Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(12): 36259–36272.
DOI:10.3934/math.20241720
Received: 15 October 2024
Revised: 11 December 2024
Accepted: 20 December 2024
Published: 30 December 2024

Research article

On generalized biderivations of Banach algebras

Berna Arslan*

Department of Mathematics, Aydın Adnan Menderes University, Aydın, Turkey

* **Correspondence:** Email: byorganci@adu.edu.tr.

Abstract: The aim of this paper is to introduce the concept of generalized biderivations of unital Banach algebras and prove some results concerning generalized biamenability of unital Banach algebras. Let A and B be unital Banach algebras, and let X be a unital A-B-module. Let $T = Tri(A, X, B)$ be the corresponding triangular Banach algebra. We also study the generalized biamenability of triangular Banach algebras and show that if $X = \{0\}$ and T is generalized biamenable, then A and B are both generalized biamenable.

Keywords: generalized biderivation; inner generalized biderivation; generalized biamenability; weak generalized biamenability; triangular Banach algebra
Mathematics Subject Classification: 15A78, 46H25

1. Introduction and preliminaries

Let A be a unital Banach algebra with unit e_A and X be a unitary Banach A-bimodule in the sense that $e_A \cdot x = x \cdot e_A = x$ for every $x \in X$. We say that a linear map $d : A \to X$ is a *derivation* if $d(ab) = d(a) \cdot b + a \cdot d(b)$ for all $a, b \in A$. For each $x \in X$, the mapping $d_x : A \to X$, $d_x(a) := a \cdot x - x \cdot a$ is a bounded derivation, called an inner derivation.

We can define the right and left actions of A on the dual space X^* of X via

$$(a \cdot f)(x) = f(xa), \quad (f \cdot a)(x) = f(ax)$$

for each $a \in A$, $x \in X$, $f \in X^*$.

A Banach algebra A is called amenable if for each Banach A-bimodule X, the only bounded derivations from A to X^* are inner derivations. The notion of an amenable Banach algebra was introduced by Johnson in [11]. For more details about this notion, see [14]. A Banach algebra A is weakly amenable, if every bounded derivation from A to A^* is an inner derivation. The concept of weak amenability of Banach algebras was introduced by Bade, Curtis, and Dales [1] for commutative Banach algebras and then by Johnson [12] for a general Banach algebra.

A bilinear mapping $D : A \times A \to X$ is called a biderivation if it is a derivation in each argument; that is, for every $b \in A$, the maps $a \mapsto D(a, b)$ and $a \mapsto D(b, a)$ are derivations. Consider the subspace $Z(A, X) = \{x \in X \mid a \cdot x = x \cdot a, \forall a \in A\}$ of X. Then, for each $x \in Z(A, X)$, the mapping $D_x : A \times A \to X$ defined by $D_x(a, b) = x[a, b] = x(ab - ba)$ $(a, b \in A)$ is an example of a biderivation and called an inner biderivation. In [7], Bresar proved that all biderivations on noncommutative prime rings are inner. For more applications and details about biderivations, see [8]. Also see [5, 9], where the structures of biderivations on triangular algebras and generalized matrix algebras were studied, along with the conditions under which these biderivations are inner.

Although derivations and biderivations, as well as inner derivations and inner biderivations, appear similar, there are fundamental differences between them. These differences become more evident when a biderivation is required to be an inner biderivation. One reason is that biderivations depend on two components, while another is that inner biderivations must involve elements from $Z(A, X)$. Similarly, amenability and weak amenability differ from biamenability and weak biamenability, as explained in [3, 4].

In [4], Barootkoob and Mohammadzadeh introduced the concept of biamenability of Banach algebras and demonstrated that, although amenability and biamenability of Banach algebras share some superficial similarities, they exhibit notably distinct and, in some cases, contrasting properties. Specifically, it was shown that commutative Banach algebras and the unitization of Banach algebras are not biamenable, even if they are amenable. Furthermore, it was established that $B(H)$, the algebra of all bounded operators on an infinite-dimensional Hilbert space H, is biamenable but not amenable. A complete characterization of biderivations and inner biderivations on triangular Banach algebras has been provided in [2]. Additionally, a result concerning the weak biamenability of triangular Banach algebras was established in [2].

Bresar introduced the concept of generalized derivations in [6]. The notion of generalized amenability of Banach algebras was later investigated in [13, 15], where the authors provided various results specifically for triangular Banach algebras.

Let A be a unital Banach algebra and X be a unitary Banach A-bimodule. A linear mapping $g : A \to X$ is said to be a generalized derivation if

$$g(ab) = g(a) \cdot b + a \cdot g(b) - a \cdot g(e_A) \cdot b$$

for all $a, b \in A$. The generalized derivation $g_{x,y} : A \to X$ is called an inner generalized derivation if there exist $x, y \in X$ such that $g_{x,y}(a) := x \cdot a + a \cdot y$. Similar to the amenability of a Banach algebra, we say A is generalized amenable if for each Banach A-bimodule X, the only bounded generalized derivations from A to X^* are inner generalized derivations.

A bilinear mapping $G : A \times A \to X$ is called a generalized biderivation if it is a generalized derivation in each argument; that is,

$$G(ab, c) = G(a, c) \cdot b + a \cdot G(b, c) - a \cdot G(e_A, c) \cdot b,$$

and

$$G(a, bc) = G(a, b) \cdot c + b \cdot G(a, c) - b \cdot G(a, e_A) \cdot c$$

for all $a, b, c \in A$.

