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Research article

On Roman balanced domination of graphs

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Abstract: Let G be a graph with vertex set V. A function $f : V \to \{-1, 0, 2\}$ is called a Roman balanced dominating function (RBDF) of G if $\sum_{u\in N_G[v]} f(u) = 0$ for each vertex $v \in V$. The maximum (resp. minimum) Roman balanced domination number $\gamma^M_{Rb}(G)$ (resp. $\gamma^m_{Rb}(G)$) is the maximum (resp. minimum) value of $\sum_{v\in V} f(v)$ among all Roman balanced dominating functions f. A graph G is called Rd-balanced if $\gamma^M_{Rb}(G) = \gamma^m_{Rb}(G) = 0$. In this paper, we obtain several upper and lower bounds on $\gamma^M_{Rb}(G)$ and $\gamma^m_{Rb}(G)$ and further determine several classes of Rd-balanced graphs.

Keywords: Roman balanced dominating function; Roman balanced domination number; Rd-balanced graph
Mathematics Subject Classification: 05C69

1. Introduction

Let $G = (V, E)$ be a graph. For a $v \in V$, we denote by $N(v)$ and $N[v]$ the neighbour set and closed neighbour set of v, i.e., $N(v) = \{u \in V | uv \in E\}$ and $N[v] = \{v\} \cup N(v)$. The size of $N(v)$ is denoted by $d(v)$ and refers to the *degree* of v in G. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For two subsets A and B, we denote by $E(A, B)$ the set of the edges between A and B in G. In addition, we denote by $d_{G[A]}(v)$ and $E_{G[A]}$ the degree of v and the set of edges in the induced graph $G[A]$, respectively. For $S \subset V$ and a function $f : V \to R$, we write $f(S) = \sum_{v\in S} f(v)$.

Graph domination is one of the fundamental concepts in graph theory and has wide applications. The notion of a *dominating set* can also be modeled as a function $f : V \to \{0, 1\}$ such that $f(N[v]) \geq 1$ for each $v \in V$. Motivated by the problem of defending the Roman Empire [17], Cockayne et al. [9] defined the notion of *Roman dominating function (RDF)* on a graph $G = (V, E)$ by $f : V \to \{0, 1, 2\}$ such that $f(N[v]) \geq 1$ for each $v \in V$ and each vertex u with $f(u) = 0$ has a neighbor v with $f(v) = 2$. In recent years, this concept received further development, like total Roman dominating function [3], perfect and weak Roman dominating function [5, 10, 11, 15], global Roman dominating function [6, 7, 13, 14] and so on. For more related results, see [1, 2, 4, 8, 12, 16].

The notion of a *balanced dominating function (BDF)* is defined by a function $f : V \to \{-1, 0, 1\}$ that satisfies $f(N[u]) = 0$ for each $u \in V$, which was first introduced by Xu et al. [19] in 2021. The balanced domination number of G is therefore defined as the maximum weight of all BDF's in G and is denoted by $\gamma_{\mathrm{b}}(G)$. Soon after, Xu and his collaborators proposed the notion of a *balanced cycle dominating function*; see [18].

Inspired by these aforementioned results, we define a new dominating function of a graph G, called the *Roman balanced dominating function (RBDF)*, by $f : V \to \{-1, 0, 2\}$ which satisfies $f(N[v]) = 0$ for each $v \in V$. The maximum (resp. minimum) weight of all RBDF's on G refers to the maximum (resp. minimum) Roman balanced domination number, denoted by $\gamma_{\mathrm{Rb}}^{\mathrm{M}}(G)$ (resp. $\gamma_{\mathrm{Rb}}^{\mathrm{m}}(G)$), that is,

$$\gamma_{\mathrm{Rb}}^{\mathrm{M}}(G) = \max\{f(V) : f \text{ is an RBDF of } G\},$$
$$\gamma_{\mathrm{Rb}}^{\mathrm{m}}(G) = \min\{f(V) : f \text{ is an RBDF of } G\}.$$

By the definition of RBDF, the function $f = 0$ is trivially an RBDF for any G. Thus, for any G, we have $\gamma_{\mathrm{Rb}}^{\mathrm{M}}(G) \geq 0$ and $\gamma_{\mathrm{Rb}}^{\mathrm{m}}(G) \leq 0$. In particular, if $\gamma_{\mathrm{Rb}}^{\mathrm{M}}(G) = \gamma_{\mathrm{Rb}}^{\mathrm{m}}(G) = 0$ then we call G *Rd-balanced*. Of course, there exists some graphs that are not Rd-balanced; see Figure 1, where the black vertices are presented as 2 and the white vertices are presented as -1.

