chi_{PC_g(L,\Phi)}(\Delta_{n-1}) \frac{\varphi(T)^\gamma}{\gamma}. \end{aligned}\tag{4.24}$$

From the compactness of both $\Re$ and I_i; $i \in \mathbb{N}_{1,r}$, we have

$$\chi\{\Re(W_m) : m \geq 1\} = 0 \text{ and } \chi\{\sum_{k=1}^{k=i} I_k(W_m(\theta_k^-)) : m \geq 1\} = 0.$$

Thus, from (4.21-4.24) it follows that

$$\chi_{PC_g(L,\Phi)}(\Delta_n) \leq \kappa\chi_{PC_g(L,\Phi)}(\Delta_{n-1}) + 2\kappa\chi_{PC_g(L,\Phi)}(\Delta_{n-1}) \frac{\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)} + \epsilon$$

$$= \chi_{PC_g(L,\Phi)}(\Delta_{n-1})[\kappa + 2\kappa \frac{\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)}] + \epsilon.$$

Since ϵ is arbitrary, we get

$$\chi_{PC_g(L,\Phi)}(\Delta_n) \leq \chi_{PC_g(L,\Phi)}(\Delta_{n-1})[\kappa + 2\kappa \frac{\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)}].$$

Since this relation is true for each n, we get

$$\chi_{PC_g(L,\Phi)}(\Delta_n) \leq \chi_{PC_g(L,\Phi)}(\Delta_1)[\kappa + 2\kappa \frac{\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)}]^{n-1}.$$

But, (3.6) gives that $\kappa + 2\kappa \frac{\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)} < 1$, so the relationship (4.16) is achieved, and Δ is compact. ¶

Applying Schauder's fixed point theorem to the mapping $\Upsilon : \Delta \to \Delta$, we conclude that Υ has a fixed point, which will be the solution for Problem (2.1). □

Remark 3. *If $g(\upsilon) = 1$ and $\varphi(\upsilon) = \upsilon$; $\upsilon \in L$, $\mathfrak{R}(\mathfrak{I}) = 0, \forall \mathfrak{I} \in PC_g(L, \Phi)$ and there are no impulses ($I_i(x) = 0, \forall x \in \Phi$, and $\forall i \in \mathbb{N}_{1,r}$), then Conditions (4.7) reduces to*

$$\frac{1-\gamma}{M(\gamma)}\varrho_2 + \frac{\varrho_2 T^\gamma}{M(\gamma)\Gamma(\gamma)} < 1,$$

This inequality appears in the literature, see for example, Theorem 3.1 in [29].

In the following, we give another existence result for Problem (2.1).

Theorem 2. *If $(Hz)^*, (H\mathfrak{R})^*$, and $(HI)^*$ are satisfied, then Problem (2.1) has a unique solution under the condition that*

$$[\xi_2 + r\xi_3 + \xi_1(\frac{(1-\gamma)\rho_1}{M(\gamma)} + \frac{\rho_1 \varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)})] < 1. \tag{4.25}$$

Proof. We are going to show that the function $\Upsilon : PC_g(L, \Phi) \to PC_g(L, \Phi)$, defined by (4.8), is a contraction. Let $\mathfrak{I}, \varsigma \in PC_g(L, \Phi)$. For any $\upsilon \in [0, \theta_1]$, we have

$$\begin{aligned}
&\|g(\upsilon)\Upsilon(\mathfrak{I})(\upsilon) - g(\upsilon)\Upsilon(\varsigma)(\upsilon)\| \\
&\leq \|\mathfrak{R}(\mathfrak{I}) - \mathfrak{R}(\varsigma)\| + \frac{(1-\gamma)\rho_1}{M(\gamma)}\xi_1\|\mathfrak{I} - \varsigma\|_{PC_g(L,\Phi)} \\
&+ \frac{\gamma}{M(\gamma)\Gamma(\gamma)} \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1}\varphi'(s)g(s)\|z(s, \mathfrak{I}(s)) - z(s, \varsigma(s))\|ds \\
&\leq \xi_2\|\mathfrak{I} - \varsigma\|_{PC_g(L,\Phi)} + \frac{(1-\gamma)\rho_1}{M(\gamma)}\xi_1\|\mathfrak{I} - \varsigma\|_{PC_g(L,\Phi)} + \|\mathfrak{I} - \varsigma\|_{PC_g(L,\Phi)}\frac{\rho_1\xi_1\varphi(\upsilon)^\gamma}{M(\gamma)\Gamma(\gamma)} \\
&= \|\mathfrak{I} - \varsigma\|_{PC_g(L,\Phi)}[\xi_2 + \frac{(1-\gamma)\rho_1}{M(\gamma)}\xi_1 + \frac{\rho_1\xi_1\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)}].
\end{aligned}$$

Let $\upsilon \in (\theta_i, \theta_{i+1}],\ i \in \mathbb{N}_{1,r}$. In view $(HI)^*$

$$\sum_{k=1}^{k=i} \|I_k(\mathfrak{I}(\theta_k^-) - I_k\varsigma(\theta_k^-)\| \leq r\xi_3\|\mathfrak{I} - \varsigma\|_{PC_g(L,\Phi)}.$$

But

$$\|g(\upsilon)\Upsilon(\Im)(\upsilon) - g(\upsilon)\Upsilon(\varsigma)(\upsilon)\| = \|\Im - \varsigma\|_{PC_g(L,\Phi)}.[\xi_2 + r\xi_3 + \frac{(1-\gamma)\rho_1}{M(\gamma)}\xi_1 + \frac{\xi_1\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)}].$$

Therefore,

$$\|\Upsilon(\Im) - \Upsilon(\varsigma)\|_{PC_g(L,\Phi)} \le \|\Im - \varsigma\|_{PC_g(L,\Phi)}[\xi_2 + r\xi_3 + \xi_1(\frac{(1-\gamma)\rho_1}{M(\gamma)} + \frac{\rho_1\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)})]$$

By (4.25), Υ is a contraction. By Banach fixed-point theorem, it has a unique fixed point that gives a solution to Problem (2.1). □

5. Existence of solutions of Problem (2.2)

Lemma 5. *Let $\Im : L \to \Phi$, $W : L \to \Phi$ continuous with $W(\theta_i) = 0, \forall i \in \mathbb{N}_{0,r}$.*

(1) If $\Im \in PCH^1((0,T),\Phi)$ is a solution for the following initial problem:

$$\begin{cases} {}^{ABC}D^{\gamma,\varphi,g}_{\theta_i,\upsilon}\Im(\upsilon) = W(\upsilon), \upsilon \in (\theta_i, \theta_{i+1}), i \in \mathbb{N}_{0,r}, \\ \Im(0) = \Im_0 g^{-1}(0) - g^{-1}(0)\Re(\Im), \\ \Im(\theta_i^+) = \Im(\theta_i^-) + g^{-1}(\theta_i^-)I_i(\Im(\theta_i^-)),\ i \in \mathbb{N}_{1,r}, \end{cases} \tag{5.1}$$

then, for any $\upsilon \in L$

$$\Im(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) +{}^{AB} I^{\gamma,\varphi,g}_{0,\upsilon}W(\upsilon), \upsilon \in [0,\theta_1] \\ \\ g^{-1}(\upsilon)\Im_0 - g^{-1}(\upsilon)\Re(\Im)) + g^{-1}(\upsilon)\sum_{k=1}^{k=i} g(\theta_k)\,{}^{AB}I^{\gamma,g,\varphi}_{\theta_{k-1},\theta_k}W(\upsilon) \\ +g^{-1}(\upsilon)\sum_{k=1}^{k=i} I_k(\Im(\theta_k^-)) +{}^{AB} I^{\gamma,g,\varphi}_{\theta_i,\upsilon}W(\upsilon), \upsilon \in (\theta_i, \theta_{i+1}],\ i \in \mathbb{N}_{1,r}. \end{cases} \tag{5.2}$$

(2) If $W \in PCH^1(L,\Phi)$ and $\Im$ satisfies (5.2), then $\Im \in PCH^1(L,\Phi)$ and $\Im$ is a solution for (5.1)

Proof.

(1) Let $\upsilon \in (0,\theta_1)$. Then,

$$\begin{cases} {}^{ABC}D^{\gamma,\varphi,g}_{0,\upsilon}\Im(\upsilon) = W(\upsilon), \upsilon \in (0,\theta_1], \\ \Im(0) = \Im_0 g^{-1}(0) - g^{-1}(0)\Re(\Im), \end{cases}$$

As in the proof of Lemma (4), we get that for $\upsilon \in (0,\theta_1]$

$$\Im(\upsilon) = \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) +{}^{AB} I^{\gamma,\varphi,g}_{0,\upsilon}W(\upsilon). \tag{5.3}$$

Next, consider the fractional differential equation:

$$\begin{cases} {}^{ABC}D^{\gamma,\varphi,g}_{\theta_1,\upsilon}\Im(\upsilon) = W(\upsilon), \upsilon \in (\theta_1,\theta_2], \\ \Im(\theta_1^+) = \Im(\theta_1^-) + g^{-1}(\theta_1)I_1(\Im(\theta_1^-)). \end{cases}$$

Applying ${}^{AB}I^{\gamma,\varphi,g}_{\theta_1,\upsilon}$ on both sides of ${}^{ABC}D^{\gamma,\varphi,g}_{\theta_1,\upsilon}\Im(\upsilon) = W(\upsilon)$ and using Lemma 3, we obtain

$$\Im(\upsilon) = c_1 g^{-1}(\upsilon) +{}^{AB} I^{\gamma,\varphi,g}_{\theta_1,\upsilon}W(\upsilon), \upsilon \in (\theta_1,\theta_2]. \tag{5.4}$$

Using the boundary condition $\mathfrak{I}(\theta_1^+) = \mathfrak{I}(\theta_1^-) + g^{-1}(\theta_1)I_1(\mathfrak{I}(\theta_1^-))$, we get

$$c_1 g^{-1}(\theta_1) = \mathfrak{I}_0 g^{-1}(\theta_1) - g^{-1}(\theta_1)\mathfrak{R}(\mathfrak{I}) +^{AB} I_{0,\theta_1}^{\gamma,\varphi,g} W(\upsilon) + g^{-1}(\theta_1)I_1(\mathfrak{I}(\theta_1^-)),$$

which gives that

$$c_1 = \mathfrak{I}_0 - \mathfrak{R}(\mathfrak{I}) + g(\theta_1)^{AB} I_{0,\theta_1}^{\gamma,\varphi,g} W(\upsilon) + I_1(\mathfrak{I}(\theta_1^-)),$$

Substituting the value of c_1 in (5.3), we obtain for $\upsilon \in (\theta_1, \theta_2]$,

$$\begin{aligned}\mathfrak{I}(\upsilon) = {} & \mathfrak{I}_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\mathfrak{R}(\mathfrak{I}) + g^{-1}(\upsilon)g(\theta_1)^{AB} I_{0,\theta_1}^{\gamma,\varphi,g} W(\upsilon) \\ & + g^{-1}(\upsilon)I_1(\mathfrak{I}(\theta_1^-)) +^{AB} I_{\theta_1,\upsilon}^{\gamma,\varphi,g} W(\upsilon).\end{aligned} \tag{5.5}$$

Next, consider the fractional differential equation:

$$\begin{cases} {}^{ABC}D_{\theta_2,\upsilon}^{\gamma,\varphi,g}\mathfrak{I}(\upsilon) = W(\upsilon), \upsilon \in (\theta_2, \theta_3], \\ \mathfrak{I}(\theta_1^+) = \mathfrak{I}(\theta_1^-) + g^{-1}(\theta_1)I_1(\mathfrak{I}(\theta_1^-)). \end{cases}$$

Applying ${}^{AB}I_{\theta_2,\upsilon}^{\gamma,\varphi,g}$ to both sides of ${}^{ABC}D_{\theta_2,\upsilon}^{\gamma,\varphi,g}\mathfrak{I}(\upsilon) = W(\upsilon)$ and using Lemma 3, we obtain

$$\mathfrak{I}(\upsilon) = c_2 g^{-1}(\upsilon) +^{AB} I_{\theta_2,\upsilon}^{\gamma,\varphi,g} W(\upsilon), \upsilon \in (\theta_1, \theta_2]. \tag{5.6}$$

Using the boundary condition $\mathfrak{I}(\theta_2^+) = \mathfrak{I}(\theta_2^-) + g^{-1}(\theta_2)I_2(\mathfrak{I}(\theta_2^-))$, we get

$$\begin{aligned}c_2 g^{-1}(\theta_2) = {} & \mathfrak{I}_0 g^{-1}(\theta_2) - g^{-1}(\theta_2)\mathfrak{R}(\mathfrak{I}) + g^{-1}(\theta_2)g(\theta_1)^{AB} I_{0,\theta_1}^{\gamma,\varphi,g} W(\upsilon) \\ & + g^{-1}(\theta_2)I_1(\mathfrak{I}(\theta_1^-)) +^{AB} I_{\theta_1,\theta_2}^{\gamma,\varphi,g} W(\upsilon) + g^{-1}(\theta_2)I_2(\mathfrak{I}(\theta_2^-)),\end{aligned}$$

which gives us

$$c_2 = \mathfrak{I}_0 - \mathfrak{R}(\mathfrak{I}) + g(\theta_1)^{AB} I_{0,\theta_1}^{\gamma,\varphi,g} W(\upsilon) + I_1(\mathfrak{I}(\theta_1^-)) + g(\theta_2)^{AB} I_{\theta_1,\theta_2}^{\gamma,\varphi,g} W(\upsilon) + I_2(\mathfrak{I}(\theta_2^-)),$$

Substituting the value of c_2 in (5.6), yields for $\upsilon \in (\theta_1, \theta_2]$

$$\begin{aligned}\mathfrak{I}(\upsilon) = {} & \mathfrak{I}_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\mathfrak{R}(\mathfrak{I}) + g^{-1}(\upsilon)g(\theta_1)^{AB} I_{0,\theta_1}^{\gamma,\varphi,g} W(\upsilon) + g^{-1}(\upsilon)I_1(\mathfrak{I}(\theta_1^-)) \\ & + g^{-1}(\upsilon)g(\theta_2)^{AB} I_{\theta_1,\theta_2}^{\gamma,\varphi,g} W(\upsilon) + g^{-1}(\upsilon)I_2(\mathfrak{I}(\theta_2^-)) +^{AB} I_{\theta_2,\upsilon}^{\gamma,\varphi,g} W(\upsilon).\end{aligned}$$

By repeating the same procedures, we get Eq (5.2).

