Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(12): 36191–36201.
DOI: 10.3934/math.20241718
Received: 14 October 2024
Revised: 18 November 2024
Accepted: 29 November 2024
Published: 27 December 2024

Research article

On The oscillatory behavior of solutions to a class of second-order nonlinear differential equations

Maged Alkilayh*

Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia

* **Correspondence:** Email: m.alkilayh@qu.edu.sa.

Abstract: In this paper, we obtain some oscillatory properties for the noncanonical second-order differential equation with mixed neutral terms. We established our results first by transforming the equation into canonical type and then by using the Riccati technique to get new oscillatory properties for the considered equation. We obtained these results to extend and simplify existing criteria in the literature. We discussed some examples to illustrate the effectiveness of our main results.

Keywords: mixed neutral; differential equations; oscillation conditions; second-order; delay
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

The study of the oscillatory behavior of delays in differential equations has advanced significantly during the last few decades; see [1–3]. It has been shown to be quite helpful in the engineering and applied sciences fields to build a range of applied mathematical models that faithfully represent real-world events, see [4–6]. DDEs of neutral type are involved in the delay system of second-order equations issues; which were studied by some authors; see [7, 8].

The theory of oscillatory conditions generally appears from a large adequate instance of drug organization, physics, engineering, the trash hold hypothesis in biology, and air ship control. Some studies have studied the numerical aspects and computational methods of some types of equations;see [2, 3]. Impulsive conditions play an important role in modeling phenomena, principally in relating the dynamics of the population's area under discussion to unexpected modification.

In this work, we focus our attention on the oscillation of the second-order nonlinear differential equation with the form

$$\left(f(s)\left|y'(s)\right|^{\mu-1}y'(s)\right)' + q(s)\left|w(h(s))\right|^{\mu-1}w(h(s)) = 0,\ s \geq s_0, \tag{1.1}$$
$$y(s) = w(s) + \delta_1(s)\,w(\sigma_1(s)) + \delta_2(s)\,w(\sigma_2(s)),$$

where $\delta_1, \delta_2, q \in C([s_0,\infty),[0,\infty)), f \in C([s_0,\infty),(0,\infty))$, and $\eta(s_0) = \int_{s_0}^{\infty} f^{-1/\mu}(\upsilon)\,\mathrm{d}\upsilon < \infty$. Moreover, μ is a quotient of odd positive integers, $\sigma_1, \sigma_2, h \in C([s_0,\infty),\mathbb{R}), \sigma_1(s) \leq s, \sigma_2(s) \geq s, h(s) \leq s, \lim_{s\to\infty}\sigma_1(s) = \lim_{s\to\infty}\sigma_2(s) = \lim_{s\to\infty}h(s) = \infty$ and $q(s)$ is not identically zero for large s.

By a solution of (1.1), we mean a nontrivial function $w(s) \in C^1([s_0,\infty),\mathbb{R})$, which has the properties $f(s)(y'(s))^{\mu} \in C^1([s_0,\infty),\mathbb{R})$, and $w(s)$ satisfies (1.1) on $[s_w,\infty)$, then y is a solution of (1.1). In our study, we focus on the solutions that satisfy $\sup\{|w(s)| : s_w \leq s < \infty\} > 0$, for every $s_w \geq s_0$. If the solution to (1.1) is neither positive in the end nor negative in the end, it is said to be oscillatory. Otherwise, this solution is called non-oscillatory. Therefore, we say that this equation is oscillatory if it has oscillatory solutions.

Exploring Laplace differential equations yields a multitude of essential utilities in mechanical systems, electrical circuits, and the regulation of chemical processes. In addition, their utility extends to ecological systems, epidemiology, and the modeling of population dynamics.

Investigations by some authors in [9–11] have yielded techniques and methodologies aimed at enhancing the oscillatory behavior of solutions for differential equations. For a recent review of the main results in the framework of asymptotic properties for differential equations of different order with neutral term, we refer the reader to [12–14]. Furthermore, the work [15–18] has expanded this inquiry to include differential equations of neutral variety. In recent years, there has also been a significant exploration of oscillation behaviors in fourth-order delay differential equations, as evidenced by studies such as [19–22]. Now, in some detail, Graef et al. [5] improved some results for mixed-type equations.

$$(f(s)(w(s) + \delta_1(s)w(\sigma_1(s)) + \delta_2(s)w(\sigma_2(s)))')' + b(s)y'(s) + q(s)w(h(s)) = 0,$$

where

$$\int_{s_0}^{\infty} \frac{1}{f(s)} \exp\left(-\int_{s_0}^{s} \frac{b(\upsilon)}{f(\upsilon)}\mathrm{d}\upsilon\right)\mathrm{d}s = \infty.$$

Li et al. [20,23] introduced good conditions concerning solutions to second-order differential equations featuring a mixed term in the canonical case.

$$\left(f(s)|y'(s)|^{\mu-1}y'(s)\right)' + q(s)|w(h(s))|^{\mu-1}w(h(s)) = 0,$$

where

$$y(s) = w(s) + \sum_{i=1}^{m} b_i(s)w(a_i(s)).$$

Thandapani et al. [21] and Thandapani and Rama [22] established new conditions to improve and extend some of the results for the oscillation of the equation of second-order

$$(f(s)(y'(s))^{\mu})' + q_1(s)w^{\delta}(s-\sigma_1) + q_2(s)w^{\sigma}(s+\sigma_2) = 0,$$

where $y(s) = w(s) + \delta_1(s)w(s-h_1) + \delta_2(s)w(s+h_2)$.

