Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(12): 36158–36166.
DOI: 10.3934/math.20241716
Received: 15 August 2024
Revised: 16 December 2024
Accepted: 19 December 2024
Published: 27 December 2024

Review

Another special role of L^∞-spaces-evolution equations and Lotz' theorem

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Abstract: In this paper, we review the underappreciated theorem by Lotz that tells us that every strongly continuous operator semigroup on a Grothendieck space with the Dunford-Pettis property is automatically uniformly continuous. A large class of spaces that carry these geometric properties are $\mathrm{L}^\infty(\Omega, \Sigma, \mu)$ for non-negative measure spaces. This shows once again that L^∞-spaces have to be treated differently.

Keywords: L^∞-spaces; Lotz' theorem; Grothendieck spaces; Dunford-Pettis property
Mathematics Subject Classification: 47D03, 47D06

Introduction

The role of the space of essential bounded functions $\mathrm{L}^\infty(\Omega)$ on a measure space (Ω, Σ, μ) seems always to be special. Already during the mathematics study at the university, we recognize that the space $\mathrm{L}^\infty(\Omega)$ needs some special attention in comparison to the other L^p-spaces for $p \in [1, \infty)$. One recognizable fact is that if $p \in (1, \infty)$ and (Ω, Σ, μ) is an arbitrary measure space, one has that the dual space $(\mathrm{L}^p(\Omega))'$ is isomorphic to $\mathrm{L}^q(\Omega)$ whenever $\frac{1}{p} + \frac{1}{q} = 1$. Naively, one would expect, after making the convention that $\frac{1}{\infty} = 0$, that $(\mathrm{L}^1(\Omega))'$ is isomorphic to $\mathrm{L}^\infty(\Omega)$ and $(\mathrm{L}^\infty(\Omega))'$ is isomorphic to $\mathrm{L}^1(\Omega)$. However, this is unfortunately not true. The first setback is that $(\mathrm{L}^1(\Omega))'$ is (isometric) isomorphic to $\mathrm{L}^\infty(\Omega)$ if and only if (Ω, Σ, μ) is localizable, see for example [22, Theorem 243G]. Unfortunately, it becomes even worse with the dual of $\mathrm{L}^\infty(\Omega)$. In fact, the dual space $(\mathrm{L}^\infty(\Omega))'$ contains $\mathrm{L}^1(\Omega)$ but is, however, much bigger. For example, if (Ω, Σ, μ) is σ-finite, then $(\mathrm{L}^\infty(\Omega))'$ is isomorphic to the space $\mathrm{ba}(\Omega, \Sigma, \mu)$ consisting of all finitely additive finite signed measures on Σ, which are absolutely continuous with respect to μ. A general respresentation theorem for the dual of $(\mathrm{L}^\infty(\Omega))$ can, for example, be found here [46, Theorem 2.3] or here [18, Theorem IV.8.16]. Even so, when considering interpolation of L^p-spaces, the space L^∞ again plays a special role, see for example [6].

Despite the challenging behavior, the space $\mathrm{L}^\infty(\Omega)$ has some applications in economies with infinitely many commodities [8] and is still part of active research; see for example [37, 45], just to mention a few. Another big reasons why $L^\infty(\Omega)$ is an important space to consider is that it is a Banach space, i.e., a complete normed space. This is the reason why it a priori qualifies for the theory of so-called C_0-*semigroups*. Those objects serve as solutions of abstract Cauchy problems that are linked to partial differential equations.

This article is structured as follows: In Section 1 we introduce the concept of evolution equations, abstract Cauchy problems and operator semigroups. In particular, we work out the difference between uniformly continuous semigroups and strongly continuous semigroups. In Section 2 we discuss Grothendieck space with the Dunford-Pettis property and formulate Lotz' theorem which then applies to L^∞-spaces.

It is noteworthy that the results we discuss in this review article are not due to the author, but it should rather showcase the major role of L^∞-spaces in different areas of mathematics and, in particular, when it comes to evolution equations by means of Lotz' theorem.

1. Evolution equations and operator semigroups

A lot of well-known partial differential equations modeling physical systems, such as the heat equation, the Schrödinger equation or the wave equation, use temporal change of states. The term evolution equation is an umbrella term for such equations that can be interpreted as differential laws describing the development of a system or as a mathematical treatment of motion in time. With a solution of an evolution equation, one can predict the future of the corresponding physical system which makes it deterministic. One can find various monographs on the theory of evolution equations, see for example [20, 23, 34], just to mention a few. At the same time, there are also monographs consisting only of mathematical applications of evolution equations in physics, life sciences, optimization, inverse problems, as well as machine learning, cf. [32, 47, 48], which in fact emphasizes the strength of the theory.