Example 1. *Let $A = M_2(\mathbb{C})$ be the unital Banach algebra of all 2×2 upper triangular matrices over the complex field $\mathbb{C}$. We define a map $G : A \times A \to A$ by*

$$G\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}\right) = \begin{pmatrix} 0 & bf + ae \\ 0 & 0 \end{pmatrix}.$$

Then, we see that G is a generalized biderivation.

Lemma 1. *Let A be a unital Banach algebra and X be a unitary Banach A-bimodule. Suppose that $x, y \in X$. Then the mapping $G_{x,y} : A \times A \to X$ defined by*

$$G_{x,y}(a, b) := a \cdot x \cdot b + b \cdot y \cdot a$$

is a generalized biderivation.

Proof. It is clear that $G_{x,y}$ is bilinear. Also, we have

$$\begin{aligned} G_{x,y}(ab, c) &= (ab) \cdot x \cdot c + c \cdot y \cdot (ab) \\ &= (ab) \cdot x \cdot c + c \cdot y \cdot (ab) + a \cdot x \cdot (cb) - a \cdot x \cdot (cb) + (ac) \cdot y \cdot b - (ac) \cdot y \cdot b \\ &= a \cdot x \cdot (cb) + c \cdot y \cdot (ab) + (ab) \cdot x \cdot c + (ac) \cdot y \cdot b - a \cdot x \cdot (cb) - (ac) \cdot y \cdot b \\ &= (a \cdot x \cdot c + c \cdot y \cdot a) \cdot b + a \cdot (b \cdot x \cdot c + c \cdot y \cdot b) - a \cdot (x \cdot c + c \cdot y) \cdot b \\ &= G_{x,y}(a, c) \cdot b + a \cdot G_{x,y}(b, c) - a \cdot G_{x,y}(e_A, c) \cdot b, \end{aligned}$$

and

$$\begin{aligned} G_{x,y}(a, bc) &= a \cdot x \cdot (bc) + (bc) \cdot y \cdot a \\ &= a \cdot x \cdot (bc) + (bc) \cdot y \cdot a + b \cdot y \cdot (ac) - b \cdot y \cdot (ac) + (ba) \cdot x \cdot c - (ba) \cdot x \cdot c \\ &= a \cdot x \cdot (bc) + b \cdot y \cdot (ac) + (ba) \cdot x \cdot c + (bc) \cdot y \cdot a - (ba) \cdot x \cdot c - b \cdot y \cdot (ac) \\ &= (a \cdot x \cdot b + b \cdot y \cdot a) \cdot c + b \cdot (a \cdot x \cdot c + c \cdot y \cdot a) - b \cdot (a \cdot x + y \cdot a) \cdot c \\ &= G_{x,y}(a, b) \cdot c + b \cdot G_{x,y}(a, c) - b \cdot G_{x,y}(a, e_A) \cdot c \end{aligned}$$

for all $a, b, c \in A$. Hence, $G_{x,y}$ is a generalized biderivation. □

We call the map of the form $G_{x,y}$ given in Lemma 1 an inner generalized biderivation. We denote by $GZ^1(A, X)$ and $GN^1(A, X)$ the linear spaces of all bounded generalized biderivations and inner generalized biderivations from $A \times A$ into X, respectively. Also we call the quotient space

$$GH^1(A, X) := GZ^1(A, X)/GN^1(A, X),$$

the first generalized bicohomology group from $A \times A$ into X. Similar to the definition of biamenability [2, 4] of Banach algebras, we now define the concept of generalized biamenability of unital Banach algebras as follows. The Banach algebra A is said to be generalized biamenable if every bounded generalized biderivation $G : A \times A \to X^*$ is an inner generalized biderivation; i.e., $GH^1(A, X^*) = \{0\}$. A Banach algebra A is called weakly generalized biamenable if every bounded generalized biderivation from $A \times A$ to A^* is an inner generalized biderivation.

In this paper, we first prove some theorems on generalized biamenability of Banach algebras. And we also give some results characterizing all generalized biderivations and inner generalized biderivations on triangular Banach algebras.

2. Results

In this section, let A be a unital Banach algebra with the unit e_A and X be a unitary Banach A-bimodule.

Theorem 1. *Let A be a Banach algebra and consider $\mathbb{C}$ as a Banach A-bimodule. If there is a nonzero generalized derivation $g : A \to \mathbb{C}$, then generalized biamenability of A implies generalized amenability of A.*

