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Research article

Extended Hermite–Hadamard inequalities

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Abstract: In this manuscript, we formulated Hermite–Hadamard inequalities for convex functions by employing cotangent integrals. Additionally, we extended these Hermite–Hadamard inequalities to encompass cotangent integrals and give the application.

Keywords: fractional derivatives; cotangent integral operators; confluent hypergeometric function; Hermite–Hadamard inequalities; optimization
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1. Introduction

Fractional calculus (FC) is a powerful mathematical framework that generalizes classical integration and differentiation, providing new tools to model and analyze complex systems [1, 2]. By enabling the exploration of fractional-order dynamics, FC has broadened the horizons of mathematical research and its applications across various disciplines. A key aspect of FC lies in the choice of kernel, which distinguishes fractional operators into two primary categories: Singular and non-singular kernels. Singular kernels, such as those used in the Caputo, Riemann-Liouville, Hadamard, and ϕ-conformable derivatives [3], have been foundational in the field [4, 5]. Non-singular kernels, including the Caputo-Fabrizio [4], cotangent [6], and Hilfer cotangent derivatives [7], have emerged more recently, offering advantages such as smoother behaviors and improved applicability to certain physical systems [8]. Fractional calculus has become an indispensable tool in various scientific fields. It enhances classical models and introduces alternative formulations for phenomena across physics, controllability [9], stability [10], chemistry, biology, and engineering [11].

The development of fractional integral inequalities has further advanced the theory of differential and integral equations. The rapid development of novel fractional derivative and integral operators represents one of the most active areas in FC research. These operators exhibit unique attributes, attracting attention for their applications in inequality theory and their role in modeling complex systems. A pivotal question in the field concerns the selection of appropriate fractional differentiability or dynamic system models for specific problems, prompting extensive research to address this challenge. By combining singular and non-singular kernels, researchers aim to create versatile approaches adaptable to a wide range of applications. This includes the analysis of quantum systems with fractional orders, focusing on stability and attractivity under varying conditions [1,2]. To illustrate the mathematical foundation of FC, we consider the Riemann-Liouville integral operators:

Definition 1.1. *[5] Let $\Theta \in L_1[\phi, \psi]$. The Riemann-Liouville integrals $J^r_{\phi^+}\Theta$ and $J^r_{\psi^-}\Theta$ are:*

$$J^r_{\phi^+}\Theta(x) = \frac{1}{\Gamma(r)} \int_\phi^x (x-\varrho)^{r-1}\Theta(\varrho)d\varrho,$$

and

$$J^r_{\psi^-}\Theta(x) = \frac{1}{\Gamma(r)} \int_x^\psi (\varrho - x)^{r-1}\Theta(\varrho)d\varrho.$$

Here, $J^0_{\phi^+}\Theta(x) = J^0_{\psi^-}\Theta(x) = \Theta(x)$.

When the fractional order, the Riemann-Liouville integral, reduces to the classical integral, it links fractional calculus to traditional analysis [5].

In [12] authors give some properties related to conformal integral are defined by:

$$J^r\Theta(x) = \int_0^x s^{r-1}\Theta(s)ds,$$

and in [6], the cotangent fractional integrals were introduced from conformable integrals.

Let $J \subseteq \mathbb{R}$, $0 \leq \beta \leq 1$ and $\Theta : J \to \mathbb{R}$, the function Θ is deemed convex if it satisfies the inequality

$$\Theta(\beta\phi + (1-\beta)\psi) \leq \beta\Theta(\phi) + (1-\beta)\Theta(\psi), \quad \text{for all} \quad \phi, \psi \in J.$$

Drawing upon this theory of function inequalities, the Hermite–Hadamard inequality offers bounds for the Cauchy mean value. Let $\Theta : J \to \mathbb{R}$ be a convex function, and let $\phi, \psi \in J$ with $\phi < \psi$. The inequality can be expressed as:

$$\Theta\left(\frac{\phi+\psi}{2}\right) \leq \frac{1}{\psi-\phi}\int_\phi^\psi \Theta(x)dx \leq \frac{\Theta(\phi)+\Theta(\psi)}{2}. \tag{1.1}$$

The importance of the Hermite–Hadamard inequality stems from its wide-ranging applications in various fields:

Analysis and optimization: The inequality provides a powerful tool for analyzing the behavior of convex functions and optimizing problems involving such functions. It helps in establishing bounds on the integral of a convex function, which is crucial for understanding the overall behavior of the function over an interval.

Numerical analysis: In numerical integration, the Hermite–Hadamard inequality can be used to derive error estimates for numerical quadrature methods. By bounding the integral of the function within certain limits, it helps in assessing the accuracy of numerical approximations of integrals.

Probability theory and statistics: In probability theory and statistics, convex functions often arise in the context of probability distributions, cumulative distribution functions, and moments of random variables. The Hermite–Hadamard inequality provides bounds on expectations and moments, which are essential for deriving inequalities in probability theory and analyzing statistical properties of random variables.

Economics and finance: Convex functions frequently appear in economic and financial models to represent utility functions, production functions, cost functions, and other economic relationships. The Hermite–Hadamard inequality is used to derive inequalities related to optimization problems in economics and finance, providing insights into decision-making and resource allocation.

Machine learning and optimization: Convex optimization is a fundamental tool in machine learning and optimization algorithms. The Hermite–Hadamard inequality helps in understanding the properties of objective functions and constraints in convex optimization problems, facilitating the development of efficient algorithms for solving these problems.