(2) Suppose that $\mathfrak{I}$ is defined by (5.2) and $W(0) = 0$. Let $\upsilon \in (0, \theta_1)$. Then,

$$\mathfrak{I}(\upsilon) = \mathfrak{I}_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\mathfrak{R}(\mathfrak{I}) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W(\upsilon).$$

Applying Lemma 3 and noting that ${}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\, g^{-1}(\upsilon) = 0$, we obtain ${}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\mathfrak{I}(\upsilon) = W(\upsilon)$. Similarly, if $\upsilon \in (\theta_i, \theta_{i+1})$; $i \in i \in \mathbb{N}_{1,r}$, then

$$\begin{aligned}\mathfrak{I}(\upsilon) = {} & g^{-1}(\upsilon)\mathfrak{I}_0 - g^{-1}(\upsilon)\mathfrak{R}(\mathfrak{I})) + g^{-1}(\upsilon) \sum_{k=1}^{k=i} g(\theta_k)\, {}^{AB}I_{\theta_{k-1},\theta_k}^{\gamma,g,\varphi} W(\upsilon) \\ & + g^{-1}(\upsilon) \sum_{k=1}^{k=i} I_k(\mathfrak{I}(\theta_k^-)) +^{AB} I_{\theta_i,\upsilon}^{\gamma,g,\varphi} W(\upsilon).\end{aligned}$$

Again, since $W \in PCH^1(L,\Phi)$, $W(\theta_i) = 0$ and ${}^{ABC}D^{\gamma,\varphi,g}_{0,\upsilon}\, g^{-1}(\upsilon) = 0$, it follows by Lemma 3,

$$ {}^{ABC}D^{\gamma,\varphi,g}_{\theta_i,\upsilon}\Im(\upsilon) = W(\upsilon), $$

giving us $\Im \in H^1((0,T),\Phi)$. In addition, it is easy to check that

$$ \Im(\theta_i^+) = \Im(\theta_i^-) + g^{-1}(\theta_i^-)I_i(\Im(\theta_i^-)),\ i \in \mathbb{N}_{1,r} $$

Therefore, $\Im$ is a solution for (5.1).

□

Remark 4.

- *We can't omit the assumption $W(0) = 0$, in the second assertion of Lemma 3, since* ${}^{ABC}D^{\gamma}_{0,\upsilon}\ {}^{AB}I^{\gamma}_{0,\upsilon}W(\upsilon) = W(\upsilon) - g(0)W(0)g^{-1}(\upsilon) \neq W(\upsilon)$.
- *If W is continuous and not in $H^1((0,T),\Phi)$, then Eq (4.6) does not lead to the existence of* ${}^{ABC}D^{\gamma}_{0,\upsilon}\Im(\upsilon)$. *Thus, without the assumption $W \in H^1((0,T),\Phi)$, we can not conclude that ${}^{ABC}D^{\gamma}_{0,\upsilon}\Im(\upsilon)$ exists.*

Using Lemma (5), we give the formula for a solution to Problem (2.2).

Corollary 2. *For any $x \in PC_g(L,\Phi)$, let*

$$ S^2_{\Psi(.,x(.))} = \{z \in L^2(J,\Phi) : z(s) \in \Psi(s, x(s)), a, e.\}. $$

A function $\Im \in PCH^1((0,T),\Phi)$ is a solution of Problem (2.2) if it satisfies the fractional integral equation:

$$ \Im(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(x) + \frac{1-\gamma}{M(\gamma)}W_0(\upsilon) \\ + \frac{\gamma}{M(\gamma)\Gamma(\gamma)g(\upsilon)}\int_0^{\upsilon}(\varphi(\upsilon)-\varphi(s))^{\gamma-1}g(s)\varphi'(s)W_0(s)ds, \upsilon \in [0,\theta_1] \\ \\ g^{-1}(\upsilon)\Im_0 - g^{-1}(\upsilon)\Re(x) + g^{-1}(\upsilon)\sum_{k=1}^{k=i} g(\theta_k)\ {}^{AB}I^{\gamma,g,\varphi}_{\theta_{k-1},\theta_k}W_i(\upsilon) \\ + g^{-1}(\upsilon)\sum_{k=1}^{k=i} I_k(\Im(\theta_k^-)) + {}^{AB}I^{\gamma,g,\varphi}_{\theta_i,\upsilon}W_i(\upsilon), \upsilon \in (\theta_i,\theta_{i+1}],\ i \in \mathbb{N}_{1,r}. \end{cases} \tag{5.7} $$

where $W_i(\upsilon) = \int_{\theta_i}^{\upsilon} z(s)ds, z \in S^2_{\Psi(.,x(.))}, \upsilon \in (\theta_i,\theta_{i+1}], i \in \mathbb{N}_{0,r}$.

We need to the following fixed points theorems for multivalued functions.

Let $U \in P_{cc}(\Phi)$, ρ a non-singular measure of noncompactness defined on subsets of $\Im$, $\Pi : U \to P_{ck}(\Im)$ a closed multifunction and $Fix(\Pi) = \{x \in \Im : x \in \Pi(x)\}$.

Lemma 6. *[[37], Corollary 3.3.1]. If $\Pi : U \to P_{ck}(U)$ is ρ−condensing then $Fix(\Pi)$ is not empty.*

Lemma 7. *[[37], Proposition.3.5.1]]. In addition to the assumptions of Lemma (6), if ρ is a monotone measure of noncompactness defined on U and $Fix(\Pi)$ is a bounded, then it is compact.*

For more information about multi-valued functions, we refer the reader to [38].

Theorem 3. *If $(H\Psi)$, $(H\Re)$ and (HI) hold, then the solution set for Problem (2.2) is non-empty and compact, provided that*

$$ c + \frac{\rho_1\rho_2(1-\gamma)}{M(\gamma)} + \frac{2\rho_1\rho_2\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)} + \lambda r + \frac{\rho_1\rho_3\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)} < 1, \tag{5.8} $$

Proof. Due to (i), (ii) and (iii) of $(H\Psi)$, the set $S^2_{\Psi(.,x(.))}$ is not empty, and so, a multivalued function $\Xi : PC_g(L,\Phi) \to 2^{PC_g(L,\Phi)} - \{\phi\}$, where ϕ is the empty set, can be defined by: $\Im \in R(x)$ if and only if

$$\Im(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(x) +^{AB} I^{\gamma,\varphi,g}_{0,\upsilon} W_0(\upsilon), \upsilon \in [0,\theta_1] \\ \\ g^{-1}(\upsilon)\Im_0 - g^{-1}(\upsilon)\Re(x) + g^{-1}(\upsilon)\sum_{k=1}^{k=i} g(\theta_k)\, {}^{AB}I^{\gamma,g,\varphi}_{\theta_{k-1},\theta_k} W_i(\upsilon) \\ +g^{-1}(\upsilon)\sum_{k=1}^{k=i} I_k(\Im(\theta_k^-)) +^{AB} I^{\gamma,g,\varphi}_{\theta_i,\upsilon} W_i(\upsilon), \upsilon \in (\theta_i,\theta_{i+1}],\ i \in \mathbb{N}_{1,r}. \end{cases} \tag{5.9}$$

where $W_i(\upsilon) = \int_{\theta_i}^{\upsilon} z(s)ds,\ z \in S^2_{\Psi(.,x(.))}, \upsilon \in (\theta_i,\theta_{i+1}], i \in \mathbb{N}_{0,r}$. Note that, if $\Im$ is a fixed point for R, then

$$\Im(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) +^{AB} I^{\gamma,\varphi,g}_{0,\upsilon} W_0(\upsilon), \upsilon \in [0,\theta_1] \\ \\ g^{-1}(\upsilon)\Im_0 - g^{-1}(\upsilon)\Re(\Im)) + g^{-1}(\upsilon)\sum_{k=1}^{k=i} g(\theta_k)\, {}^{AB}I^{\gamma,g,\varphi}_{\theta_{k-1},\theta_k} W_i(\upsilon) \\ +g^{-1}(\upsilon)\sum_{k=1}^{k=i} I_k(\Im(\theta_k^-)) +^{AB} I^{\gamma,g,\varphi}_{\theta_i,\upsilon} W_i(\upsilon), \upsilon \in (\theta_i,\theta_{i+1}],\ i \in \mathbb{N}_{1,r}. \end{cases}$$

where $W_i(\upsilon) = \int_{\theta_i}^{\upsilon} z(s)ds, z(s) \in \Psi(s,\Im(s)), a.e.; \upsilon \in (\theta_i,\theta_{i+1}], i \in \mathbb{N}_{0,r}$, which gives that $W_i(\upsilon) \in \int_{\theta_i}^{\upsilon} \Psi(s,\Im(s))ds; \upsilon \in (\theta_i,\theta_{i+1}], i \in \mathbb{N}_{0,r}$ and $W(\theta_i) = 0, \forall i \in \mathbb{N}_{0,r}$, and therefore, by the second statement of Lemma (5), $\Im$ is a solution for Problem (2.2). So, our aim is using Lemma (5), to show that Ξ has a fixed point. The proof will divided into several claims.

Claim 5. *There is a natural number ζ_0 such that $\Xi(\Delta_{\zeta_0}) \subseteq \Delta_{\zeta_0}$.*

Pf: If this was not true, then for every natural number n there are $x_n, \Im_n$ with $\|\Im_n\|_{PC_g(L,\Phi)} > n$, $\|x_n\|_{PC_g(L,\Phi)} \le n$ and $\Im_n \in \Xi(x_n)$. By definition of Ξ,

$$\Im_n(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(x_n) +^{AB} I^{\gamma,\varphi,g}_{0,\upsilon} W_{0,n}(\upsilon), \upsilon \in [0,\theta_1] \\ \\ g^{-1}(\upsilon)\Im_0 - g^{-1}(\upsilon)\Re(x_n) + g^{-1}(\upsilon)\sum_{k=1}^{k=i} g(\theta_k)\, {}^{AB}I^{\gamma,g,\varphi}_{\theta_{k-1},\theta_k} W_{i,n}(\upsilon) \\ +g^{-1}(\upsilon)\sum_{k=1}^{k=i} I_k(\Im_n(\theta_k^-)) +^{AB} I^{\gamma,g,\varphi}_{\theta_i,\upsilon} W_{i,n}(\upsilon), \upsilon \in (\theta_i,\theta_{i+1}],\ i \in \mathbb{N}_{1,r}. \end{cases} \tag{5.10}$$

where $W_{i,n}(\upsilon) = \int_{\theta_i}^{\upsilon} z_n(s)ds, z_n \in S^2_{\Psi(.,x_n.))}, \upsilon \in (\theta_i,\theta_{i+1}], i \in \mathbb{N}_{0,r}$. Using *(iii)* of $(H\Psi)$, we get for any $\upsilon \in L$,

$$\|W_{i,n}(\upsilon)\| \le \int_{\theta_i}^{\upsilon} \|z_n(s)\|ds \le (1+n)\int_{\theta_i}^{\upsilon} \tau(s)ds \le (1+n)\|\tau\|_{L^2(L,\mathbb{R}^+)} \le (1+n)\rho_2. \tag{5.11}$$

Let $\upsilon \in [0,\theta_1]$. Making use of (5.9), (5.10) and $(H\Re)$, we obtain

$$\begin{aligned} \|g(\upsilon)\Im_n(\upsilon)\| &\le \|\Im_0\| + \ cn + d + \frac{1-\gamma}{M(\gamma)}\rho_1\rho_2(1+n) + \frac{\gamma\rho_1\rho_2(1+n)}{M(\gamma)\Gamma(\gamma)}\int_0^{\upsilon} (\varphi(\upsilon)-\varphi(s))^{\gamma-1}\varphi'(s)ds \\ &\le \|\Im_0\| + \ cn + d + \frac{1-\gamma}{M(\gamma)}\rho_1\rho_3(1+n) + \frac{\rho_1\rho_2(1+n)\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}. \end{aligned} \tag{5.12}$$

Let $\upsilon \in (\theta_i\ \theta_{i+1}], i \in \mathbb{N}_{1,r}$. By (5.9), (5.10), $(H\Re)$ and (HI), we obtain

$$\|g(\upsilon)\Im_n(\upsilon)\| \le \|\Im_0\| + \ cn + d + \rho_1(\frac{1-\gamma}{M(\gamma)}(1+n)\rho_2$$

$$
\begin{aligned}
&+\frac{\gamma}{M(\gamma)\Gamma(\gamma)}\int_{\theta_{k-1}}^{\theta_k}(\varphi(\upsilon)-\varphi(s))^{\gamma-1}\varphi'(s)g(s)W(s)ds) \\
&+\lambda r\zeta+\frac{\gamma}{M(\gamma)\Gamma(\gamma)}\int_{\theta_i}^{\upsilon}(\varphi(\upsilon)-\varphi(s))^{\gamma-1}\varphi'(s)g(s)W(s)ds \\
&\leq \|\mathfrak{I}_0\|+\ cn+d+\frac{\rho_1\rho_2(1-\gamma)(1+n)}{M(\gamma)} \\
&+\frac{\rho_1\rho_2 2(1+n)\varphi(\upsilon)^{\gamma}}{M(\gamma)\Gamma(\gamma)}+\lambda rn+\frac{\rho_1\rho_2(1+n)\varphi(\upsilon)^{\gamma}}{M(\gamma)\Gamma(\gamma)}. \qquad (5.13)
\end{aligned}
$$

From (5.12) and (5.13), it follows that

$$
\begin{aligned}
n<\left\|\mathfrak{I}_n\right\|_{PC_g(J,\Phi)} &\leq \|\mathfrak{I}_0\|+\ cn+d+\frac{\rho_1\rho_2(1-\gamma)(1+n)}{M(\gamma)} \\
&+\frac{2\rho_1\rho_2(1+n)\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}+\lambda rn+\frac{\rho_1\rho_2(1+n)\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}.
\end{aligned}
$$

Dividing both sides of this relation by n and then letting $n\to\infty$, we get

$$
1<c+\frac{\rho_1\rho_2(1-\gamma)}{M(\gamma)}+\frac{2\rho_1\rho_3\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}+\lambda r+\frac{\rho_1\rho_2\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)},
$$

which contradicts (5.8). ¶

Claim 6. *If $x_k\in\Delta_{\zeta_0}$, $\mathfrak{I}_n\in\Xi(x_n)$, $x_n\to x$ and $\mathfrak{I}_n\to\mathfrak{I}$, in $PC_g(L,\Phi)$, then $\mathfrak{I}\in\Xi(x)$.*

Pf: From the definition of Ξ,

$$
\mathfrak{I}_n(\upsilon)=\begin{cases}
\mathfrak{I}_0 g^{-1}(\upsilon)-g^{-1}(\upsilon)\mathfrak{R}(x_n)+^{AB}I_{0,\upsilon}^{\gamma,\varphi,g}W_{0,n}(\upsilon),\upsilon\in[0,\theta_1] \\
\\
g^{-1}(\upsilon)\mathfrak{I}_0-g^{-1}(\upsilon)\mathfrak{R}(x_n)+g^{-1}(\upsilon)\sum_{k=1}^{k=i}g(\theta_k)\,^{AB}I_{\theta_{k-1},\theta_k}^{\gamma,g,\varphi}W_{i,n}(\upsilon) \\
+g^{-1}(\upsilon)\sum_{k=1}^{k=i}I_k(\mathfrak{I}_n(\theta_k^-))+^{AB}I_{\theta_i,\upsilon}^{\gamma,g,\varphi}W_{i,n}(\upsilon),\upsilon\in(\theta_i,\theta_{i+1}],\ i\in\mathbb{N}_{1,r},
\end{cases}
$$

where $W_{i,n}(\upsilon)=\int_{\theta_i}^{\upsilon}z_n(s)ds, z_n\in S^2_{\Psi(.,x_n(.))}, \upsilon\in(\theta_i,\theta_{i+1}], i\in\mathbb{N}_{0,r}$.

