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Research article

Vector fields on bifurcation diagrams of quasi singularities

Fawaz Alharbi[1] **and Yanlin Li**[2,*]

[1] Department of Mathematics, College of Sciences, Umm Al-Qura University, Makkah 21955, Saudi Arabia

[2] School of Mathematics, Hangzhou Normal University, Hangzhou 311121, China

* **Correspondence:** Email: liyl@hznu.edu.cn.

Abstract: We describe the generators of the vector fields tangent to the bifurcation diagrams and caustics of simple quasi boundary singularities. As an application, submersions on the pair (G, B), which consists of a cuspidal edge G in $\mathbb{R}^3$ that contains a distinguishing regular curve B, are classified. This classification was used as a means to investigate the contact that a general cuspidal edge G equipped with a regular curve $B \subset G$ has with planes. The singularities of the height functions on (G, B) are discussed and they are related to the curvatures and torsions of the distinguished curves on the cuspidal edge. In addition to this, the discriminants of the versal deformations of the submersions that were accomplished are described and they are related to the duality of the cuspidal edge.

Keywords: bifurcation diagram; caustic; vector field; cuspidal edge; contact; curvatures and torsions; height function; contact; deformations; discriminat; daul
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1. Introduction

One of the core tools in singularity theory is to classify functions on a certain space equipped with a distinguished hyperspace in it. The infinitesimal level problems of this kind require finding diffeomorphisms of the ambient space such that this hypersurface is preserved. In order to construct these diffeomorphisms, it is necessary to provide a description of the generators of vector fields that are parallel to the hypersurface. Many authors have studied algorithm and algebraic aspects of such vector fields (see for example [1–3]) to classify singularities of maps (functions) between two manifolds that can be constructed from the differential geometry point of view (see e.g. [4, 5]. Further motivations of the topics can be found in various relevant papers with differential geometry [6–8] and submanifolds theory [9–11]. The classification can help study manifolds via other functions such as the height function and distance squared function. In many cases, this hypersurface appears as a discriminant (or bifurcarion diagram) or caustics of versal deformation of classes with respect to a standard equivalence relation.

In a series of papers [12–14], a new non-standard equivalence relation, on a space $\mathbb{R}^n$ equipped with a variety Γ, are studied, and, consequently, simple classes were obtained. Classification of projections of Lagrangian manifolds endowed with a hypersurface Γ is accomplished through the utilization of these classes. As a result of the classification, the bifurcation diagrams and caustics of versal unfolding of simple classes were described in [15], which were conduct in a different manner. In particular, let $G(z,u) = \widetilde{G}(z,u) + u_0$, with $z \in \mathbb{R}^n$ and $u = (u_0, u_1 \dots, u_s)$ as parameters, be a versal unfolding of the simple $g(z) = G(z,0)$ with respect to the quasi equivalence relation. Then, the respective bifurcation diagram in the space of parameters consists of two components W_0, which is the standard discriminant given by the equations $G = 0$ and $\frac{\partial G}{\partial z} = 0$ and W_1, which is contained in W_0 and it is determined by constraints that define Γ. The caustics is in the unfolding base $\widetilde{u} = (u_1, \dots, u_s)$ (which does not include λ_0),and consists of two parts Σ_0 which represents the singular set image of W_0 under the projection $\pi : u \to \widetilde{u}$ and $\Sigma_1 = \pi(W_1)$. The preceding construction yields that the bifurcation diagrams is a pair $W = (W_0, W_1)$, where W_0 is a hypersurface in $\mathbb{R}^s_u$ and $W_1 \subset W_0$, while the caustics is the union $\Sigma^* = \Sigma_0 \cup \Sigma_1$ with $dim(\Sigma_0) = dim(\Sigma_1)$.

In this work, in Section 2, we calculate the generators of the vector fields that are parallel to the quasi bifurcation diagrams and caustics, obtained in [15]. This implies that, for the bifurcation diagrams, we seek vector fields that preserve not only W_0 but also the points of W_0, and for the caustics, we seek vector fields that preserve both Σ_0 and Σ_1.

Singularity theory techniques and differential geometry tools can be employed to comprehend the geometry of an object by examining its interaction with planar objects, such as planes or lines. In order to determine the former, it is necessary to analyze the singularities of the height functions along particular directions, which define the object's contact with the plane orthogonal to that direction. Many authors have investigated the contact with planes of singular surfaces, including the cross-cap [16], the swallowtail [17], the cuspidal edge [18], and the folded cuspidal edges [19].

Thus, as an application, in Section 3, we consider a cuspidal edge G equipped with a distinguished regular curve$B$ in it. The object appears as a bifurcation diagram of the quasi boundary class B_3. We then apply the module of vector fields obtained in Section 2 to classify submersions on the pair (G, B). Then, we use such classification to study the contact of a general cuspidal edge equipped with a regular curve in it by studying the singularities of height function on (G, B). There are two distinguished regular curves, Σ_G (the singular set) and B. Finally, we examine the duality of the two curves by describing the versal deformation of the generic submersions that are obtained.

2. Preliminaries

Let $\mathbb{K}$ denote the real number $\mathbb{R}$ or the complex numbers $\mathbb{C}$ with local coordinates z. The set of all smooth function germs from $(\mathbb{K}^n, 0)$ to $\mathbb{K}$ is denoted by $\mathcal{E}_n$ (or $\mathcal{E}_z$), and the maximal ideal in this set is denoted by $\mathcal{M}_n$. Let θ_n represent the module over $\mathcal{E}_n$ consisting of all vector fields formed on $(\mathbb{K}^n, 0)$. Let $\mathbb{K}[z]$ be the polynomial ring or formal power series over $\mathbb{K}$.

Let $V \subset (\mathbb{K}^n, 0)$ be an analytic variety. The ideal of germs that vanish on V is denoted by $I(V)$.

Definition 1. *If $\xi(I(V)) \subseteq I(V)$, then a vector field $\xi \in \theta_n$ is considered to be tangent to V or to preserve V. The module of vector fields of this nature is represented by $Der(-logV)$.*

It is important to note that if $\xi \in Der(-logV)$, then it can be integrated to generate a diffeomorphism $\varphi : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ that preserves V, i.e., $\varphi(V) \subseteq V$.

Also note that Definition (1) implies that if $V = (h_1, h_2, \ldots, h_s)$, where $h_i \in \mathcal{M}_n$, then

$$Der(-logV) = \{\xi \in \theta_n : \exists\, f_{ij} \in \mathcal{E}_n \text{ such that } \xi(h_i) = \sum_{i=1}^{s} f_{ij} h_i,\ j = 1, \ldots, s\}.$$

Definition 2. *Let ζ be a vector field on $(\mathbb{K}^p)$ and $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p)$ be a smooth map germ. Then, ζ is said to be liftable over f if there exist a vector field η on $(\mathbb{K}^n)$ such that $df \circ \eta = \zeta \circ f$. In this case, η is said to be lowerable.*

The concepts of liftable and tangent vector fields on the discriminant are identical for stable map germs when $\mathbb{K} = \mathbb{C}$ (see [20]). In fact, Arnold, in [2], showed that there are liftable vector fields that are not tangent vector fields when $\mathbb{K} = \mathbb{R}$.

Let V_1 be an $\mathbb{R}-$analytic variety in $(\mathbb{R}^r, 0)$ in local coordinates $(u_0, \ldots, u_{r-1})$. Assume that $V_0 \subseteq V_1$ of codimension 1. Denote by $\widetilde{V}$ the pair consisting of V_1 and a distinguished sub-variety V_0 in it and set $\widetilde{V} = (V_1, V_0)$. Thus, we may assume that the pair represents a variety equipped with a boundary.

Definition 3. *A diffeomorphism $\phi : (\mathbb{R}^r, 0) \to (\mathbb{R}^r, 0)$ will be said to preserve $\widetilde{V}$ if and only if $\phi(V_1) \subseteq V_1$ and $\phi(V_0) \subseteq V_0$.*

Definition 4. *A vector field $\xi \in \theta_r$ is considered tangent to $\widetilde{V}$ if and only if the following conditions are fulfilled.*

(1) $\xi\,(I(V_1)) \subseteq I(V_1)$,
(2) $\xi\,(I(V_0)) \subseteq I(V_0)$.

The module of all vector fields satisfying the given conditions will be represented as $Der(-log\widetilde{V})$ over the $\mathcal{E}_r$-module, that is

$$Der(-log\widetilde{V}) = \left\{\xi = \sum_{i=0}^{r-1} \dot{u}\frac{\partial}{\partial u_i} \in \theta_r : \xi g_1 \in I(V_1), \xi g_2 \in I(V_0)\ \forall\ g_1 \in I(V_1), g_2 \in I(V_0)\right\},$$

and it is commonly referred to as the stationary algebra *of $\widetilde{V}$.*

Remark 1. If ξ belongs to $Der(-log\widetilde{V})$, then ζ conserves $\widetilde{V}$ and, as a result, is tangent to it. Furthermore, $Der(-log\widetilde{V})$ is the Lie algebra associated with the group of diffeomorphisms that preserve $(\widetilde{V}, 0)$ in the space $(\mathbb{R}^n, 0)$.

3. Vector fields on quasi bifurcation diagrams

Consider the coordinate space $\mathbb{R}^n$ in local coordinates $z = (x, y_1, \ldots, y_{n-1})$ equipped with a smooth hypersurface $\Gamma = \{x = 0\}$, which is referred to as a boundary.