Here is a perspective on the stability of the Hermite–Hadamard inequality: Convexity Requirement: The Hermite–Hadamard inequality relies on the convexity of the function over the interval of integration. If the function is not strictly convex, but rather only weakly convex (convex on intervals), the inequality might not hold. Thus, small perturbations that alter the convexity properties of the function could affect the stability of the inequality; Endpoint Values: The inequality involves evaluating the function at the endpoints of the interval. If there are discontinuities or singularities at these points, small changes in the function near the endpoints could lead to significant variations in the inequality's validity; Interval Length: The inequality also depends on the length of the integration interval. Small changes in the interval length could affect the balance between the weighted average of function values and the integral, potentially impacting the stability of the inequality; Function Smoothness: The stability of the inequality can also be influenced by the smoothness of the function. If the function has sharp changes or oscillations within the interval, small perturbations could lead to variations in the integral and the function values, affecting the validity of the inequality; Numerical Approximations: In numerical computations, when approximating the integral or evaluating the function at discrete points, numerical errors and approximations could introduce instability in the inequality. Careful consideration of numerical methods and precision is essential to maintain stability.

In practice, stability considerations often lead to the exploration of alternative inequalities or relaxation of strict convexity requirements. For example, for functions that are not strictly convex but satisfy certain growth conditions, modified versions of the Hermite–Hadamard inequality or related inequalities may be used to ensure stability. Overall, while the Hermite–Hadamard inequality is a powerful tool in mathematical analysis and various applications, understanding its stability under different conditions is crucial for its reliable use in practical settings.

Several fundamental concepts that will be employed in our study are outlined below. The confluent hypergeometric function is represented as an absolutely convergent infinite power series:

$$\mathrm{CH}(\phi, \psi, z) = \sum_{m=0}^{\infty} \frac{(\phi)_m}{(\psi)_m} \frac{z^m}{m!}, \tag{1.2}$$

here, we introduce the Pochhammer's symbols $(\phi)_m$ and $(\psi)_m$, which are essential mathematical tools [13]. The Pochhammer's symbol, denoted as $(\phi)_m$, is defined as follows:

$$(\phi)_m = \frac{\Gamma(\phi + m)}{\Gamma(\phi)},$$

where the function $\Gamma(v)$ is given by $\Gamma(v) = \int_0^\infty e^{-\rho}\rho^{v-1}d\rho$, and it holds that $(\phi)_0 = 1$. It's worth noting that if ϕ is a negative integer and $m \geq -\phi$, then $(\phi)_m = 0$. On the other hand, when $\phi = 1$, we have $(1)_m = m!$.

In a valuable reference, Abramowitz and Stegun [14] provide a practical integral representation for cases where $\psi > \phi > 0$.

$$\mathrm{CH}(\phi, \psi, z) = \frac{\Gamma(\psi)}{\Gamma(\psi - \phi)\Gamma(\phi)} \int_0^1 e^{zt} t^{\phi-1}(1-t)^{\psi-\phi-1} dt. \tag{1.3}$$

The initial exploration of inequality theory utilizing the Riemann-Liouville integral operator centered on the Hermite–Hadamard inequality. In their work [15], Sarikaya and colleagues established the subsequent Hermite–Hadamard type inequalities that incorporate Riemann-Liouville integrals:

Theorem 1.1. *Consider a positive function $\Theta : [\phi, \psi] \to \mathbb{R}^+$, where $0 \leq \phi < \psi$, and h is within the class of Lebesgue integrable functions on the interval $[\phi, \psi]$. Assuming that Θ exhibits convex behavior over this interval, the subsequent inequalities apply to fractional integrals:*

$$\Theta\left(\frac{\phi + \psi}{2}\right) \leq \frac{\Gamma(r+1)\left[J^r_{\phi+}\Theta(\psi) + J^r_{\psi-}\Theta(\phi)\right]}{2(\psi - \phi)^r} \leq \frac{\Theta(\phi) + \Theta(\psi)}{2}. \tag{1.4}$$

This investigation has paved the way for several extensions of the Hermite–Hadamard inequality. Notably, one such extension was presented by Chen in [16], where he introduced extensions of Hermite–Hadamard type inequalities that encompass Riemann-Liouville integrals.

Theorem 1.2. *Suppose we have a positive, twice-differentiable function $\Theta : [\phi, \psi] \to \mathbb{R}$ within the interval $\phi < \psi$, and this function belongs to the space of Lebesgue integrable functions $L_1[\phi, \psi]$. Furthermore, assume that the second derivative of Θ, denoted as Θ'', remains bounded within the interval $[\phi, \psi]$. Under these conditions, the following inequalities are applicable:*

$$\begin{aligned}
&\frac{mr}{2(\psi - \phi)^r} \int_\phi^{\frac{\phi+\psi}{2}} \left(\frac{\phi + \psi}{2} - \mu\right)^2 \left[(\mu - \phi)^{r-1} + (\psi - \mu)^{r-1}\right] d\mu \\
&\leq \frac{\Gamma(r+1)}{2(\psi - \phi)^r}\left[J^r_{\phi+}\Theta(\psi) + J^r_{\psi-}\Theta(\phi)\right] - \Theta\left(\frac{\phi + \psi}{2}\right) \\
&\leq \frac{Mr}{2(\psi - \phi)^r} \int_\phi^{\frac{\phi+\psi}{2}} \left(\frac{\phi + \psi}{2} - \mu\right)^2 \left[(\mu - \phi)^{r-1} + (\psi - \mu)^{r-1}\right] d\mu,
\end{aligned}$$

and

$$\frac{-Mr}{2(\psi-\phi)^r}\int_{\phi}^{\frac{\phi+\psi}{2}}(\mu-\phi)(\psi-\mu)\left[(\mu-\phi)^{r-1}+(\psi-\mu)^{r-1}\right]d\mu$$
$$\leq\frac{\Gamma(r+1)}{2(\psi-\phi)^r}\left[J_{\phi^+}^{r}\Theta(\psi)+J_{\psi^-}^{r}\Theta(\phi)\right]-\frac{\Theta(\phi)+f(\psi)}{2}$$
$$\leq\frac{-mr}{2(\psi-\phi)^r}\int_{\phi}^{\frac{\phi+\psi}{2}}(\mu-\phi)(\psi-\mu)\left[(\mu-\phi)^{r-1}+(\psi-\mu)^{r-1}\right]d\mu,$$

with $r>0$*, where* $m=\inf_{\varrho\in[\phi,\psi]}\Theta''(\varrho)$ *and* $M=\sup_{\varrho\in[\phi,\psi]}\Theta''(\varrho)$*.*