Proof. Let X be a Banach A-bimodule and $g' : A \to X^*$ be a bounded generalized derivation. Then,

$$G : A \times A \to X^*,\ G(a,b) = g(a)g'(b)$$

is a bounded generalized biderivation. Indeed, we have

$$\begin{aligned} G(ab,c) &= g(ab)g'(c) \\ &= (g(a)b)g'(c) + (ag(b))g'(c) - (ag(e_A)b)g'(c), \end{aligned}$$

and

$$\begin{aligned} & G(a,c) \cdot b + a \cdot G(b,c) - a \cdot G(e_A,c) \cdot b \\ = \ & (g(a)g'(c)) \cdot b + a \cdot (g(b)g'(c)) - a \cdot (g(e_A)g'(c)) \cdot b \end{aligned}$$

for all $a, b, c \in A$. Since $\mathbb{C}$ is a Banach A-bimodule, we get

$$G(ab,c) = G(a,c) \cdot b + a \cdot G(b,c) - a \cdot G(e_A,c) \cdot b$$

for all $a, b, c \in A$. Similarly, we have

$$\begin{aligned} G(a,bc) &= g(a)g'(bc) \\ &= g(a)(g'(b) \cdot c) + g(a)(b \cdot g'(c)) - g(a)(b \cdot g'(e_A) \cdot c), \end{aligned}$$

and

$$\begin{aligned} & G(a,b) \cdot c + b \cdot G(a,c) - b \cdot G(a,e_A) \cdot c \\ = \ & (g(a)g'(b)) \cdot c + b \cdot (g(a)g'(c)) - b \cdot (g(a)g'(e_A)) \cdot c \end{aligned}$$

for all $a, b, c \in A$. Hence, it is clear that

$$G(a,bc) = G(a,b) \cdot c + b \cdot G(a,c) - b \cdot G(a,e_A) \cdot c.$$

Then, we see that G is a generalized biderivation. Hence, there are $f, h \in X^*$ such that

$$g(a)g'(b) = G(a,b) = a \cdot f \cdot b + b \cdot h \cdot a$$

for all $a, b \in A$. Therefore, for every $b \in A$ and for some $a \in A$ such that $g(a) \neq 0$, we have

$$g^{'}(b) = \left(\frac{a \cdot f}{g(a)}\right) \cdot b + b \cdot \left(\frac{h \cdot a}{g(a)}\right).$$

So, we get that g' is inner, and then A is generalized amenable. □

A question naturally arises from the generalized biamenability of a Banach algebra, can we then determine whether another Banach algebra is generalized biamenable? The answer is affirmative in the following cases.

Theorem 2. *If $\theta : A \to B$ is a continuous homomorphism of Banach algebras with the dense range and A is generalized biamenable, then so is B.*

Proof. Let X be a Banach B-bimodule. Consider X as an A-bimodule with module actions $a \cdot x = \theta(a)x$ and $x \cdot a = x\theta(a)$ for each $a \in A$ and $x \in X$. Now, for each $G \in GZ^1(B, X^*)$, $G \circ (\theta \times \theta) \in GZ^1(A, X^*)$ and generalized biamenability of A implies that

$$(G \circ (\theta \times \theta))(a, b) = G(\theta(a), \theta(b)) = \theta(a) \cdot f \cdot \theta(b) + \theta(b) \cdot h \cdot \theta(a) \quad (a, b \in A)$$

for some $f, h \in X^*$. By density we conclude that

$$G(a', b') = a' \cdot f \cdot b' + b' \cdot h \cdot a'$$

for all $a', b' \in B$. □

An immediate consequence of this theorem is the following:

Corollary 1. *Let A be a generalized biamenable Banach algebra and I be a closed ideal in A. Then the Banach algebra A/I is generalized biamenable.*

Proof. The canonical mapping $A \to A/I$ is a contractive, surjective homomorphism, and therefore continuous. □

3. Results for triangular Banach algebras

Let A and B be Banach algebras, and suppose that X is A-B-module; that is, X is a Banach space, a left A-module, a right B-module, and the actions of A and B are continuous in that

$$||a \cdot x \cdot b|| \leq ||a||||x||||b||$$

for each $a \in A$, $x \in X$, $b \in B$. If A has a unit 1_A and B has a unit 1_B, then X is said to be *unital* in the sense that $1_A \cdot x = x \cdot 1_B = x$ for every $x \in X$. We define the corresponding triangular Banach algebra

$$T := Tri(A, X, B) = \left\{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \mid a \in A, x \in X, b \in B \right\}$$

with the usual 2×2 matrix addition and multiplication. The norm on T is

$$\left\| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right\| := ||a|| + ||x|| + ||b||.$$

Moreover, if the Banach algebras A, B, and the A-B-module X are unital, then T is unital. The dual of triangular Banach algebra T is

$$T^* = \left\{ \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \mid f \in A^*, h \in X^*, g \in B^* \right\},$$

where $\begin{pmatrix} f & h \\ 0 & g \end{pmatrix}\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\right) := f(a) + h(x) + g(b)$. T^* is a triangular T-bimodule with respect to the following module actions

$$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} := \begin{pmatrix} a \cdot f + x \cdot h & b \cdot h \\ 0 & b \cdot g \end{pmatrix},$$

$$\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \cdot \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} := \begin{pmatrix} f \cdot a & h \cdot a \\ 0 & h \cdot x + g \cdot b \end{pmatrix}$$

for every $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in T$ and $\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \in T^*$. Such algebras were introduced by Forrest and Marcoux in [10].

We will assume that the corner Banach algebras A and B are unital and that X is a unital A-B-module and T is associated triangular Banach algebra. We begin with the following theorem.

Theorem 3. *Let $\delta_A : A \times A \to A^*$ and $\delta_B : B \times B \to B^*$ be bounded generalized biderivations. Then the bilinear mapping $G : T \times T \to T^*$ defined by*

$$G\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) = \begin{pmatrix} \delta_A(a, a') & 0 \\ 0 & \delta_B(b, b') \end{pmatrix}$$

is a bounded generalized biderivation. Furthermore, G is inner if and only if δ_A and δ_B are inner.