It follows from (iii) in $(H\Psi)$, that $\|z_n(\upsilon)\|\leq\tau(\upsilon)(1+\zeta_0), a.e,$. Thus, $\{z_n: n\geq 1\}$ is weakly compact in $L^2(L,\Phi)$. By Mazure's lemma, we can find a subsequence $(z_n^*), n\geq 1$ of convex combinations of (z_n) and converging almost everywhere to a function $z\in L^2(L,\Phi)$. By the upper semicontinuity of $\Psi(\upsilon,.), a,e.$, it follows that $z\in S^2_{\Psi(,.x(.))}$. Set $W_{i,n}^*(\upsilon)=\int_{\theta_i}^{\upsilon}z_n^*(s)ds;\upsilon\in(\theta_i,\theta_{i+1}]$. Then, $W_{i,n}^*(\upsilon)\to W_i(\upsilon)=\int_{\theta_i}^{\upsilon}z(s)ds\in\int_{\theta_i}^{\upsilon}\Psi(s,x(s))ds$. From the continuity of both $\mathfrak{R}$ and I_i, we have

$$
\mathfrak{I}(\upsilon)=\begin{cases}
\mathfrak{I}_0 g^{-1}(\upsilon)-g^{-1}(\upsilon)\mathfrak{R}(x)+^{AB}I_{0,\upsilon}^{\gamma,\varphi,g}W_0(\upsilon),\upsilon\in[0,\theta_1] \\
\\
g^{-1}(\upsilon)\mathfrak{I}_0-g^{-1}(\upsilon)\mathfrak{R}(x)+g^{-1}(\upsilon)\sum_{k=1}^{k=i}g(\theta_k)\,^{AB}I_{\theta_{k-1},\theta_k}^{\gamma,g,\varphi}W(\upsilon) \\
+g^{-1}(\upsilon)\sum_{k=1}^{k=i}I_k(\mathfrak{I}(\theta_k^-))+^{AB}I_{\theta_i,\upsilon}^{\gamma,g,\varphi}W_i(\upsilon),\upsilon\in(\theta_i,\theta_{i+1}],\ i\in\mathbb{N}_{1,r},
\end{cases}
$$

which implies to $\mathfrak{I}\in\Xi(x)$.¶

Claim 7. $\Xi(x)$; $x\in\Delta_{\zeta_0}$ *is compact.*

Pf: Suppose that $(\Im_k)$ is a sequence in $\Xi(x) : x \in \Delta_{\zeta_0}$. By arguing as in Claim 2, there is a subsequence of $(\Im_k)$ converging to $\overline{\Im} \in \Xi(x)$.¶

Claim 8. *The set $D_1|_{[\theta_i,\theta_{i+1}]}$ is equicontinuous for any $i \in \mathbb{N}_{0,r}$, where*

$$\begin{aligned} D_1|_{[\theta_i,\theta_{i+1}]} = \{&\Im^* \in C([\theta_i,\theta_{i+1}],\Phi) : \Im^*(\upsilon) = g(\upsilon)\Im(\upsilon), \upsilon \in (\theta_i,\theta_{i+1}], \\ &\Im^*(\theta_i) = \lim_{\upsilon\to\theta_i^+} g(\upsilon)\Im(\upsilon), \Im \in D_1\}. \end{aligned} \tag{5.14}$$

Pf: Let $\Im \in \Xi(x);\ x \in \Delta_\zeta$. Then

$$\Im(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(x) + {}^{AB}I_{0,\upsilon}^{\gamma,\varphi,g} W_0(\upsilon), \upsilon \in [0,\theta_1] \\ \\ g^{-1}(\upsilon)\Im_0 - g^{-1}(\upsilon)\Re(x) + g^{-1}(\upsilon)\sum_{k=1}^{k=i} g(\theta_k)\, {}^{AB}I_{\theta_{k-1},\theta_k}^{\gamma,g,\varphi} W(\upsilon) \\ +g^{-1}(\upsilon)\sum_{k=1}^{k=i} I_k(\Im(\theta_k^-)) + {}^{AB}I_{\theta_i,\upsilon}^{\gamma,g,\varphi} W_i(\upsilon), \upsilon \in (\theta_i,\theta_{i+1}],\ i \in \mathbb{N}_{1,r}. \end{cases}$$

where $W_i(\upsilon) = \int_{\theta_i}^{\upsilon} z(s)ds,\ z \in S^2_{\Psi(.,x(.))}, \upsilon \in (\theta_i,\theta_{i+1}], i \in \mathbb{N}_{0,r}$. So,

$$\Im^*(\upsilon) = g(\upsilon)\Im(\upsilon) = \begin{cases} \Im_0 - \Re(x) + g(\upsilon)^{AB}I_{0,\upsilon}^{\gamma,\varphi,g} W_0(\upsilon), \upsilon \in [0,\theta_1] \\ \\ \Im_0 - \Re(x)) + \sum_{k=1}^{k=i} g(\theta_k)^{AB}I_{\theta_{k-1},\theta_k}^{\gamma,g,\varphi} W_i(\upsilon) \\ + \sum_{k=1}^{k=i} I_k(W_i(\theta_k^-)) + g(\upsilon)^{AB}I_{\theta_i,\upsilon}^{\gamma,g,\varphi} W_i(\upsilon), \upsilon \in (\theta_i,\theta_{i+1}], i \in \mathbb{N}_{1,r}. \end{cases} \tag{5.15}$$

and $\Im^*(\theta_i) = \lim_{\upsilon\to\theta_i^+} g(\upsilon)\Im(\upsilon)$. From (3.1), for any $\upsilon \in (\theta_i,\theta_{i+1}],\ i \in \mathbb{N}_{0,r}$ we have

$$\|g(\upsilon)W_i(\upsilon)\| \le \rho_1 \int_{\theta_i}^{\upsilon} \|z(s)\|ds \le \rho_1(1+\zeta_0)\int_{\theta_i}^{\upsilon} \tau(s)ds \le \rho_1(1+\zeta_0)\|\tau\|_{L^1(L,\mathbb{R}^+)} \le \rho_1\rho_2(1+\zeta_0). \tag{5.16}$$

Case 1. $i = 0$. Let $\upsilon, \upsilon+\delta$ be two points in $[0,\theta_1]$. From the continuity of W_0 and the previous inequality, we get

$$\begin{aligned} \lim_{\delta\to 0}\|\Im^*(\upsilon+\delta) - \Im^*(\upsilon)\| \le{}& \frac{1-\gamma}{M(\gamma)}\lim_{\delta\to 0}[g(\upsilon)\|W_0(\upsilon+\delta) - W_0(\upsilon)\| + \|W_0(\upsilon+\delta)\|\|g(\upsilon+\delta) - g(\upsilon)\|] \\ &+ \lim_{\delta\to 0}\frac{\gamma}{M(\gamma)\Gamma(\gamma)}\|\int_0^{\upsilon+\delta}(\varphi(\upsilon+\delta) - \varphi(s))^{\gamma-1}\varphi'(s)g(s)W_0(s)ds \\ &- \int_0^{\upsilon}(\varphi(\upsilon) - \varphi(s))^{\gamma-1}\varphi'(s)g(s)W_0(s)ds\| \\ &\le \frac{\gamma}{M(\gamma)\Gamma(\gamma)}\lim_{\delta\to 0}[\int_0^{\upsilon}|(\varphi(\upsilon+\delta) - \varphi(s))^{\gamma-1}\varphi'(s) \\ &- (\varphi(\upsilon) - \varphi(s))^{\gamma-1}\varphi'(s)|\ \|g(s)W_0(s)\|ds] \\ &+ \frac{\gamma}{M(\gamma)\Gamma(\gamma)}\lim_{\delta\to 0}\int_{\upsilon}^{\upsilon+\delta}(\varphi(\upsilon+\delta) - \varphi(s))^{\gamma-1}\varphi'(s)\|g(s)W(s)\|ds \\ &\le \frac{\gamma\rho_1\rho_2(1+\zeta_0)}{M(\gamma)\Gamma(\gamma)}\lim_{\delta\to 0}[\int_0^{\upsilon}|(\varphi(\upsilon+\delta) - \varphi(s))^{\gamma-1}\varphi'(s) - (\varphi(\upsilon) - \varphi(s))^{\gamma-1}\varphi'(s)|ds \\ &+ \frac{\gamma\rho_1\rho_2(1+\zeta_0)}{M(\gamma)\Gamma(\gamma)}\lim_{\delta\to 0}\int_{\upsilon}^{\upsilon+\delta}(\varphi(\upsilon) - \varphi(s))^{\gamma-1}\varphi'(s)ds = 0, \end{aligned} \tag{5.17}$$

independently of x.

Case 2. If $\upsilon, \upsilon+\delta$ are in $(\theta_i, \theta_{i+1}), i \in \mathbb{N}_{1,r}$, then by using similar arguments to those in Case 1, we obtain $\lim_{\delta\to 0} \|\mathfrak{I}^*(\upsilon+\delta) - \mathfrak{I}^*(\upsilon)\| = 0$.

Case 3. If $\upsilon = \theta_i, i \in \mathbb{N}_{1,r}$ and $\delta > 0$. As in Case 1, we obtain

$$\begin{aligned}\lim_{\delta\to 0}\|\mathfrak{I}^*(\theta_i+\delta) - \mathfrak{I}^*(\theta_i)\| &= \lim_{\delta\to 0}\lim_{\lambda\to\theta_i^+}\|\mathfrak{I}(\theta_i+\delta) - \mathfrak{I}(\lambda)\| \\ &\le \lim_{\delta\to 0}\lim_{\lambda\to\theta_i^+}\|g(\theta_i+\delta)W_i((\theta_i+\delta)) - g(\lambda)W_i(\lambda)\| \\ &+ \frac{\gamma}{M(\gamma)\Gamma(\gamma)}\lim_{\delta\to 0}\lim_{\lambda\to\theta_i^+}\|\int_0^{\upsilon+\delta}(\varphi(\upsilon+\delta)-\varphi(s))^{\gamma-1}\varphi'(s)g(s)W_i(s)ds \\ &- \int_0^{\upsilon}(\varphi(\upsilon)-\varphi(s))^{\gamma-1}\varphi'(s)g(s)W(s)ds\| = 0.\end{aligned}$$

Establishing the claim. ¶

Claim 9. *The set $D = \cap_{k=1}D_k$ is compact, where $D_1 = \Xi(\Delta_{\zeta_0})$ and $D_k = \Xi(D_{k-1}), k \ge 2$.*

Pf: From Claims 1 and 2, it follows that (D_k) is a non-increasing sequence of bounded closed convex subsets. So, by the Generalized Cantor's intersection property, D will be compact, if we prove that

$$\lim_{n\to\infty}\varkappa_{PC_g}(D_n) = 0. \tag{5.18}$$

Let $\epsilon > 0$ and $n \ge 1$ be fixed. As in Claim 4, in Theorem 1,

$$\chi_{PC_g(L,\Phi)}(D_n) \le 2\max_{\upsilon\in L}\chi\{g(\upsilon)\mathfrak{I}_m(\upsilon) : m \ge 1\} + \epsilon, \tag{5.19}$$

Since $\mathfrak{I}_m \in D_n = \Xi(D_{n-1})$, there is $x_m \in D_{n-1}$with $\mathfrak{I}_m \in \Xi(x_m)$, and hence

$$g(\upsilon)\mathfrak{I}_m(\upsilon) = \begin{cases}\mathfrak{I}_0 - \mathfrak{R}(x_m) + \frac{1-\gamma}{M(\gamma)}g(\upsilon)W_{0,m}(\upsilon) \\ +\frac{\gamma}{M(\gamma)\Gamma(\gamma)}\int_0^{\upsilon}(\varphi(\upsilon)-\varphi(s))^{\gamma-1}g(s)\varphi'(s)W_m(s))ds, \upsilon \in [0,\theta_1], \\ \\ \mathfrak{I}_0 - \mathfrak{R}(x_m) + \sum_{k=1}^{k=i} g(\theta_k)\, {}^{AB}I^{\gamma,g,\varphi}_{\theta_{k-1},\theta_k}W_{i,m}(\upsilon) \\ +\sum_{k=1}^{k=i} I_k(W_{i,m}(\theta_k^-)) \\ +g(\upsilon)^{AB}I^{\gamma,g,\varphi}_{\theta_i,\upsilon}W_{i,m}(\upsilon), \upsilon \in (\theta_i,\theta_{i+1}],\ i \in \mathbb{N}_{1,r}.\end{cases}$$

where $W_{i,m}(\upsilon) = \int_{\theta_i}^{\upsilon} z_m(s)ds,\ z_m \in S^2_{\Psi(.,x_m(.))}, \upsilon \in (\theta_i,\theta_{i+1}], i \in \mathbb{N}_{0,r}$. and $m \ge 1$. Note that, in view of (3.2), for $\upsilon \in L$,

$$\begin{aligned}\chi\{g(\upsilon)W_{i,m}(\upsilon) : m \ge 1\} &\le \chi\{g(\upsilon)\int_{s_i}^{\upsilon} z_m(s)ds : m \ge 1\} \\ &\le 2g(\upsilon)\int_{s_i}^{\upsilon}\chi\{z_m(s) : m \ge 1\}ds \\ &\le 2g(\upsilon)\int_{s_i}^{\upsilon}\chi\{\Psi(s,\{x_m(s) : m \ge 1\})\}ds \\ &\le 2g(\upsilon)\int_{s_i}^{\upsilon}\eta(s)\chi\{g(s)x_m(s) : m \ge 1\}\end{aligned}$$

$$\leq 2\rho_1\chi\chi_{PC_g(L,\Phi)}(D_{n-1})\int_{s_i}^{\upsilon}\eta(s)ds$$
$$\leq 2\rho_1\rho_3\chi_{PC_g(L,\Phi)}(D_{n-1}). \tag{5.20}$$