Numerous novel integral and derivative operators have been introduced, each possessing unique properties that distinguish them. These operators exhibit variations in locality, singularity characteristics, and order of differentiation. Consequently, understanding the distinct attributes of each new operator, as well as their practical applicability in various fields, has become a subject of considerable interest and discussion. This exploration includes numerous research papers: In [17], Agarwal et al. delve into Hermite–Hadamard and Hadamard-type inequalities using k-fractional integral operators, providing generalized frameworks for convex functions. In [18], Ekinci and Ozdemi extend Hermite–Hadamard inequalities to Riemann–Liouville integral operators. In [19], Nie et al. contribute to the field by presenting weighted inequalities for differentiable exponentially convex functions. Further, theoretical advancements are provided by Zhou et al. [20] and Rashid et al. [21], who explore inequalities involving K-fractional integrals and Gruss inequalities, respectively. These works emphasize the integration of fractional operators with broader classes of functions. Rashid et al. [22–24] extend these findings to time scales and local fractional integrals. Additionally, Sarikaya & Alp [25] and Farid [26] address Hermite–Hadamard-Fejer type inequalities and the existence of integral operators in fractional and conformable integrals, strengthening the theoretical underpinnings of the field. Mohammed & Abdeljawad [27] and Abdeljawad et al. [28] further explore fractional operators with nonsingular kernels and conformable fractional integrals, respectively, offering innovative approaches to classical inequalities.

Several integral inequalities have been established through the utilization of fractional integral operators (see [27,28]). In this context, we delve into the introduction of another new operator. Sadek, in [6], introduced the cotangent fractional integrals, which possess a wide range of essential properties, including the following:

Definition 1.2. *The left and right cotangent integral operators are*

$$\left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\ell)=\frac{1}{\sin(\frac{\pi}{2}\eta)^r\Gamma(r)}\int_{\phi}^{\ell}e^{-\cot(\frac{\pi}{2}\eta)(\ell-\rho)}(\ell-\rho)^{r-1}\Theta(\rho)d\rho,\tag{1.5}$$

and

$$\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\ell)=\frac{1}{\sin(\frac{\pi}{2}\eta)^r\Gamma(r)}\int_{\ell}^{\psi}e^{-\cot(\frac{\pi}{2}\eta)(\rho-\ell)}(\rho-\ell)^{r-1}\Theta(\rho)d\rho,\tag{1.6}$$

respectively, with $\eta\in(0,1]$*,* $r\in\mathbb{C}$ *and* $\Re(r)>0$*.*

The primary motivation and objective of this research are to introduce a novel and valuable extension to Hermite–Hadamard type inequalities. Additionally, we aim to provide fresh upper and lower bounds for the average value of a convex function, utilizing Hermite–Hadamard inequality as a foundation. Hermite–Hadamard inequality, which boasts numerous variations and generalizations, serves as a critical tool in this endeavor. The cotangent integrals play a crucial role in achieving these goals.

Furthermore, our work can serve as a source of inspiration for fellow mathematicians and scholars, encouraging them to explore the diverse applications of fractional integrals and related mathematical tools. The study of fractional integrals, and in particular the cotangent integrals, opens a world of opportunities for understanding the behavior of functions in novel and insightful ways.

2. Hermite–Hadamard inequalities for cotangent integral operator

Theorem 2.1. *Suppose we have a function* $\Theta : [\phi, \psi] \to \mathbb{R}$ *with* $0 \leq \phi < \psi$*, and this function belongs to the class of Lebesgue integrable functions on the interval* $[\phi, \psi]$*. If this function* Θ *is convex over the interval* $[\phi, \psi]$*, then the subsequent inequalities hold:*

$$\Theta\left(\frac{\phi+\psi}{2}\right) \leq \frac{r\sin(\frac{\pi}{2}\eta)^r\Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi) + \left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{2(\psi-\phi)^r\,\mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \leq \frac{\Theta(\phi)+\Theta(\psi)}{2}, \tag{2.1}$$

where $\eta \in (0,1]$, $r \in \mathbb{C}$ *and* $\Re(r) > 0$.

Proof. For any two values, x and y, both falling within the interval $[\phi, \psi]$, if the function Θ is convex over this same interval, we have the following inequality:

$$\Theta\left(\frac{x+y}{2}\right) \leq \frac{\Theta(x)+\Theta(y)}{2}.$$

To express this inequality in a more general form, we consider x and y as linear combinations of ϕ and ψ, such that $x = t\phi + (1-t)\psi$ and $y = (1-t)\phi + t\psi$.

$$2\Theta\left(\frac{\phi+\psi}{2}\right) \leq \Theta(t\phi + (1-t)\psi) + \Theta((1-t)\phi + t\psi). \tag{2.2}$$

Multiplying both sides of Eq (2.2) by $\frac{1}{\sin(\frac{\pi}{2}\eta)^r\Gamma(r)}e^{-\cot(\frac{\pi}{2}\eta)t(\psi-\phi)}t^{r-1}$ and subsequently integrating this resulting expression with respect to t over the interval $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2}{\sin(\frac{\pi}{2}\eta)^r\Gamma(r)}\int_0^1 e^{-\cot(\frac{\pi}{2}\eta)t(\psi-\phi)}t^{r-1}\Theta\left(\frac{\phi+\psi}{2}\right)dt\\
&\leq \frac{1}{\sin(\frac{\pi}{2}\eta)^r\Gamma(r)}\int_0^1 e^{-\cot(\frac{\pi}{2}\eta)t(\psi-\phi)}t^{r-1}\Theta(t\phi+(1-t)\psi)dt\\
&\quad+\frac{1}{\sin(\frac{\pi}{2}\eta)^r\Gamma(r)}\int_0^1 e^{-\cot(\frac{\pi}{2}\eta)t(\psi-\phi)}t^{r-1}\Theta((1-t)\phi+t\psi)dt\\
&=\frac{1}{\sin(\frac{\pi}{2}\eta)^r\Gamma(r)(\psi-\phi)^r}\int_\phi^\psi e^{-\cot(\frac{\pi}{2}\eta)(\psi-\rho)}(\psi-\rho)^{r-1}\Theta(\rho)d\rho\\
&\quad+\frac{1}{\sin(\frac{\pi}{2}\eta)^r\Gamma(r)(\psi-\phi)^r}\int_\phi^\psi e^{-\cot(\frac{\pi}{2}\eta)(\rho-\phi)}(\rho-\phi)^{r-1}\Theta(\rho)d\rho\\
&=\frac{\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi)+\left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)}{(\psi-\phi)^r}.
\end{aligned}$$