Recall from [12] that on the Γ, every simple function germ g can be stably transformed via the quasi equivalence relation into one of the subsequent germs:

$$B_k : g_1(x, y_1) = \pm x^k \pm y_1^2, \quad \text{where} \quad k \geq 2,$$

or

$$F_{p,k} : g_2(x, y_1) = \pm(x \pm y_1^p)^2 \pm y_1^k, \quad \text{where} \quad k > p \geq 2.$$

The tangent space to the quasi boundary equivalence singularity of g at g is

$$TQ\Gamma_g = \left\{ \frac{\partial g}{\partial x}\Big(xA + \sum_{i=1}^{n} \frac{\partial g}{\partial z_i} B_i\Big) + \sum_{j=1}^{n-1} \frac{\partial g}{\partial y_j} E_j : \quad A, B_i, E_j \in \mathcal{E}_z \right\}.$$

Let $G(z, u)$ be a deformation of $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, where $u = (u_0, u_1, \ldots, u_{r-1}) \in \mathbb{R}^r$ are parameters. Set $G_u(z) = G(z, u)$; so that, $G_0 = g$.

The initial speeds of G are defined by

$$\dot{G}_i = \frac{\partial G}{\partial u_i}(z, 0), \quad \forall\, i \in \{0, 1, 2, \ldots, r-1\}.$$

The subsequent result is an adaptation of Theorem 3 from [16].

Proposition 1. *A deformation G of a function g is considered versal in regard to the quasi equivalence if and only if*

$$TQ\Gamma_g + \mathbb{R}\{\dot{G}_0, \ldots \dot{G}_{r-1}\} = \mathcal{E}_z.$$

Assume that the elements $\omega_0, \ldots, \omega_{r-1} \in \mathcal{E}_z$ form a basis of the quotient space $\mathcal{E}_z/TQ\Gamma_g$. Then, Proposition 1 implies that a miniversal deformation of a function germ g may take the form:

$$G(z, u) = g(z) + \sum_{i=0}^{r-1} u_i \omega_i(z). \tag{3.1}$$

Therefore, the formulas of quasi boundary versal deformations of $g_1 \in B_k$ and $g_2 \in F_{p,k}$ are

$$G_k(z, u) = \pm y_1^2 \pm x^k + \sum_{i=0}^{k-1} u_i x^i$$

and

$$G_{p,k}(z, u) = \pm(x \pm y_1^p + \sum_{j=k-1}^{p+k-2} u_j y_1^{j-(k-1)})^2 \pm y_1^k + \sum_{i=0}^{k-2} u_i y_1^i,$$

respectively.

Remark 2. The versal deformation of the class $F_{2,3}$ can be written equivalently as

$$G(x, y_1, u) = \pm x^2 \pm y_1^3 + u_0 + u_1 x + u_2 y + u_3 x y_1.$$

Definition 5. *[15] The quasi bifurcation diagram of a germ g with $G(z, u)$ being its quasi versal deformation, is the pair $\mathcal{W}(g) = (\mathcal{W}_1, \mathcal{W}_0)$, where*

$$\mathcal{W}_1 = \{u : G = \frac{\partial G}{\partial z_i} = 0\},$$

and

$$\mathcal{W}_0 = \{u : G = \frac{\partial G}{\partial z_i} = x = 0\}.$$

Note that $\mathcal{W}_0$ is contained in $\mathcal{W}_1$ and it satisfies the constraint $x = 0$. Thus, in particular, the bifurcation diagrams of the classes B_k is $\mathcal{W}(B_k) = (\mathcal{W}_1, \mathcal{W}_0)$, where

$$\mathcal{W}_1 = \{(u_0, \dots, u_{k-1}) : u_0 = (\pm 1 \mp k)x^k - \sum_{i=2}^{k-1}(i-1)u_i x^i, u_1 = \mp k x^{k-1} - \sum_{j=2}^{k-1} j u_j x^{j-1}, \quad x \in \mathbb{R}\},$$

and

$$\mathcal{W}_0 = \{(u_0, \dots, u_{k-1}) : u_0 = 0, u_1 = 0\},$$

On the other hand, the bifurcation diagram of the class $F_{2,3}$ is $\mathcal{W}(F_{2,3}) = (\mathcal{W}_1, \mathcal{W}_0)$, where

$$\mathcal{W}_1 = \{(u_0, u_1, u_2, u_3) : u_0 = \pm x^2 \pm 2y_1^3 + u_3 x y_1, u_1 = \mp 2x - u_2 y_1, u_2 = \mp 3y_1^2 - u_3 x, \quad x, y_1, u_3 \in \mathbb{R}\},$$

and

$$\mathcal{W}_0 = \{(u_0, u_1, u_2, u_3) : u_0 = \pm 2y_1^3, u_1 = -u_2 y_1, u_2 = \mp 3y_1^2, \quad y_1, u_3 \in \mathbb{R}\}.$$

Theorem 3.1. *The stationary algebra of $\mathcal{W}(B_k)$, for $k = 2, 3, 4$, and $\mathcal{W}(F_{2,3})$ is described as follows.*

(1) $Der(-log\mathcal{W}(B_2))$ is generated by

$$\begin{aligned}
\xi_1 &= 0\frac{\partial}{\partial u_0} + (u_0 - 4u_1^2)\frac{\partial}{\partial u_1},\\
\xi_2 &= 2u_0\frac{\partial}{\partial u_0} + u_0^2\frac{\partial}{\partial u_1},\\
\xi_3 &= 2u_1\frac{\partial}{\partial u_0} + u_0 u_1\frac{\partial}{\partial u_1}.
\end{aligned}$$

(2) $Der(-log\mathcal{W}(B_3))$ is generated by

$$\begin{aligned}
\xi_1 &= 3u_0\frac{\partial}{\partial u_0} + 2u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2},\\
\xi_2 &= u_0 u_1\frac{\partial}{\partial u_0} + 2u_0 u_2\frac{\partial}{\partial u_1} + 3u_0\frac{\partial}{\partial u_2},\\
\xi_3 &= 3u_0 u_2\frac{\partial}{\partial u_0} + 18u_0\frac{\partial}{\partial u_1} + (12u_1 - 3u_2^2)\frac{\partial}{\partial u_2},\\
\xi_4 &= (u_1^2 - 3u_0 u_2)\frac{\partial}{\partial u_0} + (3u_1 - u_2^2)\frac{\partial}{\partial u_2}.
\end{aligned}$$

(3) $Der(-log\mathcal{W}(B_4))$ is generated by

$$\begin{aligned}
\xi_1 &= 4u_0\frac{\partial}{\partial u_0} + 3u_1\frac{\partial}{\partial u_1} + 2u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3},\\
\xi_2 &= u_0 u_1\frac{\partial}{\partial u_0} + 2u_0 u_2\frac{\partial}{\partial u_1} + 3u_0 u_3\frac{\partial}{\partial u_2} + 4u_3\frac{\partial}{\partial u_3},\\
\xi_3 &= 2u_0 u_3\frac{\partial}{\partial u_0} + 24u_0\frac{\partial}{\partial u_1} + (18u_1 - 2u_2 u_3)\frac{\partial}{\partial u_2} + (12u_2 - 4u_3^2)\frac{\partial}{\partial u_3},\\
\xi_4 &= 4u_0 u_2\frac{\partial}{\partial u_0} + 18u_0 u_3\frac{\partial}{\partial u_1} + (24u_0 - 4u_2^2 + 12u_1 u_3)\frac{\partial}{\partial u_2} + (18u_1 - 2u_2 u_3)\frac{\partial}{\partial u_3},\\
\xi_5 &= (3u_1^2 - 8u_0 u_2)\frac{\partial}{\partial u_0} \qquad + (9u_1 u_3 - 4u_3^2)\frac{\partial}{\partial u_2} + (12u_1 - 2u_2 u_3)\frac{\partial}{\partial u_3}.
\end{aligned}$$

(4) $Der(-log\mathcal{W}(F_{2,3}))$ is generated by

$$\xi_1 = 6u_0\frac{\partial}{\partial u_0} + 3u_1\frac{\partial}{\partial u_1} + 4u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3},$$

$$\xi_2 = (3u_1^2 - u_2u_3^2)\frac{\partial}{\partial u_0} + (6u_1 - u_3^3)\frac{\partial}{\partial u_1} + 6u_1u_3\frac{\partial}{\partial u_2} + 6u_3\frac{\partial}{\partial u_3},$$

$$\xi_3 = (-16u_1u_2u_3 + 6u_0u_3^2)\frac{\partial}{\partial u_0} + (-16u_2u_3 - 5u_1u_3^3)\frac{\partial}{\partial u_1} + (24u_1^2 - 4u_2u_3^2)\frac{\partial}{\partial u_2} + (48u_1 + u_3^3)\frac{\partial}{\partial u_3},$$

$$\xi_4 = (-16u_1u_2^2 - 18u_0u_2u_3)\frac{\partial}{\partial u_0} + (-32u_2^2 + 3u_1u_2u_3 - 24u_0u_3^2)\frac{\partial}{\partial u_1} + (72u_0u_1 - 12u_2^2u_3)\frac{\partial}{\partial u_2} + (144u_0 + u_2u_3^2)\frac{\partial}{\partial u_3},$$

$$\xi_5 = (-64u_2^2 + 48u_1u_2u_3 - 18u_0u_3^2)\frac{\partial}{\partial u_0} + (-32u_2u_3 + 23u_1u_3^2)\frac{\partial}{\partial u_1} + (288u_0 - 72u_1^2 + 12u_2u_3^2)\frac{\partial}{\partial u_2} + 5u_3^3\frac{\partial}{\partial u_3},$$

$$\xi_6 = (9u_1^3 - 8u_2^2u_3 + 3u_1u_2u_3^2)\frac{\partial}{\partial u_0} + (72u_0 - 2u_2u_3^2)\frac{\partial}{\partial u_1} + (36u_0u_3 + 9u_1^2u_3 + 3u_2u_3^3)\frac{\partial}{\partial u_2} + (8u_2 - 6u_1u_3)\frac{\partial}{\partial u_3}.$$

Proof. Let $p_1, p_2 \in \mathcal{E}_r$. Assume that p_1 is the defining equation of $\mathcal{W}_1$ and p_1, p_2 are the defining equations of $\mathcal{W}_0$. Let $I(\mathcal{W}_1)$ be the ideal generated by p_1 and $I(\mathcal{W}_0)$ is the ideal generated by p_1 and p_2.