Now, as in (4.24), Relation (5.20) leads to

$$\chi\{\int_0^{\upsilon}(\varphi(\upsilon)-\varphi(s))^{\gamma-1}g(s)\varphi'(s)W_{0,m}(s))ds : k\geq 1\}\leq 4\rho_1\rho_3\chi_{PC_g(L,\Phi)}(D_{n-1})\frac{\varphi(\upsilon)^{\gamma}}{\gamma}, \upsilon\in[0,\theta_1], \tag{5.21}$$

$$\chi\{\int_{\theta_{i-1}}^{\theta_i}(\varphi(\upsilon)-\varphi(s))^{\gamma-1}g(s)\varphi'(s)W_{i,m}(s))ds : k\geq 1\}\leq 4\rho_1\rho_3\chi_{PC_g(L,\Phi)}(D_{n-1})\frac{\varphi(\upsilon)^{\gamma}}{\gamma},\ i\in\mathbb{N}_{1,r}, \tag{5.22}$$

and

$$\chi\{\int_{\theta_i}^{\upsilon}(\varphi(\upsilon)-\varphi(s))^{\gamma-1}g(s)\varphi'(s)W_{i,m}(s))ds : k\geq 1\}\leq 4\rho_1\rho_3\chi_{PC_g(L,\Phi)}(D_{n-1})\frac{\varphi(T)^{\gamma}}{\gamma}, \upsilon\in(\theta_i,\theta_{i+1}],\ i\in\mathbb{N}_{0,r}. \tag{5.23}$$

Thus, by (5.21) it follows that for $\upsilon\in[0,\theta_1]$

$$\chi\{g(\upsilon)\Im_m(\upsilon) : m\geq 1\}\leq \frac{1-\gamma}{M(\gamma)}2\rho_1\rho_3\chi_{PC_g(L,\Phi)}(D_{n-1})+\frac{4\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}\rho_1\rho_3\chi_{PC_g(L,\Phi)}(D_{n-1})$$
$$=\chi_{PC_g(L,\Phi)}(D_{n-1})\rho_1\rho_3[\frac{2(1-\gamma)}{M(\gamma)}+\frac{4\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}]. \tag{5.24}$$

Next, from the compactness of both $\Re$ and I_i; $i\in\mathbb{N}_{1,r}$ it follows that

$$\chi\{\Re(x_m) : m\geq 1\}=0\ \text{ and }\chi\{\sum_{k=1}^{k=i}I_k(\Im_m(\theta_k^-)) : m\geq 1\}=0,$$

and so, by (5.22) and (5.23), we get for $\upsilon\in(\theta_i,\theta_{i+1}],\ i\in\mathbb{N}_{1,r}$,

$$\chi\{g(\upsilon)\Im_m(\upsilon) : m\geq 1\}\leq 8\rho_1\rho_3\chi_{PC_g(L,\Phi)}(D_{n-1})\frac{\varphi(T)^{\gamma}}{\gamma}. \tag{5.25}$$

Relations (5.19), (5.24) and (5.25) give

$$\chi_{PC_g(L,\Phi)}(D_n)\leq\chi_{PC_g(L,\Phi)}(D_{n-1})2\rho_1\rho_3[\frac{2(1-\gamma)}{M(\gamma)}+\frac{12\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}]+\epsilon.$$

Since ϵ is arbitrary, we get

$$\chi_{PC_g(L,\Phi)}(D_n)\leq\chi_{PC_g(L,\Phi)}(D_{n-1})2\rho_1\rho_3[\frac{2(1-\gamma)}{M(\gamma)}+\frac{12\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}].$$

Since this relation is true for each n, we get

$$\chi_{PC_g(L,\Phi)}(D_n)\leq\chi_{PC_g(L,\Phi)}(D_1)[2\rho_1\rho_3(\frac{2(1-\gamma)}{M(\gamma)}+\frac{12\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)})]^{n-1}. \tag{5.26}$$

Taking the limit as $n\to\infty$ in (5.26), while keeping in mind (3.3), we get (5.18) and D is compact.¶ Applying Lemma (6), it follows that there is $\Im\in D$ such that $\Im\in\Xi(\Im)$ and as discussed above, such $\Im$ is a solution for Problem (2.2). In addition, by arguing as in Claim 1, one can show that the set of fixed points of Ξ is bounded, hence, by Lemma (7), the solution set for Problem (2.2) is compact. □

6. Existence of solutions of Problem (2.3)

Lemma 8. *Let $\Im : L \to \Phi$, $\Re_i : [b_i, \tau_i] \times \Phi \to \Phi; i \in \mathbb{N}_{1,r}$, and $W : L \to \Phi$ be continuous with $W(0) = 0$.*

(1) If $\Im \in PCH^{1,*}((0,T), \Phi)$ is a solution to the fractional differential equation

$$\begin{cases} {}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) = W(\upsilon), \upsilon \in \cup_{i\in\mathbb{N}_{0,r}} L_i, \\ \Im(\upsilon) = \Re_i(\upsilon, \Im(b_i^-)), \upsilon \in [b_i, \tau_i];\ i \in \mathbb{N}_{1,r}, \\ \Im(0) = \Im_0 g^{-1}(0) - g^{-1}(0)\Re(\Im), \end{cases} \tag{6.1}$$

then

$$\Im(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W(\upsilon), \upsilon \in [0, b_1], \\ \Re_i(\upsilon, \Im(b_i^-)), \upsilon \in (b_i, \tau_i], i \in \mathbb{N}_{1,r}, \\ \Re_i(\tau_i, \Im(b_i^-)) - {}^{AB}I_{0,\tau_i}^{\gamma,\varphi,g}\ W(\upsilon) \\ +^{AB}I_{0,\upsilon}^{\gamma,\varphi,g} W(\upsilon), \upsilon \in (\tau_i, b_{i+1}], i \in \mathbb{N}_{1,r}. \end{cases} \tag{6.2}$$

(2) If $W \in PCH^{1,*}(L, \Phi)$, and $\Im$ is defined by (6.2), then $\Im \in PCH^{1,*}(L, \Phi)$ and $\Im$ is a solution of (6.1).

Proof.

(1) Suppose $\Im$ is a solution of (6.1) and $\upsilon \in (0, b_1]$. Then, ${}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) = W(\upsilon)$. By applying the operator ${}^{AB}I_{0,\upsilon}^{\gamma}$ to both sides of this equation and using the first statement of Lemma (3), we get

$$\Im(\upsilon) = c_0 + {}^{AB}I_{0,\upsilon}^{\gamma,\varphi,g} W(\upsilon).$$

From the boundary condition, $\Im(0) = \Im_0 g^{-1}(0) - g^{-1}(0)\Re(\Im)$, it follows that $c_0 = \Im_0 g^{-1}(0) - g^{-1}(0)\Re(\Im)$, and hence,

$$\Im(\upsilon) = \Im_0 g^{-1}(\upsilon) - g^{-1}(0)\Re(\Im) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W(\upsilon); \upsilon \in [0, b_1]$$

likewise,

$$\Im(\upsilon) = c_i +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W(\upsilon), \upsilon \in (\tau_i, b_{i+1}], i \in \mathbb{N}_{1,r}.$$

Since $\Im$ is continuous at $\tau_i; i \in \mathbb{N}_{1,r}$, it follows that

$$\Re_i(\tau_i, \Im(b_i^-)) = c_i + {}^{AB}I_{0,\tau_i}^{\gamma,\varphi,g} W(\upsilon),$$

hence, $c_i = \Re_i(\tau_i, \Im(b_i^-)) - {}^{AB}I_{0,\tau_i}^{\gamma} W(\upsilon)$, and thus

$$\Im(\upsilon) = \Re_i(\tau_i, \Im(b_i^-)) -^{AB} I_{0,\tau_i}^{\gamma,\varphi,g} W(\upsilon) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W(\upsilon), \upsilon \in (\tau_i, b_{i+1}), i \in \mathbb{N}_{1,r}.$$

Therefore, $\Im$ satisfies (6.2).

(2) Suppose that $\Im$ satisfies (6.2). Then

$$\Im(\upsilon) = \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W(\upsilon), \upsilon \in [0, b_1]$$

Since ${}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}(\Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im)) = 0$ and $W(0) = 0$, it follows from the second statement of Lemma (3), that $\Im \in H^1((0,b_1),\Phi)$ and ${}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) = W(\upsilon); \forall \upsilon \in [0,b_1]$. Let $\upsilon \in (\tau_i, b_{i+1}], i \in \mathbb{N}_{1,r}$. From (6.2), we have

$$\Im(\upsilon) = \Re_i(\tau_i, \Im(b_i^-)) - {}^{AB}I_{0,\tau_i}^{\gamma,\varphi,g}W(\upsilon) + {}^{AB}I_{0,\upsilon}^{\gamma,\varphi,g}W(\upsilon).$$

Since ${}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}(\Re_i(\tau_i,\Im(b_i^-)) - {}^{AB}I_{0,\tau_i}^{\gamma,\varphi,g}W(\upsilon)) = 0$, it follows, from the second statement of Lemma (3), that $\Im \in H^1((\tau_i, b_{i+1})),\Phi)$ and ${}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) = W(\upsilon); \forall \upsilon \in [0,b_1]$.
Let $\upsilon \in (\tau_i, b_{i+1}], i \in \mathbb{N}_{1,r}$. From (6.2), we have ${}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) = W(\upsilon); \forall \upsilon \in (\tau_i, b_{i+1}]$. Since $\Re_i$ are continuous for any $i \in \mathbb{N}_{1,r}$, $\Im \in PCH^{1,*}(L,\Phi)$. Obviously,$\Im(0) = \Im_0 g^{-1}(0) - g^{-1}(0)\Re(\Im)$ and for any $i \in \mathbb{N}_{1,r}$, thus

$$\Im(\tau_i^+) = \Re_i(\tau_i, \Im(b_i^-)) = \Im(\tau_i^-),$$

proving the continuity of $\Im$ at τ_i.

□

From Lemma (8), we obtain in the following corollary.

Corollary 3. *A function $\Im \in PCH^{1,*}(L,\Phi)$ is a solution to Problem (2.3) if $z(0,\Im(0)) = 0$ and $\Im$ satisfies the fractional integral equation:*

$$\Im(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) + \frac{1-\gamma}{M(\gamma)} z(\upsilon,\Im(\upsilon)) \\ + \frac{\gamma}{M(\gamma)\Gamma(\gamma)g(\upsilon)} \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s,\Im(s))ds, \upsilon \in [0,b_1], \\ \\ \Re_i(\upsilon, \Im(b_i^-)), \upsilon \in (b_i, \tau_i], i \in \mathbb{N}_{1,r}, \\ \\ \Re_i(\tau_i, \Im(b_i^-)) - {}^{AB}I_{0,\tau_i}^{\gamma,\varphi,g} z(\upsilon,\Im(\upsilon)) \\ + {}^{AB}I_{0,\upsilon}^{\gamma,\varphi,g} z(\upsilon,\Im(\upsilon)), \upsilon \in (\tau_i, b_{i+1}], i \in \mathbb{N}_{1,r}. \end{cases} \tag{6.3}$$

In the following theorem, we prove the existence and uniqueness of solutions for Problem (2.3).

Theorem 4. *Assume that $(H\Re_i)$, $(H\Re)^*$, and $(Hz)^{**}$ hold. If $g : L \to [1,\infty)$, then Problem (2.3) has a unique solution under the condition*

$$\rho_1\xi_4 + \xi_1\left(\frac{2(1-\gamma)}{M(\gamma)} + \frac{\varphi(T)^\gamma}{M(\gamma)g(\upsilon)\Gamma(\gamma)} + \frac{\rho_1\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)}\right) < 1. \tag{6.4}$$

Proof. Consider the operator $\Theta : PC_g^*(L,\Phi) \to PC_g^*(L,\Phi)$ defined by

$$\Theta(\Im)(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) + {}^{AB}I_{0,\upsilon}^{\gamma,\varphi,g} z(\upsilon,\Im(\upsilon)), \upsilon \in [0,b_1], \\ \\ \Re_i \upsilon, \Im(b_i^-)), \upsilon \in (b_i,\tau_i], i \in \mathbb{N}_{1,r}, \\ \\ \Re_i(\tau_i, \Im(b_i^-)) - {}^{AB}I_{0,\tau_i}^{\gamma,\varphi,g} z(\upsilon,\Im(\upsilon)) + {}^{AB}I_{0,\upsilon}^{\gamma,\varphi,g} z(\upsilon,\Im(\upsilon)), \upsilon \in (\tau_i, b_{i+1}], i \in \mathbb{N}_{1,r}. \end{cases} \tag{6.5}$$

Note that if $\Im$ is a fixed point for Θ, then

$$\Im(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) + {}^{AB}I_{0,\upsilon}^{\gamma,\varphi,g} W(\upsilon), \upsilon \in [0,b_1], \\ \Re_i(\upsilon, \Im(b_i^-)), \upsilon \in (b_i,\tau_i], i \in \mathbb{N}_{1,r}, \\ \Re_i(\tau_i, \Im(b_i^-)) - {}^{AB}I_{0,\tau_i}^{\gamma,\varphi,g} W(\upsilon) \\ + {}^{AB}I_{0,\upsilon}^{\gamma,\varphi,g} W(\upsilon), \upsilon \in (\tau_i, b_{i+1}], i \in \mathbb{N}_{1,r}, \end{cases}$$

where $W(\upsilon) = z(\upsilon, \Im(\upsilon))$. From $(Hz)^*(i)$ and the second statement of Lemma 8, it follows that $\Im$ is a solution of Problem (2.3). So, we only need to show that the the function Θ is a contraction. Let $\Im, \varsigma \in PC_g^*(L, \Phi)$. For any $\upsilon \in [0, b_1]$, from (6.3), (6.5) and $(H\mathfrak{R})^*$, it follows that

$$\begin{aligned}\|g(\upsilon)\Theta(\Im)(\upsilon) - g(\upsilon)\Theta(\varsigma)(\upsilon)\| &\leq \|\mathfrak{R}(\Im) - \mathfrak{R}(\varsigma)\| + \frac{1-\gamma}{M(\gamma)}\xi_1\|g(\upsilon)\Im(\upsilon) - g(\upsilon)\varsigma(\upsilon)\| \\ &+ \frac{\gamma}{M(\gamma)g(\upsilon)\Gamma(\gamma)}\int_0^{\upsilon}(\varphi(\upsilon) - \varphi(s))^{\gamma-1}\varphi'(s)g(s)\|z(s, \Im(s)) - z(s, \varsigma(s)\|ds \\ &\leq \xi_2\|\Im - \varsigma\|_{PC_g(L,\Phi)} + \xi_1\|\Im - \varsigma\|_{PC_g(L,\Phi)} + \|\Im - \varsigma\|_{PC_g(L,\Phi)}\frac{\xi_1\varphi(T)^{\gamma}}{M(\gamma)g(\upsilon)\Gamma(\gamma)} \\ &= \|\Im - \varsigma\|_{PC_g(L,\Phi)}[\xi_2 + \xi_1 + \frac{\xi_1\varphi(T)^{\gamma}}{M(\gamma)g(\upsilon)\Gamma(\gamma)}].\end{aligned}$$