This manuscript aims to broaden the scope of inquiry by incorporating second-order differential equations with mixed neutral terms under condition $\eta(s_0) < \infty$, using some methods. In this context, the paper introduces innovative criteria for analyzing oscillatory solutions of Eq (1.1). Three examples are shown to illustrate the results that we obtained.

The following notation will be used in the remaining sections of this work:

$$M(\varrho, v) = \int_{\varrho}^{v} f^{-1/\mu}(\xi)\,\mathrm{d}\xi,$$

$$x_1(s) := 1 - \delta_1(s)\frac{M(\sigma_1(s), \infty)}{M(s, \infty)} - \delta_2(s),$$

and

$$x_2(s) := 1 - \delta_1(s) - \delta_2(s)\frac{M(s_1, \sigma_2(s))}{M(s_1, s)}.$$

2. Main results

In this section, we first introduce some important lemmas and then discuss our main results about (1.1).

Lemma 2.1. *[23] The equation*

$$\eta(\upsilon^*) \leq \max_{\varrho\in\mathbb{R}} \eta(\upsilon) = QL + \frac{\mu^\mu}{(\mu+1)^{\mu+1}} Q^{\mu+1} N^{-\mu}, N > 0,$$

is holds, If $\eta(\upsilon) := Q\upsilon - N(\upsilon - L)^{(\mu+1)/\mu}$*, where* Q, N *and* L *are real constants.*

Lemma 2.2. *Assume that* w *eventually is a positive solution of Eq (1.1). Then either*

$$\left(\frac{y(s)}{M(s, \infty)}\right)' \geq 0, \tag{2.1}$$

or

$$\left(\frac{y(s)}{M(s_1, s)}\right)' \leq 0, \ \textit{for all } s \geq s_1. \tag{2.2}$$

Proof. Suppose that w eventually is a positive solution of Eq (1.1). Obviously, for all $s \geq s_1$, $y(s) \geq w(s) > 0$ and $f(s)(y'(s))^\mu$ is decreasing. Since

$$(f(s)(y'(s))^\mu)' = -q(s)\, w^\mu(h(s)) \leq 0.$$

Then, y' is either eventually negative or eventually positive.

Assume first that $y' < 0$ on $[s, \infty)$. Since

$$y(s) \geq -\int_s^\infty f^{-1/\mu}(\upsilon) f^{1/\mu}(\upsilon)\, y'(\upsilon)\,\mathrm{d}\upsilon \geq -M(s, \infty) f^{1/\mu}(s)\, y'(s), \tag{2.3}$$

and so

$$\left(\frac{y(s)}{M(s, \infty)}\right)' = \frac{M(s, \infty)\, y'(s) + f^{-1/\mu}(s)\, y(s)}{(M(s, \infty))^2} \geq 0.$$

Assume now that $y' > 0$ on $[s_1, s)$, we obtain

$$y(s) \geq \int_{s_1}^{s} f^{-1/\mu}(\upsilon) f^{1/\mu}(\upsilon)\, y'(\upsilon)\,\mathrm{d}\upsilon \geq M(s_1, s) f^{1/\mu}(s)\, y'(s),$$

it follows that

$$\left(\frac{y(s)}{M(s_1,s)}\right)' = \frac{M(s_1,s)\,y'(s) - f^{-1/\mu}(s)\,y(s)}{M^2(s_1,s)} \le 0.$$

Thus, the proof is complete. □

Theorem 2.1. *Let $x_2(s) \ge x_1(s) > 0$. If*

$$\limsup_{s\to\infty} \int_{s_1}^{s} \frac{1}{f^{1/\mu}(\varrho)} \left(\int_{s_1}^{\varrho} M^{\mu}(\upsilon,\infty)\, q(\upsilon)\, x_1^{\mu}(h(\upsilon))\,\mathrm{d}\upsilon \right)^{1/\mu} \mathrm{d}\varrho = \infty, \tag{2.4}$$

then, (1.1) is oscillatory.