Let $\Theta(\mathcal{W}_1)$ be the module of all vector fields $\xi = \sum_{i=1}^{r} \xi_i \frac{\partial}{\partial u_i}$ on $\mathbb{R}^r$ such that $\xi(I(\mathcal{W}_1)) \subseteq I(\mathcal{W}_1)$. To find $\Theta(\mathcal{W}_1)$, we have to solve the equation

$$\sum_{i=1}^{r} \xi_i \frac{\partial p_1}{\partial u_i} = qp_1,$$

for ξ_i and $q \in \mathcal{E}_r$. Now consider the map $\phi : \mathcal{E}_r^{r+1} \to \mathbb{R}$, given by

$$\Phi(\xi, q) = \sum_{i=1}^{r} \xi_i \frac{\partial h_1}{\partial u_i} - qh_1,$$

where $\xi = (\xi_1, \ldots, \xi_r) \in \mathcal{E}_r^r$ and $q \in \mathcal{E}_r^1$. Let $K = \ker\Phi$ and $\pi : \mathcal{E}_r^{r+1} \to \mathcal{E}_r^r$ be defined by $\pi(\xi, q) = \xi$. Then $\Theta(\mathcal{W}_1) = K$. Using the syzygies that are supplied in the Singular software package, we are able to obtain the K.

Next, we are looking for the module $\Theta(\mathcal{W}_1)$ of all vector fields such that $\xi(I(\mathcal{W}_0)) \subseteq I(\mathcal{W}_0)$. This implies that, for each $j = 1, 2$, we have to solve

$$\sum_{i=1}^{r} \xi_i \frac{\partial p_j}{\partial u_i} = \sum_{i=1}^{2} q_i p_i,$$

for $\xi = \sum\limits_{i=1}^{r} \xi_i \frac{\partial}{\partial u_i}$ and q_i.

For $j = 1, 2$, let $\Phi_j : \mathcal{E}_r^{r+2} \to \mathbb{R}$ be the map that is defined as

$$\Phi_j(\xi, \tilde{q}) = \sum_{i=1}^{r} \xi_i \frac{\partial p_j}{\partial u_i} - \sum_{i=1}^{2} q_i p_i,$$

where $\xi = (\xi_1, \ldots, \xi_r) \in \mathcal{E}_r^r$ and $\tilde{q} = (q_1, q_2) \in \mathcal{E}_r^2$. Let $K_j = \ker \Phi_j$. Set $\pi : \mathcal{E}_r^{r+2} \to \mathcal{E}_r^r$ be defined by $\pi(\xi, \tilde{q}) = \xi$. Let $S_j = \pi(K_j)$. Therefore,

$$\Theta(\mathcal{W}_0) = S_1 \bigcap S_2.$$

Again we use the syzygies to obtain the K_i. Finally, we have

$$\Theta(\mathcal{W}) = \Theta(\mathcal{W}_1) \bigcap \Theta(\mathcal{W}_0).$$

All of the vector fields ξ that are created by this approach can be verified to be liftable. This means that there is a vector field η on $\mathbb{R}^{r-1}$ that is such that $dp_i(\eta) = \xi \circ p_i$, and as a result, they are tangent to $\mathcal{W}$. □

In (3.1), if we set $\omega_0 = 1$ and $\omega_i \in \mathcal{M}_z$, then the space $\mathbb{R}^{r-1}$ in the local coordinates $u_1, \ldots, u_{r-1}$ is known as the principle of a shortened quasi-boundary versal deformation of g.

Consider the projection map $\pi : \mathbb{R}^r \to \mathbb{R}^{r-1}, (u_0, u_1, \ldots u_{r-1}) \mapsto (u_1, \ldots u_{r-1})$.

Definition 6. *[15] The caustic of a a versal deformation of a function g with respect to the quasi equivalence relation is a hypersurface in* $\mathbb{R}^{r-1}$ *which is a union*

$$\Sigma_1 \cup \Sigma_0,$$

which will be denoted by Σ^*, *where* $\Sigma_1 0 =$ *the* π*-image of the singular points of* $\mathcal{W}_1$ *and* $\Sigma_0 = \pi(\mathcal{W}_0)$.

Recall from [15] (applying Definition [6]) that the caustic of B_k is a union of two hypersurfaces which are tangent to each other, the first one is the set

$$\Sigma_1 = \{(u_1, \ldots, u_{k-1} : u_1 = \mp k x^{k-1} - \sum_{i=2}^{k-1} j u_i x^{i-1}, u_2 = \mp \frac{k(k-1)}{2} x^{k-2} - \sum_{j=3}^{k-1} \frac{j(j-1)}{2} u_j x^{j-2} \quad x \in \mathbb{R}\},$$

that is a cylindrical generalized swallow tail over a line. The second one of them is the submanifold of the greatest dimension that crosses the edge of Σ_0., in particular:

$$\Sigma_0 = \{u_1, \ldots, u_{k-1} : u_1 = 0\}.$$

Furthermore, the caustic of the class $F_{2,3}$ is a union of the smooth surface

$$\Sigma_1 = \{(u_1, u_2, u_3) : u_1 = \mp 2x - \frac{u_3^3}{12}, u_2 = \mp \frac{3}{144} u_3^4 - u_3 x, \quad x, u_3 \in \mathbb{R}\},$$

and Whitney Umbrella

$$\Sigma_0 = \{(u_1, u_2, u_3) : u_1 = -u_3 y_1, u_2 = \mp 3 y_1^2, \quad y_1, u_3 \in \mathbb{R}\}.$$

Theorem 3.2. *The module $Der(-log(\Sigma^*)$ is generated by the vector fields*

(1) For the caustic of B_3

$$\xi_1 = 2u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2},$$
$$\xi_2 = 2u_1u_2\frac{\partial}{\partial u_1} + 3u_1\frac{\partial}{\partial u_2}.$$

(2) For the caustic of B_4

$$\xi_1 = 3u_1\frac{\partial}{\partial u_1} + 2u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3},$$
$$\xi_2 = (-4u_2^2 + 9u_1u_2)\frac{\partial}{\partial u_2} + (12u_1 - 2u_2u_3)\frac{\partial}{\partial u_2},$$
$$\xi_2 = (8u_1 - 3u_2u_3)\frac{\partial}{\partial u_2} + (16u_2 - 6u_3^2)\frac{\partial}{\partial u_2}.$$

(3) For the caustic of $F_{2,3}$

$$\xi_1 = (48u_1 - 12u_2u_3)\frac{\partial}{\partial u_1} + (24u_2 + 2u_3^3)\frac{\partial}{\partial u_2} + 0\frac{\partial}{\partial u_3},$$
$$\xi_2 = (6u_2^2 - 2u_1u_3^2)\frac{\partial}{\partial u_1} + (-4u_1u_3 - 2u_2u_3^2)\frac{\partial}{\partial u_2} + 12u_2\frac{\partial}{\partial u_3},$$
$$\xi_3 = (12u_2u_3)\frac{\partial}{\partial u_1} + (12u_2 - 2u_3^3)\frac{\partial}{\partial u_2} + 12u_3\frac{\partial}{\partial u_3},$$
$$\xi_4 = (3u_2^2u_3 + u_1u_3^3)\frac{\partial}{\partial u_1} + (6u_2^2 + 2u_1u_3^2)\frac{\partial}{\partial u_2} + 0\frac{\partial}{\partial u_3}.$$

Proof. Assume that $\Theta(\Sigma_1)$ and $\Theta(\Sigma_0)$ are the modules of vector fields that are that are tangential to Σ_1 and Σ_0, respectively.

To determine $\Theta(\Sigma_i), i = 0, 1$, we employ similar processes as those used in the proof of Theorem 3.1 to determine $\Theta(\mathcal{W}_1)$.

Hence, we have,

$$\Theta(\Sigma^*) = \Theta(\Sigma_1)\bigcap\Theta(\Sigma_0).$$

All such vector fields $\xi \in \Theta(\Sigma^*)$ are liftable and as a result, they are tangent to Σ^*. □

Next, we investigate whether or not $\mathcal{W}$ and Σ^* are considered to be free divisors. Recall that a hypersurface $V = \{h = 0\} \subset (\mathbb{K}^n, 0)$ with a reduced defining ideal $I(V)$, is called *free divisor*, in the sense of Saito, if $Der(-logV)$ is a free $\mathcal{E}_n$-module, necessarily, so its rank is equal to n. The following criterion was established also by K. Saito and is now commonly referred to after him.

Proposition 2. *[21][Saito's Criterion]* Let $h \in \mathbb{K}[z]$ be reduced. Then, h defines a free divisor if and only if there exists $n \times n$ matrix H with entries in $\mathbb{K}[z]$ such that

$$det(H) = h \quad \text{and} \quad (\nabla h)H \equiv 0 \; mod(h).$$

Here, $\nabla h = \left(\frac{\partial h}{\partial z_1}, \ldots, \frac{\partial h}{\partial z_n}\right)$ and the last condition just expresses that each entry of the (row) vector $(\nabla h)H$ is divisible by h in $\mathbb{K}[z]$. The columns of H can then be viewed as the coefficients of a basis, with respect to the partial derivatives $\frac{\partial}{\partial z_i}$, of the logarithmic vector fields along the divisor $h = 0$. The matrix H is called a discriminant or Saito matrix of h.

