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Research article

On fuzzy sub-semi-rings of nexuses

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Abstract: In this paper, we first constructed a semi-ring on a nexus and then defined a fuzzy sub-semi-ring associated with a nexus N. We investigated some properties and applications. Fuzzy versions of some well-known crisp concepts are provided over a nexus. We verified some applications of this fuzzy on semi-ring N. We obtained some relationships between sub-semi-ring and fuzzy sub-semi-ring of N. However, these relationships were not true for ideals. We put a condition on fuzzy sub-semi-ring so that these relationships were true for ideals. We defined strong fuzzy sub-semi-ring on N. For strong fuzzy sub-semi-ring on N and for every $\alpha \in [0, \mu(0)]$, the level set μ^{α} was an ideal of N. For some strong fuzzy sub-semi-rings μ, we verified when μ^{α} was a prime ideal of N. In the following, for a semi-ring homomorphism $f : N \longrightarrow M$, we showed that if $\mu \in FSUB_S(N)$, then $f(\mu) \in FSUB_S(M)$ and if $\mu \in FSUB_S(M)$ then $f \circ \mu \in FSUB_S(N)$. Finally, we verified some concepts of fuzzy quotient of a nexus semi-ring.

Keywords: nexus; semi-ring; fuzzy sub-semi-ring; semi-ring homomorphism; fuzzy quotient
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1. Introduction

In 1980, Haristchain [10], in order to be able to conveniently handle the vast amount of varied data that defines a spatial structure, a sophisticated form of database was evolved, which called a *plenix* (also see, [13, 19]). In 1984, Nooshin [18] defined the notion of a *nexus* as a mathematical object that represents the constitution of a plenix, and a nexus was defined axiomatically by using the concept of the *address set*. In 2009, Bolourian [5] introduced the notion of *nexus algebras* as an abstract algebraic structure and investigated them. Many authors have worked on nexuses and sub nexuses. In 2019, Bolourian et al. [6] constructed a moduloid on a nexus. In 2020, Kamrani et al. gave a structure of moduloid on a *nexus* and verified the concepts of *sub moduloids*, *finitely generated sub moduloids*, and *prime sub moduloids* on a *nexus* [14, 15]. In 2020, Norouzi et al. worked on sub nexuses on

N-structure [20].

Many familiar concepts in an abstract algebra are studied deeply in the context of nexus algebra [8, 9, 11, 17, 26, 27]. In 1965, L. A. Zadeh [29] introduced the concept of fuzzy sets. Later in 1971, A. Rosenfeld [24] used this concept to define a fuzzy sub-groupoid and a fuzzy subgroup. In 1982, W. J. Liu [16] studied fuzzy invariant subgroups, fuzzy ideals and proved some fundamental properties. In 1988, U. M. Swamy and K. L. N. Swamy [28] defined the concept of fuzzy prime ideals of rings and got some useful results. In 1991, S. Abou-Zaid [2] introduced the notion of fuzzy sub-near-rings and ideals. In 2020, S. Abdurrahman [1] defined the notion of fuzzy sub-semi-ring and investigated the related properties. Moreover, they introduced the ideal of a sub-semi-ring induced from the level set. They also characterized fuzzy sub-semi-rings.

In 2011, A. Saeidi Rashkolia and A. Hasankhani [25] introduced fuzzy sub nexuses and investigated these properties. In 2012, D. Afkhami Taba et al. [3], as a generalization of fuzzy nexuses, defined soft nexuses. In 2014, H. Hedayati and A. Asadi [12] discussed normal, maximal, and product fuzzy sub nexuses of nexuses. In 2015, A. A. Estaji et al. [7] defined the fuzzy sub-nexuses over a nexus and the notion of prime fuzzy sub-nexuses. For some recent papers, see references [4, 22, 23]. Many authors work in fuzzy ideals. In [23], the authors defined q-rung orthopair fuzzy ideals, and also introduced the notion of q-rung orthopair fuzzy cosete. In [21], the authors defined the concept of Pythagorean fuzzy cosets of a Pythagorean fuzzy ideal and proved that the set of all Pythagorean fuzzy cosets of a Pythagorean fuzzy ideal forms a ring under cetain binary operations. In this paper, we construct a semi-ring on a nexus and then define a fuzzy sub-semi-ring on related nexus N and verify some applications of it. We obtain some relationship between sub-semi-ring and fuzzy sub-semi-ring of N. In Proposition 3.12, we show that A is a sub-semi-ring of N if and only if χ_A is a fuzzy sub-semi-ring of N. In Proposition 3.14, we show that μ is a fuzzy sub-semi-ring of N if and only if, for every $\alpha \in [0, \mu(0)]$, μ^α is a sub-semi-ring of N. Since the Propositions 3.12 and 3.14 are not true for ideals and since ideals in semi-rings are very important than sub-semi-rings, we put a condition on fuzzy sub-semi-ring such that Propositions 3.12 and 3.14 be true for ideals. We define this fuzzy and named it by a strong fuzzy sub-semi-ring. Also, for a strong fuzzy sub-semi-ring, for every $a \in N$, we have $\mu(0) \geqslant \mu(a)$. In [28], the authors define fuzzy ideals in rings, but in this paper, the notion of a fuzzy sub-semi-ring is different from it because N is not a ring (see Remark 4.5). The set of all fuzzy sub-semi-rings of N is denoted by $FSUB_S(N)$, and the set of all strong fuzzy sub-semi-rings of N is denoted by $FSUB_T(N)$. In the following, for a semi-ring homomorphism $f : N \longrightarrow M$, we show that if $\mu \in FSUB_S(N)$ then $f(\mu) \in FSUB_S(M)$ and if $\mu \in FSUB_S(M)$, then $f \circ \mu \in FSUB_S(N)$. Finally, we verify some concepts of fuzzy quotient of a nexus semi-rings.

2. Preliminaries

Now, we review the basic definitions and some elementary aspects that are necessary for this paper.

An address is a sequence of $\mathbb{N}^*$ such that $a_k = 0$ implies that $a_i = 0$, for all $i \geqslant k$. The sequence of zero is called the empty address and denoted by (). In other words, every non-empty address is of the form $(a_1, a_2, \ldots, a_n, 0, 0, \ldots)$, where a_i and $n \in \mathbb{N}$, and. it is denoted by $(a_1, a_2, \ldots, a_n)$.

Definition 2.1. ([3, 6, 18]) A set N of addresses is called a nexus if

(I) $(a_1, a_2, \ldots, a_{n-1}, a_n) \in N$ implies $(a_1, a_2, \ldots, a_{n-1}, t) \in N, \forall\ 0 \leqslant t \leqslant a_n$,

(II) $\{a_i\}_{i=1}^{\infty} \in N, a_i \in \mathbb{N}$ implies $\forall n \in \mathbb{N}, \forall\, 0 \leqslant t \leqslant a_n, (a_1, \ldots, a_n - t) \in N$.

Let $a \in N$. The level of a is said to be:

(I) n, if $a = (a_1, a_2, \ldots, a_n)$, for some $a_k \in \mathbb{N}$,
(II) ∞, if a is an infinite sequence of N,
(III) 0, if $a = ()$.

The level of a is denoted by $l(a)$ and $stem\, a = a_1$. We put $st(N) = sup\{i \in \mathbb{N} : (i) \in N\}$.

Let $a = \{a_i\}$ and $b = \{b_i\}$, $i \in \mathbb{N}$, be two addresses. Then $a \leqslant b$, if $l(a) = 0$ or if one of the following cases is satisfied:

(I) if $l(a) = 1$, that is $a = (a_1)$, for some $a_1 \in N$ and $a_1 \leqslant b_1$,
(II) if $1 < l(a) < \infty$, then $l(a) \leqslant l(b)$ and $a_{l(a)} \leqslant b_{l(a)}$ and for any $1 \leqslant i < l(a)$, $a_i = b_i$,
(III) if $l(a) = \infty$, then $a = b$.

Definition 2.2. ([29]) Let N be a set. A fuzzy subset of N is a mapping $\mu : N \longrightarrow [0, 1]$. If μ and ν are fuzzy subsets of N such that $\nu(x) \leqslant \mu(x)$ for all $x \in N$, we write $\nu \leqslant \mu$ or $\nu \subseteq \mu$ and say that ν is contained in μ or ν is a fuzzy subset of μ.

Definition 2.3. ([25]) Let μ be a fuzzy subset of a nexus N. Then μ is called a fuzzy sub-nexus of N, if $a \leqslant b$ implies that $\mu(b) \leqslant \mu(a)$, for all $a, b \in N$.

3. Fuzzy sub-semi-rings related to a nexus

In this section, at first we construct a semi-ring on a nexus N.

For $() \neq a, b \in N$, let $a = \{a_t\}_{t=1}^{n}$ and $b = \{b_t\}_{t=1}^{m}$. We define binary operations "+" and "·" on N as following:

(I) $() + a = a + () = a$,
(II) $a + b = (a_1 \vee b_1)$,
(III) $a \cdot b = (a_1, \cdots, a_{i-1}, a_i \wedge b_i)$,

where $i\{a, b\} = \min\{t : a_t \neq b_t\}$ (briefly, $i\{a, b\} := i$). If there is no such that i, then $a = b$ and $i\{a, a\} = l(a)$.