Proof. Given that $w(s)$ constitutes the final positive solution of (1.1). Then $w(\sigma_1(s))$, $w(\sigma_2(s))$ and $w(h(s))$ are positive. From (1.1) and $y(s) = w(s) + \delta_1(s)\,w(\sigma_1(s)) + \delta_2(s)\,w(\sigma_2(s))$, we see $y(s) \ge w(s) > 0$ and $f(s)\,(y'(s))^{\mu}$ is nonincreasing. Therefore, y' is either eventually negative or eventually positive.

Suppose first that $y'(s) < 0$ on $[s_1,\infty)$. From Lemma 2.2, we have

$$y(\sigma_1(s)) \le \frac{M(\sigma_1(s),\infty)}{M(s,\infty)} y(s),$$

based on the fact that $\sigma_1(s) \le s$. Therefore,

$$\begin{aligned} w(s) &= y(s) - \delta_1(s)\,w(\sigma_1(s)) - \delta_2(s)\,w(\sigma_2(s)) \\ &\ge y(s) - \delta_1(s)\,y(\sigma_1(s)) - \delta_2(s)\,y(\sigma_2(s)) \\ &\ge \left(1 - \delta_1(s)\frac{M(\sigma_1(s),\infty)}{M(s,\infty)} - \delta_2(s)\right) y(s) \\ &= x_1(s)\,y(s). \end{aligned}$$

Hence, (1.1) becomes

$$\begin{aligned} \left(f(s)\,(y'(s))^{\mu}\right)' &\le -q(s)\,w^{\mu}(h(s)) \le -q(s)\,x_1^{\mu}(h(s))\,y^{\mu}(h(s)) \\ &\le -y^{\mu}(s)\,q(s)\,x_1^{\mu}(h(s)). \end{aligned} \tag{2.5}$$

Since $\left(f(s)\,(y'(s))^{\mu}\right)' \le 0$, we have

$$f(s)\,(y'(s))^{\mu} \le f(s_1)\,(y'(s_1))^{\mu} := -\varpi < 0, \tag{2.6}$$

for all $s \ge s_1$, from (2.3) and (2.6), we have

$$y^{\mu}(s) \ge \varpi M^{\mu}(s,\infty) \text{ for all } s \ge s_1. \tag{2.7}$$

Combining (2.5) with (2.7) yields

$$\left(f(s)\,(y'(s))^{\mu}\right)' \le -\varpi M^{\mu}(s,\infty)\,q(s)\,x_1^{\mu}(h(s)), \tag{2.8}$$

for all $s \ge s_1$. Integrating (2.8) from s_1 to s, we obtain

$$f(s)\,(y'(s))^{\mu} \le f(s_1)\,(y'(s_1))^{\mu} - \varpi \int_{s_1}^{s} M^{\mu}(\upsilon,\infty)\,q(\upsilon)\,x_1^{\mu}(h(\upsilon))\,\mathrm{d}\upsilon$$

$$\leq \quad -\varpi \int_{s_1}^{s} M^{\mu}(\upsilon,\infty)\, q(\upsilon)\, x_1^{\mu}(h(\upsilon))\, \mathrm{d}\upsilon.$$

Integrating the last inequality from s_1 to s, we get

$$y(s) \leq y(s_1) - \varpi^{1/\mu} \int_{s_1}^{s} \frac{1}{f^{1/\mu}(\varrho)} \left(\int_{s_1}^{\varrho} M^{\mu}(\upsilon,\infty)\, q(\upsilon)\, x_1^{\mu}(h(\upsilon))\, \mathrm{d}\upsilon \right)^{1/\mu} \mathrm{d}\varrho.$$

At $s \to \infty$, we arrive at a contradiction with (2.4).

Assume now that $y'(s) > 0$ on $[s_1, \infty)$. From Lemma 2.2, we arrive at

$$y(\sigma_2(s)) \leq \frac{M(s_1, \sigma_2(s))}{M(s_1, s)} y(s). \tag{2.9}$$

From the definition of y, we obtain

$$\begin{aligned} w(s) &= y(s) - \delta_1(s)\, w(\sigma_1(s)) - \delta_2(s)\, w(\sigma_2(s)) \\ &\geq y(s) - \delta_1(s)\, y(\sigma_1(s)) - \delta_2(s)\, y(\sigma_2(s)). \end{aligned} \tag{2.10}$$

Using that (2.9) and $y(\sigma_1(s)) \leq y(s)$ where $\sigma_1(s) < s$ in (2.10), we obtain

$$\begin{aligned} w(s) &\geq y(s)\left(1 - \delta_1(s) - \delta_2(s) \frac{M(s_1, \sigma_2(s))}{M(s_1, s)}\right) \\ &\geq x_2(s)\, y(s). \end{aligned} \tag{2.11}$$

Thus, (1.1) becomes

$$\begin{aligned} \left(f(s)\left(y'(s)\right)^{\mu}\right)' &\leq -q(s)\, w^{\mu}(h(s)) \leq -q(s)\, x_2^{\mu}(h(s))\, y^{\mu}(h(s)) \\ &\leq -y^{\mu}(h(s))\, q(s)\, x_2^{\mu}(h(s)). \end{aligned} \tag{2.12}$$