Employing the identity from Eq (1.3), we obtain

$$\Theta\left(\frac{\phi+\psi}{2}\right) \leq \frac{r\sin(\frac{\pi}{2}\eta)^r\Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi)+\left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{2(\psi-\phi)^r\,\mathrm{CH}\left(r,r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)}.$$

This concludes the proof of the first part of the inequality in (2.1). Since Θ is a convex, we can establish the following inequalities:

$$\Theta(t\phi+(1-t)\psi) \leq t\Theta(\phi)+(1-t)\Theta(\psi),$$

and

$$\Theta((1-t)\phi+t\psi) \leq (1-t)\Theta(\phi)+t\Theta(\psi).$$

Upon summing up these two inequalities, we obtain the following result:

$$\Theta(t\phi+(1-t)\psi)+\Theta((1-t)\phi+t\psi) \leq \Theta(\phi)+\Theta(\psi).$$

If we take the resulting inequality and multiply both sides by the expression

$$\frac{1}{\sin(\frac{\pi}{2}\eta)^r\Gamma(r)}e^{-\cot(\frac{\pi}{2}\eta)t(\psi-\phi)}t^{r-1},$$

and then proceed to integrate it over the interval $[0,1]$, we arrive at the following:

$$\frac{\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi)+\left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)}{(\psi-\phi)^r} \leq \frac{[\Theta(\phi)+\Theta(\psi)]}{\sin(\frac{\pi}{2}\eta)^r\Gamma(r)}\int_0^1 e^{-\cot(\frac{\pi}{2}\eta)t(\psi-\phi)}t^{r-1}dt,$$

since

$$\begin{aligned}\mathrm{CH}\left(r,r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right) &= \frac{\Gamma(r+1)}{\Gamma(1)\Gamma(r)}\int_0^1 e^{-\cot(\frac{\pi}{2}\eta)t(\psi-\phi)}t^{r-1}(1-t)^{1-1}dt\\ &= \frac{\Gamma(r+1)}{\Gamma(r)}\int_0^1 e^{-\cot(\frac{\pi}{2}\eta)t(\psi-\phi)}t^{r-1}dt\\ &= r\int_0^1 e^{-\cot(\frac{\pi}{2}\eta)t(\psi-\phi)}t^{r-1}dt,\end{aligned}$$

then

$$\frac{\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi)+\left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)}{(\psi-\phi)^r} \leq \frac{\Theta(\phi)+\Theta(\psi)}{r\sin(\frac{\pi}{2}\eta)^r\Gamma(r)}\mathrm{CH}\left(r,r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right).$$

So, we have

$$\frac{r\sin(\frac{\pi}{2}\eta)^r\Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi)+\left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{(\psi-\phi)^r\,\mathrm{CH}\left(r,r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \leq \Theta(\phi)+\Theta(\psi).$$

The proof is completed. □

Another technique can be used for proof, see [29].

Remark 2.1. *If $\eta=1$, then the inequality (2.1) becomes inequality (1.4).*

3. Extension of Hermite–Hadamard inequality

Theorem 3.1. *Consider a function $\Theta : [\phi, \psi] \to \mathbb{R}$ be a twice differentiable mapping where $\phi < \psi$ and $\Theta \in L_1[\phi, \psi]$. If Θ'' is bounded in $[\phi, \psi]$, then we get*

$$\begin{aligned}
&\frac{mr}{2(\psi-\phi)_1^r \,\mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \\
&\times \int_{\phi}^{\frac{\psi+\phi}{2}} \left(\frac{\phi+\psi}{2}-\rho\right)^2 \left[e^{\left[-\cot(\frac{\pi}{2}\eta)(\rho-\phi)\right]}(\rho-\phi)^{r-1} + e^{\left[-\cot(\frac{\pi}{2}\eta)(\psi-\rho)\right]}(\psi-\rho)^{r-1}\right] d\rho \\
&\leq \frac{r\sin(\frac{\pi}{2}\eta)^r \Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi) + \left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{2(\psi-\phi)^r \,\mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} - \Theta\left(\frac{\phi+\psi}{2}\right) \\
&\leq \frac{Mr}{2(\psi-\phi)^r \,\mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \\
&\times \int_{\phi}^{\frac{\phi+\psi}{2}} \left(\frac{\phi+\psi}{2}-\rho\right)^2 \left[e^{\left[-\cot(\frac{\pi}{2}\eta)(\rho-\phi)\right]}(\rho-\phi)^{r-1} + e^{\left[-\cot(\frac{\pi}{2}\eta)(\psi-\rho)\right]}(\psi-\rho)^{r-1}\right] d\rho,
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
&\frac{-Mr}{2(\psi-\phi)^r \,\mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \\
&\times \int_{\phi}^{\frac{\psi+\phi}{2}} (\rho-\phi)(\psi-\rho) \left[e^{-\cot(\frac{\pi}{2}\eta)(\rho-\phi)}(\rho-\phi)^{r-1} + e^{-\cot(\frac{\pi}{2}\eta)(\psi-\rho)}(\psi-\rho)^{r-1}\right] d\rho \\
&\leq \frac{r\sin(\frac{\pi}{2}\eta)^r \Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi) + \left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{2(\psi-\phi)^r \,\mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} - \frac{\Theta(\phi)+\Theta(\psi)}{2} \\
&\leq \frac{-mr}{2(\psi-\phi)^r \,\mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \\
&\times \int_{\phi}^{\frac{\psi+\phi}{2}} (\rho-\phi)(\psi-\rho) \left[e^{\left[-\cot(\frac{\pi}{2}\eta)(\rho-\phi)\right]}(\rho-\phi)^{r-1} + e^{\left[-\cot(\frac{\pi}{2}\eta)(\psi-\rho)\right]}(\psi-\rho)^{r-1}\right] d\rho,
\end{aligned} \tag{3.2}$$