Example 3.1. Consider a nexus:
$N = \{(), (1), (2), (3), (1, 1), (1, 2), (1, 2, 1), (2, 1), (2, 2), (2, 3), (2, 3, 1), (3, 1)\}$
with the following diagram:

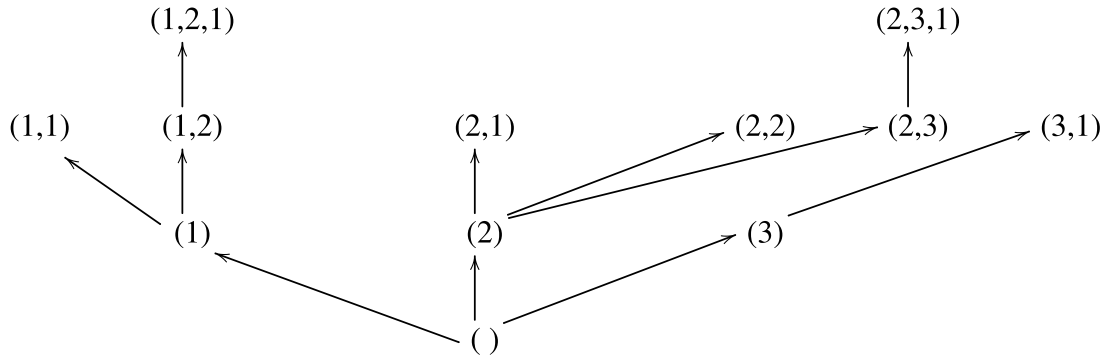

Figure 1. Diagram of N.

We have $(2,3,1)+(3)=(3)$, $(1,1)+(3,1)=(3)$, $(1,1)+(1,1)=(1)$.
$i\{(2,3),(2,3,1)\}=3$ and $(2,3)\cdot(2,3,1)=(2,3)$.
$i\{(2,3,1),(1,2,1)\}=1$ and $(2,3,1)\cdot(1,2,1)=(1)$.
$i\{(2,3),(2,3)\}=2$ and $(2,3)\cdot(2,3)=(2,3)$.

Now, we prove that $(N,+,\cdot,())$ is a semi-ring.

Lemma 3.2. *Let* $a=\{a_t\}_{t=1}^n$, $b=\{b_t\}_{t=1}^n$, $c=\{c_t\}_{t=1}^n\in N$. *Then*

(I) *if* $i\{a,b\}=i\{a,c\}=r$, *then* $i\{b,c\}\geqslant r$,
(II) *if* $i\{a,b\}=r$, $i\{a,c\}=s$ *and* $r\neq s$, *then* $i\{b,c\}=\min\{r,s\}$.

Proof. (I) Since $i\{a,b\}=i\{a,c\}=r$, for every $1\leqslant i\leqslant r-1$, we get $a_i=b_i=c_i$. Consequently, $i\{b,c\}\geqslant r$.

(II) Let $r<s$. Then $a_i=b_i$, for every $1\leqslant i\leqslant r-1$ and $b_i=c_i$, for every $1\leqslant i\leqslant s-1$. Hence $b_i=c_i$, for every $1\leqslant i\leqslant r-1$, $a_r\neq b_r$ and $a_r=c_r$. It follows that $b_r\neq c_r$, and so $i\{b,c\}=r$. By a similar argument we can see that if $s<r$, then $i\{b,c\}=s$. Therefore, $i\{b,c\}=\min\{r,s\}$. □

Theorem 3.3. *The algebra* $(N,+,\cdot,())$ *is a semi-ring.*

Proof. It is obvious that $(N,+,())$ is a commutative semi-group. Now, assume $a=\{a_t\}_{t=1}^n$, $b=\{b_t\}_{t=1}^n$, $c=\{c_t\}_{t=1}^n\in N$. Then there are two cases:

Case 1. Let $i\{a,b\}=i\{a,c\}=i\{b,c\}=r$. For this we consider 4 sub cases as follows:

Sub case 1-1. If $a\cdot b=(a_1,\cdots,a_r)$ and $(a_1,\cdots,a_r)\cdot c=(a_1,\cdots,a_r)$, then $a_r<b_r$ and $a_r<c_r$. Thus, $(a\cdot b)\cdot c=(a_1,\cdots,a_r)\cdot c=(a_1,\cdots,a_r)$. On the other hand, we have $b\cdot c=(b_1,\cdots,b_r)$ or $b\cdot c=(c_1,\cdots,c_r)$. Since $a_r<b_r$ and $a_r<c_r$, we get $a\cdot(b\cdot c)=(a_1,\cdots,a_r)$. Therefore, $(a\cdot b)\cdot c=a\cdot(b\cdot c)$.

Sub case 1-2. If $a\cdot b=(a_1,\cdots,a_r)$ and $(a_1,\cdots,a_r)\cdot c=(c_1,\cdots,c_r)$, then $a_r<b_r$ and $c_r<a_r$ and so $c_r<b_r$. Thus, $(a\cdot b)\cdot c=(a_1,\cdots,a_r)\cdot c=(c_1,\cdots,c_r)$. On the other hand, since $a_r<b_r$, $c_r<a_r$ and $c_r<b_r$, we get $a\cdot(b\cdot c)=(a_1,\cdots,a_r)\cdot(c_1,\cdots,c_r)=(c_1,\cdots,c_r)$. Therefore, $(a\cdot b)\cdot c=a\cdot(b\cdot c)$.

Sub case 1-3. If $a\cdot b=(b_1,\cdots,b_r)$ and $(b_1,\cdots,b_r)\cdot c=(b_1,\cdots,b_r)$, then $b_r<a_r$ and $b_r<c_r$. Thus, $(a\cdot b)\cdot c=(b_1,\cdots,b_r)\cdot c=(b_1,\cdots,b_r)$. On the other hand, since $b_r<a_r$ and $b_r<c_r$, we get $a\cdot(b\cdot c)=(a_1,\cdots,a_r).(b_1,\cdots,b_r)=(b_1,\cdots,b_r)$. Therefore, $(a\cdot b)\cdot c=a\cdot(b\cdot c)$.

Sub case 1-4. If $a \cdot b = (b_1, \cdots, b_r)$ and $(b_1, \cdots, b_r) \cdot c = (c_1, \cdots, c_r)$, then $b_r < a_r$ and $c_r < b_r$ and so $c_r < a_r$. Thus, $(a \cdot b) \cdot c = (b_1, \cdots, b_r) \cdot c = (c_1, \cdots, c_r)$. On the other hand, since $b_r < a_r$, $c_r < b_r$ and $c_r < a_r$, we get $a \cdot (b \cdot c) = (a_1, \cdots, a_r) \cdot (c_1, \cdots, c_r) = (c_1, \cdots, c_r)$. Therefore, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Case 2. Let $i\{a, b\} = i\{a, c\} = r$ and $i\{b, c\} = s > r$. Then $b_r = c_r$ and we consider 2 sub cases as follows:

Sub case 2-1. If $a \cdot b = (a_1, \cdots, a_r)$, then $a_r < b_r$. Since $b_r = c_r$, $(a_1, \cdots, a_r) \cdot c = (a_1, \cdots, a_r)$. Thus, $(a \cdot b) \cdot c = (a_1, \cdots, a_r) \cdot c = (a_1, \cdots, a_r)$. On the other hand, we have $b \cdot c = (b_1, \cdots, b_s)$ or $b \cdot c = (c_1, \cdots, c_s)$. Since $a_r < b_r$ and $b_r = c_r$, we get $a \cdot (b \cdot c) = (a_1, \cdots, a_r)$. Therefore, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Sub case 2-2. If $a \cdot b = (b_1, \cdots, b_r)$, then $b_r < a_r$. Since $b_r = c_r$, $(b_1, \cdots, b_r) \cdot c = (b_1, \cdots, b_r)$. Thus, $(a \cdot b) \cdot c = (b_1, \cdots, b_r) \cdot c = (b_1, \cdots, b_r)$. On the other hand, we have $b \cdot c = (b_1, \cdots, b_s)$ or $b \cdot c = (c_1, \cdots, c_s)$. Since $b_r < a_r$ and $b_r = c_r$, we get $a \cdot (b \cdot c) = (b_1, \cdots, b_r)$. Therefore, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

For distributivity, there are two cases:

Case 1. Let $a_1 \neq b_1 \vee c_1$. We have $a \cdot (b + c) = (a_1, \cdots, a_n) \cdot (b_1 \vee c_1) = (a_1 \wedge (b_1 \vee c_1))$. On the other hand, $a \cdot b + a \cdot c = ((a_1 \wedge b_1) \vee (a_1 \wedge c_1)) = (a_1 \wedge (b_1 \vee c_1))$. Therefore, $a \cdot (b + c) = a \cdot b + a \cdot c$.

Case 2. $a_1 = b_1 \vee c_1$. we consider 2 sub cases as follows:

Sub case 2-1. Let $c_1 \leqslant b_1$. Then $a_1 = b_1$. We have $a \cdot (b + c) = (a_1, \cdots, a_n) \cdot (b_1 \vee c_1) = (a_1)$. Since $c_1 \leqslant b_1$, $a \cdot b + a \cdot c = (a_1)$. Therefore, $a \cdot (b + c) = a \cdot b + a \cdot c$.

Sub case 2-2. Let $b_1 \leqslant c_1$. Then $a_1 = c_1$. We have $a \cdot (b + c) = (a_1, \cdots, a_n) \cdot (b_1 \vee c_1) = (a_1)$. Since $b_1 \leqslant c_1$, $a \cdot b + a \cdot c = (a_1)$. Therefore, $a \cdot (b + c) = a \cdot b + a \cdot c$.

