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Research article

Schur-type inequality for solitonic hypersurfaces in (k, μ)-contact metric manifolds

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Abstract: In this article, we derive a Schur-type Inequality in terms of the gradient r-Almost Newton-Ricci-Yamabe soliton in (k, μ)-contact metric manifolds. We discuss the triviality for the compact gradient r-Almost Newton-Ricci-Yamabe soliton in (k, μ)-Contact metric manifolds. In the end, we deduce a Schur-type inequality for the gradient r-Almost Newton-Yamabe soliton in (k, μ)-contact metric manifolds, static Riemannian manifolds, and normal homogeneous compact Riemannian manifolds coupled with a projected Casimir operator.

Keywords: (k, μ)-contact metric manifold; Schur-type inequality; hypersurfaces; gradient r-almost Newton-Ricci-Yamabe solitons
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1. Introduction

If the traceless Ricci tensor of an n-dimensional Riemannian manifold $(\overline{\mathcal{M}}, g)$ is identically zero, then manifold is termed Einstein. According to the classical Schur's lemma, the scalar curvature of an Einstein manifold of dimension ≥ 3 has to be constant. De Lellis and Topping recently demonstrated [1] that if a closed Riemannian manifolds $(\overline{\mathcal{M}}, g)$ ($n \geq 3$) has a non-negative Ricci curvature, then

$$\int_{\overline{\mathcal{M}}} (\mathcal{R} - \overline{\mathcal{R}})^2 dv_g \leq \frac{4n(n-1)}{(n-2)^2} \int_{\overline{\mathcal{M}}} \left| \mathcal{R}ic - \frac{\mathcal{R}}{n} g \right|^2 dv_g. \tag{1.1}$$

The average of $\mathcal{R}$ on the Riemannian manifolds $(\overline{\mathcal{M}}, g)$ is indicated by the symbol $\overline{\mathcal{R}}$. Furthermore, if and only if $(\overline{\mathcal{M}}, g)$ is Einstein, then the equality holds for (1.1).

According to convention, a Riemannian manifold $(\overline{\mathcal{M}}, g)$ is said to be closed if and only if it is compact and boundary free. In [2], Cheng established an almost-Schur lemma for closed manifolds with a Ricci curvature bounded from below by a negative constant, thereby generalizing the findings of De Lellis and Topping [1]. In other words, Cheng found a Schur-type inequality where the coefficient depended on both the Laplace operator and the Ricci curvature. The results for the $(0, 2)$-symmetric tensor were also expanded upon by Cheng [3], who provided an application for the k-scalar curvatures for closed locally conformally flat manifolds and the r-th mean curvatures of closed hypersurfaces in the space form. Regarding the latest studies conducted in this area, go to [4–6] and the associated references.

However, in contrast, the Ricci flow was studied by Hamilton in [7]. The Ricci soliton and Yamabe soliton, are the limit solutions of the Ricci flow.

The construction of Ricci-Yamabe solitons (RYS) from a Ricci-Yamabe geometric flow [8] was discussed by Siddiqi et al. [9]. The Ricci-Yamabe flow of the form (δ, ε) is another name for this. The semi-Riemannian multiple metric that gives rise to the Ricci-Yamabe flow is represented by the following

$$\partial_t g(t) = -2\delta \mathcal{R}ic(t) + \varepsilon \mathcal{R}(t) g(t), \quad g_0 = g(0), \tag{1.2}$$

where the terms $\mathcal{R}ic$ and $\mathcal{R}$ refer for the scalar curvature and the Ricci tensor, respectively. Additionally, the authors in [8] Guler treated the Ricci-Yamabe flow of type (δ, ε).

In the Ricci-Yamabe flow, a RYS is one that exclusively evolves by diffeomorphism and scales by a single parameter group. A RYS is a data $(g, F, \Lambda, \delta, \varepsilon)$ that obeys the Riemannain manifold (M, g):

$$\frac{1}{2}\mathfrak{L}_F g + \delta \mathcal{R}ic = \left(\Lambda + \frac{\varepsilon}{2}\mathcal{R}\right) g, \tag{1.3}$$

where $\mathfrak{L}_F$ shows the Lie derivative along the vector field F, and Λ, δ, and ε are real scalars. A RYS is called either *shrinking* $(\Lambda > 0)$, *expanding* $(\Lambda < 0)$, or *steady* (or $\Lambda = 0$).

Additionaly, if (1.3) holds for the Λ, δ, and ε smooth functions, then the soliton is called an almost RYS [10, 11].

If there exists a smooth function $\gamma : M \to \mathbb{R}$ such that $F = \nabla\gamma$, then the (δ, ε)-type RYS is called a *gradient* RYS of type (δ, ε), which is denoted by $(M, g, \gamma, \Lambda, \delta, \varepsilon)$, and in this case, (1.3) takes the following form:

$$\hbar ess(\gamma) + \delta \mathcal{R}ic = \left(\Lambda + \frac{\varepsilon}{2}\mathcal{R}\right) g, \tag{1.4}$$

where $\hbar ess$ is the Hessain of the function γ, and γ is called potential of the gradient RYS of type (δ, ε).

