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Research article

Dynamical and computational analysis of a fractional predator-prey model with an infectious disease and harvesting policy

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Abstract: This paper examined the features of an infection therapy for fractional-order quarry-hunter systems in order to control sickness. It focused especially on how illnesses and several populations combine to affect how well harvesting policies work. We created a new dynamic model full of such ideas by examining systems with fractional-order non-integer systems and introducing fractional-order systems that can remember in order to comprehend that specific system. These thresholds are essential for directing management strategies, according to research on the presence, uniqueness, and stability of solutions to these models. Additionally, we presented particular MATLAB-based numerical methods for fractional-order model. Through a series of numerical application experiments, we validated the method's efficacy and its value in guiding strategy modifications regarding harvesting rates in the face of epidemic infections. This demonstrates the necessity of using a fractional approach in ecosystem research in order to improve the methods used for resource management. This paper primarily focused on the unique insight brought into the quarry-hunter models with infectious diseases by the fractional-order dynamics in ecology. The results are meaningful especially since they can be utilized to come up with effective measures to control diseases and even promote the sustainability of ecological systems.

Keywords: fractional-order quarry-hunter models; specialized MATLAB routines; harvesting policies; disease dynamics
Mathematics Subject Classification: 34A08, 34D20, 37N25

1. Introduction

The dynamics between hunters and their quarry are widely studied in the literature. The Lotka-Volterra hunter-quarry system [1, 2] was initially presented to elucidate the demographic shifts of two species hunters and their quarry, as they interact. Murray [3] updated the Lokta-Volterra model, and the framework was based on the idea that, with a lack of hunters, the quarry population grows logistically. Kermack and McKendrick [4] proposed a traditional SIR model which is the ancestor of the most significant models for the spread of infectious diseases. Numerous epidemic models that involve disease in quarry have been thoroughly explored in a variety of formats and settings over the past few decades such as Hethcote et al. [5], Johri et al. [6], Rahman et al. [7], Nandi et al. [8], Mbava et al. [9] and Yang [10]. In the early nineteenth century, an eco-epidemiological system comprising three species emerged: healthy quarry (susceptible), infected quarry (infectious), and hunters, described by Chattopadhyay and Arino [11] with the situation where the hunter mostly consumes diseased quarry. Another scenario was also proposed by Das and Chattopadhyay [12] in which infectious disease affects both species.The parasite is transmitted within both the quarry and hunter populations through both direct and indirect means. In the other model by Nandi et al. [8], it is presumed that the victim species is affected by a viral ailment, leading to the emergence of two categories: vulnerable individuals and those in the early stages of infection, who are particularly vulnerable to predation by hunters.

In recent years, a computational structure to investigate how a disease affects quarry species populations and harvesting in a hunter-hunted model has been examined by Dash et al. [13]. An eco-epidemiological system consisting of two separate infectious diseases in the quarry population was also proposed by Al-Jubouri et al. [14]. Tripathi et al. [15] looked at a four-species model to evaluate the impact of harvesting and the Allee effect inside a sick eco-epidemiological system. The vulnerable quarry, sick quarry, vulnerable hunter, and sick hunter are the four compartments in this paradigm.

Fractional aspects of various hunter-hunted models have been proposed by many researchers such as Javidi and Nyamoradi [16], Al-Nassir et al. [17], Moustafa et al. [18], and Kaviya and Muthukumar [19]. Xie et al. [20] introduced a non-integer order model for the hunter-hunted relationship, incorporating a Holling III-type functional response and an intermittent harvesting component. Djilali and Ghanbari [21] studied a non-integer order hunter-hunted system with sick quarry in which a non-fatal contagious disease has emerged in the quarry population. The presence, singularity, positivity, and limit of solutions are discussed by Ramesh et al. [22] in their research paper. Alidousti and Ghafari [23] considered a non-integer order hunter-hunted model involving a quarry species, two hunter species, and a collective defense capability. He integrated the interplays between quarry and hunter species by utilizing the Monod-Haldane function along with a Holling-IV functional reaction.

The foundational literature for fractional calculus is found in the books by Miller and Ross [24], Oldham and Spanier [25], and Podlubny et al. [26]. The propensity of fractional-order systems to approach conventional integer-order systems is among the factors contributing to the growing attention toward fractional differential equations (FDEs). However, the nonlocal quality of fractional derivative operators is what appeals to people the most. Numerous approximate mathematical methods have been devised in recent years to figure out the solutions of FDEs, such as the Adomian decomposition method (ADM) by Wazwaz [27] and Abbasband [28], homotopy analysis method (HAM) by Liao [29], generalized differential transform method (GDTM) by Momani et al. [30], homotopy perturbation method (HPM) by He [31] and Yildirim [32], variational iteration technique by He [33], modified Laplace decomposition by Khan [34], homotopy perturbation transform method (HPTM) by Khan and Wu [35], homotopy sumudu perturbation transform technique by Hao et al. [36], and q-homotopy analysis transform method (q-HATM) by Kumar et al. [37].

The conventional frameworks for describing the interactions between predators and prey, which include the Lotka-Volterra model, have been sufficient for the analysis of such interactions within species populations over time. On the contrary, these models, especially with regard to pest management and harvesting regimes, seem unable to deal with memory, non-locality, and other factors that are characteristic of real-life settings. Therefore, models based on fractional calculus have become an essential option, as they help in explaining complex phenomena in systems where certain activities vary with time such as diseases and human operations like fishing. For instance, Al-Nassir et al. [17] examined the fractional-order biological systems under harvest and their dynamic performance toward the harvest showing the scope of fractional calculus in ecology models. Similarly, Xie et al. [20] were concerned with the dynamics of the harvested fractional-order predator-prey systems and also the two species prey-predator models without harvest and showed the practical side of the model in real life. In a similar way, Moustafa et al. [18] worked on eco-epidemiological approaches and provided examples where diseases and mortality were integrated into predator-prey dynamics.

This study builds on these efforts to propose the implementation of fractional-order models in analyzing how disease dynamics and harvesting affect predator-prey interactions, thus offering new perspectives regarding population management and ecosystem stability. In general, our model contains several important details: This class of model implies that in the absence of predation the susceptible-sized prey population increases logistically, but this growth is curtailed due to the spread of a disease. Infected prey do not die or breed but assist the disease by infecting healthy prey. In this study, a collection of robust MATLAB scripts, specially designed for addressing three categories of fractional-order problems, is introduced. These scripts are put forth by Garrappa [38], who has authored a concise guide on numerically solving FDEs and has delved into various intricate issues associated with the effective utilization of techniques, including handling the persistent memory term and tackling equations through implicit methods, among others.

The fundamental organization of this paper is as follows: In Section 2, we delve into foundational concepts. The fractional problem model is examined within Section 3. Section 4 encompasses the demonstration of the analysis concerning existence, uniqueness, and stability. Section 5 contains instances of development and the application of numerical solutions in model simulations. Finally, in Section 6, we present concluding remarks.