On the other hand, it is follows from (2.4) that $\int_{s_1}^{s} M^{\mu}(\upsilon,\infty)\, q(\upsilon)\, x_1^{\mu}(h(\upsilon))\, \mathrm{d}\upsilon$ must be unbounded. Further, since $M'(s,\infty) < 0$, it is easy to see that

$$\int_{s_1}^{s} q(\upsilon)\, x_1^{\mu}(h(\upsilon))\, \mathrm{d}\upsilon \to \infty \quad \text{as } s \to \infty. \tag{2.13}$$

Integrating (2.12) from s_2 to s, we get

$$\begin{aligned} f(s)\left(y'(s)\right)^{\mu} &\leq f(s_2)\left(y'(s_2)\right)^{\mu} - \int_{s_2}^{s} y^{\mu}(h(\upsilon))\, q(\upsilon)\, x_2^{\mu}(h(\upsilon))\, \mathrm{d}\upsilon \\ &\leq f(s_2)\left(y'(s_2)\right)^{\mu} - y^{\mu}(h(s_2)) \int_{s_2}^{s} q(\upsilon)\, x_2^{\mu}(h(\upsilon))\, \mathrm{d}\upsilon. \end{aligned}$$

Since $x_2(s) > x_1(s)$, we get

$$f(s)\left(y'(s)\right)^{\mu} \leq f(s_2)\left(y'(s_2)\right)^{\mu} - y^{\mu}(h(s_2)) \int_{s_2}^{s} q(\upsilon)\, x_1^{\mu}(h(\upsilon))\, \mathrm{d}\upsilon, \tag{2.14}$$

we have shown that, according to Eq (2.13), the positivity of $y'(s)$ as $s \to \infty$ is contradicted. Therefore, the proof is complete. □

Theorem 2.2. *Assume that $x_2(s) \geq x_1(s) > 0$. If*

$$\limsup_{s\to\infty} M^{\mu}(s,\infty) \int_{s_1}^{s} q(\upsilon)\, x_1^{\mu}(h(\upsilon))\, d\upsilon > 1. \tag{2.15}$$

then every solutions of (1.1) are oscillatory.

Proof. The function w is identified as the ultimate positive solution to (1.1). Then there exists $s_1 \geq s_0$ such that $w(\sigma_1(s)) > 0$, $w(\sigma_2(s)) > 0$ and $w(h(s)) > 0$ for all $s \geq s_1$. The same way we prove the Theorem 2.1, the sign of y' becomes consistently positive or negative eventually. First, let $y' < 0$. Integrating (2.5) from s_1 to s, we see that

$$\begin{aligned} f(s)(y'(s))^{\mu} &\leq f(s_1)(y'(s_1))^{\mu} - \int_{s_1}^{s} y^{\mu}(\upsilon)\, q(\upsilon)\, x_1^{\mu}(h(\upsilon))\, d\upsilon \\ &\leq -y^{\mu}(s) \int_{s_1}^{s} q(\upsilon)\, x_1^{\mu}(h(\upsilon))\, d\upsilon \end{aligned} \tag{2.16}$$

Using (2.3) in (2.16), we obtain

$$-f(s)(y'(s))^{\mu} \geq -M^{\mu}(s,\infty) f(s)(y'(s))^{\mu} \int_{s_1}^{s} q(\upsilon)\, x_1^{\mu}(h(\upsilon))\, d\upsilon. \tag{2.17}$$

Divide both sides of inequality (2.17) by $-f(s)(y'(s))^{\mu}$ and taking the limsup, we get

$$\limsup_{s\to\infty} M^{\mu}(s,\infty) \int_{s_1}^{s} q(\upsilon)\, x_1^{\mu}(h(\upsilon))\, d\upsilon \leq 1.$$

we arrive at a contradiction with (2.15).

Let $y' > 0$ on $[s_1,\infty)$. From (2.15) and $M(s,\infty) < \infty$, we have that (2.13) holds. We notice that this part of the proof is exactly like the part of Theorem 2.1, so the proof is complete. □

Theorem 2.3. *If $x_2(s) > 0$, $x_1(s) > 0$ and $f' > 0$ such that*

$$\limsup_{s\to\infty} \frac{M^{\mu}(s,\infty)}{\xi(s)} \int_{s}^{s} \left(\xi(\upsilon)\, q(\upsilon)\, x_1^{\mu}(h(\upsilon)) - \frac{f(\upsilon)}{(\mu+1)^{\mu+1}} \frac{(\xi'(\upsilon))^{\mu+1}}{(\xi(\upsilon))^{\mu}} \right) d\upsilon > 1 \tag{2.18}$$

and

$$\limsup_{s\to\infty} \int_{s}^{s} \left(\varsigma(\upsilon)\, q(\upsilon)\, x_2^{\mu}(h(\upsilon)) - \frac{1}{(\mu+1)^{\mu+1}} \frac{f(\upsilon)(\varsigma'(\upsilon))^{\mu+1}}{\varsigma^{\mu}(\upsilon)(h'(\upsilon))^{\mu}} \right) d\upsilon = \infty, \tag{2.19}$$

where the functions $\varsigma, \xi \in C^1([s_0,\infty),(0,\infty))$ and $s \in [s_0,\infty)$.