For $\upsilon \in (b_i, \tau_i]$, from (3.9) and (6.5), we have

$$\|\Theta(\Im)(\upsilon) - \Theta(\varsigma)(\upsilon)\| \leq \|\mathfrak{R}_i(\upsilon, \Im(b_i^-)) - \mathfrak{R}_i(\upsilon, \varsigma(b_i^-))\| \leq \xi_4\|\Im - \varsigma\|_{PC_g(L,\Phi)}.$$

Let $\upsilon \in (\tau_i, b_{i+1}], i \in \mathbb{N}_{1,r}$. From $(H\mathfrak{R})^*$, (6.3), (3.9), (6.5) and from the fact that $g : L \to [1, \infty)$, we obtain

$$\begin{aligned}\|g(\upsilon)\Theta(\Im)(\upsilon) - g(\upsilon)\Theta(\varsigma)(\upsilon)\| &\leq g(\upsilon)\|\mathfrak{R}_i(\tau_i, \Im(b_i^-)) - \mathfrak{R}_i(\tau_i, \varsigma(b_i^-))\| \\ &+ \frac{1-\gamma}{M(\gamma)}g(\upsilon)\|z(\tau_i, \Im(\tau_i)) - |z(\tau_i, \varsigma(\tau_i))\| \\ &+ \frac{\gamma g(\upsilon)}{M(\gamma)\Gamma(\gamma)g(\tau_i)}\|\int_0^{\tau_i}(\varphi(\tau_i) - \varphi(s))^{\gamma-1}\varphi'(s)\|g(s)\|z(s, \Im(s)) - (s, \varsigma(s)\|ds \\ &+ \frac{1-\gamma}{M(\gamma)}g(\upsilon)\|z(\upsilon, \Im(\upsilon)) - |z(\upsilon, \varsigma(\upsilon))\| \\ &+ \frac{\gamma}{M(\gamma)\Gamma(\gamma)}\int_0^{\upsilon}(\varphi(\upsilon) - \varphi(s))^{\gamma-1}\varphi'(s)\|g(s)\|z(s, \Im(s)) - z(s, \varsigma(s)\|ds \\ &\leq \rho_1\xi_4\|\Im - \varsigma\|_{PC_g(L,\Phi)} + \frac{1-\gamma}{M(\gamma)}\xi_1\|\Im - \varsigma\|_{PC_g(L,\Phi)} + \|\Im - \varsigma\|_{PC_g(L,\Phi)}\frac{\rho_1\xi_1\varphi(\upsilon)^{\gamma}}{M(\gamma)\Gamma(\gamma)} \\ &+ \frac{1-\gamma}{M(\gamma)}\xi_1\|\Im - \varsigma\|_{PC_g(L,\Phi)} + \|\Im - \varsigma\|_{PC_g(L,\Phi)}\frac{\xi_1\varphi(\upsilon)^{\gamma}}{M(\gamma)\Gamma(\gamma)} \\ &= \|\Im - \varsigma\|_{PC_g(L,\Phi)}[\rho_1\xi_4 + \xi_1(\frac{2(1-\gamma)}{M(\gamma)} + \frac{\varphi(T)^{\gamma}}{M(\gamma)g(\upsilon)\Gamma(\gamma)} + \frac{\rho_1\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)})].\end{aligned}$$

This inequality and (6.4) gives that

$$\|\Theta(\Im) - \Theta(\varsigma)\|_{PC_g^*(L,\Phi)} < \|\Im - \varsigma\|_{PC_g^*(L,\Phi)},$$

which means that Θ is a contraction, and hence, by the Banach fixed-point theorem, it has a unique fixed point which is a solution of Problem (2.3). □

7. Existence of solutions of Problem (2.4)

Lemma 9. *Let $\Im : L \to \Phi$, $\mathfrak{R}_i : L \times \Phi \to \Phi; i \in \mathbb{N}_{1,r}$, be continuous and $W : L \to \Phi$ be continuous with $W(\tau_i) = 0, i \in N_{1,r}$.*

(1) If $\Im \in PCH^{1,*}((0,T),\Phi)$ is a solution of the fractional differential equation:

$$\begin{cases} {}^{ABC}D^{\gamma,\varphi,g}_{\tau_i,\upsilon}\Im(\upsilon) = W(\upsilon), \upsilon \in \cup_{i=0}^{i=r} L_i, \\ \Im(0) = \Im_0 g^{-1}(0) - g^{-1}(0)\Re(\Im), \\ \Im(\upsilon) = \Re_i(\upsilon, \Im(b_i^-)), \upsilon \in [b_i, \tau_i];\ i \in \mathbb{N}_{1,r}. \end{cases} \tag{7.1}$$

Then,

$$\Im(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) +^{AB} I^{\gamma,\varphi,g}_{0,\upsilon} W(\upsilon), \upsilon \in [0, b_1], \\ \Re_i(\upsilon, \Im(b_i^-)), \upsilon \in M_i, i \in N_{1,r}, \\ \Re_i(\tau_i, \Im(b_i^-)) +^{AB} I^{\gamma,\varphi,g}_{\tau_i,\upsilon} W(\upsilon), \upsilon \in L_i,\ i \in N_{1,r}. \end{cases} \tag{7.2}$$

(2) If $W \in PCH^{1,*}((0,T),\Phi)$ and $\Im$ satisfy (7.2), then $\Im \in PCH^{1,*}((0,T),\Phi)$ and $\Im$ is a solution to Problem (2.4).

Proof.

(1) Suppose that $\Im \in PCH^{1,*}((0,T),\Phi)$ is a solution for (7.1). By following the same arguments in the proof of Lemma (8), we obtain

$$\Im(\upsilon) = \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(\Im) +^{AB} I^{\gamma,\varphi,g}_{0,\upsilon} W(\upsilon); \upsilon \in [0, b_1],$$

and

$$\Im(\upsilon) = c_i +^{AB} I^{\gamma,\varphi,g}_{\tau_i,\upsilon} W(\upsilon), \upsilon \in (\tau_i, b_{i+1}); i \in \mathbb{N}_{1,r}.$$

Since $\Im$ is continuous at $\tau_i; i \in \mathbb{N}_{1,r}$, we have

$$\Re_i(\tau_i, \Im(b_i^-)) = \Im(\tau_i^-) = \Im(\tau_i^+) = c_i + {}^{AB}I^{\gamma,\varphi,g}_{\tau_i,\tau_i} W(\upsilon) = c_i,$$

hence

$$\Im(\upsilon) = \Re_i(\tau_i, \Im(b_i^-)) +^{AB} I^{\gamma,\varphi,g}_{\tau_i,\tau_i} W(\upsilon); \upsilon \in (\tau_i, b_{i+1}), i \in \mathbb{N}_{1,r}.$$

Therefore, $\Im$ satisfies (7.2).

(2) By following the same arguments in the proof of Lemma (8), we can show that if $W \in PCH^{1,*}((0,T),\Phi)$, and $\Im$ is defined by(7.2), then $\Im \in PCH^{1,*}((0,T),\Phi)$ and $\Im$ is a solution to Problem (7.1).

□

Based on Lemma (9), we obtain the following corollary.

Corollary 4. *A function $\Im \in PCH^{1,*}(L,\Phi)$ is a solution to Problem (2.4) if it satisfies the fractional integral equation:*

$$\Im(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(0)\Re(x) + \frac{1-\gamma}{M(\gamma)} W_0(\upsilon) \\ + \frac{\gamma}{M(\gamma)\Gamma(\gamma)g(\upsilon)} \int_0^\upsilon (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s) W_0(s) ds, \upsilon \in [0, b_1], \\ \Re_i(\upsilon, x(b_i^-)), \upsilon \in M_i, i \in N_{1,r}, \\ \Re_i(\tau_i, x(b_i^-)) +^{AB} I^{\gamma,\varphi,g}_{\tau_i,\upsilon} W_i(\upsilon), \upsilon \in L_i, i \in N_{1,r}, \end{cases} \tag{7.3}$$

where $W_i(\upsilon) = \int_{\tau_i}^\upsilon z(s) ds,\ z \in S^2_{\Psi(.,x(.))}, \upsilon \in (\tau_i, b_{i+1}], i \in \mathbb{N}_{0,r}$.

In the following theorem, we give an existence result of solutions to Problem(2.4).

Theorem 5. *Assume that* $(H\Psi)$ *holds after replacing (3.3) with*

$$2\rho_1\rho_2[\frac{1-\gamma}{M(\gamma)} + \frac{2\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)}] < 1. \tag{7.4}$$

and suppose that $(H\mathfrak{R})^{**}$, $(H\mathfrak{R}_i)^*$ *hold. If there is* $\xi_5 > 0$ *such that for any* $x \in PC_g(L, \Phi)$,

$$\|\mathfrak{R}_i(\upsilon, x(b_i^-))\| \leq \xi_5 \|x\|_{PC_g(L,\Phi)}, \ \forall \upsilon \in \cup_{i=1}^r [b_i, \tau_i],$$

then the solution set to Problem (2.4) is non-empty and compact provided that

$$c + \frac{\rho_1\rho_2(1-\gamma)}{M(\gamma)} + \xi_5 + \frac{\rho_1\rho_3\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)} < 1, \tag{7.5}$$

where $\rho_2 = \|\tau\|_{L^1(L,\mathbb{R}^+)}$.

Proof. As in the proof of Theorem 3, we define a set-valued function $\Omega : PC_g^*(L, \Phi) \to 2^{PC_g^*(L,\Phi)} - \{\phi\}$, where ϕ is the empty set, as follows: $\mathfrak{I} \in \Omega(x)$ if and only if

$$\mathfrak{I}(\upsilon) = \begin{cases} \mathfrak{I}_0 g^{-1}(\upsilon) - g^{-1}(0)\mathfrak{R}(x) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W_0(\upsilon), \upsilon \in [0, b_1], \\ \mathfrak{R}_i(\upsilon, x(b_i^-)), \upsilon \in M_i, i \in N_{1,r}, \\ \mathfrak{R}_i(\tau_i, x(b_i^-)) +^{AB} I_{\tau_i,\upsilon}^{\gamma,\varphi,g} W_i(\upsilon), \upsilon \in L_i, i \in N_{1,r}, \end{cases}$$

where $W_i(\upsilon) = \int_{\tau_i}^{\upsilon} z(s)ds, \ z \in S^2_{\Psi(.,x(.))}, \upsilon \in (\tau_i, b_{i+1}], i \in \mathbb{N}_{0,r}$.

Note that if $\mathfrak{I}$ is a fixed point for Ω, then

$$\mathfrak{I}(\upsilon) = \begin{cases} \mathfrak{I}_0 g^{-1}(\upsilon) - g^{-1}(0)\mathfrak{R}(\mathfrak{I}) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W_0(\upsilon), \upsilon \in [[0, b_1], \\ \mathfrak{R}_i(\upsilon, \mathfrak{I}(b_i^-)), \upsilon \in M_i, i \in N_{1,r}, \\ \mathfrak{R}_i(\tau_i, \mathfrak{I}(b_i^-)) +^{AB} I_{\tau_i,\upsilon}^{\gamma,\varphi,g} W_i(\upsilon), \upsilon \in L_i, i \in N_{1,r}. \end{cases}$$

where $W_i(\upsilon) = \int_{\tau_i}^{\upsilon} z(s)ds, z(s) \in \Psi(s, \mathfrak{I}(s)), a.e.$ Let $W : L \to \Phi$ be the function defined by

$$W(\upsilon) = \begin{cases} \int_0^{\upsilon} z(s)ds, \upsilon \in [0, b_1], \\ 0, \upsilon \in (b_i, \tau_i], i \in N_{1,r}, \\ \int_{\tau_i}^{\upsilon} z(s)ds, \upsilon \in (\tau_i, b_{i+1}], i \in N_{1,r}, \end{cases}$$

then $W \in PCH^{1,*}((0,T), \Phi)$, $W(\upsilon) \in \int_{\tau_i}^{\upsilon} \Psi(s, \mathfrak{I}(s))ds$ and $W(\tau_i) = 0, \forall i \in \mathbb{N}_{0,r}$, and therefore, by the second statement of Lemma (9), $\mathfrak{I}$ is a solution for Problem (2.4). We will use Lemma (5), to show that Ω has a fixed point. Since we will follow the same method as in proving Theorem 2, we omit some details and focus on the differences with that proof.

Claim 10. *There is a natural number* ζ_0^* *such that* $R(\Delta_{\zeta_0^*}) \subseteq \Delta_{\zeta_0^*}$.