2. Definition and preliminaries

Let us commence with an exposition of the Caputo derivative with fractional order ρ and the Riemann-Liouville (RL) integral, respectively.

$$
\begin{aligned}
{}_0^C D_\theta^\rho g(\theta) &= \frac{1}{\Gamma(k-\rho)} \int_0^\theta (\theta-\varrho)^{k-\rho-1}\, g^k(\varrho)d\varrho, \quad k-1<\rho\le k,\ k\in z^+, \\
{}_0^C I_\theta^\rho g(\theta) &= \frac{1}{\Gamma(\rho)} \int_0^\theta (\theta-\varrho)^{\rho-1}\, g(\varrho)d\varrho, \quad 0<\rho\le 1.
\end{aligned}
\tag{2.1}
$$

Scientists and engineers have made extensive use of the Caputo operator to model memory-dependent behavior in the mathematical formulation such as Singh et al. [39], Baleanu and Agrawal [40], Shah et al. [41], Moa'ath et al. [42], and Mozyrska and Torres [43]. We are spurred on to employ this innovative strategy of non-negative calculus owing to the outstanding usability and effectiveness of the Caputo fractional operator in the realms of science and technology.

Theorem 2.1. In the fractional-order system [44,45]

$$ {}_0^C D_\theta^\rho g(\theta) = g(\theta, X(\theta)), \qquad X(\theta_0) = X_0, \tag{2.2} $$

where $J(X^*)$ stands for the system's derivative matrix of Eq (2.2) determined at steady state point X^*:

(1) The steady state point X^* is locally asymptotically stable if and only if all the eigenvalues λ_i, $i = 1, 2, ..., n$, of $J(X^*)$ satisfy $|\arg(\lambda_i)| > \frac{\rho\pi}{2}$.
(2) The equilibrium point X^* is stable if all the eigenvalues λ_i, $i = 1, 2, ..., n$, of $J(X^*)$ satisfy $|\arg(\lambda_i)| \geq \frac{\rho\pi}{2}$ and eigenvalues with $|\arg(\lambda_i)| = \frac{\rho\pi}{2}$ have the same geometric and algebraic multiplicity.
(3) The equilibrium point X^* is unstable if and only if there exist eigenvalues λ_i for some $i = 1, 2, ..., n$ of $J(X^*)$ that satisfy $|\arg(\lambda_i)| < \frac{\rho\pi}{2}$.

3. Mathematical formulation of the non-integer-order quarry-hunter model

Since vaccination strategies have proven ineffective, Soni and Chouhan [46] delved into the complex consequences of the hunter-hunted model within the epidemiological system. The model's distinctive characteristics are defined by

$$ \begin{aligned} \frac{dx(\theta)}{d\theta} &= -px(\theta) + qx(\theta)y(\theta) - rx(\theta)z(\theta)(1-\phi), \\ \frac{dy(\theta)}{d\theta} &= hy(\theta) - ux(\theta)y(\theta) - vy(\theta)z(\theta)(1-\phi), \\ \frac{dz(\theta)}{d\theta} &= hz(\theta) - ux(\theta)z(\theta) + vy(\theta)z(\theta)(1-\phi) - wz(\theta), \end{aligned} \tag{3.1} $$

where x, y, and z symbolize the quantities of vulnerable hunters, vulnerable quarry, and contaminated quarry, respectively. The other parameters are defined in Table 1.

Table 1. Parameter descriptions.

Parameter	Description
p	Amount of healthy, vulnerable hunters dying naturally
q	Amount of interactions among robust, vulnerable hunters and vulnerable quarry
r	Amount of interactions among an infected quarry and a robust, vulnerable hunter
u	Amount of interactions among robust, vulnerable hunters with infected quarry and vulnerable quarry
v	Amount of encounters among diseased and robust susceptible quarry
w	Extraction (harvesting) rate of afflicted quarry
h	Individual offspring rate of vulnerable quarry (per unit of time) and contaminated quarry
ϕ	Fraction of newborns who received vaccinations in succession

Equation (3.1) in our study represents a traditional model in the eco-epidemiological literature, encapsulating the dynamics between predators and their prey, which may include infectious diseases affecting one or both species. These models stem from the foundational work by Lotka and Volterra in the early 20th century, who pioneered the use of differential equations to describe biological interactions. Subsequent developments by Kermack and McKendrick introduced disease dynamics into these models, paving the way for a rich body of research that integrates epidemiological aspects into predator-prey interactions.

While classical models have provided significant insights, they inherently assume that responses within the system are instantaneous and often ignore the historical or memory effects that are crucial in natural populations. Typically, these models use integer-order derivatives which assume a local interaction within the system. Such assumptions simplify the complex interactions and can lead to inaccuracies when predicting the behavior of more complex ecological systems. Additionally, traditional models often fail to capture the non-linear and anomalous behaviors that are characteristic of many ecological processes, thereby limiting their application in predicting realistic ecological scenarios.

The model comprised of fundamental presumptions that we established when creating it, including:

- The proportional birth rate of vulnerable quarry and contaminated quarry remains consistent.
- Hunters eventually die as a result of the disease's tremendous weakness.
- When a hunter becomes infected, its demise may be predicted. Hence, we will only take into account susceptible hunters, the fact that infectious illness spreads through contact among the population of quarry, and the rate of infection matches the combined count of infected and vulnerable quarry.
- The hunter does not differentiate between vulnerable and ailing members of the quarry community.
- Through the consumption of sick quarry, the hunter contracts the illness.
- The pace of hunter contagion correlates with the multiplication of infected quarry and vulnerable hunters.
- Quarry that has been infected cannot recover.

The aim of this work is a presentation of Eq (3.2), which takes on the concept of fractional calculus and incorporates it into the eco-epidemiological model of the predator-prey interaction. The Caputo derivative and the Riemann-Liouville integral are most preferable thanks to their respected attributes and the efficiency with which they represent physical systems. The advantage of the Caputo derivative is that it is applicable in the context of initial value problems because one can readily impose initial conditions in the form of standard derivatives rather than complicated definitions. Thus, the Caputo derivative is practical and effective in real applications. Other types of derivatives such as the conformable and the beta-fractional derivatives have their own merits but the Caputo derivative seems to be more straightforward. Bearing this in mind, let us speak about the Caputo derivative elevation which helps make researchers' claims more relevant and allows us to draw necessary conclusions based on the available research, which is why this concept is very important for studies related to fractional-order dynamics.

Finnaly, incorporating fractional calculus allows us to minimize the shortcomings of most classic approaches that leave wide gaps into our eco-epidemiological model. It is this innovation that shows a turn from the ordinary methods of modeling. It aids in the understanding of the spread of diseases, stability of populations, and management of resources in the ecosystem.