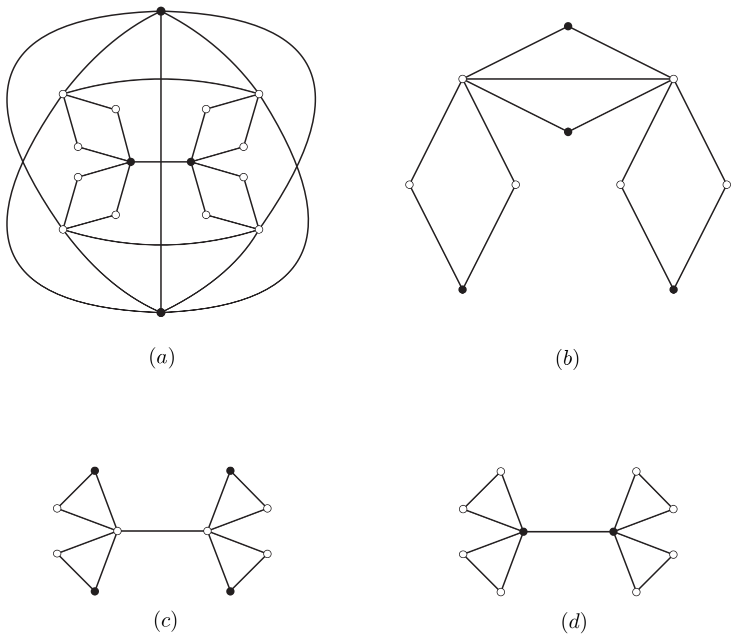

Figure 1. The examples of non-Rd-balanced graphs.

In this paper, we introduce the bounds on $\gamma_{\mathrm{Rb}}^{\mathrm{M}}(G)$ and $\gamma_{\mathrm{Rb}}^{\mathrm{m}}(G)$ with the maximal degree, the minimal degree, the order, and the size of edges of a graph G. Furthermore, we establish a relationship between the Roman balanced dominating function and the balanced dominating function, which is used for determining whether a graph is Rd-balanced or not. Finally, we give several classes of Rd-balanced graphs.

2. Main results

In this section, we consider some bounds on $\gamma_{\mathrm{Rb}}^{\mathrm{M}}$ and $\gamma_{\mathrm{Rb}}^{\mathrm{m}}$. For an RBDF f and $i \in \{-1, 0, 2\}$, let $A_i = \{v \in V : f(v) = i\}$, the size, of A_0, A_2, and A_{-1} are denoted by r, s, and t, respectively. First, we provide the range of s and t, which is useful in the following proof.

Lemma 1. *For any RBDF f,*

$$t \le \frac{4n - \sqrt{8n+1} + 1}{4} \quad \text{and} \quad s \le n + 2 - 2\sqrt{n+1}.$$

Proof. Let $n' = s + t$. Then $f(V) = 2s - t = 3s - n' \ge 3s - n$ as $n' = s + t \le n$.

For any $u \in A_{-1}$, we have $|N[u] \cap A_2| \ge 1$. Hence $|E(A_2, A_{-1})| \ge t$. That implies for $v \in A_2$, $|N[v] \cap A_{-1}| \ge \left\lceil \frac{t}{s} \right\rceil$. Note that $f(N[v]) = 0$ and $f(N[v]) = 2|N[v] \cap A_2| - |N[v] \cap A_{-1}|$. Therefore, $|N[v] \cap A_2| \ge \left\lceil \frac{t}{2s} \right\rceil$. Hence,

$$n' - t = s \ge \left\lceil \frac{t}{2s} \right\rceil \ge \frac{t}{2(n' - t)}.$$

Since $n \ge n' \ge t$, we have $t \le \frac{4n' - \sqrt{8n'+1} + 1}{4} \le \frac{4n - \sqrt{8n+1} + 1}{4}$.