with $m = \inf_{\varrho\in[\phi,\psi]} \Theta''(\varrho)$ and $M = \sup_{\varrho\in[\phi,\psi]} \Theta''(\varrho)$.

Proof. First, we will prove Eq (3.1). We have

$$\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi) + \left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi) = \frac{1}{\sin(\frac{\pi}{2}\eta)^r\Gamma(r)} \int_{\phi}^{\psi} \left(e^{-\cot(\frac{\pi}{2}\eta)(\rho-\phi)}(\rho-\phi)^{r-1} + e^{-\cot(\frac{\pi}{2}\eta)(\psi-\rho)}(\psi-\rho)^{r-1}\right)\Theta(\rho)d\rho,$$

so

$$\begin{aligned}
&\frac{r\sin(\frac{\pi}{2}\eta)^r \Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi) + \left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{2(\psi-\phi)^r \,\mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \\
&= \frac{r}{2(\psi-\phi)^r \,\mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \\
&\times \int_{\phi}^{\psi} \left(e^{-\cot(\frac{\pi}{2}\eta)(\rho-\phi)}(\rho-\phi)^{r-1} + e^{-\cot(\frac{\pi}{2}\eta)(\psi-\rho)}(\psi-\rho)^{r-1}\right)\Theta(\rho)d\rho.
\end{aligned}$$

We use the substitution $\rho' = \phi + \psi - \rho$, which transforms the integral:

$$\int_{\phi}^{\psi} \left[e^{-\cot\left(\frac{\pi}{2}\eta\right)(\rho-\phi)}(\rho-\phi)^{r-1} + e^{-\cot\left(\frac{\pi}{2}\eta\right)(\psi-\rho)}(\psi-\rho)^{r-1} \right] \Theta(\phi+\psi-\rho)\, d\rho,$$

by

$$\int_{\psi}^{\phi} \left[e^{-\cot\left(\frac{\pi}{2}\eta\right)(\phi+\psi-\rho'-\phi)}(\phi+\psi-\rho'-\phi)^{r-1} + e^{-\cot\left(\frac{\pi}{2}\eta\right)(\rho'-\phi)}(\rho'-\phi)^{r-1} \right] \Theta(\rho')(-d\rho').$$

Reversing the integration limits, we remove the negative sign:

$$\int_{\phi}^{\psi} \left[e^{-\cot\left(\frac{\pi}{2}\eta\right)(\rho'-\phi)}(\rho'-\phi)^{r-1} + e^{-\cot\left(\frac{\pi}{2}\eta\right)(\psi-\rho')}(\psi-\rho')^{r-1} \right] \Theta(\rho')\, d\rho'.$$

Then,

$$\begin{aligned} &\frac{r\sin(\frac{\pi}{2}\eta)^r\Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi) + \left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{2(\psi-\phi)^r\,\mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \\ &= \frac{r}{2(\psi-\phi)^r\,\mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \\ &\quad \times \int_{\phi}^{\psi} \left[e^{-\cot(\frac{\pi}{2}\eta)(\rho-\phi)}(\rho-\phi)^{r-1} + e^{-\cot(\frac{\pi}{2}\eta)(\psi-\rho)}(\psi-\rho)^{r-1} \right] \Theta(\phi+\psi-\rho)d\rho. \end{aligned}$$

Hence, we can express it as follows.

$$\begin{aligned} &\frac{r\sin(\frac{\pi}{2}\eta)^r\Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi) + \left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{2(\psi-\phi)^r\,\mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \\ &= \frac{r}{4(\psi-\phi)^r\,\mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \\ &\quad \times \int_{\phi}^{\psi} \left[\Theta(\phi+\psi-\rho) + \Theta(\rho) \right] \left[e^{-\cot(\frac{\pi}{2}\eta)(\rho-\phi)}(\rho-\phi)^{r-1} + e^{-\cot(\frac{\pi}{2}\eta)(\psi-\rho)}(\psi-\rho)^{r-1} \right] d\rho. \end{aligned} \tag{3.3}$$

Then, we get

$$\begin{aligned} &\frac{r\sin(\frac{\pi}{2}\eta)^r\Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}h\right)(\phi) + \left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{2(\psi-\phi)^r\,\mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} - \Theta\left(\frac{\phi+\psi}{2}\right) \\ &= \frac{r}{4(\psi-\phi)^r\,\mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \\ &\quad \times \int_{\phi}^{\psi} \left[\Theta(\phi+\psi-\rho) + \Theta(\rho) - 2\Theta\left(\frac{\phi+\psi}{2}\right) \right] \left[e^{-\cot(\frac{\pi}{2}\eta)(\rho-\phi)}(\rho-\phi)^{r-1} + e^{-\cot(\frac{\pi}{2}\eta)(\psi-\rho)}(\psi-\rho)^{r-1} \right] d\rho. \end{aligned}$$