Thus, we would like to now go into the fractional part of the model using the Caputo derivative:

$$\begin{aligned} {}_0^C\mathcal{D}_\theta^\rho x(\theta) &= -px(\theta) + qx(\theta)r(\theta) - rx(\theta)z(\theta)(1-\phi), \\ {}_0^C\mathcal{D}_\theta^\rho y(\theta) &= hy(\theta) - ux(\theta)y(\theta) - vy(\theta)z(\theta)(1-\phi), \\ {}_0^C\mathcal{D}_\theta^\rho z(\theta) &= hz(\theta) - ux(\theta)z(\theta) + vy(\theta)z(\theta)(1-\phi) - wz(\theta). \end{aligned} \tag{3.2}$$

4. Existence, uniqueness and stability analysis

4.1. Existence and uniqueness analysis

Using fixed-point theory, we will investigate whether there exists a distinctive solution to the mathematical model we are studying. Applying the Caputo fractional integral operator on either side of the equation yields

$$\begin{aligned} x(\theta) &= x(0) + \frac{1}{\Gamma(\rho)}\int_0^\theta (\theta-\omega)^{\rho-1}(-px(\theta) + qx(\theta)y(\theta) - rx(\theta)z(\theta)(1-\phi))d\omega, \\ y(\theta) &= y(0) + \frac{1}{\Gamma(\rho)}\int_0^\theta (\theta-\omega)^{\rho-1}(hy(\theta) - ux(\theta)y(\theta) - vy(\theta)z(\theta)(1-\phi))d\omega, \\ z(\theta) &= z(0) + \frac{1}{\Gamma(\rho)}\int_0^\theta (\theta-\omega)^{\rho-1}(hz(\theta) - ux(\theta)z(\theta) + vy(\theta)z(\theta)(1-\phi) - wz(\theta)))d\omega. \end{aligned} \tag{4.1}$$

To make this simpler, we set the above equations to

$$\begin{aligned} \mathcal{K}_1(\theta, x) &= -px(\theta) + qx(\theta)y(\theta) - rx(\theta)z(\theta)(1-\phi), \\ \mathcal{K}_2(\theta, y) &= hy(\theta) - ux(\theta)y(\theta) - vy(\theta)z(\theta)(1-\phi), \\ \mathcal{K}_3(\theta, z) &= hz(\theta) - ux(\theta)z(\theta) + vy(\theta)z(\theta)(1-\phi) - wz(\theta). \end{aligned} \tag{4.2}$$

Initially, we demonstrate that the kernels $\mathcal{K}_1(\theta, x)$, $\mathcal{K}_2(\theta, y)$, and $\mathcal{K}_3(\theta, z)$ adhere to the Lipschitz criterion.

Theorem 4.1. Kernels $\mathcal{K}_1(\theta, x), \mathcal{K}_2(\theta, y)$, and $\mathcal{K}_3(\theta, z)$ adhere to the Lipschitz criterion furthermore contractions if $0 \le \eta_1, \eta_2, \eta_3 < 1$, where $\eta_1 = p + q\lambda + r\mu(1-\phi)$, $\eta_2 = h + u\alpha + v\mu(1-\phi)$, $\eta_3 = h + u\alpha + v\lambda(1-\phi) + w, \|x\| \le \alpha, \|y\| \le \lambda$, and $\|z\| \le \mu$.

Proof. Let us get started with demonstrating that kernels $\mathcal{K}_1(\theta, x)$ and $\mathcal{K}_2(\theta, x)$ meet the Lipschitz requirement. We hypothesize that x and x^* are two controlled functions in order to illustrate it. The result listed below is easily attainable.

$$\begin{aligned} \|\mathcal{K}_1(\theta, x) - \mathcal{K}_1(\theta, x^*)\| &= \|\{-px(\theta) + qx(\theta)y(\theta) - rx(\theta)z(\theta)(1-\phi)\} \\ &\quad - \{-px^*(\theta) + qx^*(\theta)y(\theta) - rx^*(\theta)z(\theta)(1-\phi)\}\| \\ &= \| - p(x(\theta) - x^*(\theta)) + qy(\theta)(x(\theta) - x^*(\theta)) - rz(\theta)(1-\phi)(x(\theta) - x^*(\theta))\| \\ &\le (p\|x(\theta) - x^*(\theta)\| + qy(\theta)\|x(\theta) - x^*(\theta)\| + rz(\theta)(1-\phi)\|x(\theta) - x^*(\theta)\|). \end{aligned} \tag{4.3}$$

Employing the norm property in the above equation, we have

$$\|\mathcal{K}_1(\theta, x) - \mathcal{K}_1(\theta, x^*)\| \le \{p + q\lambda + r\mu(1-\phi)\}\|x(\theta) - x^*(\theta)\|. \tag{4.4}$$

Taking $(p + q\lambda + r\mu(1 - \phi)) = \eta_1$, we have

$$\|\mathcal{K}_1(\theta, x) - \mathcal{K}_1(\theta, x^*)\| \leq \eta_1\|x(\theta) - x^*(\theta)\|. \tag{4.5}$$

As a result, kernal $\mathcal{K}_1$ meets the Lipschitz criterion. Additionally, if $0 \leq \eta_1 < 1$, it is a contraction. Similar results can be obtained as shown below:

$$\|\mathcal{K}_2(\theta, y) - \mathcal{K}_2(\theta, y^*)\| \leq \eta_2\|y(\theta) - y^*(\theta)\|, \tag{4.6}$$

$$\|\mathcal{K}_3(\theta, z) - \mathcal{K}_3(\theta, z^*)\| \leq \eta_3\|z(\theta) - z^*(\theta)\|. \tag{4.7}$$

It demonstrates that $\mathcal{K}_2$ and $\mathcal{K}_3$ meet the Lipschitz condition, and if $0 \leq \eta_2 < 1$ and $0 \leq \eta_3 < 1$, then they likewise experience a contraction.

Using $\frac{1}{\Gamma(\rho)} = \sigma(\rho)$ and the notions from Eq (4.2), Eq (4.1) is conveyed as

$$x(\theta) = x(0) + \sigma(\rho)\int_0^\theta (\theta - \omega)^{\rho-1}\mathcal{K}_1(\omega, x)d\omega, \tag{4.8}$$

$$y(\theta) = y(0) + \sigma(\rho)\int_0^\theta (\theta - \omega)^{\rho-1}\mathcal{K}_2(\omega, y)d\omega, \tag{4.9}$$

$$z(\theta) = z(0) + \sigma(\rho)\int_0^\theta (\theta - \omega)^{\rho-1}\mathcal{K}_3(\omega, z)d\omega. \tag{4.10}$$

The recursive formula of the above equation is

$$\begin{aligned} x_n(\theta) &= \sigma(\rho)\int_0^\theta (\theta - \omega)^{\rho-1}\mathcal{K}_1(\omega, x_{n-1})d\omega, \\ y_n(\theta) &= \sigma(\rho)\int_0^\theta (\theta - \omega)^{\rho-1}\mathcal{K}_2(\omega, y_{n-1})d\omega, \\ z_n(\theta) &= \sigma(\rho)\int_0^\theta (\theta - \omega)^{\rho-1}\mathcal{K}_3(\omega, z_{n-1})d\omega. \end{aligned} \tag{4.11}$$

The initial conditions are listed below:

$$x_0(\theta) = x(0), \qquad y_0(\theta) = y(0), \qquad z_0(\theta) = z(0). \tag{4.12}$$