The above discussion and Theorem 3.1 implies that the bifurcation diagrams of the classes $B_k, k = 2, 3, 4$ and class $F_{2,3}$ are not free divisors due to the cardinality of a basis of $Der(-log\mathcal{W}(B_k))$, and $Der(-log\mathcal{W}(F_{2,3}))$, respectively. Moreover,

Proposition 3. *The caustics of* $B_i, i = 3, 4$ *are free divisors.*

Proof. The caustic of the B_3 class is a union of the parabola $3u_1 - u_2^2 = 0$ and the line $u_1 = 0$. Hence, the defining equation of $\Sigma^* \subset \mathbb{R}^2$ for B_3 is $h_1 = u_1(3u_1 - u_2^2)$. On the other hand, by similar consideration we find that the defining equation of the caustic $\Sigma^* \subset \mathbb{R}^3$ for the class B_4 is $h_2 = u_1(108u_1^2 + 32u_2^3 - 108u_1u_2u_3 - 9u_2^2u_3^2 + 27u_1u_3^3)$. Applying Saito's Criterion and Theorem 3.2, one can easily show that the caustics of $B_i, i = 3, 4$ are free divisors. □

4. Geometric cuspidal edge with smooth boundary

Let G be a general cuspidal edge in $\mathbb{R}^3$ with local coordinates $z = (u, v, w)$, having the following parametrization at the origin as in [18]:

$$f : U \subseteq \mathbb{R}^2, 0 \to \mathbb{R}^3,$$

$$(t, s) \mapsto \left(t, a(t) + \frac{1}{2}s^2, b_1(t) + s^2 b_2(t) + s^3 b_3(t, s)\right),$$

where U is an open set in $\mathbb{R}^2$ and

$$\begin{aligned}
a(t) &= \frac{1}{2}a_{20}t^2 + \frac{1}{6}a_{30}t^3 + \frac{1}{24}a_{40}t^4 + O(5);\\
b_1(t) &= \frac{1}{2}b_{20}t^2 + \frac{1}{6}b_{30}t^3 + \frac{1}{24}b_{40}t^4 + O(5);\\
b_2(t) &= \frac{1}{2}b_{12}t + \frac{1}{6}b_{22}t^2 + \frac{1}{24}b_{32}t^3 + O(4);\\
b_3(t, s) &= \frac{1}{6}b_{03} + \frac{1}{6}b_{13}t + \frac{1}{24}b_{04}s + \frac{1}{24}b_{23}t^2 + \frac{1}{24}b_{14}ts + \frac{1}{120}b_{05}s^2 + O(3),
\end{aligned}$$

and $a_{ij}, b_{ij} \in \mathbb{R}$. Let $B \subseteq G$ be a distinguished smooth curve which is parametrized by:

$$\gamma_1(t) = f(t, t) = \left(t, a(t) + \frac{t^2}{2}, b_1(t) + t^2 b_2(t) + t^3 b_3(t, t)\right),$$

i.e., the image of the line $s = t$. The pair $\widetilde{G} = (G, B)$ will be called *a geometric cuspidal edge with a smooth boundary.* Note that $\widetilde{G}$ is diffeomorphic to $\mathcal{W}(B_3)$.

The set of critical points of G is $\{s = 0\}$, and hence the singular set of G is the curve:

$$\gamma_0(t) = f(t, 0) = (t, a(t), b_1(t)),$$

which will be denoted by Σ.

Denote by $(\kappa_\Sigma(0), \kappa_B(0))$ and $(\tau_\Sigma(0), \tau_B(0))$ the pairs of the curvatures and torsions of the pair of space curves (Σ, B), respectively.

Proposition 4. *(1)* $(\kappa_\Sigma(0), \kappa_B(0)) = (\sqrt{a_{20}^2 + b_{20}^2}, \sqrt{(a_{20}+1)^2 + b_{20}^2})$.

(2) $(\tau_\Sigma(0), \tau_B(0)) = (\frac{a_{20}b_{30}-b_{20}a_{30}}{\kappa_\Sigma^2(0)}, \frac{(a_{20}+1)(b_{30}+b_{03}+3b_{12})-b_{20}a_{30}}{\kappa_B^2(0)})$.

(3) The osculating planes of Σ *and* B *at the origin are orthogonal to the vectors* $(0, -b_{20}, a_{20})$ *and* $(0, -b_{20}, a_{20}+1)$*, respectively.*

Proof. The results are obtained via the standard rules of calculating curvature and torsion on space curves. □

Note that the two curves Σ and B have a common tangent line, and hence the tangential direction, which will be denoted by T_d at 0, is parallel to the vector $(1, 0, 0)$. Note that $w = 0$ is the plane that represents the tangent cone L_d to $\widetilde{G}$.

4.1. Functions on a cuspidal edge equipped with a smooth curve

Consider the pair $(\widetilde{f}, \widetilde{\gamma}_1)$ which consists of the $\mathcal{A}$-normal form of a cuspidal edge:

$$\widetilde{f} : U \subseteq \mathbb{R}^2, 0 \to \mathbb{R}^3,$$

$$(t, s) \mapsto (t, \frac{1}{2}(s^2 - t^2), s^3 + t^3 - 2ts^2),$$

and the smooth curve $\widetilde{\gamma}_1(t) = \widetilde{f}(t, t) = (t, 0, 0)$. Let $\mathcal{V}_1 = \widetilde{f}(U \subseteq \mathbb{R}^2)$ and $\mathcal{V}_0 = \widetilde{\gamma}_1(\mathbb{R})$. Then, the pair $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_0)$ will be considered as a model of a cuspidal edge with a smooth curve, which serves as a boundary. (See Figure 1).

Figure 1. A model of a cuspidaledge with a boundary (the blue line).

Recall that the bifurcation diagrams of the B_3 class is the pair $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_0)$, where $\mathcal{W}_1$ is the cuspidal edge, which is parameterized by

$$(t, s) \mapsto (t, -3s^2 - 2ts, -2s^3 - ts^2),$$

and hence it is $\mathcal{A}$-equivalent to $\widetilde{f}$, via the right change of coordinates $t \mapsto t$, $s \mapsto s - \frac{1}{3}t$, followed by normalising the coefficients. Hence, the pair $\mathcal{V}$ is diffeomorphic to $\mathcal{W}$.

The defining equation of $\mathcal{V}_1$ is

$$K(z) = w^2 - v^3 + 4uvw + u^2v^2 + 2u^3w + u^4v = 0,$$

while the defining equations of $\mathcal{V}_0$ are $K(z) = 0, v = 0$ and $w = 0$.

Definition 7. We say that $g_1, g_2 \in \mathcal{E}_z$ are $\mathcal{R}(\mathcal{V})$-equivalent whenever $g_2 = g_1 \circ \Phi$ where $\Phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ is diffeomorphism-germ and Φ preserves $\mathcal{V}$, i.e., $\Phi(\mathcal{V}_1) \subseteq \mathcal{V}_1$ and $\Phi(\mathcal{V}_0) \subseteq \mathcal{V}_0$.

Define $\Theta(\mathcal{V})$ as the module over $\mathcal{E}_z$ of vector fields in $\mathbb{R}^3$ that is tangential to $\mathcal{V}$ and define

$$\Theta^0(V) = \{\xi \in \Theta(\mathcal{V}) : \xi(0) = 0\}.$$

Then, the tangent space and the expanded tangent space to the orbit of g at g are

$$L\mathcal{R}(\mathcal{V}).g = \{\xi(g) : \xi \in \Theta^0(\mathcal{V})\},$$

and

$$L_e\mathcal{R}(\mathcal{V}).g = \{\xi(g) : \xi \in \Theta(\mathcal{V})\},$$

respectively.

The $\mathcal{R}_e^+$-codimension of g is defined as $d(g, \mathcal{R}_e^+(V)) = dim_{\mathbb{R}} (\mathcal{M}_z / L_e\mathcal{R}(\mathcal{V}).g)$.

Theorem 4.1. *The $\mathcal{E}_z$-module $\Theta(\mathcal{V})$ is generated by*

$$\begin{aligned}
\xi_1 &= u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v} + 3w\frac{\partial}{\partial w},\\
\xi_2 &= v\frac{\partial}{\partial u} - 2uv\frac{\partial}{\partial v} + (u^2v - 2v^2)\frac{\partial}{\partial w},\\
\xi_3 &= (3w - u^3)\frac{\partial}{\partial u} - (6uw + 2u^2v)\frac{\partial}{\partial v} - 6vw\frac{\partial}{\partial w},\\
\xi_4 &= 7u^2\frac{\partial}{\partial u} + (6w + 20uv)\frac{\partial}{\partial v} + (9v^2 - 12vu^2)\frac{\partial}{\partial w}.
\end{aligned}$$

Proof. By following the same procedures as that to prove Theorem 3.1. □

Corollary 1. *For $g \in \mathcal{E}_z$, the tangent space to the orbit of g atg with respect to the $\mathcal{R}(\mathcal{V})$-equivalence relation is defined as*

$$L\mathcal{R}(\mathcal{V}).g = \{[uA_1 + vA_2 + (3u - u^3)A_3 + 7u^2A_4]\frac{\partial g}{\partial u} + [2vA_1 - 2uvA_2$$

$$-(6uw + 2u^2v)A_3 + (6w + 20uv)A_4]\frac{\partial g}{\partial v} + [3wA_1 + (u^2v - 2v^2)A_2 - 6vwA_3 + (9v^2 - 12vu^2)A_4]\frac{\partial g}{\partial w} : A_i \in \mathcal{E}_z\}.$$

We proceed to classify submersion-germs $g : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$. g from $(\mathbb{R}^3, 0)$ to $(\mathbb{R}, 0)$ in accordance with the $\mathcal{R}(\mathcal{V})$-equivalence relation, where $d(g, \mathcal{R}_e^+(V)) \leq 2$. The classification method and prenormal forms are described in the following lemma and relies on Arnold's spectral sequence. At first, we go over several concepts from [22].