The idea is to investigate evolution equations by an operator-theoretical approach by means of so-called *abstract Cauchy problems*. Those problems are of the form

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = f \in X, \end{cases} \tag{ACP}$$

where A is a linear operator and X is a Banach space, i.e., a complete normed space. In particular, we search for a function $u : \mathbb{R}_{\geq 0} \to X$ that solves (ACP). If A is a bounded linear operator on the Banach space X (we will denote this by $A \in \mathscr{L}(X)$), i.e., there exists $M \geq 0$ such that $\|Ax\| \leq M\|x\|$ for all $x \in X$, then by [20, Chapter I, Proposition 3.5], the abstract Cauchy problem (ACP) has a unique solution given by $u(t) = \mathrm{e}^{tA} f$ where

$$\mathrm{e}^{tA} := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}, \quad t \geq 0. \tag{1.1}$$

One observes that in this scenario, e^{tA} is a bounded linear operator for each $t \geq 0$, i.e., $\mathrm{e}^{tA} \in \mathscr{L}(X)$ for all $t \geq 0$, that $\mathrm{e}^{tA}\mathrm{e}^{sA} = \mathrm{e}^{(t+s)A}$, $\mathrm{e}^{0A} = \mathrm{I}$, and $\|\mathrm{e}^{tA} - \mathrm{I}\|_{\mathscr{L}(X)} \to 0$ for $t \to 0$. Here, $\|\cdot\|_{\mathscr{L}(X)}$ denotes the operator norm defined by $\|T\|_{\mathscr{L}(X)} := \sup_{\|x\|\leq 1} \|Tx\|$ for $T \in \mathscr{L}(X)$. These observations lead to the following definition, cf. [20, Chapter I, Definition 3.6].

Definition 1.1. A family $(T(t))_{t\geq 0}$ of bounded linear operators on a Banach space X is called a *uniformly continuous operator semigroup* if the following properties hold:

(a) $T(t+s) = T(t)T(s)$ and $T(0) = \mathrm{I}$ for all $t, s \geq 0$.

(b) $\|T(t) - \mathrm{I}\| \to 0$ for $t \to 0$.

We see that the family of operators $(T(t))_{t\geq 0}$ defined by $T(t) := \mathrm{e}^{tA}$ for some $A \in \mathscr{L}(X)$ is an example of a uniformly continuous semigroup according to Definition 1.3. The drawback, however, is that this is the only example, as the following result shows, cf. [20, Chapter I, Theorem 3.7].

Theorem 1.2. *Let $(T(t))_{t\geq 0}$ be a uniformly continuous operator semigroup on a Banach space X. Then there exists a unique $A \in \mathscr{L}(X)$ such that $T(t) = \mathrm{e}^{tA}$ for all $t \geq 0$.*

We notice that the exponential power series in (ACP) makes perfect sense if A is a bounded linear operator. However, not all linear operators are bounded. Consider, for example, the operator $Af := f'$ on the Banach space $X = \mathrm{C}_0(\mathbb{R})$ equipped with the supremum norm $\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)|$. Then this operator cannot be bounded. Indeed, consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined by $f_n(x) := \mathrm{e}^{-nx^2}$. Then, we observe that $\|f_n\|_\infty = 1$, but $\|Af_n\|_\infty = \|f_n'\|_\infty \to \infty$ for $n \to \infty$. For those operators, which we call unbounded operators, we have to specify a subspace of the Banach space, where the operator has a meaning. In the previous example, we can, for example choose this subspace to be $\mathrm{D}(A) = \{f \in \mathrm{C}_0(\mathbb{R}) : \ f' \in \mathrm{C}_0(\mathbb{R})\} = \mathrm{C}_0^1(\mathbb{R})$. We call those subspaces the domain of the operator. Those operators can also be used to formulate abstract Cauchy problems. For example, we consider for $f \in \mathrm{C}_0(\mathbb{R})$ the transport equation on $\mathbb{R}$ given by

$$\begin{cases} \dfrac{\partial}{\partial t} w(t,x) = \dfrac{\partial}{\partial x} w(t,x), & t \geq 0, \ x \in \mathbb{R}, \\ w(0,x) = f(x), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

Then (1.2) can also be rewritten as an abstract Cauchy problem of the form (ACP) by choosing the operator A to be $Af = f'$ with domain $\mathrm{D}(A) = \{f \in \mathrm{C}_0(\mathbb{R}) : \ f' \in \mathrm{C}_0(\mathbb{R})\} = \mathrm{C}_0^1(\mathbb{R})$ as mentioned above. Let us now connect (ACP) with operator semigroups. In particular, let us first weaken the assumption of Definition 1.3, see also [20, Chapter I, Definition 5.1].