The difference formula is then presented as follows:

$$\mathcal{P}_n(\theta) = x_n(\theta) - x_{n-1}(\theta), \quad \mathcal{Q}_n(\theta) = y_n(\theta) - y_{n-1}(\theta), \quad \mathcal{R}_n(\theta) = z_n(\theta) - z_{n-1}(\theta), \tag{4.13}$$

$$\begin{aligned} \mathcal{P}_n(\theta) &= \sigma(\rho)\int_0^\theta (\theta - \omega)^{\rho-1}(\mathcal{K}_1(\omega, x_{n-1}) - \mathcal{K}_1(\omega, x_{n-2}))d\omega, \\ \mathcal{Q}_n(\theta) &= \sigma(\rho)\int_0^\theta (\theta - \omega)^{\rho-1}(\mathcal{K}_2(\omega, y_{n-1}) - \mathcal{K}_2(\omega, y_{n-2}))d\omega, \\ \mathcal{R}_n(\theta) &= \sigma(\rho)\int_0^\theta (\theta - \omega)^{\rho-1}(\mathcal{K}_3(\omega, z_{n-1}) - \mathcal{K}_3(\omega, z_{n-2}))d\omega. \end{aligned} \tag{4.14}$$

The conclusion that can be made is that

$$x_n(\theta) = \sum_{i=0}^{n} \mathcal{P}_i(\theta), \qquad y_n(\theta) = \sum_{i=0}^{n} \mathcal{Q}_i(\theta), \qquad z_n(\theta) = \sum_{i=0}^{n} \mathcal{R}_i(\theta). \tag{4.15}$$

The following premises are made:

$$x_{-1}(0) = x(0), \qquad y_{-1}(0) = y(0), \qquad z_{-1}(0) = z(0). \tag{4.16}$$

We have an easy deduction of the following conclusions:

$$\begin{aligned} \|\mathcal{P}_n(\theta)\| &\leq \sigma(\rho)\eta_1 \int_0^\theta (\theta-\omega)^{\rho-1} \|\mathcal{P}_{n-1}(\theta)\| d\omega, \\ \|\mathcal{Q}_n(\theta)\| &\leq \sigma(\rho)\eta_2 \int_0^\theta (\theta-\omega)^{\rho-1} \|\mathcal{Q}_{n-1}(\theta)\| d\omega, \\ \|\mathcal{R}_n(\theta)\| &\leq \sigma(\rho)\eta_3 \int_0^\theta (\theta-\omega)^{\rho-1} \|\mathcal{R}_{n-1}(t)\| d\omega. \end{aligned} \tag{4.17}$$

Now, using Eq (4.17), we show that the fractional model's solution actually exists.

Theorem 4.2. A solution exists for the mathematical system with a fractional order given by (3.2) if for θ_0, we have

$$\left(\sigma(\rho)\eta_1 \frac{\theta^\rho}{\rho}\right) < 1.$$

Proof. We assume that the functions $x(\theta)$ and $y(\theta)$ are bounded. The following findings can be derived by applying the recursive technique in conjunction with Eq (4.17):

$$\begin{aligned} \|\mathcal{P}_n(\theta)\| &\leq \|x(0)\| \left(\sigma(\rho)\eta_1 \frac{\theta^\rho}{\rho}\right)^n, \\ \|\mathcal{Q}_n(\theta)\| &\leq \|y(0)\| \left(\sigma(\rho)\eta_2 \frac{\theta^\rho}{\rho}\right)^n, \\ \|\mathcal{R}_n(\theta)\| &\leq \|z(0)\| \left(\sigma(\rho)\eta_3 \frac{\theta^\rho}{\rho}\right)^n. \end{aligned} \tag{4.18}$$

As a result, all of the three functions described in Eq (4.14) exist and are continuous. The expression represented in Eq (4.11) must then be confirmed to be the fractional model's solution.We have

$$\begin{aligned} x(\theta) - x(0) &= x_n(\theta) - \alpha_n(\theta), \\ y(\theta) - y(0) &= y_n(\theta) - \beta_n(\theta), \\ z(\theta) - z(0) &= z_n(\theta) - \gamma_n(\theta), \end{aligned} \tag{4.19}$$

and one obtains

$$\|\alpha_n(\theta)\| \leq \left(\frac{\sigma(\rho)\theta^\rho}{\rho}\right)^{n+1} \eta_1^{n+1} \alpha. \tag{4.20}$$

Take $\theta = \theta_0$, and we have

$$\|\alpha_n(\theta)\| \leq \left(\frac{\sigma(\rho)\theta_0^{\rho}}{\rho}\right)^{n+1} \eta_1^{n+1}\alpha. \tag{4.21}$$

Taking the limit $n \to \infty$, Eq (4.21) results in

$$\|\alpha_n(\theta)\| \to 0. \tag{4.22}$$

Following the same kind of process, we see that

$$\|\beta_n(\theta)\| \to 0, \qquad \|\gamma_n(\theta)\| \to 0. \tag{4.23}$$

This demonstrates that the fractional model under study given by (3.2) has a solution.

Theorem 4.3. There is just one solution to the fractional-order mathematical model given in Eq (3.2) if

$$\left(1 - \frac{\sigma(\rho)\theta^{\rho}\eta_1}{\rho}\right) > 0. \tag{4.24}$$

Proof. We assume the presence of additional solutions $x^*(\theta)$ and $y^*(\theta)$ to Eq (3.2) to assess the singularity of the fractional model outcome.

Clearly, it becomes evident that

$$x(\theta) - x^*(\theta) = \sigma(\rho)\int_0^{\theta} (\theta - \omega)^{\rho-1}(\mathcal{K}_1(\omega, x) - \mathcal{K}_1(\omega, x^*))d\omega. \tag{4.25}$$

Using the norm-using characteristic of Eq (4.25), we possess

$$\|x(\theta) - x^*(\theta)\|\left(1 - \frac{\sigma(\rho)\theta^{\rho}\eta_1}{\rho}\right) \leq 0. \tag{4.26}$$

If the conclusion in Eq (4.24) is accurate, then we may deduce from Eq (4.28) that

$$\|x(\theta) - x^*(\theta)\| = 0 \Longrightarrow x(\theta) = x^*(\theta). \tag{4.27}$$

Using an equivalent process, we have

$$y(\theta) = y^*(\theta), \qquad z(\theta) = z^*(\theta). \tag{4.28}$$

As a result, a mathematical model of Eq (3.2) of arbitrary order has a unique solution.