Similarly, we have $f(N[v]) = 0$ for any $v \in A_2$. Thus, v has at least two neighbors belonging to A_{-1}, and $|E(A_{-1}, A_2)| \ge 2s$. Therefore, we can find a vertex $u \in A_{-1}$, and we have $|N[u] \cap A_2| \ge \left\lceil \frac{2s}{t} \right\rceil$. Since $f(N[u]) = 0$ and $f(N[u]) = 2|N[u] \cap A_2| - |N[u] \cap A_{-1}|$, then we have

$$t \ge |N[u] \cap A_{-1}| = 2|N[u] \cap A_2| \ge 2\left\lceil \frac{2s}{t} \right\rceil \ge \frac{4s}{t},$$

which implies that $(n' - s)^2 = t^2 \ge 4s$. Therefore, $s \le n' + 2 - 2\sqrt{n'+1} \le n + 2 - 2\sqrt{n+1}$ as $s \le n$. □

Theorem 1. *If G has n vertices, then we have*

$$\frac{(\delta - \Delta)(4n - \sqrt{8n+1} + 1)}{4(\Delta + 1)} \le \gamma_{Rb}^{m}(G) \le 0 \le \gamma_{Rb}^{M}(G) \le \frac{2(\Delta - \delta)n}{3(\delta + 1)},$$

where the first (or the last) equality holds if and only if G is Δ-regular.

Proof. Let f be a maximum RBDF of G with the weight $\gamma_{Rb}^{M}(G)$. We find $s + t \le n$ and $\gamma_{Rb}^{M}(G) = 2s - t$. By the definition of RBDF, $f(N[v]) = 0$ for any $v \in A_2$. Furthermore,

$$\begin{aligned}\sum_{v \in A_2} f(N[v]) &= \sum_{v \in A_2} (2|E(\{v\}, A_2)| + 2 - |E(\{v\}, A_{-1})|) \\ &= 4|E_{G[A_2]}| + 2s - |E(A_{-1}, A_2)|.\end{aligned}$$

Thus, $2s = |E(A_{-1}, A_2)| - 4|E_{G[A_2]}|$. It is easy to get that $t = 2|E(A_{-1}, A_2)| - 2|E_{G[A_{-1}]}|$ by the similar discussion. Thus,

$$\gamma_{Rb}^{M}(G) = 2s - t = 2|E_{G[A_{-1}]}| - 4|E_{G[A_2]}| - |E(A_{-1}, A_2)|. \tag{1}$$

Furthermore,

$$\sum_{v\in A_{-1}} d(v) = |E(A_0, A_{-1})| + |E(A_{-1}, A_2)| + 2|E_{G[A_{-1}]}| \leq \Delta t,$$
$$\sum_{v\in A_2} d(v) = |E(A_0, A_2)| + |E(A_{-1}, A_2)| + 2|E_{G[A_2]}| \geq \delta s.$$

Combining with Eq (1), we have

$$\begin{aligned}\gamma_{Rb}^{M}(G) \leq& \Delta t - |E(A_0, A_{-1})| - |E(A_{-1}, A_2)| \\ &- 2\left(\delta s - |E(A_0, A_2)| - |E(A_{-1}, A_2)|\right) - |E(A_{-1}, A_2)| \\ =& \Delta t - 2\delta s - |E(A_0, A_{-1})| + 2|E(A_0, A_2)| \\ =& \Delta t - 2\delta s,\end{aligned}$$

where the last equality holds because $0 = \sum_{v\in A_0} f(N[v]) = |E(A_0, A_{-1})| - 2|E(A_0, A_2)|$ for any $v \in A_0$.

Note that $\gamma_{Rb}^{M}(G) = 2s - t$. We have $2s - t \leq \Delta t - 2\delta s$, which means $s \leq \frac{\Delta+1}{2\delta+2} \cdot t$. Then

$$\gamma_{Rb}^{M}(G) = 2s - t \leq \frac{(\Delta - \delta)t}{\delta + 1}.$$

Since $\gamma_{Rb}^{M}(G) \geq 0$, we have $2s \geq t$, i.e. $t \leq \frac{2}{3}n$. Hence, $\gamma_{Rb}^{M}(G) \leq \frac{2(\Delta-\delta)n}{3(\delta+1)}$, as desired.