Definition 1.3. A family $(T(t))_{t\geq 0}$ of bounded linear operators on a Banach space X is called a *strongly continuous operator semigroup*, or *C_0-semigroup*, if the following properties hold:

(a) $T(t+s) = T(t)T(s)$ and $T(0) = \mathrm{I}$ for all $t, s \geq 0$.

(b) $\|T(t)x - x\| \to 0$ for all $x \in X$ whenever $t \to 0$.

Example 1.4. For obvious reasons, uniformly continuous semigroups (Definition 1.3) are C_0-semigroups. However, the converse is not generally true. For example, we can consider the translation semigroup on the Banach space $X = \mathrm{C}_0(\mathbb{R})$ defined by

$$(T(t)f)(x) = f(x+t), \quad t \geq 0, \ f \in \mathrm{C}_0(\mathbb{R}), \ x \in \mathbb{R}. \tag{1.3}$$

This operator semigroup is strongly continuous but not uniformly continuous (and hence also not of the form e^{tA}).

To each C_0-semigroup, one is able to define a generator, cf. [20, Chapter II, Definition 1.2].

Definition 1.5. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. The *(infinitesimal) generator* of $(T(t))_{t\geq 0}$ is defined by

$$Ax := \lim_{t\to 0} \frac{T(t)x - x}{t}, \quad \mathrm{D}(A) := \left\{ x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

Remark 1.6. (a) The expression in Definition 1.5 is inspired by the first derivative in $t = 0$ as one has that $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathrm{e}^{tA} = A$.

(b) Let us go back to Example 1.4. For the translation semigroup on $\mathrm{C}_0(\mathbb{R})$ as defined by (1.3), one obtains the operator $Af = f'$ and $\mathrm{D}(A) = \mathrm{C}_0^1(\mathbb{R})$, which we already saw earlier in this paper.

(c) As a matter of fact, a C_0-semigroup is uniformly continuous if and only if the generator A is a bounded linear operator, see also [20, Chapter II, Corollary 1.5].

In general, abstract Cauchy problems of the form (ACP) can be studied for general Banach spaces. We will see that the Banach space $\mathrm{L}^\infty(\Omega)$ has special properties when it comes to C_0-semigroups, as it belongs to a class of Banach spaces carrying particular geometric characteristics.

2. Lotz's theorem

2.1. Grothendieck spaces with the Dunford-Pettis property

Before we can discuss the meaning of Lotz's theorem, we need some clarification of notations, which will be crucial in order to understand Lotz's theorem.

Definition 2.1. Let X be a Banach space

(a) X is called a *Grothendieck space* if every weak* convergent sequence in X' converges also for the weak topology on X'.

(b) X is said to have the *Dunford-Pettis property* if every weakly compact operator from X into any Banach space maps weakly compact sets into norm compact sets.

Remark 2.2. (i) Grothendieck spaces attracted a lot of interest [12, 15, 24, 25, 36], especially as there are connections to the study of finitely additive vector-valued measures and there are still many open questions, e.g., whether there exist an internal characterisation of Grothendieck spaces or whether non-reflexive Grothendieck spaces contain a copy of c_0.

(ii) The Dunford-Pettis property has been defined by Grothendieck [25] after preparatory work of Dunford and Pettis [17]. Throughout, there has been a lot of research on Banach spaces with the Dunford-Pettis property [13, 16]. An important equivalent-and also more accessible-definition that avoids weak compactness is the following: if $x_n \to x$ weakly in X and $\varphi_n \to \varphi$ weakly in X', then $\varphi_n(x_n) \to \varphi(x)$, see for example [11].

We are now able to formulate and understand Lotz's theorem, cf. [30, Theorem 6] or [31, Theorem 3], which actually solves a question stated by Kishimoto and Robinson [28, p. 75].

Theorem 2.3. *Let X be a Grothendieck space with the Dunford-Pettis property. If $(T(t))_{t\geq 0}$ is C_0-semigroup on X, then $(T(t))_{t\geq 0}$ is uniformly continuous, i.e., there exists $A \in \mathscr{L}(E)$ such that $T(t) = \mathrm{e}^{tA}$ for all $t \geq 0$.*

2.2. *Consequences for* L^∞*-spaces*

Now, we will also see why L^∞-spaces are of particular interest. For a non-negative measure space (Ω, Σ, μ) the space $\mathrm{L}^\infty(\Omega, \Sigma, \mu)$ in fact becomes a Grothendieck space with the Dunford-Pettis property, and hence Theorem 2.3 applies.