Let us assume that we have a certain appropriate Newton diagram Γ that is a subset of the non-negative integers $\mathbb{Z}^n_{\geq 0}$. Each face Γ_i of Γ corresponds to a specific quasihomogeneity type $\boldsymbol{\alpha}_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in})$. In this type, the monomials $\boldsymbol{x}^{\boldsymbol{k}} = x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$ with exponents lying on Γ_i have a degree of one, meaning that $\langle \boldsymbol{k}, \boldsymbol{\alpha}_i \rangle = \alpha_{i1} k_1 + \cdots + \alpha_{in} k_n = 1$.

A monomial, denoted as $\boldsymbol{x}^{\boldsymbol{k}}$, is considered to have a Newton degree of d if d is the minimum value obtained from the inner product of $\boldsymbol{k}$ and $\boldsymbol{\alpha}_j$. The monomials of Newton degree d are precisely those whose exponents are contained in the diagram $d\Gamma$, which is created by scaling Γ by a factor of d.

The smallest of the Newton degrees of the monomials that appear in a power series is known as the Newton order d. An ideal $\mathcal{S}_j$ in the ring $\mathcal{E}_x$ is formed by the series of order at least d. The Newton filtration in $\mathcal{E}_x$ is generated by the ideals $\mathcal{S}_j$. More precisely, $\mathcal{S}_0 = \mathcal{E}_x$, and $\mathcal{S}_k \subseteq \mathcal{S}_l$ whenever $k > l$.

The principal part of a power series g of order d is the sum of the terms of Newton degree d.

Let $g \in \mathcal{E}_z$. We may decompose g into its principal portion g_0 of Newton degree being N and greater order elements $\widetilde{g}$ as $g = g_0 + \widetilde{g}$. It is assumed that the $\mathcal{R}^+_e$-codimension of g_0 is finite, meaning that $d(g_0, \mathcal{R}^+_e(\mathcal{V})) < \infty$. The subsequent result is a rendition of Lemma 8.1 in [12] and Lemma 2.10 in [13].

Lemma 1. *Consider a monomial basis of the linear space $\mathcal{E}_z / L\mathcal{R}(\mathcal{V}).g_0$ and let $\rho_1(z), \rho_2(z), \ldots, \rho_s(z)$ be the subset of generators that have Newton degrees greater than N.*

Assume that for every $\omega \in \mathcal{S}_\beta \setminus \mathcal{S}_{>\beta}$, $\beta > N$:

(1) There is a vector field $\xi = \dot{u}\frac{\partial}{\partial u} + \dot{v}\frac{\partial}{\partial v} + \dot{w}\frac{\partial}{\partial w} \in \Theta(\mathcal{V})$, such that

$$\omega = \frac{\partial g_0}{\partial u}\dot{u} + \frac{\partial g_0}{\partial v}\dot{v} + \frac{\partial g_0}{\partial w}\dot{w} + \widehat{\omega} + \sum_{i=1}^{s} c_i \rho_i(z),$$

where $\widehat{\omega} \in \mathcal{S}_{>\beta}$ and $c_i \in \mathbb{R}$.

(2) Furthermore, given any δ, if $N < \delta < \beta$, and for every $\psi \in \mathcal{S}_\delta$, the following statement

$$E(\psi, \omega) = \frac{\partial \psi}{\partial u}\dot{u} + \frac{\partial \psi}{\partial v}\dot{v} + \frac{\partial \psi}{\partial w}\dot{w},$$

belongs to $\mathcal{S}_\beta$.

Then any germ $g = g_0 + \widetilde{g}$ is $\mathcal{R}(\mathcal{V})$-equivalent to a germ $g_0 + \sum_{i=1}^{s} d_i \rho_i$, where $d_i \in \mathbb{R}$.

Remark 3. By eliminating the prerequisite that g_0 has a finite codimension, the proof of Lemma 1 demonstrates that any function $g = g_0 + \widetilde{g}$ is $\mathcal{R}(\mathcal{V})$-equivalent to a comparable form:

$$g_0 + \sum d_i \rho_i + \Lambda,$$

where Λ is a member of a relatively large power of the maximal ideal.

Definition 8. *The families of germs of functions $H_1, H_2 : (\mathbb{R}^3 \times \mathbb{R}^l, (0,0)) \to (\mathbb{R}, 0)$ are referred to as P-$\mathcal{R}^+(\mathcal{V})$.-equivalent if*

$$H_2(z, u) = H_1 \circ \Phi(z, u) + C(u),$$

where $\Phi : (\mathbb{R}^3 \times \mathbb{R}^l, (0,0)) \to (\mathbb{R}^3 \times \mathbb{R}^l, (0,0))$ is a germ of diffeomorphism with $\Phi(z, u) = (\varphi(z, u), \chi(u))$, and $C : (\mathbb{R}^l, 0) \to \mathbb{R}$ is a function germ.

Consider $G(z,u)$ as a deformation of $g \in \mathcal{E}_z$.Then, G is referred to as *versal* with regard to the $\mathcal{R}^+(\mathcal{V})$-equivalence relation if, for whatever other form of deformation H of g, there exists a map germ $\Phi(z,u)$ (which may not necessarily be a diffeomorphism) and C as described above, satisfying

$$H(z,u) = G \circ \Phi(z,u) + C(u).$$

Proposition 5. *A deformation G of g on $\mathcal{V}$ is $\mathcal{R}^+(\mathcal{V})$-versal provided*

$$L\mathcal{R}(\widetilde{X}).g + \mathbb{R}\{1, \dot{G}_1, \ldots \dot{G}_l\} = \mathcal{E}_z,$$

where $\dot{G}_i$ are the initial speeds of G.

The result that follows is a description of the classification of germs of submersions in accordance with the $\mathcal{R}(\mathcal{V})$-equivalence relation.

Theorem 4.2. *Let $\mathcal{V}$ be a pair consisting of the cuspidal edge $\mathcal{V}_1$, which is represented by the parametrization $\widetilde{f}(t,s) = (t, \frac{1}{2}(s^2 - t^2), s^3 + t^3 - 2ts^2)$, and a distinguished smooth curve $\mathcal{V}_0$ within it, represented by the parametrization $\widetilde{\gamma}(t)_1 = \widetilde{f}(t,t) = (t,0,0)$. Then, any function germ g at 0 that has a $\mathcal{R}^+(\mathcal{V})$-codimension of not more than 2 (with moduli) is equivalent to a function germ mentioned in the Table 1.*

Table 1. Submersion germs in $\mathcal{M}_z$ of $\mathcal{R}^+(\mathcal{V})$-codimension ≤ 2.

Equivalent Germ	Constraints	$d(g, \mathcal{R}^+(\mathcal{V}))$	mini-versal unfolding
$\pm u$	–	0	$\pm u$
$\pm v + \epsilon u^2,^1$	$\epsilon \neq 0, \pm 1$	1	$\pm v + \epsilon u^2 + \lambda_1 u$
$\pm v + \epsilon u^3,^1$	$\epsilon \neq 0$	2	$\pm v + \epsilon u^3 + \lambda_1 u + \lambda_2 u^2$
$\pm w + \epsilon u^2,^1$	$\epsilon \neq 0$	2	$\pm w + \epsilon u^2 + \lambda_1 u + \lambda_2 v$

*Note: The symbol ϵ represents a modulus, while the codimension refers to the dimension of the stratum.

Proof. The linear transformations of coordinates derived via the integration of the 1-jets of the vector fields in $\Theta(\mathcal{V})$ are:

$$\begin{aligned}
\varphi_1(z) &= (e^{c_1}u, e^{2c_1}v, e^{3c_1}w);\\
\varphi_2(z) &= (u + c_2 v, v, w);\\
\varphi_3(z) &= (u + c_3 w, v, w);\\
\varphi_3(z) &= (u, v + c_4 w, w);
\end{aligned}$$

where $c_i \in \mathbb{R}$.

Let g be decomposed into its 1-jet $g_0 = au + bv + cw$, where $a, b, c \in \mathbb{R}$ and $\widetilde{g} \in \mathcal{M}_z^2$. Using φ_i, one can show that the orbits of the space of 1-jets are $\pm u$, $\pm v$, and $\pm w$.

The subsequent conclusions may be established by applying Lemma 1 and Remark 3.

Let $g_0 = \pm u$. Then, the tangent space to the orbit of g_0 at g_0 is

$$L\mathcal{R}(\mathcal{V}).g_0 = \{uA_1 + vA_2 + (3w - u^3)A_3 + 7u^2 A_4 : A_i \in \mathcal{E}_z\}.$$

Clearly, we have *mod* $L\mathcal{R}(\mathcal{V}).g_0$: $u \equiv 0, v \equiv 0$ and $w \equiv 0$. Hence, Lemma 3 implies that g is $\mathcal{R}(\mathcal{V})$-equivalent to its principal part $g_0 = \pm u$.