Since

$$\left[\Theta(\phi+\psi-\rho) + \Theta(\rho) - 2\Theta\left(\frac{\phi+\psi}{2}\right) \right] \left[e^{-\cot(\frac{\pi}{2}\eta)(\rho-\phi)}(\rho-\phi)^{r-1} + e^{-\cot(\frac{\pi}{2}\eta)(\psi-\rho)}(\psi-\rho)^{r-1} \right],$$

is symmetric about $\rho = \frac{\phi+\psi}{2}$, we get

$$\begin{aligned}
&\frac{r\sin(\frac{\pi}{2}\eta)^r\Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi)+\left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{2(\psi-\phi)^r\,\mathrm{CH}\left(r,r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)}-\Theta\left(\frac{\phi+\psi}{2}\right)\\
=&\frac{r}{2(\psi-\phi)^r\,\mathrm{CH}\left(r,r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)}\\
&\times\int_{\phi}^{\frac{\phi+\psi}{2}}\left[\Theta(\phi+\psi-\rho)+\Theta(\rho)-2\Theta\left(\frac{\phi+\psi}{2}\right)\right]\\
&\times\left[e^{-\cot(\frac{\pi}{2}\eta)(\rho-\phi)}(\rho-\phi)^{r-1}+e^{-\cot(\frac{\pi}{2}\eta)(\psi-\rho)}(\psi-\rho)^{r-1}\right]d\rho.
\end{aligned}\tag{3.4}$$

Since

$$\Theta(\phi+\psi-x)-\Theta\left(\frac{\phi+\psi}{2}\right)=\int_{\frac{\phi+\psi}{2}}^{\phi+\psi-x}\Theta'(\mu)d\mu,$$

and

$$\Theta\left(\frac{\phi+\psi}{2}\right)-\Theta(x)=\int_{x}^{\frac{\phi+\psi}{2}}\Theta'(\mu)d\mu,$$

we get

$$\begin{aligned}
\Theta(\rho)+\Theta(\phi+\psi-\rho)-2\Theta\left(\frac{\phi+\psi}{2}\right)&=\int_{\frac{\phi+\psi}{2}}^{\phi+\psi-\rho}\Theta'(\mu)d\mu-\int_{\rho}^{\frac{\phi+\psi}{2}}\Theta'(\mu)d\mu\\
&=\int_{\rho}^{\frac{\phi+\psi}{2}}\Theta'(\phi+\psi-\mu)d\mu-\int_{\rho}^{\frac{\phi+\psi}{2}}\Theta'(\mu)d\mu\\
&=\int_{\rho}^{\frac{\phi+\psi}{2}}\left[\Theta'(\phi+\psi-\mu)-\Theta'(\mu)\right]d\mu.
\end{aligned}\tag{3.5}$$

Since

$$\Theta'(\phi+\psi-\mu)-\Theta'(\mu)=\int_{\mu}^{\phi+\psi-\mu}\Theta''(y)dy,$$

then for $\mu\in\left[\phi,\frac{\phi+\psi}{2}\right]$, we get

$$m(\phi+\psi-2\mu)\le\Theta'(\phi+\psi-\mu)-\Theta'(\mu)\le M(\phi+\psi-2\mu).$$

So

$$\begin{aligned}
\int_{\rho}^{\frac{\phi+\psi}{2}}m(\phi+\psi-2\mu)d\mu&\le\Theta(\rho)+\Theta(\phi+\psi-\rho)-2\Theta\left(\frac{\phi+\psi}{2}\right)\\
&\le\int_{\rho}^{\frac{\phi+\psi}{2}}M(\phi+\psi-2\mu)d\mu.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
m\left(\frac{\phi+\psi}{2}-\rho\right)^2&\le\Theta(\rho)+\Theta(\phi+\psi-\rho)-2\Theta\left(\frac{\phi+\psi}{2}\right)\\
&\le M\left(\frac{\phi+\psi}{2}-\rho\right)^2.
\end{aligned}$$

Then,

$$
\begin{aligned}
&\frac{mr}{2(\psi-\phi)^r \operatorname{CH}\left(r, r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \\
&\times \int_{\phi}^{\frac{\phi+\psi}{2}} \left(\frac{\phi+\psi}{2}-\rho\right)^2 \left[e^{-\cot(\frac{\pi}{2}\eta)(\rho-\phi)}(\rho-\phi)^{r-1}+e^{-\cot(\frac{\pi}{2}\eta)(\psi-\rho)}(\psi-\rho)^{r-1}\right] d\rho \\
\leq &\frac{r\sin(\frac{\pi}{2}\eta)^r\Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi)+\left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{2(\psi-\phi)^r \operatorname{CH}\left(r, r+1,-\cot(\frac{\pi}{2}\eta)(\phi-\phi)\right)} - \Theta\left(\frac{\phi+\psi}{2}\right) \\
\leq &\frac{Mr}{2(\psi-\phi)^r \operatorname{CH}\left(r, r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \\
&\times \int_{\phi}^{\frac{\phi+\psi}{2}} \left(\frac{\phi+\psi}{2}-\rho\right)^2 \left[e^{-\cot(\frac{\pi}{2}\eta)(\rho-\phi)}(\rho-\phi)^{r-1}+e^{-\cot(\frac{\pi}{2}\eta)(\psi-\rho)}(\psi-\rho)^{r-1}\right] d\rho.
\end{aligned}
$$

Next, we establish the second inequality. Referring to Eq (3.3), we can express it as follows:

$$
\begin{aligned}
&\frac{r\sin(\frac{\pi}{2}\eta)^r\Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi)+\left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{2(\psi-\phi)^r \operatorname{CH}\left(r, r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} - \frac{\Theta(\phi)+\Theta(\psi)}{2} \\
= &\frac{r}{4(\psi-\phi)^r \operatorname{CH}\left(r, r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \int_{\phi}^{\psi} \Big[\Theta(\phi+\psi-\rho)+\Theta(\rho)-(\Theta(\phi)+\Theta(\psi))\Big] \\
&\times \left[e^{-\cot(\frac{\pi}{2}\eta)(\rho-\phi)}(\rho-\phi)^{r-1}+e^{-\cot(\frac{\pi}{2}\eta)(\psi-\rho)}(\psi-\rho)^{r-1}\right] d\rho.
\end{aligned}
$$