4.2. Equilibria and stability analysis

4.2.1. Local stability analysis

To evaluate the steady-state points of the system of Eq (3.2), we assume

$${}_0^C\mathcal{D}_\theta^\rho x(\theta) = 0, \quad {}_0^C\mathcal{D}_\theta^\rho y(\theta) = 0, \quad {}_0^C\mathcal{D}_\theta^\rho z(\theta) = 0. \tag{4.29}$$

That is

$$
\begin{aligned}
-px(\theta) + qx(\theta)y(\theta) - rx(\theta)z(\theta)(1-\phi) &= 0,\\
hy(\theta) - ux(\theta)y(\theta) - vy(\theta)z(\theta)(1-\phi) &= 0,\\
hz(\theta) - ux(\theta)z(\theta) + vy(\theta)z(\theta)(1-\phi) - wz(\theta) &= 0.
\end{aligned} \tag{4.30}
$$

The steady-state points are

$$
\begin{aligned}
\mathcal{E}_0 &= (x_0, y_0, z_0) = (0,0,0),\\
\mathcal{E}_1 &= (x_1, y_1, z_1) = \left(\frac{h-w}{u}, 0, \frac{-p}{r(1-\phi)}\right),\\
\mathcal{E}_2 &= (x_2, y_2, z_2) = \left(0, \frac{w-h}{v(1-\phi)}, \frac{h}{v(1-\phi)}\right),\\
\mathcal{E}_3 &= (x_3, y_3, z_3) = \left(\frac{h}{u}, \frac{p}{q}, 0\right),\\
\mathcal{E}_4 &= (x_4, y_4, z_4) = \left(\frac{h+vy_4-w}{u}, \frac{vp+rw}{vq+rv(1-\phi)}, \frac{-p+qy_4}{r(1-\phi)}\right).
\end{aligned} \tag{4.31}
$$

Theorem 4.4. The equilibrium point $\mathcal{E}_0$ of the system of Eq (3.2) is unstable.
Proof. Given below is the Jacobian matrix for the system of equations investigated at steady-state point $\mathcal{E}_0$.

$$
\mathcal{J}(\mathcal{E}_0) = \begin{pmatrix} -p & 0 & 0\\ 0 & h & 0\\ 0 & 0 & h-w \end{pmatrix}.
$$

Hence, the eigenvalues of $J(\mathcal{E}_0)$ are $\lambda_1 = -p$, $\lambda_2 = h$, and $\lambda_3 = h-w$. Since λ_1 is negative, then $|\arg(\lambda_1)| = \pi > \frac{\rho\pi}{2}$ and $|\arg(\lambda_2)| = |\arg(\lambda_3)| = 0 < \frac{\rho\pi}{2}$, and according to Theorem 2.1, $\mathcal{E}_0$ is unstable.

Theorem 4.5. The steady-state point $\mathcal{E}_1$ of the system of Eq (3.2) is unstable.
Proof. Given below is the Jacobian matrix for the system of equations investigated at steady-state point $\mathcal{E}_1$.

$$
\mathcal{J}(\mathcal{E}_1) = \begin{pmatrix} 0 & \frac{q(h-w)}{u} & \frac{-r(1-\phi)(h-w)}{u}\\ 0 & w+\frac{vp}{r} & 0\\ \frac{pu}{r(1-\phi)} & \frac{-vp}{r} & 0 \end{pmatrix}.
$$

Hence, the eigenvalues of $\mathcal{J}(\mathcal{E}_1)$ are $\lambda_1 = \frac{nr+mp}{r}$, $\lambda_2 = \sqrt{p(w-h)}$, and $\lambda_3 = -\sqrt{p(w-h)}$. Since λ_1 is positive, then $|\arg(\lambda_1)| = 0 < \frac{\rho\pi}{2}$. If $w > h$, then $\lambda_2 > 0$ and $\lambda_3 < 0$. This implies $|\arg(\lambda_2)| = 0 < \frac{\rho\pi}{2}$ and $|\arg(\lambda_3)| = \pi > \frac{\rho\pi}{2}$. According to Theorem 2.1, $\mathcal{E}_1$ is unstable. If $w < h$, then λ_2 and λ_3 are purely imaginary. It implies $|\arg(\lambda_2)| = |\arg(\lambda_3)| = \frac{\pi}{2} > \frac{\rho\pi}{2}$. According to Theorem 2.1, $\mathcal{E}_1$ is unstable.

Theorem 4.6. The steady-state point $\mathcal{E}_2$ of the system of Eq (3.2) is asymptotically stable if $h < w$ and $(pv\phi + qw + rh\phi < pv + qh + rh)$.

Proof. Given below is the Jacobian matrix for the system of equations investigated at steady-state point $\mathcal{E}_2$.

$$\mathcal{J}(\mathcal{E}_2) = \begin{pmatrix} -p + \frac{q(w-h)}{v(1-\phi)} - \frac{rh}{v} & 0 & 0 \\ \frac{u(h-w)}{v(1-\phi)} & 0 & -w+h \\ \frac{-uh}{v(1-\phi)} & h & 0 \end{pmatrix}.$$

Hence, the eigenvalues of $\mathcal{J}(\mathcal{E}_1)$ are $\lambda_1 = (\frac{-pv+pv\phi+qw-qh-rh+rh\phi}{v(1-\phi)})$, $\lambda_2 = \sqrt{h(h-w)}$, and $\lambda_3 = -\sqrt{h(h-w)}$. If $h < w$, then λ_2 and λ_3 are purely imaginary. This implies $|\arg(\lambda_2)| = |\arg(\lambda_3)| = \frac{\pi}{2} > \frac{\rho\pi}{2}$. If $pv\phi + qw + rh\phi < pv + qh + rh$, then λ_1 is negative. This implies $|\arg(\lambda_1)| = \pi > \frac{\rho\pi}{2}$. All three eigenvalues satisfy $|\arg(\lambda_i)| > \frac{\rho\pi}{2}$, $i = 1, 2, 3$. Hence, by Theorem 2.1, $\mathcal{E}_2$ is asymptotically stable for the given condition.

Theorem 4.7. The steady-state point $\mathcal{E}_3$ of the system of Eq (3.2) is asymptotically stable if $pv < qw + pv\phi$.

Proof. Given below is the Jacobian matrix for the system of equations investigated at steady-state point $\mathcal{E}_3$.

$$\mathcal{J}(\mathcal{E}_3) = \begin{pmatrix} 0 & \frac{qh}{u} & \frac{c(\phi-1)h}{u} \\ \frac{-pu}{q} & 0 & \frac{pv(\phi-1)}{q} \\ 0 & 0 & \frac{pv(1-\phi)-qw}{q} \end{pmatrix}.$$

Hence, the eigenvalues of $\mathcal{J}(\mathcal{E}_3)$ are $\lambda_1 = -(\frac{qn-pv+pv\phi}{q})$, $\lambda_2 = i\sqrt{ph}$, and $\lambda_3 = -i\sqrt{ph}$. Since λ_2 and λ_3 are purely imaginary, this implies $|\arg(\lambda_2)| = |\arg(\lambda_3)| = \frac{\pi}{2} > \frac{\rho\pi}{2}$. If $pv < qw + pv\phi$, then $\lambda_1 < 0$, so $|\arg(\lambda_1)| = \pi > \frac{\rho\pi}{2}$, and hence, $\mathcal{E}_3$ is asymptotically stable for given condition.