Proof. It is easy to verify that G is a generalized biderivation. Also,

$$\begin{aligned} & \left\| \begin{pmatrix} \delta_A(a, a') & 0 \\ 0 & \delta_B(b, b') \end{pmatrix} \right\| \\ = \ & \|\delta_A(a, a')\| + \|\delta_B(b, b')\| \\ \leq \ & \|\delta_A\|\|a\|\|a'\| + \|\delta_B\|\|b\|\|b'\| \\ \leq \ & (\|\delta_A\| + \|\delta_B\|)\,(\|a\| + \|x\| + \|b\|)\,(\|a'\| + \|x'\| + \|b'\|) \\ \leq \ & (\|\delta_A\| + \|\delta_B\|) \left\| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right\| \left\| \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \right\|. \end{aligned}$$

Hence, G is bounded. Suppose that G is an inner generalized biderivation. Then, there exist $\begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix}, \begin{pmatrix} f_2 & h_2 \\ 0 & g_2 \end{pmatrix} \in T^*$ such that

$$\begin{aligned} G\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) &= \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix} \cdot \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \\ &\quad + \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \cdot \begin{pmatrix} f_2 & h_2 \\ 0 & g_2 \end{pmatrix} \cdot \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}. \end{aligned}$$

In particular, we have that

$$\begin{pmatrix} \delta_A(a, a') & 0 \\ 0 & 0 \end{pmatrix} = G\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix}\right)$$

$$
\begin{aligned}
&= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix} \cdot \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} f_2 & h_2 \\ 0 & g_2 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} a \cdot f_1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a' \cdot f_2 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} (a \cdot f_1) \cdot a' & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} (a' \cdot f_2) \cdot a & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} a \cdot f_1 \cdot a' + a' \cdot f_2 \cdot a & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Thus, $\delta_A(a, a') = a \cdot f_1 \cdot a' + a' \cdot f_2 \cdot a$. So δ_A is an inner generalized biderivation. Similarly, we can show that δ_B is an inner generalized biderivation.

Conversely, if $\delta_A : A \times A \to A^*$ and $\delta_B : B \times B \to B^*$ are inner generalized biderivations, then there are $f_1, f_2 \in A^*$ and $g_1, g_2 \in B^*$ such that for each $a, a' \in A$, $\delta_A(a, a') = a \cdot f_1 \cdot a' + a' \cdot f_2 \cdot a$, and for each $b, b' \in B$, $\delta_B(b, b') = b \cdot g_1 \cdot b' + b' \cdot g_2 \cdot b$. Then, we have

$$
\begin{aligned}
& G\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \right) \\
&= \begin{pmatrix} \delta_A(a, a') & 0 \\ 0 & \delta_B(b, b') \end{pmatrix} \\
&= \begin{pmatrix} a \cdot f_1 \cdot a' + a' \cdot f_2 \cdot a & 0 \\ 0 & b \cdot g_1 \cdot b' + b' \cdot g_2 \cdot b \end{pmatrix} \\
&= \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} f_1 & 0 \\ 0 & g_1 \end{pmatrix} \cdot \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} + \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \cdot \begin{pmatrix} f_2 & 0 \\ 0 & g_2 \end{pmatrix} \cdot \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}.
\end{aligned}
$$

Hence, G is an inner generalized biderivation. □

Theorem 4. *Let T^* be the triangular bimodule $\begin{pmatrix} A^* & X^* \\ 0 & B^* \end{pmatrix}$ associated to the triangular Banach algebra T. Assume that $G : T \times T \to T^*$ is a bounded generalized biderivation. Then, there exist bounded generalized biderivations $\delta_A : A \times A \to A^*$ and $\delta_B : B \times B \to B^*$, and $h_1, h_2 \in X^*$ such that*

$$
G\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \right) = \begin{pmatrix} \delta_A(a, a') + x \cdot h_2 & h_1 \cdot a + b \cdot h_2 \\ 0 & h_1 \cdot x + \delta_B(b, b') \end{pmatrix}
$$

for every $a, a' \in A$, $b, b' \in B$ and $x, x' \in X$.

Proof. Suppose G is a generalized biderivation on T. Write G as

$$
G\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \right) = \begin{pmatrix} \delta_A(a, a') + d_B(b, b') + k_1(x, x') & r_1(a, a') + r_2(b, b') + r_3(x, x') \\ 0 & d_A(a, a') + \delta_B(b, b') + k_2(x, x') \end{pmatrix},
$$

where $\delta_A : A \times A \to A^*$, $\delta_B : B \times B \to B^*$, $d_A : A \times A \to B^*$, $d_B : B \times B \to A^*$, $k_1 : X \times X \to A^*$, $k_2 : X \times X \to B^*$, $r_1 : A \times A \to X^*$, $r_2 : B \times B \to X^*$ and $r_3 : X \times X \to X^*$ are all bilinear maps. Let

$$
G\left(\begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \right) = \begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix},
$$

and

$$G\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & 1_B \end{array}\right),\left(\begin{array}{cc} a' & x' \\ 0 & b' \end{array}\right)\right)=\left(\begin{array}{cc} f_2 & h_2 \\ 0 & g_2 \end{array}\right)$$

for some $f_1, f_2 \in A^*$, $h_1, h_2 \in X^*$ and $g_1, g_2 \in B^*$. Then, we have