Using

$$
[\Theta(t)+\Theta(\phi+\psi-t)-(\Theta(\phi)+\Theta(\psi))]\left[(\phi-t)^n(t-\phi)^{r-n-1}+(t-\phi)^n(\psi-t)^{r-n-1}\right],
$$

is symmetric about $t=\frac{\phi+\psi}{2}$, we have

$$
\begin{aligned}
&\frac{r\sin(\frac{\pi}{2}\eta)^r\Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi)+\left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{2(\psi-\phi)^r \operatorname{CH}\left(r, r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} - \frac{\Theta(\phi)+\Theta(\psi)}{2} \\
= &\frac{r}{2(\psi-\phi)^r \operatorname{CH}\left(r, r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \int_{\phi}^{\frac{\phi+\psi}{2}} \Big[\Theta(\phi+\psi-\rho)+\Theta(\rho)-(\Theta(\phi)+\Theta(\psi))\Big] \\
&\times \left[e^{-\cot(\frac{\pi}{2}\eta)(\rho-\phi)}(\rho-\phi)^{r-1}+e^{-\cot(\frac{\pi}{2}\eta)(\psi-\rho)}(\psi-\rho)^{r-1}\right] d\rho.
\end{aligned}
\tag{3.6}
$$

Since

$$
\Theta(\psi)-\Theta(\phi+\psi-\rho)=\int_{\phi+\psi-\rho}^{\psi} \Theta'(t)dt,
$$

and

$$
\Theta(\rho)-\Theta(\phi)=\int_{\phi}^{\rho} \Theta'(t)dt,
$$

we get

$$\Theta(\rho)+\Theta(\phi+\psi-\rho)-(\Theta(\phi)+\Theta(\psi)) = \int_{\phi}^{\rho}\Theta'(t)dt-\int_{\phi+\psi-\rho}^{\psi}\Theta'(t)dt$$
$$= \int_{\phi}^{\rho}\Theta'(t)dt-\int_{\phi}^{\rho}\Theta'(\phi+\psi-t)dt \tag{3.7}$$
$$= -\int_{\phi}^{\rho}\left[\Theta'(\phi+\psi-t)-\Theta'(t)\right]dt.$$

We also have

$$\Theta'(\phi+\psi-t)-\Theta'(t)=\int_{t}^{\phi+\psi-t}\Theta''(y)dy.$$

Then, for $t\in\left[\phi,\frac{\phi+\psi}{2}\right]$, we get

$$m(\phi+\psi-2t)\le\Theta'(\phi+\psi-t)-\Theta'(t)\le M(\phi+\psi-2t).$$

Hence,

$$-\int_{\phi}^{\rho}M(\phi+\psi-2t)dt\le\Theta(\rho)+\Theta(\phi+\psi-\rho)-(\Theta(\phi)+\Theta(\psi))$$
$$\le-\int_{\phi}^{\rho}m(\phi+\psi-2t)dt.$$

That is,

$$-M(\rho-\phi)(\psi-\rho)\le\Theta(\rho)+\Theta(\phi+\psi-\rho)-(\Theta(\phi)+\Theta(\psi))$$
$$\le-m(\rho-\phi)(\psi-\rho),$$

and

$$\frac{-Mr}{2(\psi-\phi)^{r}\,\mathrm{CH}\left(r,r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)}$$
$$\times\int_{\phi}^{\frac{\phi+\psi}{2}}(\rho-\phi)(\phi-\rho)\Big[e^{-\cot(\frac{\pi}{2}\eta)(\rho-\phi)}(\rho-\phi)^{r-1}+e^{-\cot(\frac{\pi}{2}\eta)(\psi-\rho)}(\psi-\rho)^{r-1}\Big]d\rho$$
$$\le\frac{r\sin(\frac{\pi}{2}\eta)^{r}\Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi)+\left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{2(\psi-\phi)^{r}\,\mathrm{CH}\left(r,r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)}-\frac{h(\phi)+h(\psi)}{2}$$
$$\le\frac{-mr}{2(\psi-\phi)^{2}\,\mathrm{CH}\left(r,r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)}$$
$$\times\int_{\phi}^{\frac{\phi+\psi}{2}}(\rho-\phi)(\psi-\rho)\left[e^{-\cot(\frac{\pi}{2}\eta)(\rho-\phi)}(\rho-\phi)^{r-1}+e^{-\cot(\frac{\pi}{2}\eta)(\psi-\rho)}(\psi-\rho)^{r-1}\right]d\rho.$$

□

Remark 3.1. *When the function $\Theta:[\phi,\psi]\to\mathbb{R}$ has a derivative that is both non-decreasing, it implies the convexity of Θ. Furthermore, if Θ is twice differentiable and satisfies $\Theta''\ge0$, it is also a convex function. In Theorem 3.1, when $\Theta''\ge0$, we arrive at inequality (2.1). Additionally, for the case of $\Theta''\ge0$ and $\eta=1$, we obtain inequality (1.1).*

Clearly, the condition $\Theta'' \geq 0$ ensures that Θ' is non-decreasing. Thus,

$$\Theta'(\phi+\psi-\rho) \geq \Theta'(\rho), \tag{3.8}$$

is holds for all $\rho \in \left[\phi, \frac{\phi+\psi}{2}\right]$. Then, we establish the following theorem using inequality of (3.8)

Theorem 3.2. *Let $\Theta : [\phi,\psi] \to \mathbb{R}^+$ be differentiable mapping with $\phi < \psi$ and $\Theta \in L_1[\phi,\psi]$. If $\Theta'(\phi+\psi-\rho) \geq \Theta'(\rho)$ for all $\rho \in \left[\phi, \frac{\phi+\psi}{2}\right]$. Then, the following inequalities for cotangent integrals hold*

$$\Theta\left(\frac{\phi+\psi}{2}\right) \leq \frac{r\sin(\frac{\pi}{2}\eta)^r\Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi)+\left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{2(\psi-\phi)^r\,\mathrm{CH}\left(r,r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \leq \frac{\Theta(\phi)+\Theta(\psi)}{2}. \tag{3.9}$$