Proof. Given that $w(s)$ constitutes the final positive solution of (1.1) on $[s_0,\infty)$. Then $w(\sigma_1(s)) > 0$, $w(\sigma_2(s)) > 0$ and $w(h(s)) > 0$ for all $s \geq s_1$. From Theorem 2.1, yields that y' is eventually takes on a consistent sign, either remaining negative or remaining positive, beyond a certain point. First, assuming that $y' < 0$ on $[s_1,\infty)$. By following the approach used in the proof of the Theorem 2.1, we can obtain that y is a solution of the inequality (2.5). We now define the Riccati function as follows:

$$E(s) = \xi(s) \left(\frac{f(s)(y'(s))^{\mu}}{y^{\mu}(s)} + \frac{1}{M^{\mu}(s,\infty)} \right) \text{ on } [s_1,\infty). \tag{2.20}$$

Due to (2.3), we can notice that $E \geq 0$ on $[s_1, \infty)$. By Computing the derivative of (2.20), we can conclude at

$$\begin{aligned} E'(s) &= \frac{\xi'(s)}{\xi(s)}E(s) + \xi(s)\frac{(f(s)(y'(s))^{\mu})'}{y^{\mu}(s)} - \mu\xi(s)f(s)\left(\frac{y'(s)}{y(s)}\right)^{\mu+1} \\ &\quad + \frac{\mu\xi(s)}{f^{1/\mu}(s)M^{\mu+1}(s,\infty)} \\ &\leq \frac{\xi'(s)}{\xi(s)}E(s) + \xi(s)\frac{(f(s)(y'(s))^{\mu})'}{y^{\mu}(s)} - \frac{\mu}{(\xi(s)f(s))^{1/\mu}}\left(E(s) - \frac{\xi(s)}{M^{\mu}(s,\infty)}\right)^{(\mu+1)/\mu} \\ &\quad + \frac{\mu\xi(s)}{f^{1/\mu}(s)M^{\mu+1}(s,\infty)}. \end{aligned} \tag{2.21}$$

Combining (2.5) and (2.21), we have

$$\begin{aligned} E'(s) &\leq -\frac{\mu}{(\xi(s)f(s))^{1/\mu}}\left(E(s) - \frac{\xi(s)}{M^{\mu}(s,\infty)}\right)^{(\mu+1)/\mu} - \xi(s)q(s)x_1^{\mu}(h(s)) \\ &\quad + \frac{\mu\xi(s)}{f^{1/\mu}(s)M^{\mu+1}(s,\infty)} + \frac{\xi'(s)}{\xi(s)}E(s). \end{aligned} \tag{2.22}$$

Using Lemma 2.1 with $Q := \xi'(s)/\xi(s)$, $N := \mu(\xi(s)f(s))^{-1/\mu}$, $L := \xi(s)/M^{\mu}(s,\infty)$ and $\upsilon := E$, we get

$$\begin{aligned} \frac{\xi'(s)}{\xi(s)}E(s) - \frac{\mu}{(\xi(s)f(s))^{1/\mu}}\left(E(s) - \frac{\xi(s)}{M^{\mu}(s,\infty)}\right)^{(\mu+1)/\mu} &\leq \frac{1}{(\mu+1)^{\mu+1}}f(s)\frac{(\xi'(s))^{\mu+1}}{(\xi(s))^{\mu}} \\ &\quad + \frac{\xi'(s)}{M^{\mu}(s,\infty)}, \end{aligned}$$

which, in view of (2.22), we have

$$\begin{aligned} E'(s) &\leq \frac{\xi'(s)}{M^{\mu}(s,\infty)} + \frac{1}{(\mu+1)^{\mu+1}}f(s)\frac{(\xi'(s))^{\mu+1}}{(\xi(s))^{\mu}} - \xi(s)q(s)x_1^{\mu}(h(s)) \\ &\quad + \frac{\mu\xi(s)}{f^{1/\mu}(s)M^{\mu+1}(s,\infty)} \\ &\leq -\xi(s)q(s)x_1^{\mu}(h(s)) + \left(\frac{\xi(s)}{M^{\mu}(s,\infty)}\right)' + \frac{f(s)}{(\mu+1)^{\mu+1}}\frac{(\xi'(s))^{\mu+1}}{(\xi(s))^{\mu}}. \end{aligned} \tag{2.23}$$