Next, consider the principal part $g_0 = \pm v$. Then,

$$L\mathcal{R}(\mathcal{V}).g_0 = \{2vA_1 - 3uvA_2 - (6uw + vu^2)A_3 + (6w + 20uv)A_4 : A_i \in \mathcal{E}_z\}.$$

Therefore, we have *mod* $L\mathcal{R}(\mathcal{V}).g_0$: $v \equiv 0$ and $w \equiv 0$. It follows that

$$\mathcal{E}_z/L\mathcal{R}(\mathcal{V}).g_0 \simeq \{q(u) : q \in \mathcal{E}_u\}.$$

Using Remark 3 and taking into account the constraints on $\widetilde{g}$, the germ g is reduced to the form $h = \pm v + \widetilde{h}(u)$, where $\widetilde{h} \in \mathcal{M}_u^2$. Let $\widetilde{h} = d_2u^2 + d_3u^3 + \ldots$, $d_i \in \mathbb{R}$. If $d_2 \neq 0$, then h is $\mathcal{R}(\mathcal{V})$-equivalent to the germ $\pm v + \epsilon u^2$, where $0, \pm 1 \neq \epsilon \in \mathbb{R}$ (modulus) and its mini-versal deformation may be taken as $\pm v \pm u^2 + \lambda_1 u$. Next, if $d_2 = 0$ but $d_3 \neq 0$, then h is $\mathcal{R}(\mathcal{V})$-equivalent to the germ $\pm v + \epsilon u^3$, where $0 \neq \epsilon \in \mathbb{R}$ (modulus) and its mini-versal deformation may be taken as $\pm v + \epsilon u^3 + \lambda_1 u + \lambda_2 u^2$.

Finally, consider the 1-jet $g_0 = \pm w$. Then,

$$L\mathcal{R}(\mathcal{V}).g_0 = \{3wA_1 + (u^2v - 2v^2)A_2 - 6vwA_3 + (9v^2 - 12vu^2)A_4 : A_i \in \mathcal{E}_z\}.$$

Therefore, we have *mod* $L\mathcal{R}(\mathcal{V}).g_0$:

$$w \equiv 0, \tag{4.1}$$

$$u^2v - 2v^2 \equiv 0, \tag{4.2}$$

and

$$9v^2 - 12vu^2 \equiv 0. \tag{4.3}$$

Clearly, relations (4.2) and (4.3) are linearly independent. Hence, $v^2 \equiv 0$ and $vu^2 \equiv 0$. Consequently

$$\mathcal{E}_z/L\mathcal{R}(\mathcal{V}).g_0 \simeq \{a_1v + a_2uv + q(u) : q \in \mathcal{E}_u, a_i \in \mathbb{R}\}.$$

Using Remark 3 and taking into account the constraints on $\widetilde{g}$, the germ g is reduced to the form $h = \pm w + a_2uv + \widetilde{h}(u)$, where $\widetilde{h} \in \mathcal{M}_u^2$. If $\widetilde{h}$ contains $\tilde{d}_3u^2$, where $0 \neq \tilde{d}_3 \in \mathbb{R}$, then h is equivalent to $\pm w + \epsilon u^2$, $0 \neq \epsilon \in \mathbb{R}$ (modulus) and its mini-versal deformation may be taken as $\pm w + \epsilon u^2 + \lambda_1 u + \lambda_2 v$. If $\tilde{d}_3 = 0$, then in the most degenerate case h has codimension greater than 2. The proof of the theorem is now complete. □

4.2. *The discriminants of the deformations*

Let $F : (\mathbb{R}^3 \times \mathbb{R}^2, 0) \to \mathbb{R}; (z, \lambda) \mapsto F(z, \lambda)$ be a deformation of a germ $h(z)$ on $\mathcal{V}$ and consider the family $P(s, t, \lambda) = F(\widetilde{f}(t, s), \lambda)$, where $\widetilde{f}(s, t) = (t, \frac{1}{2}(s^2 - t^2), s^3 + t^3 - 2ts^2)$. Then, we define the following types of discriminants:

(1) The discriminant of the family P, everywhere:

$$\mathbb{D}_1 = \{(\lambda, P) : \frac{\partial P}{\partial t} = \frac{\partial P}{\partial s} = 0 \quad \text{at} \quad (t, s, \lambda)\},$$

(2) The discriminant of P, restricted to Σ:

$$\mathbb{D}_2 = \{(\lambda, P) : \frac{\partial P}{\partial t} = 0 \quad \text{at} \quad (t, 0, \lambda)\},$$

and

(3) The discriminant of P, restricted to the boundary $\mathcal{V}_0$:

$$\mathbb{D}_3 = \{(\lambda, P) : \frac{\partial P}{\partial t} = 0 \quad \text{at} \quad (t, t, \lambda)\}.$$

We shall calculate $\mathbb{D}_i, i = 1, 2, 3$ for the mini-versal deformations $F(z, \lambda)$ of the submersions $g(z) = F(z, 0)$ in Table 1.

(1) $g(z) = u$. We have $F(z, \lambda) = u$, and hence $P = t$. Note that the fiber $g = 0$ is transverse for both T_d and L_d. Clearly, $\mathbb{D}_i, i = 1, 2$, are all empty sets.

(2) $g(z) = \pm v + \epsilon u^k, k = 2, 3$. Note that the tangent plane to the fiber $g = 0$ contains T_d but is transverse L_d.

- For $k = 2$, we have $F(z, \lambda) = \pm v + \epsilon u^2 + \lambda_1 u$, and hence $P = \pm(s^2 - t^2) + \epsilon t^2 + \lambda_1 t$. The $\mathbb{D}_1$ set is a smooth surface which is parametrized by

$$(t, \lambda_2) \mapsto (2(\pm 1 - \epsilon)t, \lambda_2, (\pm 1 - \epsilon)t^2).$$

The $\mathbb{D}_2$ set coincides with $\mathbb{D}_1$. On the other hand, on the boundary we have $P = F(\widetilde{f}(t, t), \lambda) = \epsilon t^2 + \lambda_1 t$. Therefore, the $\mathbb{D}_3$ set is also a smooth surface which is parametrized by:

$$(t, \lambda_2) \mapsto (- 2\epsilon t, \lambda_2, -\epsilon t^2).$$

Note that $\mathbb{D}_1 = \mathbb{D}_2$ and $\mathbb{D}_3$ are tangent along the λ_2-axis.

- For $k = 3$, we have $F(z, \lambda) = \pm v + \epsilon u^3 + \lambda_1 u + \lambda_2 u^2$. The $\mathbb{D}_1$ and $\mathbb{D}_2$ sets are coinciding cuspidal edge, which are parameterized by:

$$(t, \lambda_2) \mapsto (- 3\epsilon t^2 - 2(\lambda_2 \mp 1)t, \lambda_2, (\pm 1 - \lambda_2)t^2 - 2\epsilon t^3).$$

The $\mathbb{D}_3$ set is also cuspidal edge, that is parametrized by:

$$(t, \lambda_2) \mapsto (- 3\epsilon t^2 - 2\lambda_2 t, \lambda_2, -\lambda_2 t^2 - 2\epsilon t^3).$$

Note that $\mathbb{D}_1 = \mathbb{D}_2$ intersects $\mathbb{D}_3$ along a curve.

(3) $g = \pm w + \epsilon u^2$. We have $F = \pm w + \epsilon u^2 + \lambda_2 u + \lambda_2 v$. We may consider the versal deformation:

$$F(z, \lambda) = \pm w + \epsilon u^2 + \lambda_1 u + \lambda_2 v.$$

The tangent plane to $g = 0$ includes both T_d and L_d in this scenario. Now, we have

$$P = \pm(s^3 + t^3 - 2ts^2) + \epsilon t^2 + \lambda_1 t + \lambda_2(s^2 - t^2).$$

Note that $\frac{\partial P}{\partial s} = 0$ if and only if $s = 0$ or $\lambda_2 = \frac{3}{2}s - 2t$. Moreover, $\frac{\partial P}{\partial t} = 0$ if and only if $\lambda_1 = 2s^2 \mp 3t^2 - 2(\epsilon + \lambda_2)t$. Hence, the $\mathbb{D}_1$ set consists of two components, the first one is a cuspidal edge which is parametrized by

$$(t, \lambda_2) \mapsto (\mp 3t^2 - 2(\epsilon + \lambda_2)t, \lambda_2, \mp 2t^3 - (\epsilon + \lambda_2)t^2),$$

and the second one is a smooth surface which is parametrized by

$$(t, s) \mapsto (\lambda_1, \lambda_2, (-\frac{3}{2} \pm 1)s^3 + (\mp 2 - 6)t^3 - \epsilon t^2 + 2ts^2 - \frac{3}{2}st^2),$$

where $\lambda_1 = \pm 2s^2 + (\mp 3 - 4)t^2 - (2\epsilon + 3s)t$ and $\lambda_2 = \frac{3}{2}s - 2t$. The $\mathbb{D}_2$ set coincides with the first part of $\mathbb{D}_1$. The $\mathbb{D}_3$ is a regular surface that may be described by a parametrization

$$(t, \lambda_2) \mapsto (-2\epsilon t, \lambda_2, -\epsilon t^2).$$

We condense the above calculation in the following.

Proposition 6. *(1) The discriminants $\mathbb{D}_1$, $\mathbb{D}_2$ and $\mathbb{D}_3$ of the singularity $g = u$ are empty sets.*

(2) The discriminants $\mathbb{D}_1$ and $\mathbb{D}_2$ of the singularity $g = \pm v + \epsilon u^2$ are coincident smooth surface, and the $\mathbb{D}_3$ is also a smooth surface that is tangent to $\mathbb{D}_1$ along a curve (Figures 2 and 3).

(3) The discriminants $\mathbb{D}_1$ and $\mathbb{D}_2$ of the singularity $g = \pm v + \epsilon u^3$ are coincident cuspidal edges, and the $\mathbb{D}_3$ is a different cuspidal edge that is tangent to $\mathbb{D}_1$ along a curve (Figure 4).

(4) The discriminants $\mathbb{D}_1$ of the singularity $g = \pm w + \epsilon u^2$ is a combination of two components: a cuspidal edge and a regular surface (Figure 5). The $\mathbb{D}_2$ is a cuspidal edge and coincides with one of the components of the $\mathbb{D}_1$. The $\mathbb{D}_3$ is a smooth surface (Figure 6).