By a similar argument, we denote by g a minimum RBDF of G and $\gamma_{Rb}^{m}(G)$ the weight. Thus

$$\sum_{v\in A_{-1}} d(v) = |E(A_0, A_{-1})| + |E(A_{-1}, A_2)| + 2|E_{G[A_{-1}]}| \geq \delta t,$$
$$\sum_{v\in A_2} d(v) = |E(A_0, A_2)| + |E(A_{-1}, A_2)| + 2|E_{G[A_2]}| \leq \Delta s.$$

Combining with (1), we have

$$\begin{aligned}\gamma_{Rb}^{m}(G) \geq& \delta t - |E(A_0, A_{-1})| - |E(A_{-1}, A_2)| \\ &- 2\left(\Delta s - |E(A_0, A_2)| - |E(A_{-1}, A_2)|\right) - |E(A_{-1}, A_2)| \\ =& \delta t - 2\Delta s - |E(A_0, A_{-1})| + 2|E(A_0, A_2)| \\ =& \delta t - 2\Delta s.\end{aligned}$$

Note that $\gamma_{Rb}^{m}(G) = 2s - t$. We have $2s - t \geq \delta t - 2\Delta s$, which means $s \geq \frac{\delta+1}{2\Delta+2} \cdot t$. Then

$$\gamma_{Rb}^{m}(G) = 2s - t \geq \frac{\delta - \Delta}{\Delta + 1} \cdot t.$$

Since $\frac{\delta-\Delta}{\Delta+1} \le 0$, then by Lemma 1, we have $\gamma_{Rb}^{m}(G) \ge \frac{(\delta-\Delta)(4n-\sqrt{8n+1}+1)}{4(\Delta+1)}$, as desired.

Finally, we consider the condition that makes the equality hold.

The sufficiency is obviously true. Now we prove the necessity. We only consider a graph G satisfying $\gamma_{Rb}^{M}(G) = \frac{2(\Delta-\delta)n}{3(\delta+1)}$. For the case $\gamma_{Rb}^{m}(G) = \frac{(\delta-\Delta)(4n-\sqrt{8n+1}+1)}{4(\Delta+1)}$, the argument is similar. Above all, we may derive that $\gamma_{Rb}^{M}(G) = \frac{2(\Delta-\delta)n}{3(\delta+1)}$ with the following conditions holding:

(i) Each vertex in A_{-1} (resp. A_2) has degree Δ (resp. δ);

(ii) $s = \frac{1}{3}n$ and $t = \frac{2}{3}n$.

Let $v \in A_2$. The degree of every vertex in A_2 is δ; that means $\delta = |E(\{v\}, A_2)| + |E(\{v\}, A_{-1})|$. Also, we know that $f(N[v]) = 2|E(\{v\}, A_2)| + 2 - |E(\{v\}, A_{-1})| = 0$. Thus $|E(\{v\}, A_{-1})| = \frac{2\delta+2}{3}$ and $|E(\{u\}, A_2)| = \frac{\Delta+1}{3}$ for every $u \in A_{-1}$ by the similar discussion. Thus for $|E(A_{-1}, A_2)|$, it is equal to

$$\sum_{v\in A_2}(|E(\{v\}, A_{-1})| = \frac{2s(\delta+1)}{3},$$
$$\sum_{v\in A_{-1}}(|E(\{u\}, A_2)| = \frac{t(\Delta+1)}{3}.$$

Note that $2s = t$. Then $\Delta = \delta$, which means G is Δ-regular. □

By the above theorem, we infer that the regular graph is Rd-balanced.

Theorem 2. *If G has $n \ge 3$ vertices, then $\gamma_{Rb}^{M}(G) \le 2n + 6 - 6\sqrt{n+1}$. The equality holds if and only if G is obtained by adding edges between K_t and $\frac{t^2}{4}K_1$ such that the degree of the vertex in $\frac{t^2}{4}K_1$ is 2 and in K_t is $\frac{t}{2}$, where $t = 2\sqrt{n+1} - 2$.*

Proof. Let f be a maximum RBDF of G with the weight $\gamma_{Rb}^{M}(G)$. By Lemma 1, we have $\gamma_{Rb}^{M}(G) = 2s - t = 3s - n' \le 2n' + 6 - 6\sqrt{n'+1} \le 2n + 6 - 6\sqrt{n+1}$.