Remark 2.4. It is worth noticing that $\mathrm{C}(K)$ is a Grothendieck space with the Dunford-Pettis property whenever K is a compact F-space, i.e., K is a compact Hausdorff space such that the closure of disjoint F_σ-sets are disjoint. This firstly has been proven by Grothendieck [25] and can also be found [5, Theorem 3.6].

Let us show that $\mathrm{L}^\infty(\Omega, \Sigma, \mu)$ is a Grothendieck space with the Dunford-Pettis property. The arguments to prove that result are beautiful and use results from other areas of mathematics such as order theory.

Lemma 2.5. *Let* (Ω, Σ, μ) *be a non-negative measure space. Then* $\mathrm{L}^\infty(\Omega, \Sigma, \mu)$ *is a Grothendieck space with the Dunford-Pettis property.*

Proof. We notice first that $\mathrm{L}^\infty(\Omega, \Sigma, \mu)$ is a σ-order complete AM-space with unit. Therefore, by the Kakutani representation theorem, it is isometric isomorph to $\mathrm{C}(K)$ for some compact σ-stonean space K, see for example [1, Theorem 3.6], [33, Theorem 2.1.3] or [38, Theorem 7.4]. By Remark 2.4 we conclude that $\mathrm{L}^\infty(\Omega, \Sigma, \mu)$ is a Grothendieck space with the Dunford-Pettis property. □

Remark 2.6. The L^∞-spaces hold a profound significance in operator theory, particularly as Grothendieck spaces. Grothendieck's inequality and its extensions provide foundational insights into the bounded linear operators acting on these spaces. This inequality has been instrumental in understanding dualities between Banach spaces, with applications extending into quantum information theory and non-commutative geometry. For instance, connections between L^∞-spaces and non-commutative versions of Grothendieck's theorem offer new insights into factorization properties and bounded bilinear forms on C^*-algebras and L^∞-spaces as von Neumann algebras, see for example [35]. This broadens their utility in analyzing operator systems and spectral properties in functional analysis. Key studies include applications to random and pseudo-random graph structures and the cut-norm problem via Grothendieck's inequality, cf. [3, 4].

Corollary 2.7. *If* $(T(t))_{t\geq 0}$ *is* C_0*-semigroup on* $\mathrm{L}^\infty(\Omega, \Sigma, \mu)$*, then* $(T(t))_{t\geq 0}$ *is uniformly continuous, i.e., there exists* $A \in \mathscr{L}(E)$ *such that* $T(t) = \mathrm{e}^{tA}$ *for all* $t \geq 0$*.*

One is able to tell much more about Grothendieck spaces with the Dunford-Pettis property in the context of evolution equations and operator semigroups. For example, van Neerven proved a partial converse of Theorem 2.3, see [41], or a refinement of Baillon's theorem on maximal regularity has been proven by Jacob, Schwenninger and Wintermayr [26]. Furthermore, von Below and Lubary investigated eigenvalues of the Laplacian on infinite networks in an L^∞-setting, cf. [42, 43]. That strongly continuous semigroups on $\mathrm{L}^\infty(\Omega, \Sigma, \mu)$ are automatically uniformly continuous, see Corollary 2.7, makes non-strongly continuous semigroups, such as bi-continuous semigroup, attractive on those spaces, cf. [14, 21, 29]. Moreover, other interesting spaces such as H^∞ also carry those geometric properties, which also attracted research interest, see for example [7, 27]. In particular, Lotz' Theorem has many applications, for example, in the area of ergodic theory [10, 39, 40], parabolic equations [9] or Markov processes [19, 44]. Generalizations in terms of locally convex Grothendieck spaces carrying the Dunford-Pettis property have been studied by Albanese, Bonet and Ricker [2].

3. Conclusions

This paper highlights the intricate and multifaceted role of L^∞-spaces within the broader landscape of mathematical analysis, particularly in the context of operator semigroups and evolution equations. The key focus was the interplay between L^∞-spaces and the theory of C_0-semigroups, which plasy a pivotal role in solving abstract Cauchy problems. By employing operator-theoretic methods, we demonstrated how L^∞-spaces fit into the framework of Grothendieck spaces with the Dunford-Pettis property, culminating in a discussion of Lotz's theorem and its implications. The examples and results presented underline not only the theoretical depth of L^∞-spaces but also their practical significance in modeling and understanding dynamic systems across various domains. These findings reaffirm the importance of L^∞-spaces in both pure and applied mathematics, serving as a bridge between abstract theory and real-world applications.

Use of Generative-AI tools declaration

The author declares that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is based on the research supported by the National Research Foundation (Grant number: SRUG220317136). It is acknowledged that opinions, findings and conclusions or recommendations expressed in any publication generated by this supported research is that of the author(s). The National Research Foundation accepts no liability whatsoever in this regard.

Conflict of interest

The author declares no conflict of interest.

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