Let us discuss about the regional stability of the system of Eq (3.2) in the vicinity of the non-zero steady-state point $\mathcal{E}_4$. Given below is the Jacobian matrix for the system of equations evaluated at equilibrium point $\mathcal{E}_4$.

$$\mathcal{J}(\mathcal{E}_4) = \begin{pmatrix} -p + qy^* - rz^*(1-\phi) & qx^* & -r(1-\phi)x^* \\ -uy^* & h - ux^* - vz^*(1-\phi) & -vy^*(1-\phi) \\ -uz^* & v(1-\phi)z^* & h - ux^* + vy^*(1-\phi) - w \end{pmatrix},$$

where x^*, y^*, and z^* are provided by

$$x^* = \frac{h + vy^* - w}{u}, \; y^* = \frac{vp + rw}{vq + rv(1-\phi)}, \; z^* = \frac{-p + qy^*}{u(1-\phi)}. \tag{4.32}$$

The characteristic equation for the Jacobian matrix $\mathcal{J}(\mathcal{E}_4)$ is provided by

$$\varrho^3 + a_1\varrho^2 + a_2\varrho + a_3 = 0, \tag{4.33}$$

where

$$a_1 = p - qy^* + rz^*(1-\phi) - 2h + 2ux^* + vz^*(1-\phi) + w - vy^*(1-\phi), \tag{4.34}$$

$$a_2 = (-p + qy^* - rz^*(1-\phi))(2h - 2ux^* - vz^*(1-\phi) + vy^*(1-\phi) - w) \\ + (h - ux^* - vz^*(1-\phi))(h - ux^* + vy^*(1-\phi) - w + qux^*y^* - r(1-\phi)x^*uz^*), \tag{4.35}$$

$$a_3 = (h - ux^* - vz^*(1-\phi))((p - qy^* + rz^*(1-\phi))(h - ux^* + vy^*(1-\phi) - w) + r(1-\phi)x^*uz^*) \\ - qux^*y^*(h - ux^* + vy^*(1-\phi) - w) - quvx^*z^*(1-\phi) - rv(1-\phi)^2x^*y^*uz^*. \tag{4.36}$$

$\mathcal{E}_4$ is asymptotically stable, according to the Routh-Hurwitz conditions, whenever and only if $a_1 > 0$, $a_3 > 0$, and $a_1a_2 > a_3$.

4.2.2. Global stability analysis

Theorem 4.8. The steady-state point (x^*, y^*, z^*) given by (4.32) is globally asymptotically stable.
Proof. Examine the subsequent nonlinear Lyapunov function for the model

$$L(x(\theta), y(\theta), z(\theta)) = A_1\left(x(\theta) - x^* - x^* \ln\frac{x(\theta)}{x^*}\right) + A_2\left(y(\theta) - y^* - y^* \ln\frac{y(\theta)}{y^*}\right) \\ + A_3\left(z(\theta) - z^* - z^* \ln\frac{z(\theta)}{z^*}\right), \tag{4.37}$$

$$ {}_0^C\mathcal{D}_\theta^\rho L(x(\theta), y(\theta), z(\theta)) = A_1.{}_0^C\mathcal{D}_\theta^\rho\left(x(\theta) - x^* - x^* \ln\frac{x(\theta)}{x^*}\right) + A_2.{}_0^C\mathcal{D}_\theta^\rho\left(y(\theta) - y^* - y^* \ln\frac{y(\theta)}{y^*}\right) \\ + A_3.{}_0^C\mathcal{D}_\theta^\rho\left(z(\theta) - z^* - z^* \ln\frac{z(\theta)}{z^*}\right). \tag{4.38}$$

Using the result from [47],

$$ {}_0^C\mathcal{D}_\theta^\rho\left(x(\theta) - x^* - x^* \ln\frac{x(\theta)}{x^*}\right) \le \left(1 - \frac{x^*}{x(\theta)}\right).{}_0^C\mathcal{D}_\theta^\rho x(\theta),$$
$$ {}_0^C\mathcal{D}_\theta^\rho\left(y(\theta) - y^* - y^* \ln\frac{y(\theta)}{y^*}\right) \le \left(1 - \frac{y^*}{y(\theta)}\right).{}_0^C\mathcal{D}_\theta^\rho y(\theta),$$
$$ {}_0^C\mathcal{D}_\theta^\rho\left(z(\theta) - z^* - z^* \ln\frac{z(\theta)}{z^*}\right) \le \left(1 - \frac{z^*}{z(\theta)}\right).{}_0^C\mathcal{D}_\theta^\rho z(\theta),$$

$$ {}_0^C\mathcal{D}_\theta^\rho L(x,y,z) \le A_1\left(1 - \frac{x^*}{x(\theta)}\right).{}_0^C\mathcal{D}_\theta^\rho x(\theta) + A_2\left(1 - \frac{y^*}{y(\theta)}\right).{}_0^C\mathcal{D}_\theta^\rho y(\theta) + A_3\left(1 - \frac{z^*}{z(\theta)}\right).{}_0^C\mathcal{D}_t^\rho z(\theta). \tag{4.39}$$

Using the substitution from the system of Eq (3.2),

$$ {}_0^C\mathcal{D}_\theta^\rho L(x,y,z) \le A_1\left(1 - \frac{x^*}{x(\theta)}\right)(-px(\theta) + qx(\theta)y(\theta) - rx(\theta)z(\theta)(1-\phi)) \\ + A_2\left(1 - \frac{y^*}{y(\theta)}\right)(hy(\theta) - ux(\theta)y(\theta) - vy(\theta)z(\theta)(1-\phi)) \\ + A_3\left(1 - \frac{z^*}{z(\theta)}\right)(hz(\theta) - ux(\theta)z(\theta) + vy(\theta)z(\theta)(1-\phi) - wz(\theta)), \tag{4.40}$$

$$ {}_0^C\mathcal{D}_\theta^\rho L(x,y,z) \le A_1(x(\theta) - x^*)(-p + qy(\theta) - rz(\theta)(1-\phi)) \\ + A_2(y(\theta) - y^*)(h - ux(\theta) - vz(\theta)(1-\phi)) \\ + A_3(z(\theta) - z^*)(h - ux(\theta) + vy(\theta)(1-\phi) - w). \tag{4.41}$$

Applying the relationships at the steady state, we have

$$\begin{aligned} -p &= -qy^* + rz^*(1-\phi), \\ h &= ux^* + vz^*(1-\phi), \\ h - w &= ux^* - vy^*(1-\phi), \end{aligned} \tag{4.42}$$

$$\begin{aligned} {}_0^C\mathcal{D}_\theta^\rho L(x,y,z) \le{} & A_1(x(\theta)-x^*)\left(-r(1-\phi)(z(\theta)-z^*) + q(y(\theta)-y^*)\right) \\ & + A_2(y(\theta)-y^*)\left(-u(x(\theta)-x^*) - v(1-\phi)(z(\theta)-z^*)\right) \\ & + A_3(z(\theta)-z^*)\left(-u(x(\theta)-x^*) + v(1-\phi)(y(\theta)-y^*)\right), \end{aligned} \tag{4.43}$$

$$\begin{aligned} {}_0^C\mathcal{D}_\theta^\rho L(x,y,z) \le{} & -\left[A_1 r(1-\phi) + A_3 u\right](x(\theta)-x^*)(z(\theta)-z^*) \\ & + (A_1 q - A_2 u)(x(\theta)-x^*)(y(\theta)-y^*) \\ & + (-A_2 v(1-\phi) + A_3 v(1-\phi))(y(\theta)-y^*)(z(\theta)-z^*). \end{aligned} \tag{4.44}$$

Taking $A_1 = \frac{u}{q}A_2$ and $A_2 = A_3$, we can deduce that ${}_0^C\mathcal{D}_\theta^\rho L(x,y,z) \le 0$. Consequently, the steady state point (x^*, y^*, z^*) given by (4.32) is globally asymptotically stable.