Originated with concepts by Cunha et al. ([12, 13]), consider a connected and oriented hypersurface called $\mathcal{M}^n$ that is submerged in a Riemannian manifold $\mathcal{N}^{n+1}$ of dimension $(n + 1)$.

For some $0 \leq r \leq n$, we declare that $\mathcal{M}^n$ is a gradient r-Almost Newton-Ricci-Yamabe solitons (gradient r-ANRYS) exists, which retains the following equation:

$$\delta \mathcal{R}ic + \mathcal{P}_{\mathrm{r}} \circ \hbar\mathrm{ess}(\gamma) = \left(\Lambda + \frac{\varepsilon}{2}\mathcal{R}\right) g, \tag{1.5}$$

where g denotes the Riemannian metric brought about by immersion. In addition, $\mathcal{P}_{\mathrm{r}} \circ \hbar\mathrm{ess}(\gamma)$ illustrates the tensor generated by

$$\mathcal{P}_{\mathrm{r}} \circ \hbar\mathrm{ess}(\gamma)(\mathrm{U}, \mathrm{V}) = \langle \mathcal{P}_{\mathrm{r}} \nabla_{\mathrm{U}} \nabla\gamma, \mathrm{V} \rangle,$$

for the tangent vector fields $U, V \in \mathfrak{X}(\mathcal{M})$.

Moreover, Siddiqi et al. ([14–17]) studied the same notion r-almost Newton Ricci soliton (r-ANRS), which is merely close to this topic. The study of Eq (1.5) is fascinating since a gradient r-ANRYS reduces to a gradient RYS when $r = 0$. A gradient r-ANRYS is trivial whenever the potential γ is constant. It is considered nontrivial if not. Furthermore, we refer to the gradient r- Newton Ricci-Yamabe soliton (r-ANRYS) when Λ is a constant.

Furthermore, the idea of almost Ricci solitons in a K-contact structure was started by Sharma [18]. Additionally, several properties of (k, μ)-contact and K-contact geometries were explored by Sharam [19] and Tripathi [20]. Moreover, (k, μ)-paracontact metric manifolds with almost conformal Ricci solitons were examined by the Siddiqi in [21].

The aim of the present paper is to obtain a Schur-type inequality in terms of the gradient r-ANRYS solitons in (k, μ)-contact metric manifolds.

2. Preliminaries

An almost contact structure (ϕ, ξ, η), with a $(1, 1)$-tensor field ϕ, ξ is a vector field with dual 1-form η, and; for any vector field U on $\overline{\mathcal{M}}$ [22], it is admitted to a $(2n + 1)$-dimensional smooth manifold $\overline{\mathcal{M}}$ if it admits one as follows:

$$\phi^2 U = -U + \eta(U)\xi, \tag{2.1}$$

$$\phi(\xi) = 0, \quad \eta(\phi) = 0, \quad \eta(\xi) = 1. \tag{2.2}$$

If and only if the $(1, 2)$-type torsion tensor $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi$ identically vanishes, then an almost contact structure is considered normal [23], where $[\phi, \phi]$ indicates the Nijenhuis tensor of ϕ.

Let $\overline{\mathcal{M}}$ be a nearly contact manifold [24] with a Riemannian metric g

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \tag{2.3}$$

for all $U, V \in \chi(\overline{\mathcal{M}})$. Then $(\overline{\mathcal{M}}, g)$ refers to an almost contact metric manifold. An almost contact structure is said to be a contact structure if [19]

$$g(U, \phi V) = d\eta(U, V). \tag{2.4}$$

The characteristic vector field is ξ, and the 1-form η is referred to as a contact form. The $(1, 1)$-tensor field h is given as follows:

$$\mathcal{L}_\xi \phi = 2h,$$

where $\mathcal{L}$ indicates the Lie derivative along the direction of the vector field ξ.

Blair pointed out that the tensor h is a symmetric operator in [25]. Then the following requirements are met by h:

$$h\phi = -\phi h, \qquad h\xi = 0, \tag{2.5}$$

$$\nabla_U \xi = -\phi U - \phi h U, \tag{2.6}$$

where $Tr(\phi h) = 0 = Tr(h)$, and $U \in \chi(\overline{\mathcal{M}})$.

Obviously, the tensor is h and the Levi-Civita connection of ∇ for every $U \in \chi(\overline{\mathcal{M}})$ in $\overline{\mathcal{M}}$ Riemannian manifold. The tensor h is said to be a K-contact manifold if and only if ξ is a Killing vector field. This is evident from the fact that the tensor h satisfies $h = 0$ in this case [26]:

A almost contact manifold is considered Sasakian if and only if the subsequent circumstance is met [23]

$$(\nabla_U \phi)V = g(U, V) - \eta(V)U, \tag{2.7}$$

for any $U, V \in \chi(\overline{\mathcal{M}})$.

A normal contact metric manifold is said to be Sasakian if the following holds

$$R(U, V)\xi = -[\eta(V)U - \eta(U)V], \tag{2.8}$$

for any $U, V \in \chi(\overline{\mathcal{M}})$, contrastingly, the contact metric geometry within Eq (2.8) does not infer that the contact manifold is Sasakian manifold [19].