Pf: If this is not true, then for every natural number n there are $x_n, \mathfrak{I}_n$ with $\|\mathfrak{I}_n\|_{PC_g(L,\Phi)} > n$, $\|x_n\|_{PC_g(L,\Phi)} \leq n$ and $\mathfrak{I}_n \in R(x_n)$. By the definition of R,

$$\mathfrak{I}_n(\upsilon) = \begin{cases} \mathfrak{I}_0 g^{-1}(\upsilon) - g^{-1}(0)\mathfrak{R}(x_n) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W_{0,n}(\upsilon), \upsilon \in [0, b_1], \\ \mathfrak{R}_i(\upsilon, x_n(b_i^-)), \upsilon \in M_i, i \in N_{1,r}, \\ \mathfrak{R}_i(\tau_i, x_n(b_i^-)) +^{AB} I_{\tau_i,\upsilon}^{\gamma,\varphi,g} W_{i,n}(\upsilon), \upsilon \in L_i, i \in N_{1,r}. \end{cases} \tag{7.6}$$

where $W_{i,n}(\upsilon) = \int_{\tau_i}^{\upsilon} z_n(s)ds, z_n \in S^2_{\Psi(.,x_n(.))}, \upsilon \in (\tau_i, b_{i+1}], i \in \mathbb{N}_{0,r}$. Using (iii) of $(H\Psi)$, we get for any $\upsilon \in L$, and any $i \in N_{0,r}$.

$$\|W_{i,n}(\upsilon)\| \leq \int_{\tau_i}^{\upsilon} \|z_n(s)\|ds \leq (1+n)\int_{\tau_i}^{\upsilon} \tau(s)ds \leq (1+n)\|\tau\|_{L^2(L,\mathbb{R}^+)} \leq (1+n)\rho_2. \tag{7.7}$$

As in (5.12), we obtain

$$\|g(\upsilon)\Im_n(\upsilon)\| \leq \|\Im_0\| + \ cn + d + \frac{1-\gamma}{M(\gamma)}\rho_1\rho_2(1+n) + \frac{\rho_1\rho_2(1+n)\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}, \forall \upsilon \in [0, b_1]. \tag{7.8}$$

If $\upsilon \in (b_i, \tau_i]$, then by $(H\mathfrak{R}_i)^*$,

$$\|\Im_n(\upsilon)\| = \|\mathfrak{R}_i(\upsilon, x_n(b_i^-))\| \leq \xi_5\|x_n\|_{PC_g(L,\Phi)}. \tag{7.9}$$

Let $\upsilon \in (\tau_i\ b_{i+1}], i \in \mathbb{N}_{1,r}$. Then, from (7.3), (7.6), (7.7) and $(H\mathfrak{R})$, we obtain that

$$\|g(\upsilon)\Im_n(\upsilon)\| \leq \xi_5\|x_n\|_{PC_g(L,\Phi)} + \rho_1(\frac{1-\gamma}{M(\gamma)}(1+n)\rho_2 + \frac{\rho_1\rho_2(1+n)\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}. \tag{7.10}$$

Inequalities (7.6) and (7.8–7.10) gives

$$n < \left\|\Im_n\right\|_{PC_g(J,\Phi)} \leq \|\Im_0\| + \ cn + d + \frac{1-\gamma}{M(\gamma)}\rho_1\rho_2(1+n) + \xi_5 n + \frac{\rho_1\rho_2(1+n)\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)},$$

thus

$$1 < c + \frac{\rho_1\rho_2(1-\gamma)}{M(\gamma)} + \xi_5 + \frac{\rho_1\rho_2\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)},$$

which contradicts (5.8). ¶

Claim 11. *If $x_k \in \Delta_{\zeta_0^*}, \Im_n \in \Omega(x_n), x_n \to x$ and $\Im_n \to \Im$, in $PC_g(L, \Phi)$, then $\Im \in \Omega(x)$.*

Pf:

From the definition of Ω,

$$\Im_n(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(0)\mathfrak{R}(x_n) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W_{0,n}(\upsilon), \upsilon \in [0, b_1], \\ \mathfrak{R}_i(\upsilon, x_n(b_i^-)), \upsilon \in M_i, i \in N_{1,r}, \\ \mathfrak{R}_i(\tau_i, x_n(b_i^-)) +^{AB} I_{\tau_i,\upsilon}^{\gamma,\varphi,g} W_{i,n}(\upsilon), \upsilon \in L_i, i \in N_{1,r}, \end{cases}$$

where $W_{i,n}(\upsilon) = \int_{\tau_i}^{\upsilon} z_n(s)ds; \upsilon \in (\tau_i, b_{i+1}],\ i \in \mathbb{N}_{0,r},\ z_n \in S^2_{\Psi(.,x_n(.))}$. It follows by (iii) in $(H\Psi)$, that $\|z_n(\upsilon)\| \leq \tau(\upsilon)(1+\zeta_0^*), a.e,$. Thus, $\{z_n : n \geq 1\}$ is weakly compact in $L^2(L, \Phi)$. By Mazure's lemma, we can find, without loss of generality, a sub sequence $(z_n^*), n \geq 1$ of convex combinations of (z_n) converging almost everywhere to a function $z \in L^2(L, \Phi)$. By the upper semicontinuity of $\Psi(\upsilon, .), a, e.,$ it follows that $z \in S^2_{\Psi(.,.x(.))}$. Set $W_{i,n}^*(\upsilon) = \int_{\tau_i}^{\upsilon} z_n^*(s)ds; \upsilon \in (\tau_i, b_{i+1}]$ Then, $W_{i,n}^*(\upsilon) \to W_i(\upsilon) = \int_{\tau_i}^{\upsilon} z(s)ds \in \int_{\tau_i}^{\upsilon} \Psi(s, x(s))ds$.

In addition, from the continuity of both $\mathfrak{R}$ and $\mathfrak{R}_i(\upsilon, .); \upsilon \in L$, it follows that

$$\Im(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(0)\mathfrak{R}(x) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W_0(\upsilon), \upsilon \in [0, b_1], \\ \mathfrak{R}_i(\upsilon, x(b_i^-)), \upsilon \in M_i, i \in N_{1,r}, \\ \mathfrak{R}_i(\tau_i, x(b_i^-)) +^{AB} I_{\tau_i,\upsilon}^{\gamma,\varphi,g} W_i(\upsilon), \upsilon \in L_i, i \in N_{1,r}. \end{cases}$$

which implies $\Im \in \Omega(x)$. ¶

Claim 12. *For any $x \in \Delta_{\zeta_0^*}$, the set $\Omega(x)$ is compact.*

The proof is exactly as the proof of Claim 3, in the prrof of Theorem 3.

Claim 13. *Let $\Lambda_1 = \Omega(\Delta_{\zeta_0^*})$. The set of functions*

$$\Lambda_1|_{[\tau_i,b_{i+1}]} = \{\mathfrak{I}^* \in C([\tau_i, b_{i+1}], \Phi) : \mathfrak{I}^*(\upsilon) = g(\upsilon)\mathfrak{I}(\upsilon), \upsilon \in (\tau_i, b_{i+1}],$$
$$\mathfrak{I}^*(\tau_i) = \lim_{\upsilon\to\tau_i^+} g(\upsilon)\mathfrak{I}(\upsilon), \mathfrak{I} \in \Lambda_1\}, i \in \mathbb{N}_{0,r},$$

and

$$\Lambda_1|_{[b_i,\tau_i]} = \{\mathfrak{I}^* \in C([b_i, \tau_i], \Phi) : \mathfrak{I}^*(\upsilon) = \mathfrak{I}(\upsilon), \upsilon \in (b_i, \tau_i],$$
$$\mathfrak{I}^*(b_i) = \lim_{\upsilon\to b_i^+} \mathfrak{I}(\upsilon), \mathfrak{I} \in \Lambda_1\}, i \in \mathbb{N}_{1,r}.$$

are equicontinuous in $C([\tau_i, b_{i+1}], \Phi)$ and $C([b_i, \tau_i], \Phi)$ respectively.

Pf: From the definition of Ω and from the assumption that for any $i \in \mathbb{N}_{1,r}$ and any $\upsilon \in L$, the function $x \to \mathfrak{R}_i(\upsilon, x)$ is uniformly continuous, we obtain the equicontinuity of $\Lambda_1|_{[b_i,\tau_i]}$.

Now, let $\mathfrak{I}^* \in \Lambda_1|_{[\tau_i,b_{i+1}]}$. Then

$$\mathfrak{I}^*(\upsilon) = \begin{cases} \mathfrak{I}_0 - \mathfrak{R}(x) + g(\upsilon)^{AB}I_{0,\upsilon}^{\gamma,\varphi,g}W_0(\upsilon), \upsilon \in [0, b_1], \text{if } i = 0, \\ g(\upsilon)\mathfrak{R}_i(\tau_i, x(b_i^-)) +^{AB} I_{\tau_i,\upsilon}^{\gamma,\varphi,g}W_i(\upsilon), \upsilon \in (\tau_i, b_{i+1}], i \in N_{1,r}, \end{cases} \tag{7.11}$$

and $\mathfrak{I}^*(\tau_i) = \lim_{\upsilon\to\tau_i^+} \mathfrak{I}^*(\upsilon)$, where $W_i(\upsilon) = \int_{\tau_i}^{\upsilon} z(s)ds,\ z \in S^2_{\Psi(.,x(.))}, \upsilon \in (\tau_i, b_{i+1}], i \in \mathbb{N}_{0,r}$. From (7.7), for any $\upsilon \in (\tau_i, b_{i+1}],\ i \in \mathbb{N}_{0,r}$, we have

$$\begin{aligned} \|g(\upsilon)W_i(\upsilon)\| &\le \rho_1 \int_{\tau_i}^{\upsilon} \|z(s)\|ds \le \rho_1(1 + \zeta_0) \int_{\theta_i}^{\upsilon} \tau(s)ds \\ &\le \rho_1(1 + \zeta_0)\|\tau\|_{L^1(L,\mathbb{R}^+)} \le \rho_1\rho_2(1 + \zeta_0^*). \end{aligned} \tag{7.12}$$

Using this inequality and the same arguments as in (5.17), we get

$$\lim_{\delta\to 0} \|\mathfrak{I}^*(\upsilon + \delta) - \mathfrak{I}^*(\upsilon)\| = 0, \forall \upsilon \in (\tau_i, b_{i+1}],\ i \in \mathbb{N}_{0,r}.$$

If $\upsilon = \tau_i,\ i \in \mathbb{N}_{1,r}$ and $\delta > 0$, then $\lim_{\delta\to 0} \|\mathfrak{I}^*(\tau_i+\delta)-\mathfrak{I}^*(\tau_i)\| = \lim_{\delta\to 0} \lim_{\lambda\to\theta_i^+} \|\mathfrak{I}(\tau_i+\delta)-\mathfrak{I}(\lambda)\| = 0$.

establishing the claim. ¶

Claim 14. *The set $\Lambda = \cap_{n=1}\Lambda_n$ is compact, where $\Lambda_{k+1} = R(\Lambda_k), k \ge 1$.*

Pf: As in the proof of the theorem 3, it is sufficient to prove that,

$$\lim_{n\to\infty} \varkappa_{PC_g}(\Lambda_n) = 0. \tag{7.13}$$

To prove (7.13), let $\epsilon > 0$, and $n \ge 1$ be fixed. Since the sets $\Lambda_1|_{[\tau_i,b_{i+1}]}, i \in \mathbb{N}_{0,r}$ and $\Lambda_1|_{[b_i,\tau_i]}$ are equicontinuous, it follows as in Claim 4, in Theorem 1, that

$$\chi_{PC_g(L,\Phi)}(\Lambda_n) \le 2 \max_{\upsilon\in L} \chi\{g(\upsilon)\mathfrak{I}_m(\upsilon) : m \ge 1\} + \epsilon, \tag{7.14}$$

Since $\Im_m \in \Lambda_n = \Omega(\Lambda_{n-1})$, there is $x_m \in \Lambda_{n-1}$with $\Im_m \in \Omega(x_m)$, and hence

$$g(\upsilon)\Im_m(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(0)\Re(x_m) +^{AB} I_{0,\upsilon}^{\gamma,\varphi,g} W_{0,m}(\upsilon), \upsilon \in [0, b_1], \\ \Re_i(\upsilon, x_m(b_i^-)), \upsilon \in M_i, i \in N_{1,r}, \\ \Re_i(\tau_i, x_m(b_i^-)) +^{AB} I_{\tau_i,\upsilon}^{\gamma,\varphi,g} W_{i,m}(\upsilon), \upsilon \in L_i, i \in N_{1,r}. \end{cases} \tag{7.15}$$

where $W_{i,m}(\upsilon) = \int_{\tau_i}^{\upsilon} z_m(s)ds,\ z_m \in S^2_{\Psi(.,x_m(.))}, \upsilon \in (\tau_i, b_{i+1}], i \in \mathbb{N}_{0,r}$ and $m \geq 1$.

Let $\upsilon \in (\tau_i, b_{i+1}], i \in \mathbb{N}_{0,r}$ be fixed. Using (3.2), we get

$$\begin{aligned} \chi\{g(\upsilon)W_{i,m}(\upsilon) : m \geq 1\} &\leq \chi\{g(\upsilon)\int_{\tau_i}^{\upsilon} z_m(s)ds : m \geq 1\} \leq 2g(\upsilon)\int_{\tau_i}^{\upsilon} \chi\{z_m(s) : m \geq 1\}ds \\ &\leq 2g(\upsilon)\int_{\tau_i}^{\upsilon} \chi\{\Psi(s, \{x_m(s) : m \geq 1\})\}ds \leq 2g(\upsilon)\int_{\tau_i}^{\upsilon} \tau(s)\chi\{g(s)x_m(s) : m \geq 1\} \\ &\leq 2\rho_1\chi\chi_{PC_g(L,\Phi)}(\Lambda_{n-1})\int_{s_i}^{\upsilon} \tau(s)ds \leq 2\rho_1\rho_2\chi_{PC_g(L,\Phi)}(\Lambda_{n-1}). \end{aligned} \tag{7.16}$$

Thus

$$\begin{aligned} &\chi\{\int_{\tau_i}^{\upsilon} (\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s)W_{i,m}(s))ds : m \geq 1\} \\ &\leq 2\int_{\tau_i}^{\upsilon} (\varphi(\upsilon) - \varphi(s))^{\gamma-1}\varphi'(s)\chi\{g(s)W_{i,m}(s) : m \geq 1\}ds \\ &\leq 4\rho_1\rho_2\chi_{PC_g(L,\Phi)}(\Lambda_{n-1})\frac{\varphi(T)^\gamma}{\gamma}, \upsilon \in (\tau_i, b_{i+1}],\ i \in \mathbb{N}_{0,r}. \end{aligned} \tag{7.17}$$

By (7.15) and (7.17), it follows that for $\upsilon \in [0, \theta_1]$

$$\begin{aligned} \chi\{g(\upsilon)^{AB}I_{\tau_i,\upsilon}^{\gamma,\varphi,g}W_{i,m}(\upsilon) : m \geq 1\} &\leq \frac{1-\gamma}{M(\gamma)}2\rho_1\rho_2\chi_{PC_g(L,\Phi)}(\Lambda_{n-1}) + \frac{4\varphi(b)^\gamma}{M(\gamma)\Gamma(\gamma)}\rho_1\rho_2\chi_{PC_g(L,\Phi)}(\Lambda_{n-1}) \\ &= \chi_{PC_g(L,\Phi)}(\Lambda_{n-1})\rho_1\rho_2[\frac{2(1-\gamma)}{M(\gamma)} + \frac{4\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)}]. \end{aligned} \tag{7.18}$$