Since the operations $+$ and $\cdot$ are commutative, also we have $(b + c) \cdot a = b \cdot a + c \cdot a$.
Therefore, $(N, +, \cdot, ())$ is a semi-ring. □

Example 3.4. Let $N = \{(), (1), (2), (1, 1), (2, 1)\}$. By defined the binary operations "+ " and "·" on N, we have: for every $a \in N$

$a + () = () + a = a$,
$(1) + (1) = (1) + (1, 1) = (1, 1) + (1, 1) = (1)$,
$(2) + a = (2, 1) + a = (2)$,
$a \cdot () = () \cdot a = ()$ and $a \cdot (1) = (1) \cdot a = (1)$,
$(2) \cdot (2) = (2) \cdot (2, 1) = (2)$,
$(2) \cdot (1, 1) = (1, 1) \cdot (2, 1) = (1)$,
$(1, 1) \cdot (1, 1) = (1, 1)$,
$(2, 1) \cdot (2, 1) = (2, 1)$.

For the associativity + we have:
$[(1) + (2)] + (1, 1) = (2) + (1, 1) = (2) = (1) + (2) = (1) + [(2) + (1, 1)]$,
$[(1) + (2)] + (2, 1) = (2) + (2, 1) = (2) = (1) + (2) = (1) + [(2) + (2, 1)]$,
$[(1) + (1, 1)] + (2, 1) = (1) + (2, 1) = (2) = (1) + (2) = (1) + [(1, 1) + (2, 1)]$,
$[(2) + (1, 1)] + (2, 1) = (2) + (2, 1) = (2) = (2) + (2) = (2) + [(1, 1) + (2, 1)]$.

For the associativity · we have:
$[(1) \cdot (2)] \cdot (1, 1) = (1) \cdot (1, 1) = (1) = (1) \cdot (1) = (1) \cdot [(2) \cdot (1, 1)]$,
$[(1) \cdot (2)] \cdot (2, 1) = (1) \cdot (2, 1) = (1) = (1) \cdot (2) = (1) \cdot [(2) \cdot (2, 1)]$,
$[(1) \cdot (1, 1)] \cdot (2, 1) = (1) \cdot (2, 1) = (1) = (1) \cdot (1) = (1) \cdot [(1, 1) \cdot (2, 1)]$,
$[(2) \cdot (1, 1)] \cdot (2, 1) = (1) \cdot (2, 1) = (1) = (2) + (1, 1) = (2) \cdot [(1, 1) \cdot (2, 1)]$.

For the distributivity · over + we have:
$(1) \cdot [(2) + (1, 1)] = (1) \cdot (2) = (1) = (1) + (1) = (1) \cdot (2) + (1) \cdot (1, 1)$,

$(2) \cdot [(1) + (1, 1)] = (2) \cdot (1) = (1) = (1) + (1) = (2) \cdot (1) + (2) \cdot (1, 1),$
$(1, 1) \cdot [(1) + (2)] = (1.1) \cdot (1) = (1) = (1) + (1) = (1, 1) \cdot (1) + (1, 1) \cdot (2),$
$(1) \cdot [(2) + (2, 1)] = (1) \cdot (2) = (1) = (1) + (1) = (1) \cdot (2) + (1) \cdot (2, 1),$
$(2) \cdot [(1) + (2, 1)] = (2) \cdot (2) = (2) = (1) + (2) = (2) \cdot (1) + (2) \cdot (2, 1),$
$(2, 1) \cdot [(1) + (2)] = (2, 1) \cdot (2) = (2) = (1) + (2) = (2, 1) \cdot (1) + (2, 1) \cdot (2),$
$(1) \cdot [(1, 1) + (2, 1)] = (1) \cdot (2) = (1) = (1) + (1) = (1) \cdot (1, 1) + (1) \cdot (2, 1),$
$(1, 1) \cdot [(1) + (2, 1)] = (1, 1) \cdot (2) = (1) = (1) + (1, 1) = (1, 1) \cdot (1) + (1, 1) \cdot (2, 1),$
$(2, 1) \cdot [(1) + (1, 1)] = (2, 1) \cdot (1) = (1) = (1) + (1, 1) = (2, 1) \cdot (1) + (2, 1) \cdot (1, 1),$
$(2) \cdot [(1, 1) + (2, 1)] = (2) \cdot (2) = (2) = (1) + (2) = (2) \cdot (1, 1) + (2) \cdot (2, 1),$
$(1, 1) \cdot [(2) + (2, 1)] = (1, 1) \cdot (2) = (1) = (1) + (1, 1) = (1, 1) \cdot (2) + (1, 1) \cdot (2, 1),$
$(2, 1) \cdot [(2) + (1, 1)] = (2, 1) \cdot (2) = (2) = (2) + (1, 1) = (2, 1) \cdot (2) + (2, 1) \cdot (1, 1).$
Then $(N, +, \cdot, ())$ is a semi-ring.

Remark 3.5. Notice that the semi-ring $(N, +, \cdot, ())$ can not be a ring. Since, if for every $() \neq a \in N$, there exists $() \neq b \in N$, such that $a + b = b + a = ()$, then $a = b = ()$, which is a contradiction.

Definition 3.6. Let $() \in X \subseteq N$. We say that X is a sub-ring of N, if for every $a, b \in X$, $a + b, a \cdot b \in X$.

In the sequel, for briefly, we denote the semi-ring $(N, +, \cdot, ())$ related to a nexus N only by N and put $0 := ()$.

Example 3.7. Let $N = \langle (1, 4), (1, 2, 3) \rangle$ and $X = \{(), (1), (1, 2), (1, 3), (1, 2, 3)\}$. X is a sub-semi-ring of N but since $(1, 2, 2) \cdot (1, 2, 3) = (1, 2, 2) \notin X$, X is not an ideal of N.

Definition 3.8. Let $() \in I \subseteq N$. We say that I is an ideal of N, if it satisfies the following conditions:

(I) for every $a, b \in I$, $a + b \in I$,
(II) for every $a \in I$ and every $b \in N$, $a \cdot b \in I$.

An ideal I of N is prime if for every $a, b \in N$, $a \cdot b \in I$ implies $a \in I$ or $b \in I$.

Example 3.9. Let $N = \langle (1, 4), (1, 2, 3) \rangle$ and $I = \{(), (1), (1, 2), (1, 3), (1, 2, 1), (1, 2, 2), (1, 2, 3)\}$. I is an ideal of N.

Definition 3.10. A fuzzy subset μ of N is called a fuzzy sub-semi-ring of N if it satisfies the following conditions: for all $a, b \in N$,

(I) $\mu(a + b) \geqslant \min\{\mu(a), \mu(b)\}$,
(III) $\mu(a \cdot b) \geqslant \min\{\mu(a), \mu(b)\}$.

The set of all fuzzy sub-semi-rings of N, is denoted by $FSUB_S(N)$.

Example 3.11. We define the fuzzy subset μ of N as follows:

(i) For $a = (a_1, a_2, \ldots, a_n) \in N$ of level n, $\mu(a) := \dfrac{1}{a_1 a_2 \ldots a_n}$, otherwise

$$\mu(v) = \begin{cases} 1 & v = 0; \\ 0 & l(v) = \infty. \end{cases}$$

Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_m)$ and $a_1 \geqslant b_1$. Then $\mu(a + b) = \mu((a_1)) = \dfrac{1}{a_1} \geqslant \dfrac{1}{a_1 \ldots a_n}$. Hence $\mu(a + b) \geqslant \mu(a) \geqslant \min\{\mu(a), \mu(b)\}$. Let $i\{a, b\} = i$ and $b_i \leqslant a_i$. Hence $a \cdot b = (b_1, \ldots, b_i)$ and so $\mu(a \cdot b) \geqslant \dfrac{1}{b_1 \ldots b_m} = \mu(b) \geqslant \min\{\mu(a), \mu(b)\}$. Therefore, $\mu \in FSUB_S(N)$.

(ii) For $a = (a_1, a_2, \ldots, a_n) \in N$, $\mu(a) := \dfrac{1}{l(a)}$, otherwise

$$\mu(v) = \begin{cases} 1 & v = 0; \\ 0 & l(v) = \infty. \end{cases}$$

Since $l(a + b) = 1$, $\mu(a + b) = 1 \geqslant \min\{\mu(a), \mu(b)\}$. Clearly $l(a \cdot b) \leqslant l(a)$, now we have $\mu(a \cdot b) \geqslant \mu(a) \geqslant \min\{\mu(a), \mu(b)\}$. Therefore, $\mu \in FSUB_S(N)$.

(iii) For $a = (a_1, a_2, \ldots, a_n) \in N$, $\mu(a) := \dfrac{1}{stem\, a}$, otherwise

$$\mu(v) = \begin{cases} 1 & v = 0; \\ 0 & l(v) = \infty. \end{cases}$$

Let $a, b \in N$ with $stem\, a = a_1$, $stem\, b = b_1$ and $a_1 \geqslant b_1$. Hence $stem\,(a + b) = a_1$ and $stem\, a \cdot b = b_1$. So $\mu(a + b) = \mu(a) \geqslant \min\{\mu(a), \mu(b)\}$ and $\mu(a \cdot b) = \mu(b) \geqslant \min\{\mu(a), \mu(b)\}$. Therefore, $\mu \in FSUB_S(N)$.