It is widely understood that the tangent sphere bundle of a flat Riemannian manifold can admit a contact structure which fulfills $R(X, Y)\xi = 0$. Blair et al. [26] studied the (k, μ)-nullity condition in a contact metric manifold and provided numerous motivations for investigating it, they saw it as a extension of both Sasakian manifold and $R(U, V)\xi = 0$ case.

The $(k,\, u)$-nullity distribution $N(k, \mu)$ of a contact metric manifold $\overline{\mathcal{M}}$ is defined by the following [26, 27]

$$N(k, \mu) : p \longrightarrow N_p(k, \mu) = \left\{ Z \in T_pM | R(U, V)W = (kI + \mu h)(g(V, W)U - g(U, W)V) \right\},$$

for all the vector fields $U, V, W \in T\overline{\mathcal{M}}$, and $(k, \mu) \in \mathbb{R}^2$.

In a (k, μ)-contact manifold $(\overline{\mathcal{M}}^{2n+1}, \phi, \xi, \eta, g)$, $n > 1$, the following relationships are valid [28]:

$$\phi^2 = \frac{h^2}{(k-1)}, \quad k \le 1, \tag{2.9}$$

$$(\nabla_U \phi)V = g[U + hU, V]\xi - \eta(V)[U + hU], \tag{2.10}$$

$$R(\xi, U)V = k[g(U, V) - \eta(V)U] + \mu[g(hU, V)\xi - \eta(V)hU], \tag{2.11}$$

for any vector fields $U, V \in \chi(\overline{\mathcal{M}})$. Making use of (2.6), we have the following

$$(\nabla_U \eta)V = g(U, \phi V) + g(\phi hU, V). \tag{2.12}$$

3. r-almost Newton-Ricci-Yamabe soliton on hypersurface

Let $\mathcal{M}^n$ be a connected and oriented hypersurface that is immersed in a (k, μ)-contact metric manifold $\overline{\mathcal{M}}^{(2n+1)}$. The Gauss formula for immersion is well known to be given by the following

$$\mathcal{R}^\sharp(U, V)W = \langle \mathcal{A}U, W\rangle \mathcal{A}V - \langle \mathcal{A}V, W\rangle \mathcal{A}U + (\mathcal{R}(U, V)W)^\top,$$

for the tangent vector fields $U, V, W \in \mathfrak{X}(\mathcal{M})$, where $(\,)^{\top}$ stands for a vector field's tangential component in $\mathfrak{X}(\mathcal{M})$ along $\mathcal{M}^n$.

In this instance, $R^{\sharp}$ and $\mathcal{R}$ signify the curvatures of $\overline{\mathcal{M}}^{2n+1}$ and $\mathcal{M}^n$, respectively, and $\mathcal{A} : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ signifies the second fundamental form of $\mathcal{M}^n$ in $\overline{\mathcal{M}}^{2n+1}$. Specifically, the scalar curvature $\mathcal{R}$ of the hypersurface fulfills the following requirements

$$\mathcal{R}^{\sharp} = \sum_{1 \le i,j \le n} \langle \mathcal{R}^{\sharp}(v_i, v_j)v_j, v_i \rangle + n^2\mathcal{H}^2 - |\mathcal{A}|^2, \tag{3.1}$$

where the Hilbert-Schmidt norm is denoted by $|\cdot|$ and an orthonormal frame on $T\mathcal{M}$ is represented by $\{v_1, \ldots, v_n\}$. Then the following value is calculated:

$$\mathcal{R}^{\sharp} = 2n(2n - 2 + k - n\mu) + n^2\mathcal{H}^2 - |\mathcal{A}|^2, \tag{3.2}$$

for n algebraic invariants, which are the fundamental symmetric type functions $\mathcal{R}_r$ of the hypersurface's primary curvatures $k_1, \ldots, k_n$, associated with the second fundamental form shape operator $\mathcal{A}$ of the hypersurface $\mathcal{M}^n$.

$$\mathcal{R}_0 = 1 \text{ and } \mathcal{R}_r = \sum_{i_1 < \ldots < i_r} k_{i_1} \cdots k_{i_r}.$$

The r-th mean curvature is represented by the following equation:

$$\binom{n}{r}\mathcal{H}_r = \mathcal{R}_r.$$

If $r = 1$, then we turn up the mean curvature $\mathcal{H}_1 = \mathcal{H} = \frac{1}{n}tr(\mathcal{A}) =$ of $\mathcal{M}^n$.