From the compactness of both $\Re$ and $\Re_i(\upsilon, .)$, it follows from (7.14), (7.15) and (7.18) that

$$\chi_{PC_g(L,\Phi)}(D_n) \leq \chi_{PC_g(L,\Phi)}(\Lambda_{n-1})\rho_1\rho_2[\frac{2(1-\gamma)}{M(\gamma)} + \frac{4\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)}] + \epsilon.$$

Since ϵ is arbitrary, we get

$$\chi_{PC_g(L,\Phi)}(D_n) \leq \chi_{PC_g(L,\Phi)}(D_{n-1})2\rho_1\rho_2[\frac{1-\gamma}{M(\gamma)} + \frac{2\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)}].$$

Since this relation is true for each n, we get

$$\chi_{PC_g(L,\Phi)}(\Delta_n) \leq \chi_{PC_g(L,\Phi)}(\Delta_1)[4\rho_1\rho_2(\frac{1-\gamma}{M(\gamma)} + \frac{2\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)})]^{n-1}.$$

Taking the limit as $n \to \infty$ while keeping in mind (7.3), we obtain (7.13) and Λ is compact. ¶

Applying Lemma (6), we have that there is $\Im \in \Lambda$ such that $\Im \in \Omega(\Im)$ and as we pointed out above, such $\Im$ is a solution for Problem (2.4). In addition, by arguing as in Claim 1, one can show that the set of fixed points of Ω is bounded, and hence by Lemma (7), the set of solutions of Problem (2.4) is compact. □

8. Ulam-Hyers stability of solitions to Problem (2.1)

Definition 3. *[39] Problem(2.1) is Ulam-Hyers stable if there is a* $C > 0$ *such that for each* $\epsilon > 0$ *and each solution* $y \in PCH^1((0,T),\Phi)$ *of the inequality*

$$\begin{cases} \|y(v) - \Im_0 g^{-1}(\upsilon) + g^{-1}(\upsilon)\Re(y) - \frac{1-\gamma}{M(\gamma)} z(\upsilon, y(\upsilon)) \\ -\frac{\gamma}{M(\gamma)}\frac{1}{g(\upsilon)\Gamma(\gamma)}\int_0^{\upsilon}(\varphi(\upsilon)-\varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s,y(s))ds\| \leq \epsilon, \upsilon \in [0,\theta_1] \\ \|y(v) - g^{-1}(\upsilon)\Im_0 + g^{-1}(\upsilon)\Re(y)) + g^{-1}(\upsilon)\sum_{k=1}^{k=i} I_k(y(\theta_k^-)) \\ +\frac{1-\gamma}{M(\gamma)} z(\upsilon, y(\upsilon)) \\ +\frac{\gamma}{M(\gamma)}\frac{1}{g(\upsilon)\Gamma(\gamma)}\int_0^{\upsilon}(\varphi(\upsilon)-\varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s,y(s))ds\| \leq \epsilon, \\ \upsilon \in (\theta_i, \theta_{i+1}],\ i \in \mathbb{N}_{1,r}, \end{cases} \tag{8.1}$$

there is a solution $x \in PCH^1((0,T),\Phi)$ *to Problem (2.1) such that*

$$\|x - y\|_{PC_g(L,\Phi)} \leq C\epsilon. \tag{8.2}$$

Theorem 6. *Under the assumptions of Theorem (2), Problem (2.1) is Ulam-Hyers stable.*

Proof. Let

$$C = \frac{\rho_1}{1 - [\xi_2 + r\xi_3 + \frac{(1-\gamma)\rho_1}{M(\gamma)}\xi_1 + \frac{\rho_1\xi_1\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)}]}. \tag{8.3}$$

From the relation (4.25), we obtain that C is well defined. Suppose that $y \in PCH^1((0,T),\Phi)$ is a solution to the inequality (9.1) and define $x : [0,T] \to \Phi$ by

$$x(\upsilon) = \begin{cases} \Im_0 g^{-1}(\upsilon) - g^{-1}(\upsilon)\Re(x) \\ +\frac{1-\gamma}{M(\gamma)} z(\upsilon, x(\upsilon)) \\ +\frac{\gamma}{M(\gamma)}\frac{1}{g(\upsilon)\Gamma(\gamma)}\int_0^{\upsilon}(\varphi(\upsilon)-\varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s,x(s))ds, \upsilon \in [0,\theta_1] \\ g^{-1}(\upsilon)\Im_0 - g^{-1}(\upsilon)\Re(x)) \\ +g^{-1}(\upsilon)\sum_{k=1}^{k=i} I_k(x(\theta_k^-)) \\ +\frac{1-\gamma}{M(\gamma)} z(\upsilon, x(\upsilon)) \\ +\frac{\gamma}{M(\gamma)}\frac{1}{g(\upsilon)\Gamma(\gamma)}\int_0^{\upsilon}(\varphi(\upsilon)-\varphi(s))^{\gamma-1} g(s)\varphi'(s) z(s,x(s))ds, \\ \upsilon \in (\theta_i, \theta_{i+1}],\ i \in \mathbb{N}_{1,r}. \end{cases} \tag{8.4}$$

By Corollary (1), $x \in PCH^1((0,T),\Phi)$ and is a solution to Problem (2.1). We show the existence of $C > 0$ such that (8.2) hold. For $\upsilon \in [0,\theta_1]$, we have

$$\begin{aligned} \|g(\upsilon)y(v) - g(\upsilon)x(v)\| &\leq g(\upsilon)\|y(v) - x(v)\| \\ &\leq g(\upsilon)\epsilon + \|\Re(y) - \Re(x)\| + \frac{(1-\gamma)g(v)}{M(\gamma)}|z(\upsilon, y(\upsilon)) - z(\upsilon, x(\upsilon))\| \\ &+ \frac{\gamma}{M(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^{\upsilon}(\varphi(\upsilon)-\varphi(s))^{\gamma-1} g(s)\varphi'(s)\|z(s,y(s)) - z(s,y(s))|ds \\ &\leq \rho_1\epsilon + \|y - x\|_{PC_g(L,\Phi)}[\xi_2 + \frac{(1-\gamma)\rho_1}{M(\gamma)}\xi_1 + \frac{\rho_1\xi_1\varphi(T)^\gamma}{M(\gamma)\Gamma(\gamma)}]. \end{aligned} \tag{8.5}$$

For $\upsilon \in (\theta_i, \theta_{i+1}],\ i \in \mathbb{N}_{1,r}$. In view $(HI)^*$

$$\sum_{k=1}^{k=i} \|I_k(y(\theta_k^-) - I_k x(\theta_k^-)\| \leq r\xi_3 \|\Im - \varsigma\|_{PC_g(L,\Phi)}.$$

Therefore,

$$\begin{aligned}\|g(\upsilon)y(v) - g(\upsilon)x(v)\| &\leq g(\upsilon)\|y(v) - x(v)\| \\ &\leq g(\upsilon)\epsilon + \|\Re(y) - \Re(x)\| + \frac{(1-\gamma)g(v)}{M(\gamma)}|z(\upsilon, y(\upsilon)) - z(\upsilon, x(\upsilon))\| \\ &+ \frac{\gamma}{M(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^{\upsilon}(\varphi(\upsilon) - \varphi(s))^{\gamma-1} g(s)\varphi'(s)\|z(s, y(s)) - z(s, y(s))|ds \\ &\leq \rho_1\epsilon + \|y - x\|_{PC_g(L,\Phi)}[\xi_2 + r\xi_3 + \frac{(1-\gamma)\rho_1}{M(\gamma)}\xi_1 + \frac{\rho_1\xi_1\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}]. \end{aligned} \tag{8.6}$$

From (8.5) and (8.6), it follows that

$$\|x - y\|_{PC_g(L,\Phi)} \leq \rho_1\epsilon + \|y - x\|_{PC_g(L,\Phi)}[\xi_2 + r\xi_3 + \frac{(1-\gamma)\rho_1}{M(\gamma)}\xi_1 + \frac{\rho_1\xi_1\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}].$$

So,

$$\|x - y\|_{PC_g(L,\Phi)} \leq \frac{\rho_1\epsilon}{1 - [\xi_2 + r\xi_3 + \frac{(1-\gamma)\rho_1}{M(\gamma)}\xi_1 + \frac{\rho_1\xi_1\varphi(T)^{\gamma}}{M(\gamma)\Gamma(\gamma)}]} = C\epsilon,$$

which shows that Problem (2.1) is stable in the sense of Ulan-Hyers. □

Similarly, we can study the stability of solutions for the other problems.

9. Examples

Example 1. *Let* Φ *be a Hilbert space,* $L = [0, 1]$, *and* $\theta_0 = 0, \theta_1 = \frac{1}{4}, \theta_2 = \frac{1}{2}, \theta_3 = \frac{3}{4}, \theta_4 = 1, r = 3$. *Suppose that* $g : L \to (0, \infty)$ *is continuously differentiable with* $g^{-1}(\upsilon) = \frac{1}{g(\upsilon)}; \upsilon \in L$ *and* $\varphi : L \to R$ *is a strictly increasing and continuously differentiable function with* $\varphi'(\upsilon) \neq 0, \forall \upsilon \in L$. *If* $\Psi : L \times \Phi \to \Phi$, $\Re$*:*$PC_g(L, \Phi) \to \Phi$, *and* $I_i : \Phi \to \Phi$ *are such that for any* $\Im \in PC_g(L, \Phi)$,

$$z(\upsilon, \Im(\upsilon)) = \pounds_1 \int_0^{\upsilon} g(s)\Im(s)\sin s\, ds; \upsilon \in L, \tag{9.1}$$

$$\Re(\Im) = \sum_{i=1}^{i=4} c_i g(\theta_i)\Im(\theta_i), \tag{9.2}$$

and

$$I_i(\Im(\upsilon)) = \xi_3 g(\upsilon)\Im(\upsilon), \tag{9.3}$$

where, c_i, $\pounds_1$ *and* ξ_3 *are positive real numbers. We have*

(i) If $\Im \in PC_g(L, \Phi)$ *and* $W(\upsilon) = z(\upsilon, \Im(\upsilon)); \upsilon \in L$, *then* $W(0) = 0$ *and* $W'(\upsilon) = \pounds_1 g(\upsilon)\Im(\upsilon)\sin\upsilon; \upsilon \in L$.
Since g *and* $\Im$ *are bounded on* L, *then* $W \in PCH^1(L, \Phi)$.

(ii) For any $\upsilon \in L$, and any $\Im, \varsigma \in PC_g(L,\Phi)$, we have

$$\|z(\upsilon,\Im(\upsilon)) - z(\upsilon,\varsigma(\upsilon))\| \leq £_1 \int_0^\upsilon \|g(s)\Im(s)\sin s - g(s)\varsigma(s)\sin s\| ds \leq \upsilon £_1 \|\Im - \varsigma\|_{PC_g(L,\Phi)},$$

$$\begin{aligned}\|\Re(\Im) - \Re(\varsigma)\| &\leq \sum_{i=1}^{i=4} c_i g(\theta_i)\|\Im(\theta_i) - \varsigma(\theta_i)\| \leq \sum_{i=1}^{i=4} c_i g(\theta_i)\|\Im(\theta_i) - \varsigma(\theta_i)\| \\ &\leq c|\sum_{i=1}^{i=4} g(\theta_i)\|\Im(\theta_i) - \varsigma(\theta_i)\|| \leq c\ \|\Im - \varsigma\|_{PC_g(L,\Phi)},\end{aligned}$$

and for any $i = 1,2,3,4$,

$$\left\|I_i(\Im(\theta_i) - I_i(\varsigma(\theta_i))\right\| \leq \xi_3 g((\theta_i)\|\Im(\theta_i) - \varsigma(\theta_i)\| \leq \xi_3\|\Im - \varsigma\|_{PC_g(L,\Phi)}.$$

where $c = \sum_{i=1}^{i=4} c_i$. So, $(H\Psi)^*$ $(H\Re)^*$ and $(HI)^*$are satisfied with $\xi_1 = £_1$ and $\xi_2 = c$.

By applying Theorem 2, with $\Im_0 = 0$, there is a unique solution for the problem:

$$\begin{cases} {}^{ABC}D_{0,\upsilon}^{\gamma,\varphi,g}\Im(\upsilon) = £_1 \int_0^\upsilon g(s)\Im(s)\sin s\, ds, \upsilon \in (\theta_i,\theta_{i+1}), i \in \mathbb{N}_{0,r}, \\ \Im(0) = g^{-1}(0)\Im_0 - g^{-1}(0)\sum_{i=1}^{i=4} c_i g(\theta_i)\Im(\theta_i), \\ \Im(\theta_i^+) = \Im(\theta_i^-) + \xi_3 g(\theta_i^-)\Im(\theta_i^-),\ i \in \mathbb{N}_{1,r}, \end{cases} \tag{9.4}$$

provided that

$$c + 3\xi_3 + £_1(1 + \frac{\rho_1\varphi(1)^\gamma}{M(\gamma)g(\upsilon)\Gamma(\gamma)}) < 1. \tag{9.5}$$

where $\Psi, \Re, I_i; i = 1,2,3,4$ are defined by (9.1- 9.3). By choosing φ, g, c, ξ_3 and $£_1$ appropriately, we obtain (9.4).