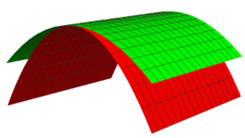

Figure 2. The discriminants $\mathbb{D}_1 = \mathbb{D}_2$ and $\mathbb{D}_3$ of the singularity $g = v + \epsilon u^2$.

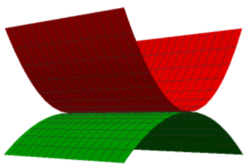

Figure 3. The discriminants $\mathbb{D}_1 = \mathbb{D}_2$ and $\mathbb{D}_3$ of the singularity $g = -v + \epsilon u^2$.

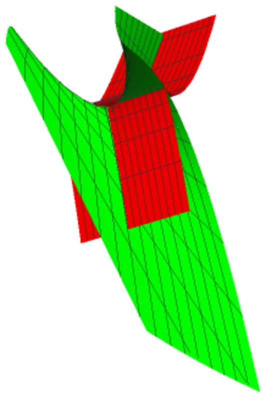

Figure 4. The discriminants $\mathbb{D}_1 = \mathbb{D}_2$ and $\mathbb{D}_3$ of the singularity $g = \pm v + \epsilon u^2$.

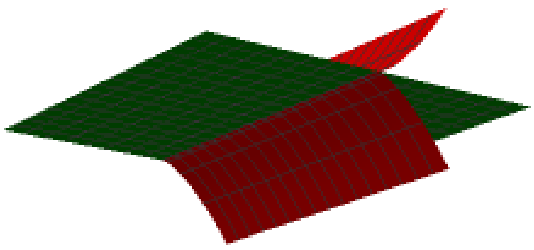

Figure 5. The $\mathbb{D}_1$ and $\mathbb{D}_2$ sets of the singularity $g = \pm w + \epsilon u^3$.

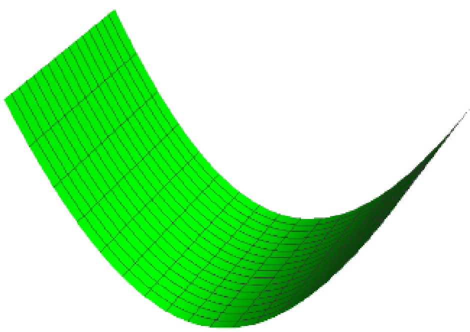

Figure 6. The $\mathbb{D}_3$ set of the singularity $g = \pm w + \epsilon u^2$.

4.3. The height functions on a geometric cuspidal edge with a smooth curve

Let $H : \widetilde{G} \times S^2 \to \mathbb{R}, H((t,s),\boldsymbol{\eta}) = H_{\boldsymbol{\eta}}(t,s) = f(t,s)\cdot\boldsymbol{\eta}$, be a family of height functions on $\widetilde{G}$, where S^2 is the 2-sphere and $\boldsymbol{\eta} = (\eta_1,\eta_2,\eta_3) \in \mathbb{R}^3$. Then, $H_{\boldsymbol{\eta}}$ measures the contact of $\widetilde{G}$ with the plane π_p which is orthogonal to the vector $\boldsymbol{\eta}$ at the point $p \in \widetilde{G}$. Generically, the submersions g, which are obtained in Theorem 4.2, describe explicitly such contact. The contact between the fiber $g = 0$ and the model $\mathcal{V}$ is equivalent to the contact between $\widetilde{G}$ and π_p.

We discuss the contact of π_p with $\widetilde{G}$ along Σ and B at the origin.

Note that the restriction of $H_{\boldsymbol{\eta}}$ along Σ and B are

$$H_{\boldsymbol{\eta}}(t,0) = t\eta_1 + \frac{1}{2}(a_{20}\eta_2 + b_{20}\eta_3)t^2 + \frac{1}{6}(a_{30}\eta_2 + b_{30}\eta_3)t^3 + O(4),$$

and

$$H_{\boldsymbol{\eta}}(t,t) = t\eta_1 + \frac{1}{2}[(a_{20}+1)\eta_2 + b_{20}\eta_3]t^2 + \frac{1}{6}[a_{30}\eta_2 + (b_{30}+3b_{12}+b_{03})\eta_3]t^3 + O(4),$$

respectively.

Clearly, $H_{\boldsymbol{\eta}}$ is singular when $\eta_1 = 0$. Further, the contact of $\widetilde{G}$ with π_p is measured by the zero of $g = u$ with $\mathcal{V}$ at the origin when π_p is transverse to T_d. The following is a description of the remaining cases in which π_p is a part of the pencil family of planes that is not transverse to T_d:

Theorem 4.3. *If π_p is not the tangential cone to $\widetilde{G}$, then the contact of π_p with $\widetilde{G}$ is equivalent to that of the zero of $g = \pm v + \epsilon u^k$ (where $k = 2,3$ and $\epsilon \neq 0,\pm 1$) with the representation $\mathcal{V}$. Moreover,*

(1) the plane π_p has an A_1-contact with Σ and B if and only if π_p is not the osculating of neither Σ nor B.

(2) π_p has an A_1-contact with Σ and an A_2-contact with B if and only if and only if π_p is not the osculating of Σ but π_p coincides with the osculating plane of B and $\tau_B(0) \neq 0$.

Proof. Among the submersions listed in Theorem 4.2, the tangent plane to the zero fiber of $g = \pm v + \epsilon u^k$, (where $k = 2,3$), contains T_d and is transverse to the tangent cone of $\widetilde{G}$. As a result, the contact of π_p with $\widetilde{G}$ is equivalent to that of $g = 0$ with the representation $\mathcal{V}$.

(1) Let $k = 2$. Then, the contact of the tangential line along Σ of and B is measured by the type of

$$g(\widetilde{f}(t,0)) = (\mp 1 + \epsilon)t^2,$$

and

$$g(\widetilde{f}(t,t) = \epsilon t^2,$$

respectively. So, it is of type A_1 along the two curves. Now, consider the restrictions $H_{\boldsymbol{\eta}}(t,0)$ and $H_{\boldsymbol{\eta}}(t,t)$ on Σ and B, respectively. Then, the plane π_p has an A_1-contact with Σ_0 if and only if $\eta_2 a_{20} + \eta_3 b_{20} \neq 0$, which implies that $(\eta_2,\eta_3) \neq (-b_{20}, a_{20})$. On the other hand, π_p has an A_1-contact with B if and only if $\eta_2(a_{20}+1) + \eta_3 b_{20} \neq 0$, which implies that $(\eta_2,\eta_3) \neq (-b_{20}, a_{20}+1)$. Geometrically, this means that π_p is not the osculating plane of neither Σ nor B at p.

(2) Let $k = 3$. Then, the contact of the tangential line along Σ and B is measured by the type of

$$g(\widetilde{f}(t,0)) = \mp t^2 + \epsilon t^3,$$

and

$$g(\widetilde{f}(t,t)) = \epsilon t^3,$$

respectively. So, it is of type A_1 along Σ and of type A_2 along B. Now, consider the restriction $H_{\boldsymbol{\eta}}(t,t)$ along B. Then, the plane π_p has an A_2-contact with B if and only if

$$(a_{20}+1)\eta_2 + b_{20}\eta_3 = 0, \tag{4.4}$$

and

$$a_{20}\eta_2 + (b_{30}+3b_{12}+b_{03})\eta_3 \neq 0. \tag{4.5}$$

The constraint (4.4) implies that $(\eta_2,\eta_3) = (-b_{20}, a_{20}+1)$, which means that π_p is the osculating plane of B. On the other hand, the constraint (4.5) becomes

$$-a_{20}b_{20} + (b_{30}+3b_{12}+b_{03})(a_{20}+1) \neq 0, \tag{4.6}$$

which implies that $\tau_B(0) \neq 0$.

□

Theorem 4.4. *If π_p is the tangent cone to $\widetilde{G}$, then the contact of π_p at $\widetilde{G}$ is equivalent to that of the zero fiber of $g = \pm w + \epsilon u^2$ ($\epsilon \neq 0, \pm 1$) with the representation $\mathcal{V}$. Furthermore, the plane π_p has an A_1-contact with both Σ and B, and it is not the osculating of neither Σ nor B.*

Proof. The tangent to the zero of $g = \pm w + \epsilon u^2$, ($\epsilon \neq 0, \pm 1$), is the same as the tangent cone of $\widetilde{G}$ for the submersions shown in Theorem 1. Hence, the contact of π_p with $\widetilde{G}$ is the same as that of $g = 0$ with the model $\mathcal{V}$. Note here that $\boldsymbol{\eta} = (0,0,1)$. On the other hand, the contact of the tangential line along Σ and B is measured by the singularity of

$$g(\widetilde{f}(t,0)) = \pm t^2 + \epsilon t^2,$$

and

$$g(\widetilde{f}(t,t)) = \epsilon t^2,$$

respectively, where $\epsilon \neq 0. \pm 1$. So, it is of type A_1. The corresponding height function restricted to Σ and B has an A_1 singularity if and only if $\eta_3 b_{20} \neq 0$, which means that $b_{20} \neq 0$, and hence does not coincide with the osculating plane of both Σ and B. □

4.4. The dual of a geometric cuspidal edge with a smooth curve

The discriminants may be used for the examination of the dual of the cuspidal edge equipped with a smooth curve as explained below.