Integrating (2.23) from s_2 to s, we arrive at

$$\begin{aligned} \int_{s_2}^{s}\left(\xi(\upsilon)q(\upsilon)x_1^{\mu}(h(\upsilon)) - \frac{f(\upsilon)}{(\mu+1)^{\mu+1}}\frac{(\xi'(\upsilon))^{\mu+1}}{(\xi(\upsilon))^{\mu}}\right)\mathrm{d}\upsilon &\leq \left.\left(\frac{\xi(s)}{M^{\mu}(s,\infty)} - E(s)\right)\right|_{s_2}^{s} \\ &\leq -\left.\left(\xi(s)\frac{f(s)(y'(s))^{\mu}}{y^{\mu}(s)}\right)\right|_{s_2}^{s}. \end{aligned} \tag{2.24}$$

From (2.3), we have

$$-\frac{f^{1/\mu}(s)y'(s)}{y(s)} \leq \frac{1}{M(s,\infty)},$$

which, in view of (2.24), implies

$$\frac{M^{\mu}(s,\infty)}{\xi(s)}\int_{s_2}^{s}\left(\xi(\upsilon)\,q(\upsilon)\,x_1^{\mu}(h(\upsilon))-\frac{f(\upsilon)}{(\mu+1)^{\mu+1}}\frac{(\xi'(\upsilon))^{\mu+1}}{(\xi(\upsilon))^{\mu}}\right)\mathrm{d}\upsilon\le 1.$$

Applying the limit superior to both sides, we are led to a contradiction with (2.18).

Now, let $y'(s)>0$ on $[s_1,\infty)$ .Let the function

$$D(s)=\varsigma(s)\frac{f(s)(y'(s))^{\mu}}{y^{\mu}(h(s))},\ \text{ on }\ [s_1,\infty),\tag{2.25}$$

we see that $E\ge 0$ on $[s_1,\infty)$. Differentiating (2.25), we arrive at

$$D'(s)=\frac{\varsigma'(s)}{\varsigma(s)}D(s)+\varsigma(s)\frac{(f(s)(y'(s))^{\mu})'}{y^{\mu}(h(s))}-\mu\varsigma(s)f(s)\frac{(y'(s))^{\mu}y'(h(s))h'(s)}{y^{\mu+1}(h(s))}.\tag{2.26}$$

Combining (2.12) and (2.26), we have

$$D'(s)\le\frac{\varsigma'(s)}{\varsigma(s)}D(s)-\varsigma(s)q(s)x_2^{\mu}(h(s))-\mu\varsigma(s)f(s)\frac{(y'(s))^{\mu}u'(h(s))h'(s)}{y^{\mu+1}(h(s))}.$$

Since $(f(s)(y'(s))^{\mu})'<0$ and $h(s)\le s$, we arrive at

$$D'(s)\le\frac{\varsigma'(s)}{\varsigma(s)}D(s)-\varsigma(s)q(s)x_2^{\mu}(h(s))-\mu\varsigma(s)f(s)h'(s)\frac{(y'(s))^{\mu+1}}{y^{\mu+1}(h(s))},$$

from (2.25), we have

$$D'(s)\le\frac{\varsigma'(s)}{\varsigma(s)}D(s)-\varsigma(s)q(s)x_2^{\mu}(h(s))-\frac{\mu h'(s)}{\varsigma^{1/\mu}(s)f^{1/\mu}(s)}D^{(\mu+1)/\mu}(s).$$

Using the inequality

$$Fv-\varpi v^{(\mu+1)/\mu}\le\frac{\mu^{\mu}}{(\mu+1)^{\mu+1}}\frac{F^{\mu+1}}{\varpi^{\mu}},\ \varpi>0,\tag{2.27}$$

with $F=\varsigma'(s)/\varsigma(s)$, $\varpi=\mu h'(s)/\left(\varsigma^{1/\mu}(s)f^{1/\mu}(s)\right)$ and $v=D$, we have

$$D'(s)\le-\varsigma(s)q(s)x_2^{\mu}(h(s))+\frac{1}{(\mu+1)^{\mu+1}}\frac{f(s)(\varsigma'(s))^{\mu+1}}{\varsigma^{\mu}(s)(h'(s))^{\mu}}.\tag{2.28}$$

Integrating (2.28) from s_2 to s, we arrive at

$$\int_{s_2}^{s}\left(\varsigma(\upsilon)q(\upsilon)x_2^{\mu}(h(\upsilon))-\frac{1}{(\mu+1)^{\mu+1}}\frac{f(\upsilon)(\varsigma'(\upsilon))^{\mu+1}}{\varsigma^{\mu}(\upsilon)(h'(\upsilon))^{\mu}}\right)\mathrm{d}\upsilon\le D(s_2).$$

We are taking lim sup on both sides of this inequality, we find a contradiction with (2.19), and therefore the proof is finished. □

3. Numerical examples

Some examples are provided to demonstrate the significance of our results.