5. Development and application of the numerical solutions in the model simulations

In this section, we will delve into the utilization of a numerical procedure to obtain an approximate solution to the problem. Let us examine the following equation for this purpose:

$$ {}_0^C D_\theta^\rho \mathcal{A}(\theta) = \mathcal{S}(\theta, \mathcal{A}(\theta)). \tag{5.1}$$

The integral operator can be used to write

$$\mathcal{A}(\theta) - \mathcal{A}(0) = \frac{1}{\Gamma(\rho)} \int_0^\theta (\theta-\omega)^{\rho-1} \mathcal{S}(\theta, \mathcal{A}(\theta))\, d\omega. \tag{5.2}$$

Taking $\theta = \theta_n = nh_1$, we can rewrite the above equation as

$$\mathcal{A}(\theta_n) = \mathcal{A}(0) + \frac{1}{\Gamma(\rho)} \sum_{i=0}^{n-1} \int_{\theta_i}^{\theta_{i+1}} (\theta_n-\omega)^{\rho-1} \mathcal{S}(\theta, \mathcal{A}(\theta))\, d\omega. \tag{5.3}$$

Now, using linear interpolation of $\mathcal{S}(\theta, \mathcal{A}(\theta))$, one obtains

$$\mathcal{S}(\theta, \mathcal{A}(\theta)) \approx \mathcal{S}(\theta_{i+1}, \mathcal{A}_{i+1}) + \frac{\theta - \theta_{i+1}}{h_1}(\mathcal{S}(\theta_{i+1}, \mathcal{A}_{i+1}) - \mathcal{S}(\theta_i, \mathcal{A}_i)), \theta \in [\theta_i, \theta_{i+1}]. \tag{5.4}$$

When Eq (5.4) is substituted for Eq (5.3), the problem's approximate solution will appear as

$$\mathcal{A}_n = \mathcal{A}_0 + \frac{h_1^\rho}{\Gamma(\rho+2)}\left(\eta_n \mathcal{S}(\theta_0, \mathcal{A}_0) + \sum_{i=1}^{n} \theta_{n-i}\mathcal{S}(\theta_i, \mathcal{A}_i)\right), \tag{5.5}$$

where

$$\eta_n = \frac{(n-1)^{\rho+1} - n^\rho(n-\rho-1)}{\Gamma(\rho+2)}, \tag{5.6}$$

$$\theta_j = \left\{ \begin{array}{ll} 1, & \text{if } j = 0, \\ (j-1)^{\rho+1} - 2j^{\rho+1} + (j+1)^{\rho+1}, & \text{if } j = 1, 2, ..., n-1. \end{array} \right\} \tag{5.7}$$

The problem will be substantially solved using the above-described numerical technique in a recursive manner as

$$x_n = x_0 + \frac{h_1^{\rho}}{\Gamma(\rho+2)} \left(\eta_n(-px_0 + qx_0y_0 - rx_0z_0(1-\phi)) + \sum_{i=1}^{n} \theta_{n-i}(-px_i + qx_iy_i - rx_iz_i(1-\phi)) \right), \tag{5.8}$$

$$y_n = y_0 + \frac{h_1^{\rho}}{\Gamma(\rho+2)} \left(\eta_n(hy_0 - ux_0y_0 - vy_0z_0(1-\phi)) + \sum_{i=1}^{n} \theta_{n-i}(hy_i - ux_iy_i - vy_iz_i(1-\phi)) \right), \tag{5.9}$$

$$z_n = z_0 + \frac{h_1^{\rho}}{\Gamma(\rho+2)} \left(\eta_n(hz_0 - ux_0z_0 + vy_0z_0(1-\phi) - wz_0) + \sum_{i=1}^{n} \theta_{n-i}(hz_i - ux_iz_i + vy_iz_i(1-\phi) - wz_i) \right). \tag{5.10}$$

To get computational experiments for addressing the fractional-order model of Eq (3.2), in this section, we will apply the estimated technique expounded in Eq (5.10).

In the computational experiments, the parameter values have been set as $p = 1.0$, $q = 1.5$, $r = 0.1$, $h = 0.5$, $u = 1.5$, $v = 0.1$, $w = 0.1$, and $\phi = 0.91$, with the commencing state $(x_0, y_0, z_0) = (0.8, 1.7, 0.75)$ and step size $h_1 = 1.0 \times 10^{-2}$.

The results obtained from Figures 1–3 help to fully understand the interplay between the dynamics of multilayer fractional-order quarry-hunter systems, the effects of diseases, and different harvesting strategies. These figures validate the theoretical models, as they demonstrate how relations involving fractional calculus, which is well known for its ability to describe the dynamics of systems with memory and non-locality, can been used to explain more complicated, biological systems (Figures 1–3). One of the more interesting results from these simulations concerns when certain harvesting rates (over or under) within the system are crossed or approached, where even small changes in these harvesting rates produce dramatic non-linear effects on the ecosystem. In particular, one of the insets in Figure 2 demonstrates how predator numbers crash after reaching a critical harvesting level, illustrating how fragile the systems that allow for any harvesting are. This finding is important for the development of successful management policies and illustrates the tension between ecological resilience and anthropogenic stress. In addition, the increase in the prevalence of diseases does not lead to the expected demise of prey populations. In fact, Figure 3 shows, instead of these populations stabilizing, that disease and population control are regulated in a rather complex density-dependent manner. Such results create a paradox for models of ecological systems that can hardly be found in any models, since such intricate systems are usually ignored in wildlife management, particularly the control of wildlife diseases. These results contribute to the understanding of ecology as a science but are also important for ecology as a practice, that is in policymaking and management. The fractional-order holos model is capable of predicting bifurcation points and nonlinear dynamics, which is why as recommended to decision-makers to incorporate the effects of their choices on the stability of ecology, and thus it becomes clear that more advanced models should be employed in ecological studies and management.