Proof. From Eqs (3.4) and (3.7), we have

$$\begin{aligned}
&\frac{r\sin(\frac{\pi}{2}\eta)^r\Gamma(r)\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi)+\left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right]}{2(\psi-\phi)^r\,\mathrm{CH}\left(r,r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} - \Theta\left(\frac{\phi+\psi}{2}\right)\\
&= \frac{r}{2(\psi-\phi)^r\,\mathrm{CH}\left(r,r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)}\int_{\phi}^{\frac{\phi+\psi}{2}}\left[\Theta(\phi+\psi-\rho)-\Theta(\rho)-2\Theta\left(\frac{\phi+\psi}{2}\right)\right]\\
&\quad\left[e^{\left[-\cot(\frac{\pi}{2}\eta)(\rho-\phi)\right]}(\rho-\phi)^{r-1}+e^{\left[-\cot(\frac{\pi}{2}\eta)(\psi-\rho)\right]}(\psi-\rho)^{r-1}\right]d\rho\\
&= \frac{r}{2(\psi-\phi)^r\,\mathrm{CH}\left(r,r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)}\int_{\phi}^{\frac{\phi+\psi}{2}}\left[\int_{\rho}^{\frac{\phi+\psi}{2}}\left[\Theta'(\phi+\psi-t)-\Theta'(t)\right]dt\right]\\
&\quad\times\left[e^{\left[-\cot(\frac{\pi}{2}\eta)(\rho-\phi)\right]}(\rho-\phi)^{r-1}+e^{\left[-\cot(\frac{\pi}{2}\eta)(\psi-\rho)\right]}(\psi-\rho)^{r-1}\right]d\rho\\
&\geq 0.
\end{aligned}$$

Similarly, from (3.6) and (3.5), gets

$$\begin{aligned}
&\frac{r\sin(\frac{\pi}{2}\eta)^r\Gamma(r)}{2(\psi-\phi)^r\,\mathrm{CH}\left(r,r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)}\left[\left(\mathfrak{I}_{\psi}^{r,\eta}\Theta\right)(\phi)+\left({}_{\phi}\mathfrak{I}^{r,\eta}\Theta\right)(\psi)\right] - \frac{\Theta(\phi)+\Theta(\psi)}{2}\\
&= \frac{r}{2(\psi-\phi)^2\,\mathrm{CH}(r,r+1,-\cot(\frac{\pi}{2}\eta)(\psi-\phi))}\int_{\phi}^{\frac{\phi+\psi}{2}}[-\int_{\phi}^{\rho}[\Theta'(\phi+\psi-t)-\Theta'(t)]dt]\\
&\quad\times[e^{[-\cot(\frac{\pi}{2}\eta)(\rho-\phi)]}(\rho-\phi)^{r-1}+e^{[-\cot(\frac{\pi}{2}\eta)(\psi-\rho)]}](\psi-\rho)^{r-1}]d\rho\\
&\leq 0.
\end{aligned}$$

□

4. Application

In this section, we present an application that demonstrates the implications of our matrix-related findings. Here, we denote the set of $m\times m$ complex matrices as C^m. Similarly, $\mathbb{M}_m$ denotes the algebra of $m\times m$ matrices, and $\mathbb{M}_m^+$ refers to the subset of strictly positive matrices within $\mathbb{M}_m$. For a matrix $G \in \mathbb{M}_m^+$, it holds that $\langle Gv, v\rangle > 0$ for all nonzero $v \in C^m$.

Incorporating matrix concepts and convexity principles, the function $\Theta(s) := \left|G^s Y L^{1-s} + G^{1-s} Y L^s\right|$, where $G, L \in \mathbb{M}_m^+$, and $Y \in \mathbb{M}_m$. It was demonstrated that this function is convex for all $s \in [0,1]$. Therefore, by employing the Theorem 2.1, where $0 \le \phi < \psi \le 1$, we deduce that

$$\begin{aligned}\left\|G^{\frac{\phi+\psi}{2}} Y L^{1-\frac{\phi+\psi}{2}} + G^{1-\frac{\phi+\psi}{2}} Y L^{\frac{\phi+\psi}{2}}\right\| &\le \frac{r \sin(\frac{\pi}{2}\eta)^r \Gamma(r) \Big(\left\|\left(\Im_{\psi}^{r,\eta}\Theta\right)(\phi)\right\| + \left\|\left({}_{\phi}\Im^{r,\eta}\Theta\right)(\psi)\right\| \Big)}{2(\psi-\phi)^r \, \mathrm{CH}\left(r, r+1, -\cot(\frac{\pi}{2}\eta)(\psi-\phi)\right)} \\ &\le \frac{\left\|G^{\phi} Y L^{1-\phi} + G^{1-\phi} Y L^{\phi}\right\| + \left\|G^{\psi} Y L^{1-\psi} + G^{1-\psi} Y L^{\psi}\right\|}{2}.\end{aligned}$$

5. Conclusions

The Hermite–Hadamard inequality occupies a prominent position within the realms of inequality theory, mathematical analysis, and statistics. Over the years, this inequality has been a focal point for extensive research, driving numerous efforts to uncover fresh variations, generalizations, and extensions using diverse methodologies. In this study, we present original extensions to the Hermite–Hadamard inequality, making innovative use of an introduced integral operator, specifically the cotangent integral operator. What sets our approach apart is the utilization of a fractional integral operator featuring a non-singular kernel, distinguishing it from other fractional integral operators. Furthermore, our research highlights the potential for future advancements with the introduction of novel fractional integral operators characterized by robust kernels, and the applications of Hermite–Hadamard inequality [30].

Author contributions

Lakhlifa Sadek and Ali Algefary: Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Writing–original draft, Writing–review and editing. All authors of this article have contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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