Example 3.1. *Consider the neutral equation*

$$\left(s^2\left(w(s)+\frac{1}{3}w\left(\frac{s}{2}\right)+\frac{1}{2}w(3s)\right)'\right)'+\frac{t_0}{s}w\left(\frac{s}{3}\right)=0,\ s>0, t_0>1. \tag{3.1}$$

Let $\mu=1,\ f(s)=s^2,\ \delta_1(s)=1/3,\ \delta_2(s)=1/2,\ \sigma_1(s)=s/2,\ \sigma_2(s)=3s, h(s)=s/3,\ q(s)=t_0/s$ *and*

$$\eta(s)=\int_{s_0}^{\infty} f^{-1/\mu}(\upsilon)\,\mathrm{d}\upsilon=\frac{1}{s}.$$

Moreover, we find

$$\begin{aligned}&\limsup_{s\to\infty}\int_{s_1}^{s}\frac{1}{f^{1/\mu}(\varrho)}\left(\int_{s_1}^{\varrho}M^{\mu}(\upsilon,\infty)\,q(\upsilon)\,x_1^{\mu}(h(\upsilon))\,\mathrm{d}\upsilon\right)^{1/\mu}\mathrm{d}\varrho\\&=\limsup_{s\to\infty}\int_{s_1}^{s}\frac{1}{\upsilon^2}\left(\int_{s_1}^{\varrho}\int_{s}^{\infty}\varkappa^{-2}\left(\frac{t_0}{\upsilon}\left(1-\frac{\int_{s/2}^{\infty}r^{-2}\mathrm{d}r}{3\int_{s}^{\infty}\upsilon^{-2}\mathrm{d}\upsilon}-\frac{1}{2}\right)\right)\mathrm{d}\varkappa\mathrm{d}\varrho\right)\mathrm{d}\upsilon=\infty,\end{aligned}$$

From Theorem 2.1, Eq (3.1) oscillatory.

Example 3.2. *Let the equation*

$$\left(s^{2\mu}\left[\left(y(s)+\frac{1}{2}y\left(\frac{s}{3}\right)+\frac{1}{3}y(\sigma s)\right)'\right]^{\mu}\right)'+\xi s u\left(\frac{s}{2}\right)=0, \tag{3.2}$$

where $\xi>1, s\geq 1, \mu\in(0,1)$ *and* $\sigma>1$. *Let* $\mu=1,\ f(s)=s^2,\ \delta_1(s)=1/3,\ \delta_2(s)=1/3,\ \sigma_1(s)=s/3,\ \sigma_2(s)=\sigma s, h(s)=s/2$ *and* $q(s)=\xi s$. *So, we find*

$$\eta(s)=\int_{s_0}^{\infty}f^{-1/\mu}(\upsilon)\,\mathrm{d}\upsilon=s^{-1}.$$

If we set $\varsigma(\upsilon)=1$, *we obtain*

$$\limsup_{s\to\infty}\int_{s}^{s}\left(\varsigma(\upsilon)\,q(\upsilon)\,x_2^{\mu}(h(\upsilon))-\frac{1}{(\mu+1)^{\mu+1}}\frac{f(\upsilon)(\varsigma'(\upsilon))^{\mu+1}}{\varsigma^{\mu}(\upsilon)(h'(\upsilon))^{\mu}}\right)\mathrm{d}\upsilon=\infty,\mu<1.$$

By Theorem 2.3, every solution of (3.2) is oscillatory.

4. Conclusions

The aim of this paper is to investigate the oscillatory characteristics inherent in second-order differential equations featuring a mixed neutral term. This investigation is conducted through the application of Riccati transformations and a comparison analysis with first-order equations, ultimately leading to the derivation of oscillation criteria. The study culminates in the establishment of a central theorem related to the oscillation behavior of (1.1). In addition, three examples of the effectiveness of these criteria were discussed. As part of our future research, we will study some of the oscillatory characteristics inherent in third-order DE with neutral terms, as well as to delve into fractional-order equations. Progress is already underway in investigating these particular equation types.

Use of Generative-AI tools declaration

The author declares that no Artificial Intelligence (AI) tools were used in the creation of this article.

Acknowledgments

The researcher would like to express gratitude to the Deanship of Graduate Studies and Scientific Research at Qassim University for their financial support under grant number QU-APC-2024.

Conflict of interest

The author declares that there is no conflicts of interest.