Proposition 3.12. *Let A be a subset of N with $0 \in A$. Then A is a sub-semi-ring of N if and only if $\chi_A \in FSUB_S(N)$, where*

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \notin A. \end{cases}$$

Proof. Let A be a sub-semi-ring of N and $a, b \in N$. If $a, b \in A$, then $a \cdot b, a + b \in A$ and so $\chi_A(a \cdot b) = \chi_A(a + b) = 1$. Now, we have $1 = \chi_A(a + b) \geqslant \min\{\chi_A(a), \chi_A(b)\} = 1$ and so $1 = \chi_A(a \cdot b) \geqslant \min\{\chi_A(a), \chi_A(b)\} = 1$. If $a \notin A$, we have $\min\{\chi_A(a), \chi_A(b)\} = 0$, hence $\chi_A(a + b) \geqslant \min\{\chi_A(a), \chi_A(b)\} = 0$ and $\chi_A(a \cdot b) \geqslant \min\{\chi_A(a), \chi_A(b)\} = 0$. Thus, $\chi_A \in FSUB_S(N)$.

Conversely, let $\chi_A \in FSUB_S(N)$. Suppose that $a, b \in A$, then $\chi_A(a) = \chi_A(b) = 1$. Since $\chi_A(a + b) \geqslant \min\{\chi_A(a), \chi_A(b)\} = 1$, and $\chi_A(a \cdot b) \geqslant \min\{\chi_A(a), \chi_A(b)\} = 1$, $\chi_A(a + b) = \chi_A(a \cdot b) = 1$ and so $a + b, a \cdot b \in A$. Therefore, A is a sub-semi-ring of N. □

Definition 3.13. Let $\mu \in FSUB_S(N)$ and $\alpha \in [0, 1]$. We define $\mu^\alpha = \{x \in N : \mu(x) \geqslant \alpha\}$ and $Supp(\mu) = \{a \in N : \mu(a) \neq 0\}$.

Proposition 3.14. *Let μ be a fuzzy subset of N such that for every $a \in N$, $\mu(0) \geqslant \mu(a)$. Then $\mu \in FSUB_S(N)$ if and only if, for every $\alpha \in [0, \mu(0)]$, μ^α be a sub-semi-ring of N.*

Proof. Let μ be a fuzzy sub-semi-ring of N, $\alpha \in [0, \mu(0)]$ and $a, b \in \mu^\alpha$. Hence $0 \in \mu^\alpha$, $\mu(a) \geqslant \alpha$ and $\mu(b) \geqslant \alpha$. This shows that $\mu(a + b) \geqslant \min\{\mu(a), \mu(b)\} \geqslant \alpha$, and so $a + b \in \mu^\alpha$. Similarly, $a \cdot b \in \mu^\alpha$ and so μ^α is a sub-ring of N.

Conversely, let for every $\alpha \in [0, \mu(0)]$, μ^α be a sub-semi-ring of N. Let $a, b \in N$, $\mu(a) = \alpha$, $\mu(b) = \beta$ and $\alpha \geqslant \beta$. Then $a \in \mu^\alpha$ and $a, b \in \mu^\beta$. So $a + b \in \mu^\beta$ and $a \cdot b \in \mu^\beta$. Hence $\mu(a + b) \geqslant \beta = \min\{\mu(a), \mu(b)\}$ and $\mu(a \cdot b) \geqslant \beta = \min\{\mu(a), \mu(b)\}$. Therefore, μ is a fuzzy sub-semi-ring of N. □

The following examples shows that the Propositions 3.12 and 3.14 are not true for ideals.

Example 3.15. Let $N = \langle(1,4),(1,2,3)\rangle$ and $X = \{(),(1),(1,2),(1,3),(1,2,3)\}$. X is a sub-ring of N and so χ_X is a fuzzy sub-semi-ring of N, but $(1,2,2)\cdot(1,2,3) = (1,2,2) \notin X$. This shows that X is not an ideal of N. Hence Proposition 3.12 is not true for ideals.

Example 3.16. Let $N = \{(),(1),(2),(3),(1,2)\}$. We define the fuzzy μ on N as following:

$\mu(()) = 1, \mu((1)) = 0.95, \mu((2)) = 0.5, \mu((3)) = 0.75, \mu((1,2)) = 0.85.$

For every $a \in N$, we have

$()+a = a$, $(1)+(1) = (1)+(1,2) = (1)$, $(1)+(2) = (2)+(1,2) = (2)+(2) = (2)$, $(1)+(3) = (2)+(3) = (3)+(3) = (3)+(1,2) = (3)$, $(1,2)+(1,2) = (1,2)$. Hence for every $a,b \in N$, $\mu(a+b) \geqslant \min\{\mu(a),\mu(b)\}$.

$()\cdot a = ()$, $(1)\cdot a = (1)$, $a\cdot a = a$, $(2)\cdot(3) = (2)$, $(2)\cdot(1,2) = (3)\cdot(1,2) = (1)$. Hence for every $a,b \in N$, $\mu(a\cdot b) \geqslant \min\{\mu(a),\mu(b)\}$. Then $\mu \in FSUB_S(N)$ and so by Proposition 3.14, for every $\alpha \in [0,1]$, μ^α is a sub-semi-ring of N. We have $\mu^{0.75} = \{(),(1),(3),(1,2)\}$. Since $(2)\cdot(3) = (2) \notin \mu^{0.75}$, $\mu^{0.75}$ is not an ideal of N. Therefore, Proposition 3.14 is not true for ideals.

4. Strong fuzzy of a nexus sub-semi-ring

In Examples 3.15 and 3.16, we show that Propositions 3.12 and 3.14 are not true for ideals. Since ideals in semi-rings are more important than sub-semi-rings, in this section, we want to put a condition on fuzzy sub-semi-ring such that Propositions 3.12 and 3.14 be true for ideals. See the following definition:

Definition 4.1. A fuzzy subset μ of N is called a strong fuzzy sub-semi-ring of N if it satisfies the following conditions: for all $a,b \in N$,

(I) $\mu(a+b) \geqslant \min\{\mu(a),\mu(b)\}$,
(III) $\mu(a\cdot b) \geqslant \max\{\mu(a),\mu(b)\}$.

The set of all strong fuzzy sub-semi-rings of N is denoted by $FSUB_T(N)$.

Remark 4.2. Let N be a semi-ring and $\mu \in FSUB_T(N)$. Then, for every $a \in N$, $\mu(0) \geqslant \mu(a)$, because for every $a \in N$, $\mu(0) = \mu(0\cdot a) \geqslant \max\{\mu(a),\mu(0)\} \geqslant \mu(a)$.

It is easy to see that all fuzzy μ that are defined in Examples 3.11(i)–(iii) are strong fuzzy sub-semi-rings.

Proposition 4.3. *Let A be a subset of N with $0 \in A$. Then A is an ideal of N if and only if, $\chi_A \in FSUB_T(N)$.*

Proof. Let A be an ideal of N and $a,b \in N$. If $a,b \in A$, then $a\cdot b, a+b \in A$ and it is show that $\chi_A(a\cdot b) = \chi_A(a+b) = 1$. Hence, $1 = \chi_A(a+b) \geqslant \min\{\chi_A(a),\chi_A(b)\} = 1$ and also we have $1 = \chi_A(a\cdot b) \geqslant \max\{\chi_A(a),\chi_A(b)\} = 1$. If $a \in A$ and $b \notin A$, then $a\cdot b \in A$ and so $\chi_A(a\cdot b) = 1$. Hence, $\chi_A(a\cdot b) \geqslant \max\{\chi_A(a),\chi_A(b)\}$. Also, we have $\chi_A(a+b) \geqslant \min\{\chi_A(a),\chi_A(b)\} = 0$. If $a \notin A$ and $b \notin A$, then $\min\{\chi_A(a),\chi_A(b)\} = \max\{\chi_A(a),\chi_A(b)\} = 0$. Now, we have $\chi_A(a\cdot b) \geqslant \max\{\chi_A(a),\chi_A(b)\}$ and $\chi_A(a+b) \geqslant \min\{\chi_A(a),\chi_A(b)\}$. Therefore, $\chi_A \in FSUB_T(N)$.

Conversely, let $\chi_A \in FSUB_T(N)$. If $a,b \in A$, then $\chi_A(a) = \chi_A(b) = 1$. Since $\chi_A(a+b) \geqslant \min\{\chi_A(a),\chi_A(b)\} = 1$, then $\chi_A(a+b) = 1$ and so $a+b \in A$. Let $a \in A$ and $b \in N$, then $\max\{\chi_A(a),\chi_A(b)\} = 1$. Then $\chi_A(a\cdot b) = 1$, since $\chi_A(a\cdot b) \geqslant \max\{\chi_A(a),\chi_A(b)\} = 1$, and so $a\cdot b \in A$. Therefore, A is an ideal of N. □

Proposition 4.4. *Let μ be a fuzzy subset on N. Then $\mu \in FSUB_T(N)$ if and only if, for every $\alpha \in [0, \mu(0)]$, μ^α be an ideal of N.*

Proof. Let $\mu \in FSUB_T(N)$, $\alpha \in [0, \mu(0)]$ and $a, b \in \mu^\alpha$. Hence, $0 \in \mu^\alpha$, $\mu(a) \geqslant \alpha$ and $\mu(b) \geqslant \alpha$. So $\mu(a + b) \geqslant \min\{\mu(a), \mu(b)\} \geqslant \alpha$. Hence, $a + b \in \mu^\alpha$. Now, let $a \in \mu^\alpha$ and $b \in N$. So $\mu(a \cdot b) \geqslant \max\{\mu(a), \mu(b)\} \geqslant \alpha$. Hence, $a \cdot b \in \mu^\alpha$. Therefore, μ^α is an ideal of N.