The discussion of the equality can be divided into two parts as follows:

First, we demonstrate the necessity. Let G satisfy $\gamma_{Rb}^{M}(G) = 2n+6-6\sqrt{n+1}$. By the proof of Lemma 1, the upper bound is tight if and only if $n = n'$, $s = \frac{t^2}{4}$. That means that $A_0 = \emptyset$ and $t = 2\sqrt{n+1} - 2$. For any $u \in A_{-1}$, we observe that $2|N[u] \cap A_2| = |N[u] \cap A_{-1}| \le |A_{-1}| = t$. Thus,

$$|E(A_2, A_{-1})| = \sum_{u\in A_{-1}} |N[u] \cap A_2| = \frac{1}{2}\sum_{u\in A_{-1}} |N[u] \cap A_{-1}| \le \frac{t^2}{2}.$$

Since $|E(A_2, A_{-1})| \ge 2s = \frac{t^2}{2}$, we obtain

$$|E(A_2, A_{-1})| = \frac{t^2}{2} \quad \text{and} \quad 2|N[u] \cap A_2| = |N[u] \cap A_{-1}| = t.$$

Therefore, any vertex in A_{-1} has just $\frac{t}{2}$ neighbours in A_2. Furthermore, since $|N[u] \cap A_{-1}| = t$, we have $G[A_{-1}] = K_t$. Similarly, $2|N[v] \cap A_2| = |N[v] \cap A_{-1}| = 2$, where v is an arbitrary vertex in A_2, which means each vertex in A_2 has exactly 2 neighbours in A_{-1} and $G[A_2] = sK_1 = \frac{t^2}{4}K_1$, as desired.

Next, we demonstrate the sufficiency. Let G be generated by adding edges between K_t and $\frac{t^2}{4}K_1$, such that the degree of the vertex in $\frac{t^2}{4}K_1$ is 2 and the degree of the vertex in K_t is $\frac{t}{2}$. We know that $n = t + \frac{t^2}{4}$. Then we define f':

$$f'(v) = \begin{cases} -1, & \text{if } v \in K_t, \\ 2, & \text{otherwise.} \end{cases}$$

Obviously, f' is an RBDF of G. Then we have $\gamma^M_{Rb}(G) \geq f'(V) = \frac{t^2}{2} - t = 2n + 6 - 6\sqrt{n+1}$. Above all, we have $\gamma^M_{Rb}(G) = 2n + 6 - 6\sqrt{n+1}$. □

Theorem 3. *If G has $n \geq 3$ vertices, then $\gamma^m_{Rb}(G) \geq 3\lceil \frac{\sqrt{1+8n}-1}{4} \rceil - n$. Figure 1 (d) is an example that makes this bound tight.*

Proof. Let f be a minimum RBDF of G; the weight is $\gamma^m_{Rb}(G)$. The notation is like Lemma 1, $n' := s + t \leq n$ and $\gamma^m_{Rb}(G) = 3s - n' \geq 3s - n$.

For all $u \in A_{-1}$, we know that $|A_2 \cap N[u]| \geq 1$, hence $|E(A_2, A_{-1})| \geq t$. If $v \in A_2$, then $|N[v] \cap A_{-1}| \geq \lceil \frac{t}{s} \rceil$. Note that $0 = f(N[v]) = 2|A_2 \cap N[u]| - |N[v] \cap A_{-1}|$. Hence $|A_2 \cap N[u]| \geq \lceil \frac{t}{2s} \rceil$, thus

$$s \geq \left\lceil \frac{t}{2s} \right\rceil \geq \frac{n' - s}{2s}.$$

By the above inequality, we deduce that

$$s \geq \frac{\sqrt{1+8n'} - 1}{4}.$$

Since s is a nonnegative integer and $\gamma^m_{Rb}(G) = 3s - n' \geq 3\, \frac{\sqrt{1+8n'}-1}{4} - n'$, then for $n \geq 3$, we have $\gamma^m_{Rb}(G) \geq 3\, \frac{\sqrt{1+8n'}-1}{4} - n' \geq 3\, \frac{\sqrt{1+8n}-1}{4} - n$, hence the conclusion is true. □

By Theorem 2 and 3, let $\omega(f)$ be the weight of all RBDF's; then we have $3\lceil \frac{\sqrt{1+8n}-1}{4} \rceil - n \leq \omega(f) \leq 2n + 6 - 6\sqrt{n+1}$, which gives the range of $\omega(f)$ about the order of the graph.