$$\begin{aligned}
G\left(\left(\begin{array}{cc} 1_A & 0 \\ 0 & 1_B \end{array}\right),\left(\begin{array}{cc} a' & x' \\ 0 & b' \end{array}\right)\right) &= G\left(\left(\begin{array}{cc} 1_A & 0 \\ 0 & 0 \end{array}\right)+\left(\begin{array}{cc} 0 & 0 \\ 0 & 1_B \end{array}\right),\left(\begin{array}{cc} a' & x' \\ 0 & b' \end{array}\right)\right)\\
&= G\left(\left(\begin{array}{cc} 1_A & 0 \\ 0 & 0 \end{array}\right),\left(\begin{array}{cc} a' & x' \\ 0 & b' \end{array}\right)\right)+G\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & 1_B \end{array}\right),\left(\begin{array}{cc} a' & x' \\ 0 & b' \end{array}\right)\right)\\
&= \left(\begin{array}{cc} f_1 & h_1 \\ 0 & g_1 \end{array}\right)+\left(\begin{array}{cc} f_2 & h_2 \\ 0 & g_2 \end{array}\right)=\left(\begin{array}{cc} f_1+f_2 & h_1+h_2 \\ 0 & g_1+g_2 \end{array}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left(\begin{array}{cc} \delta_A(a,a') & r_1(a,a') \\ 0 & d_A(a,a') \end{array}\right)=G\left(\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right),\left(\begin{array}{cc} a' & x' \\ 0 & b' \end{array}\right)\right)\\
= \ & G\left(\left(\begin{array}{cc} 1_A & 0 \\ 0 & 0 \end{array}\right)\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right),\left(\begin{array}{cc} a' & x' \\ 0 & b' \end{array}\right)\right)\\
= \ & G\left(\left(\begin{array}{cc} 1_A & 0 \\ 0 & 0 \end{array}\right),\left(\begin{array}{cc} a' & x' \\ 0 & b' \end{array}\right)\right)\cdot\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right)+\left(\begin{array}{cc} 1_A & 0 \\ 0 & 0 \end{array}\right)\cdot G\left(\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right),\left(\begin{array}{cc} a' & x' \\ 0 & b' \end{array}\right)\right)\\
&-\left(\begin{array}{cc} 1_A & 0 \\ 0 & 0 \end{array}\right)\cdot G\left(\left(\begin{array}{cc} 1_A & 0 \\ 0 & 1_B \end{array}\right),\left(\begin{array}{cc} a' & x' \\ 0 & b' \end{array}\right)\right)\cdot\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right)\\
= \ & \left(\begin{array}{cc} f_1 & h_1 \\ 0 & g_1 \end{array}\right)\cdot\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right)+\left(\begin{array}{cc} 1_A & 0 \\ 0 & 0 \end{array}\right)\cdot\left(\begin{array}{cc} \delta_A(a,a') & r_1(a,a') \\ 0 & d_A(a,a') \end{array}\right)\\
&-\left(\begin{array}{cc} 1_A & 0 \\ 0 & 0 \end{array}\right)\cdot\left(\begin{array}{cc} f_1+f_2 & h_1+h_2 \\ 0 & g_1+g_2 \end{array}\right)\cdot\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right)\\
= \ & \left(\begin{array}{cc} f_1\cdot a & h_1\cdot a \\ 0 & 0 \end{array}\right)+\left(\begin{array}{cc} \delta_A(a,a') & 0 \\ 0 & 0 \end{array}\right)-\left(\begin{array}{cc} f_1+f_2 & 0 \\ 0 & 0 \end{array}\right)\cdot\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right)\\
= \ & \left(\begin{array}{cc} f_1\cdot a & h_1\cdot a \\ 0 & 0 \end{array}\right)+\left(\begin{array}{cc} \delta_A(a,a') & 0 \\ 0 & 0 \end{array}\right)-\left(\begin{array}{cc} f_1\cdot a+f_2\cdot a & 0 \\ 0 & 0 \end{array}\right)\\
= \ & \left(\begin{array}{cc} \delta_A(a,a')-f_2\cdot a & h_1\cdot a \\ 0 & 0 \end{array}\right).
\end{aligned}$$

Thus, for all $a, a' \in A$, we have $f_2 \cdot a = 0$, $r_1(a,a') = h_1 \cdot a$ and $d_A(a,a') = 0$. So $f_2 = 0$. Similarly, we have