Conversely, let for every $\alpha \in [0, \mu(0)]$, μ^α be an ideal of N. Let $a, b \in N$, $\mu(a) = \alpha$, $\mu(b) = \beta$ and $\alpha \geqslant \beta$. Then $a \in \mu^\alpha$ and $a, b \in \mu^\beta$. So, $a + b \in \mu^\beta$ and $a \cdot b \in \mu^\alpha$. Hence, $\mu(a + b) \geqslant \beta = \min\{\mu(a), \mu(b)\}$ and $\mu(a \cdot b) \geqslant \alpha = \max\{\mu(a), \mu(b)\}$. Therefore, $\mu \in FSUB_T(N)$. □

Proposition 4.5. *Let $0 \neq \mu \in FSUB_T(N)$. Then $Supp(\mu)$ is an ideal of N.*

Proof. Let $\mu(0) = 0$. Since for every $a \in N$, $\mu(0) \geqslant \mu(a)$, $\mu(a) = 0$ and so $\mu = 0$, which is a contradiction. Hence, $\mu(0) \neq 0$ and $0 \in Supp(\mu)$. Let $a, b \in Supp(\mu)$. This shows that $\mu(a + b) \geqslant \min\{\mu(a), \mu(b)\} \neq 0$ and so $a + b \in Supp(\mu)$. Now, let $a \in Supp(\mu)$ and $b \in N$. Since $\mu(a) \neq 0$, $\mu(a \cdot b) \geqslant \max\{\mu(a), \mu(b)\} \neq 0$. Then $a \cdot b \in Supp(\mu)$. Therefore, $Supp(\mu)$ is an ideal of N. □

The following example shows that the converse of Proposition 4.5, is not true.

Example 4.6. Consider N and μ of Example 3.16. We have $Supp(\mu) = N$ is an ideal of N. By Example 3.16, $\mu^{0.75}$ is not an ideal of N. Hence, by Proposition 4.4, μ is not a strong fuzzy sub-semi-ring of N.

The following example shows that $Supp(\mu)$ is an ideal of N, but μ is not a fuzzy sub-semi-ring of N.

Example 4.7. Suppose that N is a semi-ring and for every $a = (a_1, a_2, \dots, a_n) \in N$. We put $[a] = [\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}]$, where [ ] is the bracket function. We define the fuzzy subset μ on N such that for every $a \in N$,

$$\mu(a) = \begin{cases} 1 & \text{if } [a] \in \mathbb{N}; \\ (\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}) - [a] & \text{if } [a] \notin \mathbb{N}. \end{cases}$$

Suppose that $N = \langle(2, 4)\rangle$, hence $Supp(\mu) = N$ is an ideal of N. Since $[\frac{1}{2} + \frac{1}{4}] = [\frac{3}{4}] = 0$ and $[\frac{1}{2} + \frac{1}{3}] = [\frac{5}{6}] = 0$, $\mu((2, 4)) = \frac{3}{4}$ and $\mu((2, 3)) = \frac{5}{6}$. On the other hand, we have $(2) = (2, 3) + (2, 4)$ and since $[\frac{1}{2}] = 0$, $\mu((2)) = \frac{1}{2}$.

Hence, $\mu((2, 3) + (2, 4)) = \mu((2)) = \frac{1}{2} < \frac{3}{4} = \min\{\frac{3}{4}, \frac{5}{6}\} = \min\{\mu((2, 3)), \mu((2, 4))\}$. Therefore, μ is not a fuzzy sub-semi-ring of N.

Proposition 4.8. *Let μ be a fuzzy subset of N. Then μ is a strong fuzzy sub-semi-ring if and only if it is a fuzzy sub-semi-ring and a fuzzy sub-nexus of N.*

Proof. Let μ be a strong fuzzy sub-semi-ring of N. Clearly it is a fuzzy sub-semi-ring. If $a \leqslant b$, then $a \cdot b = a$. Hence $\mu(a) = \mu(a \cdot b) \geqslant \max\{\mu(a), \mu(b)\}$. So $\mu(a) \geqslant \mu(b)$ and so μ is a fuzzy sub-nexus of N.

Conversely, let μ be a fuzzy sub-semi-ring and fuzzy sub-nexus of N. For every $a, b \in N$, since $ab \leqslant a, b$, and μ is a fuzzy sub-nexus of N, we have $\mu(ab) \geqslant \mu(a)$ and $\mu(ab) \geqslant \mu(b)$. Hence, $\mu(a \cdot b) \geqslant \max\{\mu(a), \mu(b)\}$. Therefore, $\mu \in FSUB_T(N)$. □

The following example give a fuzzy sub-semi-ring μ such that it is not a fuzzy sub-nexus.

Example 4.9. Consider N and μ of Example 3.16. We have $(2) \leqslant (3)$, $0.5 = \mu((2))$ and $0.75 = \mu((3))$, but $0.5 \leqslant 0.75$. Hence, μ is not a fuzzy sub-nexus of N.

Definition 4.10. Suppose that N is a semi-ring and μ is a fuzzy subset of N. We say that μ is an integer fuzzy if for every $a, b \in N$, $\mu(a) = 0$ and $\mu(b) = 0$, imply $\mu(a \cdot b) = 0$. Let $\alpha \in [0, \mu(0))$, we say that μ is a prime fuzzy with respect to α, if for every $a, b \in N$, $\mu(a) < \alpha$ and $\mu(b) < \alpha$ imply $\mu(a \cdot b) < \alpha$.

Proposition 4.11. *Suppose that $N \neq I$ is an ideal of N, μ is a strong fuzzy sub-semi-ring of N and $\alpha \in (0, \mu(0)]$. Then*

(I) *I is a prime ideal of N if and only if χ_I is an integer fuzzy of N,*
(II) *$N \neq Supp(\mu)$ is a prime ideal of N if and only if μ is an integer fuzzy of N,*
(III) *μ^α is a prime ideal of N if and only if μ is a prime fuzzy with respect to α.*

Proof. By Definition 4.10, it is clear. □

Proposition 4.12. *If N is a cyclic nexus and μ is a strong fuzzy sub-semi-ring, then for every $\alpha \in [0, \mu(0))$, $\mu^\alpha \neq N$ is a prime ideal of N.*

Proof. Let $a, b \in N$ and $a \cdot b \in \mu^\alpha$, hence $\mu(a \cdot b) \geqslant \alpha$. Since N is cyclic nexus, let $a \leqslant b$. Hence, $\mu(a \cdot b) = \mu(a) \geqslant \alpha$ and so $a \in \mu^\alpha$. Then, μ^α is a prime ideal of N. □

Proposition 4.13. *If N is a semi-ring and consider μ of Example 3.11(iii). Then, for every $\alpha \in [0, \mu(0))$, $\mu^\alpha \neq N$ is a prime ideal of N.*

Proof. Let $a, b \in N$ and $\alpha \in [0, \mu(0))$. If $\mu(a) < \alpha$ and $\mu(b) < \alpha$, hence $stem\, a > \alpha$ and $stem\, b > \alpha$. Thus, $stem\, a \cdot b > \alpha$. Then, $\mu(a \cdot b) < \alpha$ and μ is a prime fuzzy with respect to α. Hence, by Proposition 4.11(III), μ^α is a prime ideal of N. □

Proposition 4.14. *If N is a semi-ring and consider μ of Example 3.11(ii). Then, for every $\alpha \in [0, \mu(0))$, $\mu^\alpha \neq N$ is a prime ideal of N if and only if there exists a unique $a \in N$ with $l(a) = n$ such that $\{b \in N : l(a) > n\} \subseteq q_a$.*

Proof. At first we assume that there exist unique $a \in N$ such that $l(a) = n$ and for every $b \in N$ with $l(b) > n$, $b \in q_a$. We show that $\mu^{\frac{1}{n}}$ is a prime ideal of N. By Proposition 4.11(III), it is sufficient to show that μ is a prime fuzzy with respect to $\frac{1}{n}$. Let $b, b' \in N$ with $\mu(b) < \frac{1}{n}$ and $\mu(b') < \frac{1}{n}$. Hence, $l(b) > n$ and $l(b') > n$. Thus, $b, b' \in q_a$ and $i\{b, b'\} > n$, and so $l(b \cdot b') > n$. Then, $\mu(b \cdot b') < \frac{1}{n}$. Therefore, μ is a prime fuzzy with respect to $\frac{1}{n}$ and $\mu^{\frac{1}{n}}$ is a prime ideal.

Conversely, let there exist $a, a' \in N$ such that $a \neq a'$, $l(a) = l(a') = n$, $b \in q_a$ and $b' \in q_{a'}$ with $l(b) > n$ and $l(b') > n$. Since $a \neq a'$, $l(a) = l(a') = n$, $i\{a, a'\} \leqslant n$ and since $b \in q_a$ and $b' \in q_{a'}$ with

$l(b) > n$ and $l(b') > n$, $i\{b, b'\} \leqslant n$. Hence, $l(b \cdot b') \leqslant n$, and so $\mu(b \cdot b') \geqslant \frac{1}{n}$. Then, $b \cdot b' \in \mu^{\frac{1}{n}}$, but $b, b' \notin \mu^{\frac{1}{n}}$. Therefore, $\mu^{\frac{1}{n}}$ is not prime ideal of N and the proof is complete. □

Example 4.15. Consider μ of Example 3.11(ii) and $N = \langle(1, 2, 3), (1, 1, 4)\rangle$. Let $a = (1, 2, 3)$ and $b = (1, 1, 4)$, then $\mu(a \cdot b) = \mu((1, 1)) = \frac{1}{2}$, but $\mu(a) = \mu(b) = \frac{1}{3}$. So, $a \cdot b \in \mu^{\frac{1}{2}}$, but $a, b \notin \mu^{\frac{1}{2}}$. Hence $\mu^{\frac{1}{2}}$ is not a prime ideal of N. In the other hand, since for every $a \in N$, $stem\, a = 1$, $\mu^1 = \{(), (1)\}$ and it is easy to see that $a \cdot b = (1)$ implies $a = (1)$ or $b = (1)$. So, μ^1 is a prime ideal of N.