References

1. T. Li, Y. V. Rogovchenko, Oscillation criteria for even-order neutral differential equations, *Appl. Math. Lett.*, **61** (2019), 35–41. http://doi.org/10.1016/j.aml.2016.04.012
2. M. Alkilayh, L. Reichel, Some numerical aspects of Arnoldi-Tikhonov regularization, *Appl. Numer. Math.*, **185** (2023), 503–515. http://doi.org/10.1016/j.apnum.2022.12.009
3. M. Alkilayh, L. Reichel, Q. Ye, A method for computing a few eigenpairs of large generalized eigenvalue problems, *Appl. Numer. Math.*, **183** (2023), 108–117. http://doi.org/10.1016/j.apnum.2022.08.018
4. T. Li, Y. V. Rogovchenko, On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations, *Appl. Math. Lett.*, **105** (2020), 106293. http://doi.org/10.1016/j.aml.2020.106293
5. J. R. Graef, O. Özdemir, A. Kaymaz, E. Tunç, Oscillation of damped second-order linear mixed neutral differential equations, *Monatsh. Math.*, **194** (2021), 85–104. http://doi.org/10.1007/s00605-020-01469-6
6. O. Moaaz, C. Park, A. Muhib, O. Bazighifan, Oscillation criteria for a class of even-order neutral delay differential equations, *J. App. Math. Comput.*, **63** (2020), 607–617. http://doi.org/10.1007/s12190-020-01331-w
7. R. Arul, V. S. Shobha, Oscillation of second order nonlinear neutral differential equations with mixed neutral term, *J. Appl. Math. Phys.*, **3** (2015), 1080–1089. http://doi.org/10.4236/jamp.2015.39134
8. T. S. Hassan, Kamenev-type oscillation criteria for second order nonlinear dynamic equations on time scales, *Appl. Math. Comput.*, **217** (2011), 5285–5297. http://doi.org/10.1016/j.amc.2010.11.052
9. O. Bazighifan, An approach for studying asymptotic properties of solutions of neutral differential equations, *Symmetry*, **12** (2020), 555. http://doi.org/10.3390/sym12040555
10. B. Almarri, A. H. Ali, A. M. Lopes, O. Bazighifan, Nonlinear Differential Equations with Distributed Delay: Some New Oscillatory Solutions, *Mathematics*, **10** (2022), 995. http://doi.org/10.3390/math10060995

11. A. M. Hassan, S. Affan, Oscillation criteria for second-order delay dynamic equations with a sub-linear neutral term on time scales, *Filomat*, **37** (2024), 7445–7454. http://doi.org/10.2298/FIL2322445H
12. O. Bazighifan, Kamenev and Philos-types oscillation criteria for fourth-order neutral differential equations, *Adv. Differ. Eq.*, **2020** (2020), 201. http://doi.org/10.1186/s13662-020-02661-6
13. G. E. Chatzarakis, S. R. Grace, I. Jadlovská, T. Li, E. Tunç, Oscillation criteria for third-order Emden–Fowler differential equations with unbounded neutral coefficients, *Complexity*, **2019** (2019), 5691758. http://doi.org/10.1155/2019/5691758
14. O. Bazighifan, P. Kumam, Oscillation Theorems for Advanced Differential Equations with p-Laplacian Like Operators, *Mathematics*, **8** (2020), 821. http://doi.org/10.3390/math8050821
15. O. Bazighifan, M. A. Ragusa, Nonlinear equations of fouthr-order with p-Laplacian like operators: oscillation, methods and applications, *Proc. Amer. Math. Soc.*, **150** (2022), 1009–1020. http://doi.org/10.1090/proc/15794
16. T. Li, B. Baculíková, J. Džurina, Oscillation results for second-order neutral differential equations of mixed type, *Tatra Mt. Math. Publ.*, **48** (2011), 101–116. http://doi.org/10.2478/v10127-011-0010-8
17. Y. Qi, J. Yu, Oscillation of second order nonlinear mixed neutral differential equations with distributed deviating arguments, *Bull. Malays. Math. Sci. Soc.*, **38** (2015), 543–560.
18. C. Zhang, B. Baculíková, J. Džurina, T. Li, Oscillation results for second-order mixed neutral differential equations with distributed deviating arguments, *Math. Slovaca*, **66** (2016), 615–626. http://doi.org/10.1515/ms-2015-0165
19. O. Bazighifan, Kamenev and Philos-types oscillation criteria for fourth-order neutral differential equations, *Adv. Differ. Equ.*, **2020** (2020), 201. http://doi.org/10.1186/s13662-020-02661-6
20. T. Li, M. T. Şenel, C. Zhang, *Electron. J. Differ. Equ.*, **2013** (2013), 229.
21. E. Thandapani, S. Selvarangam, M. Vijaya, R. Rama, Oscillation results for second order nonlinear differential equation with delay and advanced arguments, *Kyungpook Math. J.*, **56** (2016), 137–146. http://doi.org/10.5666/KMJ.2016.56.1.137
22. E. Thandapani, R. Rama, Comparison and oscillation theorems for second order nonlinear neutral differential equations of mixed type, *Serdica Math. J.*, **39** (2013), 1–16.
23. T. Li, Comparison theorems for second-order neutral differential equations of mixed type, *EJDE*, **2010** (2010), 167.