The r-th *Newton transformation* is defined as $\mathcal{P}_r : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ for each $0 \le r \le n$. On the hypersurface $\mathcal{M}^n$ by using the identity operator ($\mathcal{P}_0 = I$) and the recurrence relation for $1 \le r \le n$

$$\sum_{j=0}^{r} (-1)^{r-j}\binom{n}{j}\mathcal{H}_j\mathcal{A}^{(r)} = \mathcal{P}_r, \tag{3.3}$$

where j times ($\mathcal{A}^{(0)} = I$) represent the composition of $\mathcal{A}$ with r. Observe that the second order differential operator $\mathcal{L}_r$ is connected for all Newton transformation $\mathcal{P}_r$, which is defined by the following

$$\mathcal{L}_r(\omega) = \text{tr}(\mathcal{P}_r \circ \hbar\text{ess}\ \omega).$$

We observe that $\mathcal{L}_0$ is just the Laplacian operator for $r = 0$. Additionally, it is apparent that

$$\text{div}_{\mathcal{M}}(\mathcal{P}_r\nabla\omega) = \sum_{i=1}^{n} \langle (\nabla_{v_i}\mathcal{P}_r)(\nabla\omega), v_i \rangle + \sum_{i=1}^{n} \langle \mathcal{P}_r(\nabla_{v_i}\nabla\omega), v_i \rangle = \langle \text{div}_{\mathcal{M}}\mathcal{P}_r, \nabla\omega \rangle + \mathcal{L}_r(\omega), \tag{3.4}$$

where the equation for the divergence of $\mathcal{P}_r$ on $\mathcal{M}^n$ is as follows:

$$\text{div}_{\mathcal{M}}\mathcal{P}_r = \text{tr}(\nabla\mathcal{P}_r) = \sum_{i=1}^{n} (\nabla_{v_i}\mathcal{P}_r)(v_i).$$

Because $\text{div}_{\mathcal{M}}\mathcal{P}_r = 0$, the Eq (3.4) reduces to $\mathcal{L}_r(\omega) = \text{div}_{\mathcal{M}}(\mathcal{P}_r\nabla\omega,)$, in particular when the ambient space has a constant sectional curvature (see [29] for more information). The following lemma gives useful conclusions.

Lemma 3.1. *[29] If $\mathcal{M}$ has a compact support γ without a boundary is either non-compact or compact, then we have*

$$(i)\ \int_{\mathcal{M}} \mathcal{L}_r(\gamma) = 0;$$

$$(ii)\ \int_{\mathcal{M}} \gamma\mathcal{L}_r(\gamma) = -\int_{\mathcal{M}} \langle \mathcal{P}_r\nabla\gamma, \nabla\gamma\rangle.$$

Additionally,it will be suitable for our intent to deal with the so-called traceless second fundamental form of the hypersurface, which is given by $\Phi = \mathcal{A} - \mathcal{H}I$. Observe that tr $(\Phi) = 0$ and $|\Phi|^2 = \mathrm{tr}(\Phi^2) = |\mathcal{A}|^2 - n\mathcal{H}^2 \geq 0$, with an equality if and only if $\mathcal{M}^n$ is totally umbilical.

In order to wrap up this subject, let us review Yau's lemma that corresponds to Theorem 3 of [30].

Lemma 3.2. *Let ω be a nonnegative and subharmonic function on a complete Riemannian manifold $\mathcal{M}^n$. If $\omega \in \mathcal{L}^p(\mathcal{M})$, for some $p > 1$, then ω is a constant.*

Here, we adopt the symbol $\mathcal{L}^p(\mathcal{M}) = \{\omega : \mathcal{M}^n \to \mathbb{R} \mid \int_{\mathcal{M}} |\omega|^p <\}$ for each $p \geq 1$. We end these considerations by providing an example.

Example 3.1. *Let an immersion from $\mathbb{S}^n$ into $\mathbb{S}^{n+1}$. It is recognized as totally geodesic. Specifically, $P_r \equiv 0$ for all $1 \leq r \leq n$. By choosing $\Lambda = \frac{2\delta-(n-1)\varepsilon}{2(n-1)}$, we obtain the Eq* (1.5)*, which is fulfilled by the immersion.*

In addition, the Eq (1.5) becomes true if the scalar curvature of $\mathcal{M}^n$ is constant,

$$\delta\mathcal{R}ic + \mathcal{P}_{\mathrm{r}} \circ \hbar\mathrm{ess}(\gamma) = \mu g, \tag{3.5}$$

where $\mu = \Lambda - \frac{\varepsilon}{2}\mathcal{R}$. Thus, as an additional illustration of a gradient r-ANRYS, we can refer back to Example 2 of [12].

Let $\mathbb{S}^{n+1}(1)$ denote a unit sphere in the Euclidean space $\mathbb{R}^{n+1}$ and $h : \mathbb{S}^{n+1}(1) \longrightarrow \mathbb{R}^{n+1}$ be a natural embedding with induced the metric g on $\mathbb{S}^{n+1}(1)$. Then, $(\mathbb{S}^{n+1}(1), \varphi, \xi, \eta, g)$ is a contact metric manifold with the constant sectional curvature $c = 1$.

Let $i : \mathcal{M}^n \longrightarrow \mathbb{S}^{n+1}(1) \subset \mathbb{R}^{n+1}$ be an immersion of a smooth n-dimensional manifold $\mathcal{M}^n$ into the unit sphere.