Remark 5. *If $g(\upsilon) = \upsilon + 1; \upsilon \in L = [0,1]$, then $g \in H^1(L,\mathbb{R}), g(\upsilon) \leq 2$ and $\frac{1}{g(1)} \leq 1; \upsilon \in L$.*

Example 2. *Let Φ, L, $\theta_i; i = 1,2,3,4$, $r = 3$, γ, $\Im_0 \in \Phi$, K, g, φ be as in Example 1. Suppose that $K : \Phi \to \Phi$ is a linear bounded compact operator and Z is a convex compact subset of Φ with $0 \in K$. Define $\Psi : J \times \Phi \to P_{ck}(\Phi)$, $\Re$:$PC_g(L,\Phi) \to \Phi$, and $I_i : \Phi \to \Phi$ such that for any $\Im \in PC_g(L,\Phi)$,*

$$\Psi(\upsilon, x) = \frac{g(\upsilon)\rho\ \|x\|\sin\upsilon}{\sigma(1 + \|x\|)} Z; (\upsilon, x) \in L \times \Phi, \tag{9.6}$$

$$\Re(\Im) = \sum_{i=1}^{i=4} c_i g(\theta_i)K(\Im(\theta_i)), \tag{9.7}$$

and

$$I_i(\Im(\upsilon)) = \xi_3 g(\upsilon)K(\Im(\upsilon)), \tag{9.8}$$

where, $\rho > 0, \sigma = Sup\{\|z\| : z \in Z\}$. Note that for any $\Im \in PC_g(L,\Phi)$, the function $z(\upsilon) = \frac{g(\upsilon)\rho\ \Im(\upsilon)\sin\upsilon}{\sigma\ (1+\|\Im\|)} z_0$; $z_0 \in Z$ is an element of $S^2_{\Psi(.,\Im(.))}$ and $z(\upsilon) \in \Psi(\upsilon,\Im(\upsilon)); \upsilon \in J$, and hence $S^2_{\Psi(.,\Im(.))}$ is not empty. Moreover, for any $\upsilon \in L$ and any $x, y \in \Phi$, we have

$$\sup_{y\in\Psi(\upsilon,x)} \|y\| \le \frac{g(\upsilon)\rho\ \|x\|\ |\sin\upsilon|}{(1+\|x\|)} \le \rho g(\upsilon)|\sin\upsilon|.$$

and

$$H(\Psi(\upsilon,x),\Psi(\upsilon,y)) \le g(\upsilon)\rho|\sin\upsilon|\ |\frac{\|x\|}{(1+\|x\|)} - \frac{\|y\|}{(1+\|y\|)}| \le \rho g(\upsilon)|\sin\upsilon|\ \|x-y\|,$$

Thus $\Psi(\upsilon,.)$ is upper semicontinuous and for any bounded subset $B \subseteq \Phi$,

$$\varkappa(\Psi(\upsilon,B)) \le g(\upsilon)\eta(\upsilon)\varkappa(B),\ \textit{for}\ \upsilon\in L,$$

where $\eta(\upsilon)=\rho$. The assumption $(H\Psi)$ is satisfied with $\tau(\upsilon)=\rho g(\upsilon)|\sin(\upsilon)|$, for $\upsilon\in L$. Moreover,

$$\|\mathfrak{R}(\mathfrak{I})\| \le \sum_{i=1}^{i=4} c_i g(\theta_i)\|K(\mathfrak{I}(\theta_i))\| \le \sum_{i=1}^{i=4} c_i g(\theta_i)\|K\|\|\mathfrak{I}(\theta_i)\| \le c\|\mathfrak{I}\|_{PC_g(L,\Phi)},$$

and

$$\|I_i(\mathfrak{I}(\upsilon)\| \le \lambda g(\upsilon)\|\mathfrak{I}(\upsilon)\|,$$

where $c=\|K\|\ \sum_{i=1}^{i=4} c_i$ and $\lambda=\xi_3\|K\|$. By applying Theorem (3), we have that the set of solutions of following problem:

$$\begin{cases} {}^{ABC}D^{\gamma,\varphi,g}_{\theta_i,\upsilon}\mathfrak{I}(\upsilon) \in \int_{\theta_i}^{\upsilon} \frac{g(s)\rho\ \|\mathfrak{I}(s)\|\sin s}{\sigma\ (1+\|\mathfrak{I}(s)\|)} Zds, \upsilon\in(\theta_i,\theta_{i+1}), i\in\mathbb{N}_{0,r}, \\ \mathfrak{I}(0) = g^{-1}(0)\mathfrak{I}_0 - g^{-1}(0)\sum_{i=1}^{i=4} c_i g(\theta_i)(K(\mathfrak{I}(\theta_i)), \\ \mathfrak{I}(\theta_i^+) = \mathfrak{I}(\theta_i^-) + \xi_3 g(\theta_i^-)K(\mathfrak{I}(\theta_i^-)),\ i\in\mathbb{N}_{1,r}, \end{cases} \tag{9.9}$$

where Ψ is defined by (9.6) is not empty and compact provided that

$$\|K\|\ \sum_{i=1}^{i=4} c_i + \frac{\rho_1\rho_2(1-\gamma)}{M(\gamma)} + \frac{2\rho_1\rho_2\varphi(1)^\gamma}{M(\gamma)\Gamma(\gamma)} + 3\xi_3\|K\| + \frac{\rho_1\rho_2\varphi(1)^\gamma}{M(\gamma)\Gamma(\gamma)} < 1, \tag{9.10}$$

and

$$4\rho_1\rho_3(\frac{1-\gamma}{M(\gamma)} + \frac{6\varphi(1)^\gamma}{M(\gamma)\Gamma(\gamma)}) < 1, \tag{9.11}$$

By choosing ρ,φ,g,c_i,ξ_3 and K appropriately, we obtain 9.10 and (9.11).

Example 3. *Let $L=[0,1], r=4$. Consider the following partition of $L: 0=\tau_0<b_1=\frac{1}{8}<\tau_1=\frac{1}{4}<b_2=\frac{3}{8}<\tau_2=\frac{1}{2}<b_3=\frac{5}{8}<\tau_3=\frac{6}{8}<b_4=\frac{7}{8}<\tau_4=\frac{15}{16}<b_5=1$.*

Assumes that $\mathfrak{I}_0, g, \varphi, z$ and $\mathfrak{R}$ are be as in Example (1). For any $i\in\mathbb{N}$, let $\mathfrak{R}_i:[b_i,\tau_i]\times\Phi\to\Phi$, be defined as:

$$\mathfrak{R}_i(\upsilon,x) := i\upsilon p g(b_i)x;\ (\upsilon,x)\in[b_i,\tau_i]\times\Phi, i=1,2,3,4. \tag{9.12}$$

where, p is a positive real number. For any $\upsilon\in[b_i,\tau_i], i=1,2,3,4$ and any $\mathfrak{I},\varsigma\in PC^*_g(L,\Phi)$, we have

$$\begin{aligned}\|\mathfrak{R}_i(\upsilon,\mathfrak{I}(b_i^-)) - \mathfrak{R}_i(\upsilon,\varsigma(b_i^-))\| &\le 4p\|g(b_i)\mathfrak{I}(b_i^-) - g(b_i)\varsigma(b_i^-)\| \\ &\le 4p\|\mathfrak{I}-\varsigma\|_{PC_g(L,\Phi)},\end{aligned}$$

thus, $(H\mathfrak{R}_i)$ holds with $\xi_4 = 4p$. By Applying Theorem (4), the following fractional differential equation

$$\begin{cases} {}^{ABC}D^{\gamma,\varphi,g}_{0,\upsilon}\mathfrak{I}(\upsilon) = \pounds_1 \int_0^{\upsilon} g(s)\mathfrak{I}(s)\sin s\, ds, \upsilon \in (\theta_i,\theta_{i+1}), i \in \mathbb{N}_{0,r}, \\ \mathfrak{I}(\upsilon) = \mathfrak{R}_i(\upsilon, \mathfrak{I}(b_i^-)), \upsilon \in [b_i,\tau_i];\ i \in \mathbb{N}_{1,r}, \\ \mathfrak{I}(0) = g^{-1}(0)\mathfrak{I}_0 - g^{-1}(0)\sum_{i=1}^{i=4} c_i g(\theta_i)\mathfrak{I}(\theta_i), \end{cases} \tag{9.13}$$

has a solution under the condition that

$$\rho_1 4p + \pounds_1\left(\frac{2(1-\gamma)}{M(\gamma)} + \frac{\varphi(1)^{\gamma}}{M(\gamma)g(\upsilon)\Gamma(\gamma)} + \frac{\rho_1\varphi(1)^{\gamma}}{M(\gamma)\Gamma(\gamma)}\right) < 1. \tag{9.14}$$

By choosing φ, g, p and $\pounds_1$ appropriately, we can have (9.14) hold.

Example 4. *Let $\gamma, \Phi, L, r, \tau_i, b_i, L_i$, $M_i (i = 1, 2, 3, 4)$, $\mathfrak{I}_0$, g, φ and Ψ be as in Example (2.2). Let $\mathfrak{R}$ be a non-empty convex and compact subset of Φ and $K : \Phi \to \Phi$ be a linear bounded compact operator. Define $\mathfrak{R} : PC^*_g(L,\Phi) \to \Phi$ by:*

$$\mathfrak{R}(\mathfrak{I}) = \sum_{i=0}^{i=4} c_i g(\tau_i) K(\mathfrak{I}(\tau_i)), \tag{9.15}$$

where $c_i > 0$. Obviously, $\mathfrak{R}$ is continuous, compact and $\|\mathfrak{R}(x)\| \le c\|x\|_{PC^*_g(L,\Phi)}$, where $c = \sum_{i=0}^{i=4} c_i$, and hence $(H\mathfrak{R})^{**}$ holds with $c = \sum_{i=0}^{i=4} c_i$ and $d = 0$.

For $i = 1, 2, 3, 4$, define $\mathfrak{R}_i : [b_i, \tau_i] \times \Phi \to \Phi$ as:

$$\mathfrak{R}_i(\upsilon, x) := i\upsilon\, q proj_{\mathfrak{R}} x\,, \tag{9.16}$$

where, q is a positive real number and $proj_{\mathfrak{R}} x$ is the projection of the point x on $\mathfrak{R}$. Then, for any $i = 1, 2, 3, 4$, $\mathfrak{R}_i(\upsilon, .)$, $\upsilon \in [b_i, \tau_i]$ is continuous and compact, and $\|\mathfrak{R}_i(\upsilon, x)\| \le 4q\|x\|$, and hence $(H\mathfrak{R}_i)^*$ holds with $\xi_5 = 4q$. By applying Theorem (5), the set of solutions of following problem:

$$\begin{cases} {}^{ABC}D^{\gamma,\varphi,g}_{\tau_i,\upsilon}\mathfrak{I}(\upsilon) \in \int_{\tau_i}^{\upsilon} \frac{g(\upsilon)\rho}{\sigma}\frac{\|\mathfrak{I}(s)\|\sin\upsilon}{(1+\|\mathfrak{I}(s)\|)} Z ds, \upsilon \in \cup_{i=0}^{i=r} L_i, \\ \mathfrak{I}(\upsilon) = i\upsilon\, q proj_{\mathfrak{R}}\mathfrak{I}(b_i^-), \upsilon \in [b_i,\tau_i];\ i \in \mathbb{N}_{1,r}, \\ \mathfrak{I}(0) = \mathfrak{I}_0 g^{-1}(0) - g^{-1}(0)\sum_{i=0}^{i=4} c_i g(\tau_i) K(\mathfrak{I}(\tau_i)), \end{cases} \tag{9.17}$$

is not empty and compact provided that

$$2\rho_1\rho_2\left[\frac{1-\gamma}{M(\gamma)} + \frac{2\varphi(1)^{\gamma}}{M(\gamma)\Gamma(\gamma)}\right] < 1. \tag{9.18}$$

and

$$\sum_{i=0}^{i=4} c_i + \frac{\rho_1\rho_2(1-\gamma)}{M(\gamma)} + 4q + \frac{\rho_1\rho_2\varphi(1)^{\gamma}}{M(\gamma)\Gamma(\gamma)} < 1, \tag{9.19}$$

where $\rho_1 = Sup_{\upsilon\in L} g(\upsilon)$, $\rho_2 = \|\tau\|_{L^1(L,\mathbb{R}^+)}$ and q is as in (9.16). By choosing φ, g, τ and q appropriately, we can have (9.19) hold.

10. Conclusions

There are many definitions of fractional differentiations of order $\gamma \in (0,1)$, and all these definitions are reduced to the first derivative when $\gamma \to 1$. The existence of such variety contributed to the development of fractional calculus and increased its application in many fields. Researchers continue to be interested in introducing new definitions of fractional differentiation, and this is one of our goals in this work. The notion of the g-weighted φ-Atangana-Baleanu fractional derivative is introduced, which generalizes both the Atangana-Baleanu derivative proposed by Atangana- Baleanu [7], the φ Atangana-Baleanu derivative (generalized Atangana-Baleanu derivative) introduced by HoVu, Behzad Ghanbari [27] and the g-weighted Atangana-Baleanu derivative defined by Al-Refai [31]. Some properties of the introduced derivative are obtained. The existence and stability of solutions for non-local fractional differential equations and inclusions, in infinite dimensional Banach spaces, containing this new fractional derivative in the presence of instantaneous and non-instantaneous impulses are studied. The case in which the lower limit of the fractional derivative is kept at the initial point and where it is changed to the impulsive points are considered. To achieve the results, we establish the relationship between any solution to each of the four studied problems and those of its corresponding fractional integral equation. To our knowledge, there has been no previous study of the g-weighted φ-Atangana-Baleanu fractional derivative, and so, theses results are new and interesting. The used technique are based on the properties of this new fractional differential operator and appropriate fixed point theorems for single-valued functions and set-valued functions. As is pointed out in the introduction, the following results that appear in the literature are particular cases of these obtained in this study: Theorem 4.3 and Theorem 5.1 in [23], Theorem 3.1 in [31], Theorem 2.3 in [27], Theorem 3.1 in [32] , Theorem 3.2 in [33], and Theorem 2.2 in [34].

As to the directions for further research related to this paper, we suggest the following:

- Using the same technique in this paper and the arguments and methods in [28], to study the existence of solutions for Problems (2.1)–(2.4), when $\gamma \in (0,1)$ is replaced with $\gamma(\upsilon)$, where $\gamma : L \to (0,1)$.
- Using our technique and the arguments and methods in [40] to study the existence of solutions for Problems (2.1)–(2.4) in the presence of delay.
- Study the controllability of Problems (2.2) and (2.4).
- Study the stability of solutions for the Problems (2.2)–(2.4) and the controllability of Problems (2.1)–(2.4).
- Extending the obtained results in [41–43] when the fractional derivative operator in these results is replaced with the weighted generalized Atangana-Baleanu fractional derivative and the dimension of the setting space is infinite.
- Studying the numerical solutions of the considered problems.
- Study of how fractional arithmetic can be applied to the topic of uncertain semi-Markovian jump stabilization. For uncertain semi-Markovian jump stabilization, see [44].

Author contributions

M. A and A. G. I.: Conceptualization; M. A. and A. G. I.: Methodology; M. A. and A. G. I.: Validation; M. A.and A. G. I.: Investigation; M. S. A. and A.G.I.: Resources; M. A. and A. G. I.:

Writing original draft preparation; M. A. and A. G. I.: Writing review and editing; M. A.: Project administration; M. A. and A. G. I.: Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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