As pointed out in [23], an oriented plane in $\mathbb{R}^3$ in local coordinates $z = (u,v,w)$ is characterized by a unit vector $\boldsymbol{\eta}$ and a real number c. The equation of the plane can be expressed as $z \cdot \boldsymbol{\eta} = c$, where $\cdot$ represents the scalar product. It is important to observe that the pairs $(\boldsymbol{\eta}, c)$ and $(-\boldsymbol{\eta}, -c)$ represent the same plane, but with opposing orientations. A unit space curve $\gamma(t)$ can be associated with an oriented tangent plane at $t_0 \in I \subset \mathbb{R}$ by a unit vector $\boldsymbol{\eta}$ that is perpendicular to the tangent vector $T(t)$ of $\gamma(t)$ at t_0. The equation of the tangent plane is given by $z \cdot \boldsymbol{\eta} = \gamma(t_0) \cdot \boldsymbol{\eta}$. The collection of all oriented tangent planes to the curve $\gamma(t)$ is referred to as "the dual" of $\gamma(t)$. Consequently, it is associated with the set defined as follows:

$$\{(\boldsymbol{\eta}, c) \in S^2 \times \mathbb{R} : c = \gamma(t) \cdot \boldsymbol{\eta}, \quad T(t) \cdot \boldsymbol{\eta} = 0\}.$$

Define the following families:

$$\mathbb{D}_1(H) = \{\big(\boldsymbol{\eta}, H_{\boldsymbol{\eta}}(t,s)\big) \in S^2 \times \mathbb{R} : \frac{\partial H}{\partial t} = \frac{\partial H}{\partial s} = 0 \text{ at } (t, s, \boldsymbol{\eta})\},$$

$$\mathbb{D}_2(H) = \{\big(\boldsymbol{\eta}, H_{\boldsymbol{\eta}}(t,0)\big) \in S^2 \times \mathbb{R} : \frac{\partial H}{\partial t} = 0 \text{ at } (t, 0, \boldsymbol{\eta})\},$$

and

$$\mathbb{D}_3(H) = \{\big(\boldsymbol{\eta}, H_{\boldsymbol{\eta}}(t,t)\big) \in S^2 \times \mathbb{R} : \frac{\partial H}{\partial t} = 0 \text{ at } (t, t, \boldsymbol{\eta})\}.$$

Then, in accordance with [23], if the contact of $\widetilde{G}$ with π_η is characterized by that of the fiber $g = 0$ with $\mathcal{V}$, where g is defined in Theorem (Classification of germs of submersions), then $\mathbb{D}_i(H)$ is diffeomorphic to $\mathbb{D}_i(F)$, where F is a $\mathcal{R}^+(\mathcal{V})$-versal unfolding of g with 2-parameters. Therefore, we have the following:

Proposition 7. *Let $\widetilde{G}$ be a pair of a geometric cuspidal edge G in $\mathbb{R}^3$ equipped with a smooth B in it. Then, the calculations and figures in Section 4.2 give the models, up to diffeomorphisms, of $\mathbb{D}_i(H)$, $i = 1, 2, 3$.*

The above result implies that if π_p is tangent to T_d but it is transverse L_d, then $\mathbb{D}_1(H) = \mathbb{D}_2(H)$ and $\mathbb{D}_3(H)$ describe locally the dual of the curve Σ and B, respectively. On the other hand, if π_p coincides with L_d and it is tangent to T_d, then $\mathbb{D}_1(H)$ composed of two parts: one is $\mathbb{D}_2(H)$ which is the dual of Σ) and the second is the proper dual of G away from points of Σ, whereas the set $\mathbb{D}_3(H)$ describes locally the dual of the curve B.

5. Conclusions

In this paper, we calculated the generators of the vector fields that are tangent to the bifurcation diagrams and caustics of the classes B_k, $k = 2, 3, 4$ and $F_{2,3}$ with respect to the quasi equivalence which is a non-standard equivalence relation. Consequently, we considered for application the generators of the B_3-class in which case the bifurcation diagram consists of two components: a cuspidal edge in $\mathbb{R}^3$ and a smooth curve in it, which serves as a boundary and denoted it by $V = (V_1, V_0)$. Then, we classified the submersion on V with codimension less or equal 2. This model and classifications were used to study the geometry of the pair $\widetilde{G} = (G, B)$ of the geometric cuspidal edge G equipped with a distinguished curve B in it. Apart from the standard structure, $\widetilde{G}$ contains two curves: the singular pints (the ridge) Σ and the smooth curve B. Thus, we discussed and described the contact of $\widetilde{G}$ with the plane π_p at $p \in \widetilde{G}$ along the curves Σ and B via the height function on $\widetilde{G}$, using the zero fibers of the submesrion obtained on V. In particular, we distinguished two cases. First, if π_p is the tangent cone to $\widetilde{G}$, then the contact is of type A_1 along both Σ and B if and only if π_p is not the osculating plane of neither Σ nor B, and of type A_1 along Σ and A_2 along B if and only if π_p is not the osculating plane of

Σ but it coincides with the osculating plane of B and $\tau_B(0) \neq 0$ (the torsion of B at 0). Second, if π_p is not the tangent cone to $\widetilde{G}$, then the contact is of type A_2 along both Σ and B if and only if π_p is not the osculating plane of neither Σ nor B.

Subsequent study extending beyond this work may involve examining the height function on other singular hypersurfaces in $\mathbb{R}^3$ characterized by a smooth or singular boundary. When the hypersurface is equipped with a distinguished singular curve, it is more intriguing as it may involve two transversal tangential directions, such as the situation of the cuspidal edge with a singular curve (cusp) in it.

Author contributions

Yanlin Li: Conceptualization, investigation, methodology, writing-review and editing; Fawaz Alharbi: Conceptualization, investigation, methodology, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. V. M. Zakalyukin, Reconstructions of fronts and caustics depending on a parameter and versality of mappings, *J. Math. Sci.*, **27** (1984), 2713–2735. https://doi.org/10.1007/BF01084818
2. V. Arnold, Wave front evolution and equivariant Morse lemma, *Commun. Pur. Appl. Math.*, **29** (1976), 557–582. https://doi.org/10.1002/cpa.3160290603
3. J. W. Bruce, Vector fields on discriminants and bifurcation varieties, *Bull. London Math. Soc.*, **17** (1985), 257–262. https://doi.org/10.1112/blms/17.3.257
4. A. Alghanemi, A. Alghawazi, The λ-point map between two Legendre plane curves, *Mathematics*, **11** (2023), 977–997. https://doi.org/10.3390/math11040997
5. T. Fukui, M. Hasegawa, Singularities of parallel surfaces, *Tohoku Math. J.*, **64** (2012), 387–408. https://doi.org/10.2748/tmj/1347369369

6. Y. Li, E. Guler, Right conoids demonstrating a Time-like axis within minkowski Four-Dimensional space, *Mathematics*, **12** (2024), 2421. https://doi.org/10.3390/math12152421
7. Y. Li, H. Abdel-Aziz, H. Serry, F. El-Adawy, M. Saad, Geometric visualization of evolved ruled surfaces via alternative frame in Lorentz-Minkowski 3-space, *AIMS Math.*, **9** (2024), 25619–25635. https://doi.org/10.3934/math.20241251
8. Y. Li N. Turki, S. Deshmukh, O. Belova, Euclidean hypersurfaces isometric to spheres, *AIMS Math.*, **9** (2024), 28306–28319. https://doi.org/10.3934/math.20241373
9. Y. Li, M. S. Siddesha, H. A. Kumara, M. M. Praveena, Characterization of Bach and Cotton Tensors on a Class of Lorentzian Manifolds, *Mathematics*, **12** (2024), 3130. https://doi.org/10.3390/math12193130
10. Y. Li, S. Bhattacharyya, S. Azami, Li-Yau type estimation of a semilinear parabolic system along geometric flow, *J. Inequal Appl.*, **131** (2024). https://doi.org/10.1186/s13660-024-03209-y
11. Y. Li, A. K. Mallick, A. Bhattacharyya, M. S. Stankovic, A conformal η-Ricci soliton on a Four-Dimensional lorentzian Para-Sasakian manifold, *Axioms*, **13** (2024), 753. https://doi.org/10.3390/axioms13110753
12. F. Alharbi, V. Zakalyukin, Quasi corner singularities, *P. Steklov I. Math.*, **270** (2010), 1–14. https://doi.org/10.1134/S0081543810030016
13. F. Alharbi, Quasi cusp singularities, *J. Sing.*, **12** (2015), 1–18. https://doi.org/10.5427/jsing.2015.12a
14. F. Alharbi, S. Alsaeed, Quasi semi-border singularities, *Mathematics*, **7** (2019), 495. https://doi.org/10.3390/math7060495
15. F. Alharbi, Bifurcation diagrams and caustics of simple quasi border singularities, *Topo. Appl.*, **9** (2012), 381–388. https://doi.org/10.1016/j.topol.2011.09.011
16. J. W. Bruce, J. M. West, Functions on cross-caps, *Math. Proc. Cambridge*, **123** (1988), 19–39.
17. A. P. Francisco, Functions on a swallowtail, *Arxiv Prep.*, **53** (2023), 52–74. https://doi.org/10.48550/arXiv.1804.09664
18. R. O. Sinha, F. Tari, On the geometry of the cuspidal edge, *Osaka J. Math.*, **55** (2018), 393–421.
19. R. O. Sinha, K. Saji, On the geometry of folded cuspidal edges, *Rev. Mat. Complut.*, **31** (2018), 627–650. https://doi.org/10.1007/s13163-018-0257-6
20. J. Damon, A-equivalence and the equivalence of sections of images and discriminants, *Singular. Theory Appl.*, **1462** (1991), 93–121. https://doi.org/10.1007/BFb0086377
21. D. Mond, R. Buchweitz, Linear free divisors and quiver representations, *London Math. Soc. Lecture Note Ser.*, **324** (2005), 18–20.
22. V. Arnold, *Singularities of caustics and wave fronts*, Dordrecht: Kluwer Academic Publishers, 1990.
23. J. W. Bruce, P. J. Giblin, *Curves and singularities: A geometrical introduction to singularity theory*, Cambridge University Press, 1984.