For a constant $l \in \mathbb{R}^{n+1}$, according to [12], choose the functions fl and ψ_l on $\mathbb{R}^{n+1}$ such that $f_l(t) = -g(t,l) + n - 1$ and $\psi_l(t) = -f_l + c$, $\psi_l := i * f_l \in C^\infty(\mathbb{S}^{n+1})$, where $l \in \mathbb{S}^{n+1}(1)$, $l \neq 0$, $c \in \mathbb{R}$ and $t = (t_1, ..., t_{n+1}) \in \mathbb{S}^{n+1}$ is the position vector. Now, we can see that $(\mathbb{S}^{n+1}, g, \nabla\psi_l, \lambda_l)$ satisfies the following

$$\delta\mathcal{R}ic + \mathcal{P}_{\mathrm{r}} \circ \hbar\mathrm{ess}(\gamma) = \lambda_{\mathrm{l}} g, \tag{3.6}$$

where $\lambda_l = f_l = \mu = \Lambda - \frac{\varepsilon}{2}\mathcal{R}$ and $\nabla\psi_l$ is the projection of the constant vector l to the tangent space of $\mathbb{S}^{n+1}$. It is obvious that $\mathbb{S}^{n+1}$ is totally umbilical with the second fundamental form $\mathcal{A} = I$ and the r-th mean curvature $Hr = 1$. The Newton tensor $\mathcal{P}_r$ is given by

$$\mathcal{P}_r = \alpha I, \forall \quad r, \quad 0 \leq r \leq n, \tag{3.7}$$

for $\alpha = \sum_{j=0}^{r}(-1)^{r-j}\binom{n}{j}$. Thus, if we choose the function $\psi = \alpha^{-1}\psi_l$, then we can observe that the hypersurfaces satisfies Eq (3.5).

4. Results of triviality and inequlity

With the gradient r-Newton-Ricci-Yamabe soliton (gradient r-NRYS) closed and Λ constant, we spend this part to present our key findings. The (k,μ)-contact metric manifold with the constant sectional curvature c is denoted by the symbol $\overline{\mathcal{M}}_c^{n+1}$ throughout the text.

Theorem 4.1. *If $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ is a closed gradient r-NRYS immersed into a (k,μ)-contact metric manifold $\overline{\mathcal{M}}_c^{n+1}$, such that $\mathcal{P}_r$ is either bounded above or bounded below (referring to quadratic form) and any one of the following propositions is valid, then we have*

i) $\delta > \frac{-n\varepsilon}{2}$ and $\mathcal{R}^\sharp \geq 0$ and $\Lambda \geq 0$, or $\mathcal{R}^\sharp \leq 0$, $\Lambda \leq 0$,
ii) $\delta < \frac{-n\varepsilon}{2}$ and $\mathcal{R}^\sharp \geq 0$ and $\Lambda \leq 0$, or $\mathcal{R}^\sharp \leq 0$, $\Lambda \geq 0$,
iii) $\delta \neq \frac{-n\varepsilon}{2}$ and either $\mathcal{R}^\sharp \geq \frac{n\Lambda}{2\delta+n\varepsilon}$ or $\mathcal{R}^\sharp \leq \frac{n\Lambda}{2\delta+n\varepsilon}$,

where the scalar curvature $\mathcal{R}^\sharp$ is constant of $\mathcal{M}^n$ and $\mathcal{M}^n$ is trivial.

Proof. In light of Lemma 3.1 and the structural relation, we obtain

$$0 = \int_{\mathcal{M}} \mathcal{L}_r(\gamma) = \int_M \Big(n\Lambda - (2\delta + n\varepsilon)\mathcal{R}^\sharp\Big).$$

Therefore, if (i) and (ii) are true, then we derive $\mathcal{R}^\sharp = \frac{\Lambda n}{2\delta+n\epsilon} = 0$ and $\mathcal{L}_r(\gamma) = 0$ from the structural equation. There is a positive constant $C > 0$ because the quadratic form of $\mathcal{P}_r$ is bounded either above or below:

$$0 = \mathcal{L}_r(\gamma) \leq C\Delta\gamma \ \text{ or } \ 0 = \mathcal{L}_r(\gamma) \geq -C\Delta\gamma,$$

respectively. γ is a subharmonic function as a result. Hopf's theorem tells us that γ is a constant function since $\mathcal{M}$ is compact. Therefore, the soliton is trivial. Lastly,
(iii) corresponds in the same way to (i) and (ii). □

Remark 1. *The assertion (i) and (ii) in the above theorem entails that $\mathcal{M}$ is steady and $\mathcal{R}^\sharp = 0$. Since $\mathcal{M}^n$ is trivial, we get $\mathcal{R}ic \equiv 0$. Consequently, (iii) implies $\mathcal{R}^\sharp = \frac{\Lambda n}{2\delta+n\epsilon}$. Since $\mathcal{M}$ is trivial, we turn up the following*

$$\delta\mathcal{R}ic = \left(\Lambda - \frac{n\varepsilon\Lambda}{(2\delta + \varepsilon)}\right)g = \frac{2\delta\Lambda}{(2\delta + n\varepsilon)}g = \delta\frac{\mathcal{R}^\sharp}{n}g,$$

i.e., $\mathcal{M}^n$ is Einstein.