$$\begin{aligned}
&\left(\begin{array}{cc} d_B(b,b') & r_2(b,b') \\ 0 & \delta_B(b,b') \end{array}\right)=G\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & b \end{array}\right),\left(\begin{array}{cc} a' & x' \\ 0 & b' \end{array}\right)\right)\\
= \ & G\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & b \end{array}\right)\left(\begin{array}{cc} 0 & 0 \\ 0 & 1_B \end{array}\right),\left(\begin{array}{cc} a' & x' \\ 0 & b' \end{array}\right)\right)\\
= \ & G\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & b \end{array}\right),\left(\begin{array}{cc} a' & x' \\ 0 & b' \end{array}\right)\right)\cdot\left(\begin{array}{cc} 0 & 0 \\ 0 & 1_B \end{array}\right)+\left(\begin{array}{cc} 0 & 0 \\ 0 & b \end{array}\right)\cdot G\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & 1_B \end{array}\right),\left(\begin{array}{cc} a' & x' \\ 0 & b' \end{array}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&-\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \cdot G\left(\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix} \\
= &\begin{pmatrix} d_B(b,b') & r_2(b,b') \\ 0 & \delta_B(b,b') \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} 0 & h_2 \\ 0 & g_2 \end{pmatrix} \\
&-\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} f_1 & h_1 + h_2 \\ 0 & g_1 + g_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix} \\
= &\begin{pmatrix} 0 & 0 \\ 0 & \delta_B(b,b') \end{pmatrix} + \begin{pmatrix} 0 & b \cdot h_2 \\ 0 & b \cdot g_2 \end{pmatrix} - \begin{pmatrix} 0 & b \cdot h_1 + b \cdot h_2 \\ 0 & b \cdot g_1 + b \cdot g_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix} \\
= &\begin{pmatrix} 0 & 0 \\ 0 & \delta_B(b,b') \end{pmatrix} + \begin{pmatrix} 0 & b \cdot h_2 \\ 0 & b \cdot g_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & b \cdot g_1 + b \cdot g_2 \end{pmatrix} = \begin{pmatrix} 0 & b \cdot h_2 \\ 0 & \delta_B(b,b') - b \cdot g_1 \end{pmatrix}.
\end{aligned}
$$

Thus, we have $d_B(b,b') = 0$, $b \cdot g_1 = 0$, and $r_2(b,b') = b \cdot h_2$ for all $b, b' \in B$. So $g_1 = 0$. For $x \in X$, we have

$$
\begin{aligned}
&\begin{pmatrix} k_1(x,x') & r_3(x,x') \\ 0 & k_2(x,x') \end{pmatrix} = G\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \\
= &G\left(\begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \\
= &G\left(\begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \cdot \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} \cdot G\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \\
&-\begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} \cdot G\left(\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \cdot \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \\
= &\begin{pmatrix} f_1 & h_1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} k_1(x,x') & f_3(x,x') \\ 0 & k_2(x,x') \end{pmatrix} \\
&-\begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} f_1 & h_1 + h_2 \\ 0 & g_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \\
= &\begin{pmatrix} 0 & 0 \\ 0 & h_1 \cdot x \end{pmatrix} + \begin{pmatrix} k_1(x,x') & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} f_1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \\
= &\begin{pmatrix} 0 & 0 \\ 0 & h_1 \cdot x \end{pmatrix} + \begin{pmatrix} k_1(x,x') & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
= &\begin{pmatrix} k_1(x,x') & 0 \\ 0 & h_1 \cdot x \end{pmatrix}.
\end{aligned}
$$

Thus, $r_3(x,x') = 0$ and $k_2(x,x') = h_1 \cdot x$ for all $x, x' \in X$. Also, we have

$$
\begin{aligned}
&\begin{pmatrix} k_1(x,x') & r_3(x,x') \\ 0 & k_2(x,x') \end{pmatrix} = G\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \\
= &G\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \\
= &G\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix} + \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \cdot G\left(\begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \cdot G\left(\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix} \\
= &\begin{pmatrix} k_1(x,x') & r_3(x,x') \\ 0 & k_2(x,x') \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix} + \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & h_2 \\ 0 & g_2 \end{pmatrix} \\
&-\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} f_1 & h_1 + h_2 \\ 0 & g_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix} \\
= &\begin{pmatrix} 0 & 0 \\ 0 & k_2(x,x') \end{pmatrix} + \begin{pmatrix} x \cdot h_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} x \cdot h_1 + x \cdot h_2 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix} \\
= &\begin{pmatrix} 0 & 0 \\ 0 & k_2(x,x') \end{pmatrix} + \begin{pmatrix} x \cdot h_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x \cdot h_2 & 0 \\ 0 & k_2(x,x') \end{pmatrix}.
\end{aligned}
$$

Thus, $k_1(x, x') = x \cdot h_2$ and $r_3(x, x') = 0$ for all $x, x' \in X$. Hence, we have

$$
G\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) = \begin{pmatrix} \delta_A(a,a') + x \cdot h_2 & h_1 \cdot a + b \cdot h_2 \\ 0 & h_1 \cdot x + \delta_B(b,b') \end{pmatrix}
$$

for all $a, a' \in A$, $b, b' \in B$, $x, x' \in X$. So, for each $a, a' \in A$, $b' \in B$, and $x' \in X$, we have

$$
G\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) = \begin{pmatrix} \delta_A(a,a') & h_1 \cdot a \\ 0 & 0 \end{pmatrix}.
$$