Theorem 4. *If G has n vertices and m edges, and each component is none of K_1, K_2, and Star, then*

$$\gamma^m_{Rb}(G) > \frac{4}{7}(2n - 3m). \tag{2}$$

Proof. Let $f = (A_{-1}, A_0, A_2)$ be a minimum RBDF of G. Assume that G contains exactly k pendent vertices. We will show that Ineq (2) holds for $k = 0$, that is, G contains no pendent vertex. Suppose that we have finished the proof when $k = 0$. Now we use induction on n. For $k \geq 1$, let G delete all pendent vertices, denoted by G'. Since each component of G is none of K_1, K_2, and Star, so is G'. Note that for the pendent vertex v, we have $f(v) = 0$. Thus $\gamma^m_{Rb}(G) = \gamma^m_{Rb}(G') > \frac{4}{7}(2|V(G')| - 3|E(G')|)$ by the induction. Since $|V(G')| = n - k$ and $|E(G')| = m - k$, we have

$$\gamma^m_{Rb}(G) > \frac{4}{7}(2n - 3m + k) > \frac{4}{7}(2n - 3m),$$

as desired.

Now we prove Ineq (2) holds when $k = 0$. Let $A_{02} = A_0 \cup A_2$, then we have $|E(A_{-1}, A_{02})| \geq |E(A_{-1}, A_2)| \geq |A_{-1}| = t$. Further, for $v \in A_2$, there is $f(v) + 2d_{G[A_2]}(v) - d_{G[A_{-1}]}(v) = f(N_G[v]) = 0$. Thus, $d_{G[A_{-1}]}(v) = 2d_{G[A_2]}(v) + f(v) = 2d_{G[A_2]}(v) + 2$. Recall that $r = |A_0|$, $s = |A_2|$, and $t = |A_{-1}|$. Hence,

$$\begin{aligned} t \leq& |E(A_{-1}, A_2)| = \sum_{v\in A_2} d_{G[A_{-1}]}(v) = \sum_{v\in A_2}(2d_{G[A_2]}(v) + 2) \\ =& 4|E_{G[A_2]}| + 2s = 4|E_{G[A_{02}]}| + 2s - 4|E_{G[A_0]}| - 4|E(A_0, A_2)|, \end{aligned}$$

where the last equality holds because $|E_{G[A_{02}]}| = |E_{G[A_0]}| + |E_{G[A_2]}| + |E(A_0, A_2)|$. So

$$|E_{G[A_{02}]}| \geq \frac{t - 2s + 4|E_{G[A_0]}| + 4|E(A_0, A_2)|}{4}.$$

Hence,

$$\begin{aligned} m \geq& |E_{G[A_{02}]}| + |E(A_{-1}, A_{02})| \\ \geq& \frac{1}{4}(t - 2s + 4|E_{G[A_0]}| + 4|E(A_0, A_2)|) + t \\ =& \frac{1}{4}(5t - 2s + 4|E_{G[A_0]}| + 4|E(A_0, A_2)|) \\ =& \frac{1}{4}(5n - 7n_{02} + 2r + 4|E_{G[A_0]}| + 4|E(A_0, A_2)|), \end{aligned}$$

where $n_{02} = |A_{02}| = s + r$. Equivalently,

$$n_{02} \geq \frac{1}{7}(5n - 4m + 2r + 4|E_{G[A_0]}| + 4|E(A_0, A_2)|).$$

Therefore,

$$\begin{aligned} \gamma^m_{Rb}(G) =& 2s - t = 3s + r - n = 3n_{02} - 2r - n \\ \geq& \frac{3}{7}(5n - 4m + 2r + 4|E_{G[A_0]}| + 4|E(A_0, A_2)|) - 2r - n \\ =& \frac{4}{7}(2n - 3m) + \frac{4}{7}(3|E_{G[A_0]}| + 3|E(A_0, A_2)| - 2r). \end{aligned}$$

Let

$$\phi(r) = \frac{4}{7}(3|E_{G[A_0]}| + 3|E(A_0, A_2)| - 2r).$$

We only need to show $\phi(r) \geq 0$, then we have $\gamma^m_{Rb}(G) \geq \frac{4}{7}(2n - 3m)$. If $r = 0$, then $\phi(r) = 0$. Then let $r \geq 1$. If $v \in A_0$ and $d_{G[A_{02}]}(v) = 0$, according to the assumption that G contains no K_1, then we obtain