Theorem 4.2. *If $(\mathcal{M}^n, \gamma, \Lambda, \delta, \varepsilon)$ is a closed gradient r-NRYS immersed into a (k,μ)-contact metric manifold $\overline{\mathcal{M}}_c^{n+1}$, such that $\mathcal{P}_r$ is either bounded above or bounded below (referring to quadratic form) and $\delta \neq \frac{-n\varepsilon}{2}$, then the scalar curvature of $\mathcal{M}^n$ is constant, $\mathcal{M}^n$ is Einstein and trivial.*

Proof. In view of the structural equation Lemma 3.1, we own the following

$$\int_{\mathcal{M}} |n\Lambda - (2\delta + n\varepsilon)\mathcal{R}^\sharp|^2 = \int_{\mathcal{M}} \Big(n\Lambda - (2\delta + n\varepsilon)\mathcal{R}^\sharp\Big)\mathcal{L}_r(\gamma) = \Big(n\Lambda - (2\delta + n\varepsilon)\mathcal{R}^\sharp\Big)\int_{\mathcal{M}} \mathcal{L}_r(\gamma) = 0.$$

Hence, we obtain $\mathcal{R}^\sharp = \frac{2n\Lambda}{2\delta+n\varepsilon}$ and $\mathcal{L}_r(\gamma) = 0$. Adopting that $\mathcal{P}_r$ is either bounded above or bounded below (referring to quadratic form) to demonstrate that $\mathcal{M}$ is trivial, we can adopt the same steps as in the proof of Theorem 4.1. Last but not least, since $\mathcal{M}^n$ is trivial, we can move on to Remark 1 to conclude that $\mathcal{M}^n$ is Einstein. □

5. Schur-type inequality

We established a Schur-type inequality in the following Theorem.

Theorem 5.1. *If* $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ *is a closed gradient r-NRYS immersed into a* (k, μ)*-contact metric manifold* $\overline{\mathcal{M}}_c^{n+1}$*, such that* $\mathcal{P}_{\mathrm{r}}$ *is bounded below (referring to quadratic form) and* $\delta > \frac{-n\varepsilon}{2}$*, then*

$$\int_{\mathcal{M}} |\mathcal{R}^\sharp - \overline{\mathcal{R}}|^2 \leq \frac{nC}{(n-2)\left(\delta + \frac{n\varepsilon}{2}\right)} \|\overset{\circ}{\mathrm{Ric}}\|_{\mathcal{L}^2} \left\|\nabla^2\gamma - \frac{\Delta\gamma}{n}g\right\|_{\mathcal{L}^2}. \tag{5.1}$$

Proof. The contracted second Bianchi identity states the following

$$\mathrm{div}(\mathcal{R}ic) - \frac{1}{2}\nabla\mathcal{R}^\sharp = 0.$$

Hence,

$$\mathrm{div}(\overset{\circ}{\mathcal{R}ic}) = \frac{n-2}{2n}\nabla\mathcal{R}^\sharp,$$

where $\overset{\circ}{\mathcal{R}ic}$ is the traceless Ricci tensor. Since the closed gradient r-NRYS is compact and $\mathcal{P}_r$, we get

$$\begin{aligned}
\int_{\mathcal{M}} |n\Lambda - (2\delta + n\varepsilon)\mathcal{R}^\sharp|^2 &= \int_{\mathcal{M}} \left(n\Lambda - (2\delta + n\varepsilon)\mathcal{R}^\sharp\right)\mathcal{L}_r(\gamma)\\
&= \int_{\mathcal{M}} \left(n\Lambda - (2\delta + n\varepsilon)\mathcal{R}^\sharp\right)\mathrm{div}(\mathcal{P}_r\nabla\gamma)\\
&=(2\delta + n\varepsilon)\int_{\mathcal{M}} \langle\nabla\mathcal{R}^\sharp, \mathcal{P}_r\nabla\Psi\rangle \leq C(2\delta + n\varepsilon)\int_{\mathcal{M}}\langle\nabla\mathcal{R}^\sharp, \nabla\gamma\rangle\\
&=\frac{2nC(2\delta + n\varepsilon)}{n-2}\int_{\mathcal{M}}\langle\mathrm{div}(\overset{\circ}{\mathcal{R}ic}), \nabla\gamma\rangle\\
&=\frac{2nC(2\delta + n\varepsilon)}{n-2}\int_{\mathcal{M}}\langle\overset{\circ}{\mathcal{R}ic}, \nabla^2\gamma\rangle\\
&=\frac{2nC(2\delta + n\varepsilon)}{n-2}\int_{\mathcal{M}}\langle\overset{\circ}{\mathcal{R}ic}, \nabla^2\gamma - \frac{\Delta\gamma}{n}g\rangle\\
&\leq\frac{2nC(2\delta + n\varepsilon)}{n-2}\| \overset{\circ}{\mathcal{R}ic} \|_{\mathcal{L}^2}\left\|\nabla^2\gamma - \frac{\Delta\gamma}{n}g\right\|_{\mathcal{L}^2},
\end{aligned}$$