Moreover,

$$
\begin{aligned}
&\begin{pmatrix} \delta_A(a_1a_2,a') & h_1 \cdot (a_1a_2) \\ 0 & 0 \end{pmatrix} = G\left(\begin{pmatrix} a_1a_2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \\
= &G\left(\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \\
= &G\left(\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \cdot \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} \cdot G\left(\begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \\
&-\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} \cdot G\left(\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix}\right) \cdot \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix} \\
= &\begin{pmatrix} \delta_A(a_1,a') & h \cdot a_1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \delta_A(a_2,a') & h \cdot a_2 \\ 0 & 0 \end{pmatrix} \\
&-\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} f_1 & h_1 + h_2 \\ 0 & g_2 \end{pmatrix} \cdot \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix} \\
= &\begin{pmatrix} \delta_A(a_1,a') \cdot a_2 & (h \cdot a_1) \cdot a_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a_1 \cdot \delta_A(a_2,a') & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a_1 \cdot f_1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix} \\
= &\begin{pmatrix} \delta_A(a_1,a') \cdot a_2 & (h \cdot a_1) \cdot a_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a_1 \cdot \delta_A(a_2,a') & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} (a_1 \cdot f_1) \cdot a_2 & 0 \\ 0 & 0 \end{pmatrix} \\
= &\begin{pmatrix} \delta_A(a_1,a') \cdot a_2 + a_1 \cdot \delta_A(a_2,a') - a_1 \cdot f_1 \cdot a_2 & h \cdot (a_1a_2) \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

for all $a_1, a_2 \in A$. So, we have $\delta_A(a_1a_2, a') = \delta_A(a_1, a') \cdot a_2 + a_1 \cdot \delta_A(a_2, a') - a_1 \cdot f_1 \cdot a_2$ for each $a_1, a_2 \in A$. In particular, taking $a_1 = a_2 = 1_A$, we get $\delta_A(1_A, a') = \delta_A(1_A, a') + \delta_A(1_A, a') - f_1$, that is, $f_1 = \delta_A(1_A, a')$.

Similarly, we can show

$$\delta_A(a', a_1a_2) = \delta_A(a', a_1) \cdot a_2 + a_1 \cdot \delta_A(a', a_2) - a_1 \cdot \delta_A(a', 1_A) \cdot a_2.$$

Therefore, δ_A is a generalized biderivation. Further, since G is bounded, so

$$\begin{aligned}
\|\delta_A(a, a')\| &\leq \|\delta_A(a, a')\| + \|h \cdot a\| \\
&= \left\| \begin{pmatrix} \delta_A(a, a') & h \cdot a \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| G\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} \right) \right\| \\
&\leq \|G\| \left\| \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right\| \left\| \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \|G\| \, \|a\| \, \|a'\|.
\end{aligned}$$

It follows that δ_A is bounded. Also, for each $a' \in A$, $b, b' \in B$ and $x' \in X$, we have

$$G\left(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \right) = \begin{pmatrix} 0 & b \cdot h_2 \\ 0 & \delta_B(b, b') \end{pmatrix}.$$

Moreover,

$$\begin{aligned}
&\begin{pmatrix} 0 & (b_1b_2) \cdot h_2 \\ 0 & \delta_B(b_1b_2, b') \end{pmatrix} = G\left(\begin{pmatrix} 0 & 0 \\ 0 & b_1b_2 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \right) \\
= \; & G\left(\begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \right) \\
= \; & G\left(\begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \right) \cdot \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix} \cdot G\left(\begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \right) \\
& - \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix} \cdot G\left(\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}, \begin{pmatrix} a' & x' \\ 0 & b' \end{pmatrix} \right) \cdot \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix} \\
= \; & \begin{pmatrix} 0 & b_1 \cdot h_2 \\ 0 & \delta_B(b_1, b') \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix} \cdot \begin{pmatrix} 0 & b_2 \cdot h_2 \\ 0 & \delta_B(b_2, b') \end{pmatrix} \\
& - \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix} \cdot \begin{pmatrix} f_1 & h_1 + h_2 \\ 0 & g_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix} \\
= \; & \begin{pmatrix} 0 & 0 \\ 0 & \delta_B(b_1, b') \cdot b_2 \end{pmatrix} + \begin{pmatrix} 0 & b_1 \cdot (b_2 \cdot h_2) \\ 0 & b_1 \cdot \delta_B(b_2, b') \end{pmatrix} - \begin{pmatrix} 0 & b \cdot h_1 + b_1 \cdot h_2 \\ 0 & b_1 \cdot g_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix} \\
= \; & \begin{pmatrix} 0 & 0 \\ 0 & \delta_B(b_1, b') \cdot b_2 \end{pmatrix} + \begin{pmatrix} 0 & b_1 \cdot (b_2 \cdot h_2) \\ 0 & b_1 \cdot \delta_B(b_2, b') \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & b_1 \cdot g_2 \cdot b_2 \end{pmatrix} \\
= \; & \begin{pmatrix} 0 & (b_1b_2) \cdot h_2 \\ 0 & \delta_B(b_1, b') \cdot b_2 + b_1 \cdot \delta_B(b_2, b') - b_1 \cdot g_2 \cdot b_2 \end{pmatrix}
\end{aligned}$$

for all $b_1, b_2 \in B$. So, we have

$$\delta_B(b_1b_2, b') = \delta_B(b_1, b') \cdot b_2 + b_1 \cdot \delta_B(b_2, b') - b_1 \cdot g_2 \cdot b_2$$

for each $b_1, b_2 \in B$. Taking $b_1 = b_2 = 1_B$, we have $g_2 = \delta_B(1_B, b')$. Similarly, we can show

$$\delta_B(b', b_1 b_2) = \delta_B(b', b_1) \cdot b_2 + b_1 \cdot \delta_B(b', b_2) - b_1 \cdot \delta_B(b', 1_B) \cdot b_2.$$

Therefore, δ_B is a generalized biderivation. Further, since G is bounded, it is clear that δ_B is bounded. □