$d(v) \geq 1$, and all of the neighbors of v are contained in V_{-1}. However, $f(N[v]) \leq -1$ now, we obtain a contradiction. Hence, we have $v \in V_0$, and $d_{G[A_{02}]}(v) \geq 1$. Thus,

$$\begin{aligned}\phi(r) =&\frac{6}{7}(2|E_{G[A_0]}|) + |E(A_0, A_2)| + \frac{5}{7}|E(A_0, A_2)| - \frac{8}{7}r \\ =&\frac{6}{7}\sum_{v\in A_0} d_{G[A_0]}(v) + \sum_{v\in A_0} d_{G[A_2]}(v) + \frac{5}{7}|E(A_0, A_2)| - \frac{8}{7}r \\ =&\sum_{v\in A_0} d_{G[A_{02}]}(v) - \frac{1}{7}\sum_{v\in A_0} d_{G[A_0]}(v) + \frac{5}{7}|E(A_0, A_2)| - \frac{8}{7}r. \end{aligned} \quad (3)$$

Let $X = \{d_{G[A_{02}]}(v) \geq 2 : v \in A_0\}$ and $Y = \{d_{G[A_{02}]}(v) = 1 : v \in A_0\}$. Then $r = |X| + |Y|$. Combining with $|E(A_0, A_2)| = \sum_{v\in A_0} d_{G[A_2]}(v)$, we can consider Eq (3) in two parts as follows:

$$\begin{aligned}\phi(r) =& \sum_{v\in X} d_{G[A_{02}]}(v) - \frac{1}{7}\sum_{v\in X} d_{G[A_0]}(v) + \frac{5}{7}\sum_{v\in X} d_{G[A_2]}(v) - \frac{8}{7}|X| \\ &+ \sum_{v\in Y} d_{G[A_{02}]}(v) - \frac{1}{7}\sum_{v\in Y} d_{G[A_0]}(v) + \frac{5}{7}\sum_{v\in Y} d_{G[A_2]}(v) - \frac{8}{7}|Y| \\ &\geq \frac{6}{7}\sum_{v\in X} d_{G[A_{02}]}(v) - \frac{8}{7}|X| + \sum_{v\in Y}\left(d_{G[A_{02}]}(v) - \frac{1}{7}d_{G[A_0]}(v) + \frac{5}{7}d_{G[A_2]}(v)\right) - \frac{8}{7}|Y| \\ &\geq \frac{12}{7}|X| - \frac{8}{7}|X| + \sum_{v\in Y}\left(d_{G[A_{02}]}(v) - \frac{1}{7}d_{G[A_0]}(v) + \frac{5}{7}d_{G[A_2]}(v)\right) - \frac{8}{7}|Y| \\ &\geq \sum_{v\in Y}\left(d_{G[A_{02}]}(v) - \frac{1}{7}d_{G[A_0]}(v) + \frac{5}{7}d_{G[A_2]}(v)\right) - \frac{8}{7}|Y|. \end{aligned} \quad (4)$$

where the first inequality holds as $d_{G[A_{02}]}(v) \geq d_{G[A_0]}(v)$ and $d_{G[A_2]}(v) \geq 0$ for each $v \in A_0$. Now we focus on an arbitrary vertex v in Y. Then either $d_{G[A_0]}(v) = 1$ or $d_{G[A_2]}(v) = 1$. If we suppose $d_{G[A_0]}(v) = 1$, then v must be a pendent vertex, which contradicts G having no pendent vertices. So $d_{G[A_0]}(v) = 0$ and $d_{G[A_{02}]}(v) = d_{G[A_2]}(v) = 1$ for each $v \in Y$. By Ineq (4), we have

$$\phi(r) \geq |Y| + \frac{5}{7}|Y| - \frac{8}{7}|Y| = \frac{4}{7}|Y| \geq 0,$$

and so $\gamma_{Rb}^m(G) > \frac{4}{7}(2n - 3m)$, as desired. □

Theorem 5. *If f is an RBDF of G, then there is a BDF g of G' satisfying the weight of them is equal, where G' is obtained by adding edges between G and a copy G^* of G; any $v \in V$ is adjacent to the copies of all vertices in $N[v]$. Further, $\gamma_{Rb}^M(G) \leq \gamma_b(G')$ and $\gamma_{Rb}^m(G) \geq -\gamma_b(G')$.*

Proof. Denote $V(G') = \{v_1 : v \in V(G)\} \cup \{v_2 : v \in V(G^*)\}$. There are three situations. If $f(v) = 0$, we say $g(v_1) = g(v_2) = 0$. If $f(v) = -1$, then let $g(v_1) = -1$ and $g(v_2) = 0$. If $f(v) = 2$, then $g(v_1) = g(v_2) = 1$. Obviously, $g(v_1)+g(v_2) = f(v)$. Then we have $g(N[v_1]) = g(N[v_2]) = \sum_{u\in N[v]}(g(u_1) + g(u_2)) = f(N[v])$. Hence, g is a BDF of G' satisfying the weight of f and g are equal.