wherein we employed that $\langle\overset{\circ}{\mathcal{R}ic}, g\rangle = 0$. Provided that the closed gradient r-NRYS is compact, we get the following

$$n\Lambda = (2\delta + n\varepsilon)\overline{\mathcal{R}},$$

where $\overline{\mathcal{R}}$ indicates for the average of $\mathcal{R}^\sharp$. Therefore,

$$(2\delta + n\varepsilon)^2\int_{M}|\mathcal{R}^\sharp - \overline{\mathcal{R}}|^2 = \int_{\mathcal{M}}|n\Lambda - (2\delta + n\varepsilon)\mathcal{R}^\sharp|^2,$$

i.e.,

$$(2\delta + n\varepsilon)^2\int_{\mathcal{M}}|\mathcal{R}^\sharp - \overline{\mathcal{R}}|^2 \leq \frac{2nC(2\delta + n\varepsilon)}{n-2}\| \overset{\circ}{\mathcal{R}ic} \|_{\mathcal{L}^2}\left\|\nabla^2\gamma - \frac{\Delta\gamma}{n}g\right\|_{\mathcal{L}^2},$$

i.e.,

$$\int_{\mathcal{M}} |\mathcal{R}^{\sharp} - \overline{\mathcal{R}}|^2 \leq \frac{2nC}{(n-2)(2\delta + n\varepsilon)} \|(\mathring{\mathcal{R}ic})\|_{\mathcal{L}^2} \left\| \nabla^2 \gamma - \frac{\Delta\gamma}{n} g \right\|_{\mathcal{L}^2}.$$

This completes the proof. □

Remark 2. *Due to the fact that both sides of expression* (5.1) *diminish in the foregoing theorem if M^n" is Einstein, the equality is maintained. To demonstrate the rigidity would be a fascinating problem.*

Remark 3. *By the definition of Ricci-Yamabe solitons [9] if $\delta = 0$ and $\epsilon = 1$, then, we turn up the case of the gradient r-Newton-Yamabe soliton. Consequently, in view of the Theorem 5.1, we gain the Schur-type inequality in terms of the gradient r-Newton-Yamabe soliton by the follwoing:*

Corollary 5.1. *If $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ is a closed gradient r-Newton-Yamabe soliton immersed into a (k, μ)-contact metric manifold $\overline{\mathcal{M}}_c^{n+1}$, such that $\mathcal{P}_r$ is bounded below (referring to quadratic form) and $\varepsilon > 0$, then*

$$\int_{\mathcal{M}} |\mathcal{R}^{\sharp} - \overline{\mathcal{R}}|^2 \leq \frac{2n^2C}{(n-2)} \|\mathring{\mathrm{Ric}}\|_{\mathcal{L}^2} \left\| \nabla^2 \gamma - \frac{\Delta\gamma}{n} g \right\|_{\mathcal{L}^2}. \tag{5.2}$$

6. Some applications

Definition 6.1. *[31] A Riemannian manifold with a triplet $(\mathcal{M}, g, \gamma)$ is said to be static if*

$$\Delta\gamma g - \nabla^2\gamma + \gamma\mathcal{R}ic = 0, \tag{6.1}$$

where γ is a potential function.

Therefore, in the light of Theorem 5.1, and Eqs (5.1) and (6.1), we gain the Schur-type inequality for the static gradient r-NRYS immersed into into a (k, μ)-contact metric manifold $\overline{\mathcal{M}}_c^{n+1}$:

Theorem 6.1. *Let $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ be a closed static gradient r-NRYS with a potential function γ immersed into a (k, μ)-contact metric manifold $\overline{\mathcal{M}}_c^{n+1}$. Then*

$$\int_{\mathcal{M}} |\mathcal{R}^{\sharp} - \overline{\mathcal{R}}|^2 \leq \frac{nC}{(n-2)\left(\delta + \frac{n\varepsilon}{2}\right)} \|\mathring{\mathrm{Ric}}\|_{\mathcal{L}^2} \left\{ \left\| \frac{(n-1)\Delta\gamma}{n} g \right\|_{\mathcal{L}^2} + \gamma \|\mathcal{R}ic\|_{\mathcal{L}^2} \right\}. \tag{6.2}$$

Once again, in view of Corollary 5.1, we can also state the following corollaries:

Corollary 6.1. *Let $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ be a static closed gradient r-Newton-Yamabe soliton immersed into a (k, μ)-contact metric manifold $\overline{\mathcal{M}}_c^{n+1}$, and $\varepsilon > 0$. Then*

$$\int_{\mathcal{M}} |\mathcal{R}^{\sharp} - \overline{\mathcal{R}}|^2 \leq \frac{2n^2C}{(n-2)} \|\mathring{\mathrm{Ric}}\|_{\mathcal{L}^2} \left\{ \left\| \frac{(n-1)\Delta\gamma}{n} g \right\|_{\mathcal{L}^2} + \gamma \|\mathcal{R}ic\|_{\mathcal{L}^2} \right\}. \tag{6.3}$$

Second, we roughly associate the Casimir operator Ω with the Laplace-Beltrami operator Δ. This statement is stated in [32] for rank one symmetric spaces and in [33] for a Riemannian symmetric manifold of the noncompact type. For the sake of completeness, we provide the argument for a Riemannian symmetric manifold of the compact type, sometimes known as normal homogeneous compact Riemannian manifolds. It is essentially the same.