Theorem 5. *Let A and B be unital Banach algebras, and let $T = Tri(A, 0, B)$ be the corresponding triangular Banach algebra. Then,*

$$GH^1(T, T^*) \cong GH^1(A, A^*) \oplus GH^1(B, B^*).$$

Proof. Define $f : GZ^1(A, A^*) \oplus GZ^1(B, B^*) \to GH^1(T, T^*)$ by $f((\delta_A, \delta_B)) := [G']$, where $[G']$ is the equivalent class of $G' := \begin{pmatrix} \delta_A & 0 \\ 0 & \delta_B \end{pmatrix}$ in $GH^1(T, T^*)$. Clearly, f is linear. By Theorems 3 and 4, f is surjective. Also, by Theorem 3, we have

$$\begin{aligned} Kerf &= \left\{(\delta_A, \delta_B) \mid \begin{pmatrix} \delta_A & 0 \\ 0 & \delta_B \end{pmatrix} \text{ is inner}\right\} \\ &= \{(\delta_A, \delta_B) \mid \delta_A \text{ and } \delta_B \text{ are inner}\} \\ &= GN^1(A, A^*) \oplus GN^1(B, B^*). \end{aligned}$$

Therefore, we have

$$GH^1(T, T^*) \cong (GZ^1(A, A^*) \oplus GZ^1(B, B^*))/(GN^1(A, A^*) \oplus GN^1(B, B^*)).$$

Then, we have the desired result. □

Corollary 2. *$T = Tri(A, 0, B)$ is weakly generalized biamenable if and only if A and B are weakly generalized biamenable.*

Now, we give a result about generalized biamenability for triangular Banach algebras.

Theorem 6. *If $T = Tri(A, 0, B)$ is generalized biamenable, then A and B are both generalized biamenable Banach algebras.*

Proof. Suppose that $T := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is generalized biamenable. Since $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ is a closed ideal of T, the quotient algebra $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} / \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ is generalized biamenable by Corollary 1. On the other hand, $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} / \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \cong B$, thus, B is generalized biamenable. Similarly, since $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ is a closed ideal of T, we have $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} / \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \cong A$. Thus, it follows from Corollary 1 that A is generalized biamenable. □

4. Conclusions

As we mentioned in the introduction, in this paper, we studied the generalized biderivations of unital Banach algebras. Some results of this new concept have been obtained, and the author thinks that using various methods the results on generalized biamenability can be extended to areas related to Banach algebras in the future.

Use of Generative-AI tools declaration

The author declares that she has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflicts of interest.

References

1. W. G. Bade, P. C. Curtis, H. G. Dales, Amenability and weak amenability for Beurling and Lipschitz algebras, *Proc. London Math. Soc.*, **55** (1987), 359–377. https://doi.org/10.1093/plms/s3-55_2.359
2. S. Barootkoob, Characterization of some biderivations on triangular Banach algebras, *Facta Univ.-Ser. Math.*, **35** (2020), 929–937. https://doi.org/10.22190/FUMI2004929B
3. S. Barootkoob, On *n*-weak biamenability of Banach algebras, *Wavelets and Linear Algebra*, **8** (2021), 37–47.
4. S. Barootkoob, S. Mohammadzadeh, Biamenability of Banach algebras and its applications, *AUT Journal of Mathematics and Computing*, **4** (2023), 129–136. http://doi.org/10.22060/AJMC.2022.21859.1115
5. D. Benkovic, Biderivations of triangular algebras, *Linear Algebra Appl.*, **431** (2009), 1587–1602. http://doi.org/10.1016/j.laa.2009.05.029
6. M. Bresar, On the distance of the composition of two derivations to the generalized derivations, *Glasgow Math. J.*, **33** (1991), 89–93. https://doi.org/10.1017/S0017089500008077
7. M. Bresar, W. S. Martindale 3rd, C. R. Miers, Centralizing maps in prime rings with involution, *J. Algebra*, **161** (1993), 342–357. https://doi.org/10.1006/jabr.1993.1223
8. M. Bresar, Commuting maps: A survey, *Taiwan. J. Math.*, **8** (2004), 361–397. http://doi.org/10.11650/twjm/1500407660
9. Y. Du, Y. Wang, Biderivations of generalized matrix algebras, *Linear Algebra Appl.*, **438** (2013), 4483–4499. https://doi.org/10.1016/j.laa.2013.02.017
10. B. E. Forrest, L. W. Marcoux, Derivations on triangular Banach algebras, *Indiana Univ. Math. J.*, **45** (1996), 441–462. http://doi.org/10.1512/IUMJ.1996.45.1147

11. B. E. Johnson, Cohomology in Banach algebras, *Memoirs Am. Math. Soc.*, **1972** (1972), 127. http://doi.org/10.1090/memo/0127
12. B. E. Johnson, Weak amenability of group algebras, *Bull. London Math. Soc.*, **23** (1991), 281–284. https://doi.org/10.1112/blms/23.3.281
13. M. Mosadeq, On weak generalized amenability of triangular Banach algebras, *J. Math. Ext.*, **8** (2014), 27–39.
14. V. Runde, *Lectures on amenability*, Heidelberg: Springer Berlin, 1 Eds., 2002. http://doi.org/10.1007/b82937
15. A. Zohri, A. Jabbari, Generalized derivations and generalized amenability of Banach algebras, *U.P.B. Sci. Bull., Series A*, **75** (2013), 137–144.

AIMS Press