Now we choose f a maximum RBDF of G,

$$\gamma^M_{Rb}(G) = f(V(G)) = g(V(G')) \leq \gamma_b(G').$$

Similarly, we have $\gamma^m_{Rb}(G) \geq -\gamma_b(G')$. □

Remark 1. By the above theorem, if G' is d-balanced, that is $\gamma_b(G') = 0$, then we have $\gamma^M_{Rb}(G) = \gamma^m_{Rb}(G) = 0$. So G is also Rd-balanced.

Remark 2. By Theorem 2.4 of [19] and Theorem 5, $\gamma^M_{Rb}(G) \leq \gamma_b(G') \leq 2n - 2\gamma(G')$. Furthermore, there is $\gamma(G) = \gamma(G')$, thus $\gamma^M_{Rb}(G) \leq 2n - 2\gamma(G)$. The equality holds when G contains only isolated vertices.

Proposition 6. *$f(V)$ is even.*

Proof. For an RBDF, we have $\sum_{v\in A_{-1}} f(N[v]) = 0$, and

$$\sum_{v\in A_{-1}} f(N[v]) = \sum_{v\in A_{-1}} \left(f(v) + f(N_{G[A_0\cup A_2]}(v)) + f(N_{G[A_{-1}]}(v))\right).$$

Note that $f(N_{G[A_0\cup A_2]}(v))$ and $\sum_{v\in A_{-1}} f(N_{G[A_{-1}]}(v)) = 2E_{G[A_{-1}]}$ are both even. Thus,

$$f(V) = \sum_{v\in A_0\cup A_{-1}\cup A_2} f(v) \equiv \sum_{v\in A_{-1}} f(v) \equiv 0\ (mod\ 2),$$

as desired. □

Proposition 7. *A forest is Rd-balanced.*

Proof. We only need to show each tree G is Rd-balanced. Since f is an RBDF of G, it is clear that for any $v \in V$, $f(N[v]) = 0$.

Note that for each $uv \in E$, $f(u) + f(v) = 0$ if and only if both items are zero. This means every leaf $v \in G$ satisfies $f(v) = 0$, then we delete all leaves, denoted by G'. G' is still a tree, and all leaves of G' are in V_0 by a similar argument. By deleting leaves from G repeatedly, we find that every vertex in G is in V_0. Hence, G is Rd-balanced, as desired. □

In addition, the regular graph is Rd-balanced, as mentioned before.

Some d-balanced graphs are shown by Xu et al. [19]. By the same argument, we can obtain that if the graph meets one of the following conditions, it is Rd-balanced.

(i) $\Delta(G) = n - 1$;

(ii) G is a generalized ladder graph of order $2n$;

(iii) G is a complete multipartite graph;

(iv) G is a join graph of path and cycle, or path and path, or cycle and cycle.

Remark 3. We can also give a join graph G that is not Rd-balanced, for example; see Figure 2. We find that $G \vee G$ with the given labelling in G satisfies $\gamma^m_{Rb}(G \vee G) = -2$.

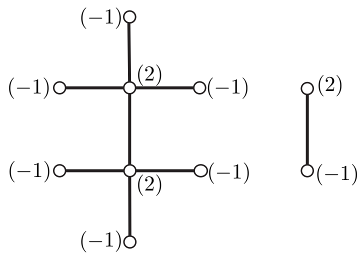

Figure 2. Graph G ($G \vee G$ is not Rd-balanced).

3. Conclusions

In this paper, firstly, we provide the notion of Roman balanced dominating function, which generates two concepts in graph theory: Roman domination function and balance domination function. Then we give some upper and lower bounds on the maximum and minimum Roman balanced domination number of the graph, in terms of the minimum degree, the maximum degree, the order, and the size of the edges of the graph. Finally, we determine some graphs that are Rd-balanced.

Author contributions

M. Y. Zhang: Conceptualization, Investigation, Methodology, Writing-original draft, Writing-review, editing. J. X. Zhang: Conceptualization, Methodology, Supervision, Writing-review, editing.

Conflict of interest

The authors declare that they have no conflicts of interest.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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