Proposition 4.16. *Suppose that $st(N) = n \neq 1$. Consider μ of Example 3.11(ii), $\delta_i = max\{l(a) : stem\, a = i\}$. Suppose that j is such that $1 \neq \delta_j = \max\{\delta_i : 1 \leqslant i \leqslant n\}$. Also, suppose that k is such that $1 \neq \delta_k = \max\{\delta_i : 1 \leqslant i \leqslant n, i \neq j\}$. Then for every $1 \leqslant m < \delta_k$, $\mu^{\frac{1}{m}}$ is not a prime ideal of N.*

Proof. Let $a, b \in N$ such that $l(a) = \delta_j$ and $l(b) = \delta_k$. Hence, $stem\, a = j$, $stem\, b = k$, $\mu(a) = \frac{1}{\delta_j}$ and $\mu(b) = \frac{1}{\delta_k}$. Then $a \cdot b = (j \wedge k)$, and so $\mu(a \cdot b) = 1 \geqslant \frac{1}{m}$, but $\mu(a) = \frac{1}{\delta_j} < \frac{1}{m}$, $\mu(b) = \frac{1}{\delta_k} < \frac{1}{m}$. Therefore, $a \cdot b \in \mu^{\frac{1}{m}}$, but $a, b \notin \mu^{\frac{1}{m}}$. Thus, $\mu^{\frac{1}{m}}$ is not a prime ideal of N. □

Example 4.17. Suppose that $N = \langle(5, 4, 3, 2, 1), (6, 2, 1, 1, 1, 1)\rangle$. By notation of Proposition 4.16, we have $st(N) = 6$, $j = 6$, $k = 5$, $\delta_6 = 6$ and $\delta_5 = 5$. Let μ as Example 3.11(ii). Now by Proposition 4.16, for every $1 \leqslant m \leqslant 4$, $\nu^{\frac{1}{m}}$, is not a prime ideal of N.

Definition 4.18. Suppose that N and M are two semi-rings, the function $f : N \longrightarrow M$ is a semi-ring homomorphism if f satisfies the following conditions:

(I) $f(0) = 0$,
(II) $f(a + b) = f(a) + f(b), \forall a, b \in N$,
(III) $f(a \cdot b) = f(a) \cdot f(b), \forall a, b \in N$.

Proposition 4.19. *Suppose that N and M are two semi-rings and $f : N \longrightarrow M$ is a semi-ring homomorphism:*

(I) *if $\mu \in FSUB_S(M)$, then $\mu \circ f \in FSUB_S(N)$ such that for every $a \in N$, $(\mu \circ f)(a) = \mu(f(a))$,*
(II) *if $\mu \in FSUB_S(N)$, then $f(\mu) \in FSUB_S(M)$ such that for every $b \in M$,*

$$f(\mu)(b) = \begin{cases} 0 & if\, b \notin Imf; \\ sup\{\mu(a) : f(a) = b\} & if\, b \in Imf. \end{cases}$$

Proof. (I) Let $a, b \in N$, then

$(\mu \circ f)(a + b) = \mu(f(a + b)) = \mu(f(a) + f(b)) \geqslant \min\{\mu(f(a)), \mu(f(b))\} = \min\{(\mu \circ f)(a), (\mu \circ f)(b)\}$.
$(\mu \circ f)(a.b) = \mu(f(a \cdot b)) = \mu(f(a) \cdot f(b)) \geqslant \min\{\mu(f(a)), \mu(f(b))\} = \min\{(\mu o f)(a), (\mu o f)(b)\}$.

Therefore, $\mu \circ f \in FSUB_S(N)$.

(II) Let $a, b \in M$. If $a \notin Imf$ or $b \notin Imf$, then $\min\{f(\mu)(a), f(\mu)(b)\} = 0$ and hence $f(\mu)(a+b) \geqslant \min\{f(\mu)(a), f(\mu)(b)\}$ and $f(\mu)(a \cdot b) \geqslant \min\{f(\mu)(a), f(\mu)(b)\}$. Now, let $a, b \in Imf$ and $f(\mu)(a) \leqslant f(\mu)(b)$. We show that $f(\mu)(a+b) \geqslant f(\mu)(a)$ and $f(\mu)(a \cdot b) \geqslant f(\mu)(a)$.

Let $\epsilon > 0$. Since $f(\mu)(a) = \sup\{\mu(x) : f(x) = a\}$ and $f(\mu)(b) = \sup\{\mu(y) : f(y) = b\}$, there exist $x', y' \in N$ such that $f(\mu)(a) - \epsilon \leqslant \mu(x') < f(\mu)(a)$ and $f(\mu)(b) - \epsilon \leqslant \mu(y') < f(\mu)(b)$. Since $f(x' + y') = a + b$, $f(\mu)(a+b) = \sup\{\mu(z) : f(z) = a + b\} \geqslant \mu(x' + y') \geqslant \min\{\mu(x'), \mu(y')\}$.

Case 1: If $\mu(x') \leqslant \mu(y')$, then $f(\mu)(a+b) \geqslant \mu(x') \geqslant f(\mu)(a) - \epsilon$.

Case 2: If $\mu(x') \geqslant \mu(y')$, then $f(\mu)(a+b) \geqslant \mu(y') \geqslant f(\mu)(b) - \epsilon \geqslant f(\mu)(a) - \epsilon$.

Since for every $\epsilon > 0$, $f(\mu)(a+b) \geqslant f(\mu)(a) - \epsilon$, we get

$$f(\mu)(a+b) \geqslant f(\mu)(a) = \min\{f(\mu)(a), f(\mu)(b)\}.$$

Similarly, since $f(x' \cdot y') = a \cdot b$, we have

$f(\mu(a \cdot b)) = \sup\{\mu(z) : f(z) = a \cdot b\} \geqslant \mu(x' \cdot y') \geqslant \min\{\mu(x'), \mu(y')\} \geqslant f(\mu)(a) - \epsilon$.

Hence, $f(\mu)(a \cdot b) \geqslant f(\mu)(a) = \min\{f(\mu)(a), f(\mu)(b)\}$.

Therefore, $f(\mu) \in FSUB_S(N)$. □

In Proposition 4.19(I), if $\mu \in FSUB_T(M)$, then by a similar argument we can show that $\mu \circ f \in FSUB_T(N)$. However, Proposition 4.19(II), is not true for $\mu \in FSUB_T(N)$. See the following example:

Example 4.20. Let $N = \langle(1,2)\rangle$, $M = \langle(2,3)\rangle$ and μ be as Example 3.11(ii). Let $f : N \longrightarrow M$ be such that $f(0) = 0$ and for every $a \in N$, $f(a) = (2)$. Let $b = (2)$ and $b' = (1)$. Since $b \cdot b' = b' \notin Imf$, we have $f(\mu)(b \cdot b') = 0 < \max\{f(\mu)(b), f(\mu)(b')\} = 1$. Then μ is a strong fuzzy sub-semi-ring of N, but $f(\mu)$ is not a strong fuzzy sub-semi-ring of M.

Let I be an ideal of N and $a \in N$. We put $a + I = \{a + b : b \in I\}$ and $\dfrac{N}{I} = \{a + I : a \in N\}$.

Lemma 4.21. *Let I be an ideal of N and $a, a' \in N$. Then*

(I) $a + I = I$ if and only if $a = ()$.
(II) $a + I = a' + I$ if and only if $stem\, a = stem\, a'$.

Proof. (I) Clearly, $() + I = I$. Let $a + I = I$. Then there exist $b \in I$ such that $a + b = ()$. Hence $a = b = ()$.

(II) Assume $a_1 = stem\, a = stem\, a' = a'_1$. Hence for every $b \in I$ with $stem\, b = b_1$, we have $a + b = (a_1 \vee b_1) = (a'_1 \vee b_1) = a' + b$. Therefore, $a + I = a' + I$.

Conversely, let $a + I = a' + I$. Then Since $() \in I$, we have $(a'_1) = a' + () \in a' + I = a + I$. Hence there exist $b \in I$, such that $(a'_1) = a + b = (a_1 \vee b_1)$, with $b_1 = stem\, b$. Then $a'_1 \geqslant a_1$. Similarly, $a_1 \geqslant a'_1$. Thus, $a_1 = a'_1$. Therefore, $stem\, a = stem\, a'$. □

Example 4.22. Consider a nexus:
$N = \{(), (1), (2), (3), (4), (1,1), (1,2), (1,1,1), (1,1,2), (1,1,3), (2), (2,1), (2,1,1), (2,1,2), (3,1), (3,2), (3,3), (4,1)\}$

with the following diagram.