In [34], Lippner et al. proved for a normal homogeneous compact Riemannian manifold $\overline{\mathcal{M}}_c^{n+1}$. Let Δ be the Laplace-Beltrami operator on $\overline{\mathcal{M}}_c^{n+1}$ and Ω be the projected *Casimir operator*. Then,

$$\Delta = \Omega. \tag{6.4}$$

Now, in view of (5.1), (5.2), (6.1), (6.2), and (6.4), we gain the Schur-type inequalities for a normal homogeneous compact Riemannian manifold in terms of the Casimir operator Ω.

Theorem 6.2. *Let $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ be a closed and normal homogeneous compact gradient r-NRYS immersed into a (k, μ)-contact metric manifold $\overline{\mathcal{M}}_c^{n+1}$, such that $\mathcal{P}_r$ is bounded below (referring to quadratic form) and $\delta > \frac{-n\varepsilon}{2}$. Then*

$$\int_{\mathcal{M}} |\mathcal{R}^\sharp - \overline{\mathcal{R}}|^2 \leq \frac{nC}{(n-2)\left(\delta + \frac{n\varepsilon}{2}\right)} \, \|\mathring{\mathrm{Ric}}\|_{\mathcal{L}^2} \left\| \nabla^2\gamma - \frac{\Omega\gamma}{n} g \right\|_{\mathcal{L}^2}. \tag{6.5}$$

Theorem 6.3. *Let $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ be a closed and normal homogeneous compact static gradient r-NRYS with a potential function γ immersed into a (k, μ)-contact metric manifold $\overline{\mathcal{M}}_c^{n+1}$. Then*

$$\int_{\mathcal{M}} |\mathcal{R}^\sharp - \overline{\mathcal{R}}|^2 \leq \frac{nC}{(n-2)\left(\delta + \frac{n\varepsilon}{2}\right)} \, \|\mathring{\mathrm{Ric}}\|_{\mathcal{L}^2} \left\{ \left\| \frac{(n-1)\Omega\gamma}{n} g \right\|_{\mathcal{L}^2} + \gamma \, \|\mathcal{R}ic\|_{\mathcal{L}^2} \right\}. \tag{6.6}$$

Corollary 6.2. *Let $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ be a closed and normal homogeneous compact gradient r-Newton-Yamabe soliton immersed into a (k, μ)-contact metric manifold $\overline{\mathcal{M}}_c^{n+1}$, such that $\mathcal{P}_r$ is bounded below (referring to quadratic form) and $\varepsilon > 0$. Then*

$$\int_{\mathcal{M}} |\mathcal{R}^\sharp - \overline{\mathcal{R}}|^2 \leq \frac{2n^2 C}{(n-2)} \, \|\mathring{\mathrm{Ric}}\|_{\mathcal{L}^2} \left\| \nabla^2\gamma - \frac{\Omega\gamma}{n} g \right\|_{\mathcal{L}^2}. \tag{6.7}$$

Corollary 6.3. *Let $(\mathcal{M}^n, g, \gamma, \Lambda, \delta, \varepsilon)$ be a statically closed and normal homogeneous compact gradient r-Newton-Yamabe soliton immersed into a (k, μ)-contact metric manifold $\overline{\mathcal{M}}_c^{n+1}$, and $\varepsilon > 0$. Then*

$$\int_{\mathcal{M}} |\mathcal{R}^\sharp - \overline{\mathcal{R}}|^2 \leq \frac{2n^2 C}{(n-2)} \, \|\mathring{\mathrm{Ric}}\|_{\mathcal{L}^2} \left\{ \left\| \frac{(n-1)\Omega\gamma}{n} g \right\|_{\mathcal{L}^2} + \gamma \, \|\mathcal{R}ic\|_{\mathcal{L}^2} \right\}. \tag{6.8}$$

7. Conclusions

In the context of a (k, μ)-contact metric manifold, this research study examined a geometric inequality known as Schur inequality in terms of r-almost Newton Ricci-Yamabe solutions. In the course of this investigation, several important results concerning trivial and compact immersed r-almost Newton Ricci-Yamabe solitons in (k, μ)-contact metric manifolds were derived and discussed. This result emphasized the use of a particular Schur's inequality in terms of the projected Casimir operator for closed and normal homogeneous compact static gradient r-almost Newton Yamabe solutions.

Author contributions

Mohd Danish Siddiqi: Conceptualization, methodology, writing-original draft preparation, writing-review and editing, supervision; Fatemah Mofarreh: Conceptualization, methodology, writing-original draft preparation, writing-review and editing, funding acquisition. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors assert that they do not have any known competing financial interests or personal relationships that could have influenced the work reported in this paper.

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