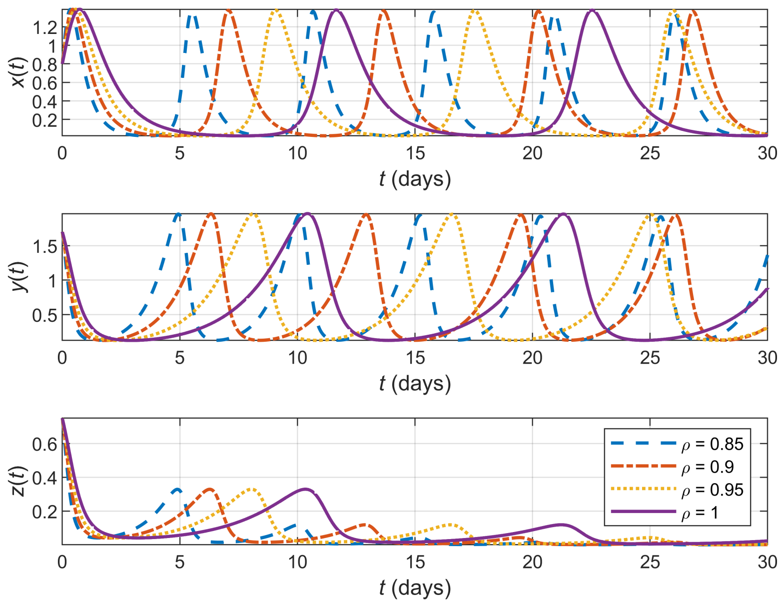

Figure 1. Influence of ρ on the response behaviour of minimal harvesting affecting the population of the robust hunter, vulnerable quarry, and sick quarry.

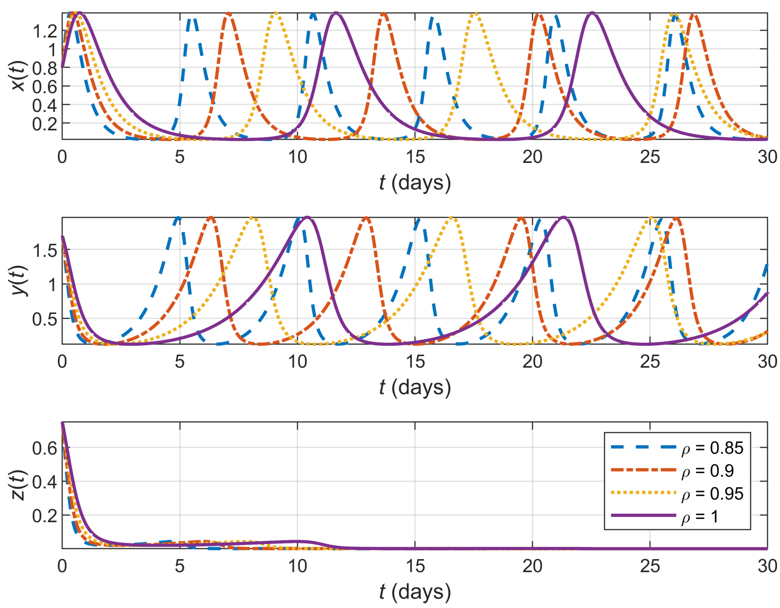

Figure 2. Influence of ρ on the response behaviour of low harvesting ($w = 0.3$) on the population of the robust hunter, vulnerable quarry and, sick quarry.

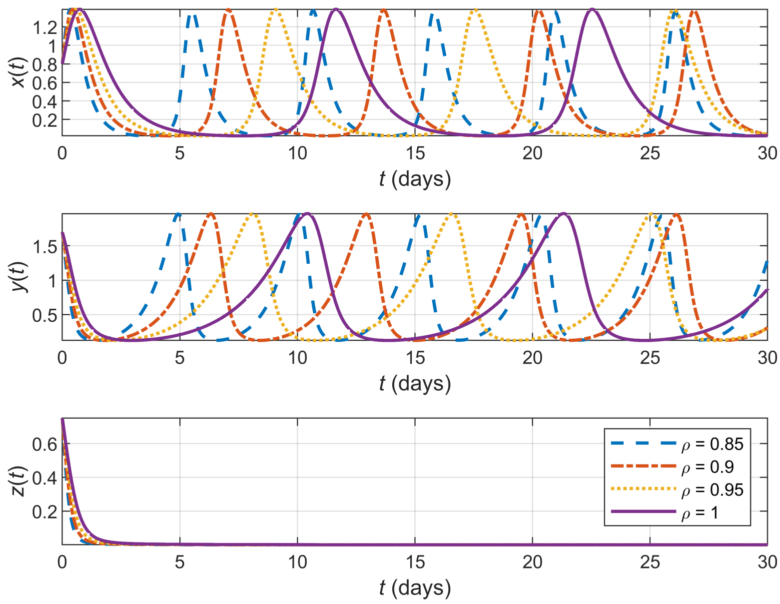

Figure 3. Influence of ρ on the response behaviour of low harvesting ($w = 0.7$) on the population of the robust hunter, vulnerable quarry ,and sick quarry.

The study we have conducted is an improvement on the use of fractional-order calculus in predator-prey dynamics and ecological modeling in general. The models that exist are very limited in their scope of ecological systems. For instance, Bandyopadhyay and Chattopadhyay [48] addressed the ratio-dependent class of functional responses in fluctuating environments; however, we are able to integrate memory-driven behavior, which is better at forecasting ecological stability. Moreover, the nonlinear models in the disease framework developed by Korobeinikov et al. [49] have been extended to historical data on disease prevalence for efficient control of the diseases. Other than that, our results are also consistent with those of Naresh et al. [50] and Venturino et al. [51], introducing a degree of complexity that is rich in detail and accounts for the effects and interactions over considerable periods of time and thus greatly adding to the conventional approaches for ecological modeling. The role of fractional dynamics in biological tissues has also been touched by Magin [52], but the present work goes further in using this principle in natural ecological models. In addition, Bate and Hilker [53] dealt with the effects of such interactions in co-epidemic models but our consideration of such an approach through fractional calculus allows such interactions to be understood differently as their effects on the present state evolve with interaction history.

6. Conclusions

In conclusion, this research paper has highlighted the dynamic impact of infectious diseases on fractional-order quarry-hunter systems and harvesting policies. Incorporating fractional calculus has enabled a deeper understanding of the dynamics, stability, and numerical solutions of these models. The review of existing literature has identified various scenarios where infectious diseases affect quarry-hunter dynamics. The investigation into the dynamics and stability of the proposed model has revealed valuable insights into the system's prolonged behavior. The presentation of numerical solution approaches, including specialized MATLAB routines, has facilitated practical implementation. This research contributes to our understanding of ecological dynamics, disease control, and the management of quarry-hunter populations. The findings have implications for ecologists, mathematicians, and policymakers, providing valuable insights for analyzing and mitigating the impact of infectious diseases on quarry-hunter relationships in real-world ecosystems.

Author contributions

Devendra Kumar and Jogendra Singh: Software, Formal analysis, Writing–review and editing; Devendra Kumar: Supervision; Devendra Kumar and Dumitru Baleanu: Project administration; Devendra Kumar, Jogendra Singh and Dumitru Baleanu: Conceptualization, Methodology, Validation, Writing–original draft. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest in this manuscript.

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