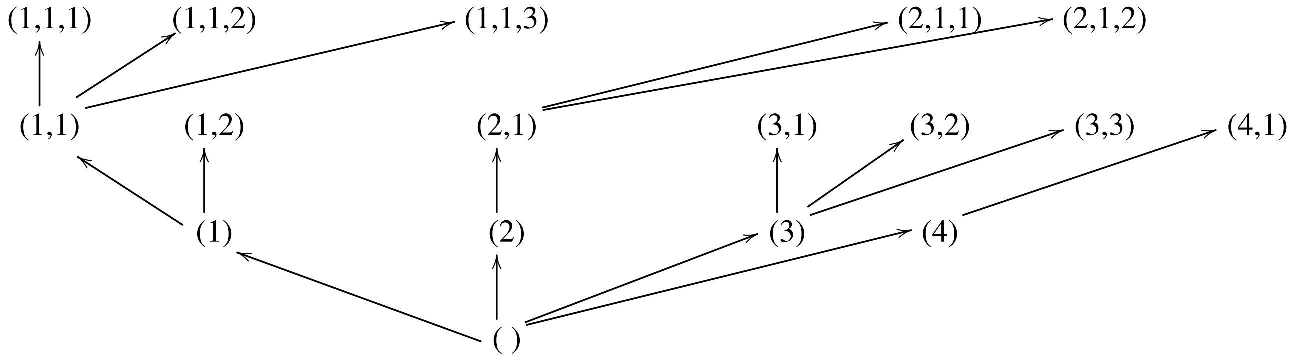

Figure 2. Diagram of N.

If $I := \{(), (1), (2), (1,1), (1,2), (2), (2,1)\}$, then I is an ideal of N. Using Lemma 4.21, we can see that

$() + I = (1,1) + I = (1,2) + I = (1,1,1) + I = (1,1,2) + I = (1,1,3) + I,$
$(2) + I = (2,1) + I = (2,1,1) + I = (2,1,2) + I,$
$(3) + I = (3,1) + I = (3,2) + I = (3,3) + I,$
$(4) + I = (4,1) + I.$
Then $\frac{N}{I} = \{I, (1) + I, (2) + I, (3) + I, (4) + I\}.$

Definition 4.23. For every $a + I, b + I \in \frac{N}{I}$, we define the binary operations " $*$ " and " $\circ$ " on $\frac{N}{I}$ with the following:

(I) $(a + I) * (b + I) = a + b + I,$
(II) $(a + I) \circ (b + I) = a \cdot b + I.$

Theorem 4.24. *Let I be an ideal of N. Then* $(\frac{N}{I}, *, \circ, I)$ *is a semi-ring.*

Proof. Let $a + I = a' + I$ and $b + I = b' + I$. By Lemma 4.21, we have $a_1 = stem\,a = stem\,a' = a'_1$ and $b_1 = stem\,b = stem\,b' = b'_1$. Hence, $a_1 \vee b_1 = a'_1 \vee b'_1$ and $a_1 \wedge b_1 = a'_1 \wedge b'_1$. This shows that $a + b + I = a' + b' + I$ and $a \cdot b + I = a' \cdot b' + I$. Therefore, the binary operations $*$ and $\circ$ are well defined. By routine calculation we can see that $(\frac{N}{I}, *, I)$ is a commutative semi-group, and we have $(() + I) * (a + I) = () + a + I = a + I$ and $(a + I) * (() + I) = a + () + I = a + I$. Thus, it is a commutative monoid. Furthermore, $(\frac{N}{I}, \circ)$ is a semi-group, in while the operation $\circ$ is distributive with respect to $*$. Therefore, $(\frac{N}{I}, *, \circ, I)$ is a semi-ring. □

Example 4.25. Let I is an ideal of N and $\frac{N}{I}$ is the quotient semi-ring. Let $f : N \longrightarrow \frac{N}{I}$ be such that for every $a \in N$, $f(a) = a + I$. Clearly f is a semi-ring epimorphism. Let μ be a fuzzy sub-semi-ring of N and $\nu : \frac{N}{I} \longrightarrow [0,1]$ be such that for every $a \in N$, $\nu(a + I) = sup\{\mu(b) : b + I = a + I\}$. Since $\nu = f(\mu)$, by Proposition 4.19(II), ν is a fuzzy sub-semi-ring of $\frac{N}{I}$.

Example 4.26. Consider a nexus $N = \{(), (1), (2), (3), (4), (1, 1), (1, 2), (1, 1, 1), (1, 1, 2), (1, 1, 3), (2), (2, 1), (2, 1, 1), (2, 1, 2), (3, 1), (3, 2), (3, 3), (4, 1)\}$ with the following diagram:

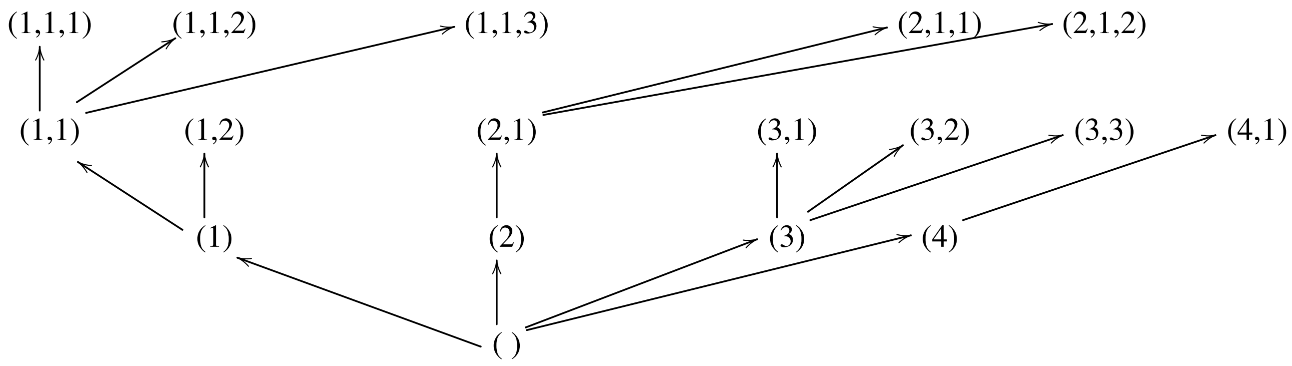

Figure 3. Diagram of N.

If $I = \{(), (1), (2), (1, 1), (1, 2), (2), (2, 1)\}$, by Lemma 4.21, we have $\frac{N}{I} = \{I, (1) + I, (2) + I, (3) + I, (4) + I\}$.

Now we define $\nu : \frac{N}{I} \longrightarrow [0, 1]$ such that $\nu(I) = 1$, $\nu((1) + I) = 1$, $\nu((2) + I) = \frac{1}{2}$, $\nu((3) + I) = \frac{1}{3}$, $\nu((4) + I) = \frac{1}{4}$.

Consider μ in Example 3.11(iii), and $f : N \longrightarrow \frac{N}{I}$ in Example 4.26. It is easy to see that $\nu = f(\mu)$ and by Example 4.26, $\nu \in FSUB_S(\frac{N}{I})$.

5. Fuzzy quotient of a nexus semi-ring

In this section, we verify some concepts of fuzzy quotient of a nexus semi-rings.

Definition 5.1. Let $a \in N$ and $\mu \in FSUB_S(N)$. We define the fuzzy $a + \mu : N \longrightarrow [0, 1]$ such that for every $x \in N$, $(a + \mu)(x) = \mu(a + x)$.

Proposition 5.2. *Suppose that $a + \mu$ is the fuzzy of Definition 5.1. Then $a + \mu \in FSUB_S(N)$.*

Proof. Let $x, x' \in N$, $stem\, a = a_1$, $stem\, x = x_1$ and $stem\, x' = x'_1$. We have

$(a+\mu)(x+x') = \mu(a+(x_1 \vee x'_1)) = \mu((a_1 \vee x_1) \vee (a_1 \vee x'_1)) = \mu((a+x)+(a+x')) \geqslant \min\{\mu(a+x), \mu(a+x')\} = \min\{(a + \mu)(x), (a + \mu)(x')\}$.

$(a + \mu)(x \cdot x') = \mu(a + (x \cdot x')) = \mu((a_1 \vee (x_1 \wedge x'_1))) = \mu((a_1 \vee x_1) \cdot (a_1 \vee x'_1)) = \mu((a + x) \cdot (a + x')) \geqslant \min\{\mu(a + x), \mu(a + x')\} = \min\{(a + \mu)(x), (a + \mu)(x')\}$

Therefore, $a + \mu \in FSUB_S(N)$. □

Definition 5.3. Let $a \in N$ and $\mu \in FSUB_S(N)$. We define $N_\mu = \{a + \mu : a \in N\}$. Also we define operations $\oplus$ and $\odot$ on N_μ such that for every $a, b \in N$, $(a+\mu) \oplus (b+\mu) = (a+b)+\mu$ and $(a+\mu) \odot (b+\mu) = (a \cdot b) + \mu$.

Proposition 5.4. *Let $\mu \in FSUB_S(N)$. Then, $(N_\mu, \oplus, \odot, 0+\mu)$ is a semi-ring.*

Proof. Suppose that $a, a', b, b' \in N$ and $a+\mu = a'+\mu$ and $b+\mu = b'+\mu$. Hence, for every $x \in N$, we have $\mu(a+x) = \mu(a'+x)$ and $\mu(b+x) = \mu(b'+x)$. Let $t \in N$. If $x = b+t$, then $\mu(a+b+t) = \mu(a'+b+t)$ and if $x+a'+t$, then $\mu(b+a'+t) = \mu(b'+a'+t)$. Now, for every $t \in N$, we have $\mu(a+b+t) = \mu(a'+b'+t)$ and so $((a+b)+\mu)(t) = ((a'+b')+\mu)(t)$. Hence $(a+b)+\mu = (a'+b')+\mu$.

Then the operation $\oplus$ is well-defined. Now we show that the operation $\odot$ is well defined. Let $stem\, a = a_1$, $stem\, b = b_1$, $stem\, a' = a'_1$, $stem\, b = b'_1$, $stem\, x = x_1$. We consider two cases: Case 1: Let $a_1 \geqslant b_1$ and $a'_1 \geqslant b'_1$. Then

$((a \cdot b)+\mu)(x) = \mu((a_1 \wedge b_1) \vee x_1) = \mu(b_1 \vee x_1) = \mu(b+x) = \mu(b'+x) = \mu(b'_1 \vee x_1) = \mu((a' \cdot b')+x) = (a' \cdot b' + \mu)(x)$.

Hence $(a \cdot b)+\mu = (a' \cdot b')+\mu$.

Case 2: Let $a_1 \geqslant b_1$ and $a'_1 \leqslant b'_1$. Then $((a \cdot b)+\mu)(x) = \mu((a \cdot b)+x) = \mu((a_1 \wedge b_1) \vee x_1) = \mu(b_1 \vee x_1) = \mu(b+x) = \mu(b'+x) = \mu((a \cdot b)+x) = \mu(a'+b'+x) = \mu(a+b+x) = \mu(a_1 \vee b_1 \vee x_1) = \mu(a+x) = \mu(a'+x) = \mu(a'_1 \vee x_1) = \mu((a'_1 \wedge b'_1) \vee x_1) = \mu((a' \cdot b')+x) = ((a' \cdot b')+\mu)(x)$.

Hence $(a \cdot b)+\mu = (a' \cdot b')+\mu$.

Then the operation $\odot$ is well-defined.

Also, for every $a, b, c \in N$, we have

$(c+\mu) \odot ((a+\mu) \oplus (b+\mu)) = (c+\mu) \odot ((a+b)+\mu)) = (c \cdot (a+b))+\mu = (c \cdot a + c \cdot b)+\mu = ((c \cdot a)+\mu) \oplus ((c \cdot b)+\mu)\oplus = ((c+\mu) \odot (a+\mu)) \oplus ((c+\mu) \odot (b+\mu))$.

Therefore, $(N_\mu, \oplus, \odot, 0+\mu)$ is a semi-ring. □

Example 5.5. Consider the semi-ring N of Example 4.26 and μ of Example 3.11(i). For every $a, b \in N$, $a+\mu = b+\mu$ if and only if, for every $x \in N$, $\mu(a+x) = \mu(b+x)$. If $x = 0$, then $\mu(a) = \mu(b)$. Let $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_m)$. So $a_1...a_n = b_1...b_m$. If $x = a$, we have $\frac{1}{a_1} = \mu((a_1)) = \mu(a+a) = \mu(a+b) = \mu((a_1 \vee b_1)) = \frac{1}{a_1 \vee b_1}$. Hence, $a_1 \geqslant b_1$. If $x = b$, similarly we have $b_1 \geqslant a_1$. Hence $a_1 = b_1$. Then $(1)+\mu = (1,1)+\mu = (1,1,1)+\mu$, $(1,2)+\mu = (1,1,2)+\mu$, $(2)+\mu = (2,1)+\mu = (2,1,1)+\mu$, $(3)+\mu = (3,1)+\mu$, $(4)+\mu = (4,1)+\mu$. Therefore, $N_\mu = \{\mu, (1)+\mu, (2)+\mu, (3)+\mu, (4)+\mu, (1,2)+\mu, (1,1,3)+\mu, (2,1,2)+\mu, (3,2)+\mu, (3,3)+\mu\}$.

Proposition 5.6. *Let $\mu, \nu \in FSUB_S(N)$ and $\nu \leqslant \mu$. Let for every $a, b \in N$, $\nu(a) = \nu(b)$ implies $\mu(a) = \mu(b)$. Then $\frac{\mu}{\nu} : N_\nu \longrightarrow [0,1]$ with $\frac{\mu}{\nu}(a+\nu) = \mu(a) \in FSUB_S(N_\nu)$.*

Proof. Let $a, b \in N$ and $a+\nu = b+\nu$. Then, for every $x \in N$, $(a+\nu)(x) = (b+\nu)(x)$ and so $\nu(a+x) = \nu(b+x)$. If $x = 0$, then $\nu(a) = \nu(b)$, and so $\mu(a) = \mu(b)$. Thus, $\frac{\mu}{\nu}$ is well-defined. Now, since $\mu \in FSUB_S(N)$, we have

$\frac{\mu}{\nu}((a+\nu) \odot (b+\nu)) = \frac{\mu}{\nu}((a \cdot b)+\nu) = \mu(a \cdot b) \geqslant \min\{\mu(a), \mu(b)\} = \min\{\frac{\mu}{\nu}(a+\nu), \frac{\mu}{\nu}(b+\nu)\}$. Similarly, $\frac{\mu}{\nu}((a+\nu) \oplus (b+\nu)) \geqslant \min\{\frac{\mu}{\nu}(a+\nu), \frac{\mu}{\nu}(b+\nu)\}$. Therefore, $\frac{\mu}{\nu} \in FSUB_S(N_\nu)$. □

Proposition 5.7. *Let $\mu, \nu \in FSUB_S(N)$ and $\nu \leqslant \mu$. Let for every $a, b \in N$, $\nu(a) = \nu(b)$ implies $\mu(a) = \mu(b)$. Then, the semi-rings N_μ and $(N_\nu)_{\frac{\mu}{\nu}}$ are isomorphic.*

Proof. Let $f : (N_\nu)_{\frac{\mu}{\nu}} \longrightarrow N_\mu$ be such that for every $a \in N$, $f((a+\nu)+\frac{\mu}{\nu}) = a+\mu$. Let $a, b \in N$ and $(a+\nu)+\frac{\mu}{\nu} = (b+\nu)+\frac{\mu}{\nu}$. If $x \in N$, $((a+\nu)+\frac{\mu}{\nu})(x+\nu) = ((b+\nu)+\frac{\mu}{\nu})(x+\nu)$,, and so $\frac{\mu}{\nu}(a+x+\nu) = \frac{\mu}{\nu}(b+x+\nu)$. Thus, $\mu(a+x) = \mu(b+x)$. Then $a+\mu = b+\mu$, and so f is well-defined. Now, we have

$f(((a+\nu)+\frac{\mu}{\nu}) \oplus ((b+\nu)+\frac{\mu}{\nu}))) = f((a+b+\nu)+\frac{\mu}{\nu})) = (a+b)+\mu = (a+\mu) \oplus (b+\mu)$,

$f(((a+\nu)+\frac{\mu}{\nu}) \odot ((b+\nu)+\frac{\mu}{\nu}))) = f((((a \cdot b)+\nu)+\frac{\mu}{\nu})) = (a \cdot b)+\mu = (a+\mu) \odot (b+\mu)$. Hence, f is semi-ring homomorphism. Clearly, f is onto. Now, let $f((a+\nu)+\frac{\mu}{\nu}) = f((b+\nu)+\frac{\mu}{\nu})$. This shows that $a+\mu = b+\mu$. Then, for every $x \in N$, $\mu(a+x) = \mu(b+x)$. Thus, $((a+\nu)+\frac{\mu}{\nu})(x+\nu) = ((b+\nu)+\frac{\mu}{\nu})(x+\nu)$. Consequently, $(a+\nu)+\frac{\mu}{\nu} = (b+\nu)+\frac{\mu}{\nu}$. Thus, f is one-to-one. Therefore, f is a semi-ring isomorphism and the semi-rings N_μ and $(N_\nu)_{\frac{\mu}{\nu}}$ are isomorphic. □

6. Conclusions and future work

We define a semi-ring on nexus N and fuzzy sub-semi-ring of this semi-ring. The set of all fuzzy sub-semi-rings of N is denoted by $FSUB_S(N)$. In Proposition 3.12, we show that A is a sub-semi-ring of N if and only if χ_A is a fuzzy sub-semi-ring of N. In Proposition 3.14, we show that μ is a fuzzy sub-semi-ring of N if and only if, for every $\alpha \in [0, \mu(0)]$, μ^α is a sub-semi-ring of N. Since the Propositions 3.12 and 3.14 are not true for ideals and since ideals in semi-rings are more important than sub-semi-rings, we put a condition on a fuzzy sub-semi-ring such that Propositions 3.12 and 3.14 are true for ideals. We define this fuzzy and named it by a strong fuzzy sub-semi-ring. The set of all strong fuzzy sub-semi-rings of N, is denoted by $FSUB_T(N)$. In the following, for a semi-ring homomorphism $f : N \longrightarrow M$, we show that if $\mu \in FSUB_S(N)$ then $f(\mu) \in FSUB_S(M)$ and if $\mu \in FSUB_S(M)$ then $f \circ \mu \in FSUB_S(N)$. Finally, we verify some concepts of fuzzy quotient of a nexus semi-ring. In future, for semi-ring related to nexus N, we define polynomial semi-ring $N[x] = \{a_0 + a_1x + \cdots + a_nx^n : n \in \mathbb{N} \cup \{0\}, a_i \in N(1 \leqslant i \leqslant n)\}$. Then, we characterize all prime ideals and prime elements of N.

Author contributions

Vajiheh Nazemi Niya: Conceptualization, Methodology, Investigation, Writing original draft preparation; Hojat Babaei: Conceptualization, Methodology, Investigation; Akbar Rezaei: Conceptualization, Methodology, Investigation, Writing review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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