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Research article

Neutrosophic modules over modules

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Abstract: A module represents a fundamental and complicated algebraic structure associated with a particular binary operation in algebraic theory. This paper introduces a new class of neutrosophic sub-module and neutrosophic R-sub-module. We extend the basic definitions in this area for the first time. Various properties of a neutrosophic R-sub-module are studied in different classes of rings. Moreover, various definitions of direct product and homomorphism of neutrosophic R-sub-modules are discussed, and results are provided.

Keywords: neutrosophic set; neutrosophic modules; modules
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1. Introduction

Uncertainty affects all aspects of human life. Zadeh [18] introduced the concept of a fuzzy set to overcome the limitations of classical set theory in dealing with such uncertainties. This approach defined a fuzzy set using a membership function with values ranging in a unit interval. However, further analysis showed that this definition fell short when addressing degrees of both membership and non-membership. To resolve this issue, Atanassov [3] developed intuitionistic fuzzy theory as an enhancement of the fuzzy set model. Although it provided a broader framework and found real-world applications [2, 17], it faced challenges in practical use. In response, Smarandache [13] introduced the concept of a neutrosophic set to handle problems involving ambiguous and inconsistent data. Since then, research has explored neutrosophic sets in various areas, including the study of algebraic structures [5, 6, 10, 15] and real-world applications, as seen in [7, 8].

The definitions of intersection and union in neutrosophic sets have been examined from three distinct angles. The initial interpretations, proposed by Smarandache [13, 15], are represented as $\cap_1$ and $\cup_1$. The second set of definitions, found in [16], are denoted as $\cap_2$ and $\cup_2$. The third approach, introduced in [19], is symbolized by $\cap_3$ and $\cup_3$. Additionally, Elrawy et al. [4] developed and explored an alternative neutrosophic sub-group and level sub-group concept, based on the first perspective.

Recently, Bal and Olgun [12] introduced neutrosophic modules using an indeterminate element, I. Also, Abed et al. [1] studied some results of the neutrosophic multiplication module. While Hameed et al. [9] introduced an approach of single-valued neutrosophic sub-modules based on the second perspective.

The investigation into the concepts of modules within the framework of neutrosophic sets is driven by three main objectives. The first is to define the neutrosophic sub-module as an algebraic structure without incorporating the indeterminate element I and based on the first perspective. The second is to examine how classical module theory can be extended to neutrosophic modules, where elements satisfy module conditions with varying levels of truth, indeterminacy, and falsity. The third objective is to establish a more adaptable framework through neutrosophic modules to address uncertain, incomplete, or conflicting information, which is crucial in fields such as artificial intelligence, economics, social sciences, and decision-making, where data often exhibit uncertainty.

Unlike classical modules, which require strict membership conditions, neutrosophic modules permit partial and uncertain membership. This flexibility results in more prosperous and versatile algebraic structures that better capture the complexity of real-world situations. Additionally, this paper introduces a novel approach to neutrosophic modules, altering the conventional perspective [12].

The study also includes the definition of neutrosophic modules over a ring and neutrosophic rings, along with an analysis of their properties. Furthermore, various properties of the direct product and homomorphism between neutrosophic modules are derived and explored.

The remainder of this article is organized as follows: Section 2 introduces essential definitions and preliminary results, laying the foundation for the paper's main contributions. Section 3 presents the concept of a neutrosophic R-sub-module along with its properties. We also derive various properties related to the direct product and homomorphism of neutrosophic modules. Finally, Section 4 summarizes the essential findings and conclusions of the study.

Table 1. Symbols and description of this article.

Symbol	Description	Symbol	Description
NS	neutrosophic set	$\mathfrak{G}, \mathfrak{H}$	classical group
R	classical ring	M	classical module over R
$\mathbb{D}$	universe set	$M(R)$	the set of R-module
$\mathfrak{R}$	neutrosophic sub-ring over R	M'	module over neutrosophic sub-ring
NSM(*R*)	the set of all neutrosophic R-sub-module	$\mathfrak{M}_M$	neutrosophic R-sub-module

2. Some basic concepts

Here, we give important concepts and outcomes as follows:

Definition 2.1. *[14, 16] An NS Ξ on a universe set $\mathbb{D}$ is defined as:*

$$\Xi = \{< \gimel, \mu(\gimel), \gamma(\gimel), \zeta(\gimel) >: \gimel \in \mathbb{D}\},$$

with $\mu, \gamma, \zeta : \mathbb{D} \to [0,1]$.

Definition 2.2. *[4] A neutrosophic subset $\mathfrak{M} = \{< \omega, \mu(\omega), \gamma(\omega), \zeta(\omega) >: \omega \in \mathfrak{G}\}$ of a group $\mathfrak{G}$ is said to be a neutrosophic subgroup of $\mathfrak{G}$ if the next axioms are met:*
(i) $\mu(\omega b) \geqslant \min(\mu(\omega), \mu(b))$,
(ii) $\mu\left(\omega^{-1}\right) \geqslant \mu(\omega)$,
(iii) $\gamma(\omega b) \leqslant \max(\gamma(\omega), \gamma(b))$,
(iv) $\gamma\left(\omega^{-1}\right) \leqslant \gamma(\omega)$,
(v) $\zeta(\omega b) \leqslant \max(\zeta(\omega), \zeta(b))$,
(vi) $\zeta\left(\omega^{-1}\right) \leqslant \zeta(\omega)$,
where $\omega, b \in \mathfrak{G}$.

Definition 2.3. *[13] Consider $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are two NSs on $\mathbb{D}$. Then:*

1. $\mathfrak{N}_1 \cap_1 \mathfrak{N}_2 = \{< \omega, \mu_1(\omega) \vee \mu_2(\omega), \gamma_1(\omega) \wedge \gamma_2(\omega), \zeta_1(\omega) \wedge \zeta_2(\omega) >: \omega \in \mathbb{D}\}$,
2. $\mathfrak{N}_1 \cup_1 \mathfrak{N}_2 = \{< \omega, \mu_1(\omega) \wedge \mu_2(\omega), \gamma_1(\omega) \vee \gamma_2(\omega), \zeta_1(\omega) \vee \zeta_2(\omega) >: \omega \in \mathbb{D}\}$.

Definition 2.4. *[11] Presume $\mathfrak{G}$ and $\mathfrak{H}$ are a group and $\mathfrak{M}_1$ and $\mathfrak{M}_2$ define on $\mathfrak{G}$ and $\mathfrak{H}$, respectively. Then*

$$\Gamma(\mu_1)(\rho) = \begin{cases} \sup\{\mu_1(\delta) : \delta \in \mathfrak{G}, \Gamma(\delta) = \rho\}, & \text{if } \Gamma^{-1}(\rho) \neq \phi, \\ 0, & \text{if } \Gamma^{-1}(\rho) = \phi. \end{cases}$$

$$\Gamma(\gamma_1)(\rho) = \begin{cases} \inf\{\gamma_1(\delta) : \delta \in \mathfrak{G}, \Gamma(\delta) = \rho\}, & \text{if } \Gamma^{-1}(\rho) \neq \phi, \\ 0, & \text{if } \Gamma^{-1}(\rho) = \phi. \end{cases}$$

$$\Gamma(\zeta_1)(\rho) = \begin{cases} \inf\{\zeta_1(\delta) : \delta \in \mathfrak{G}, \Gamma(\delta) = \rho\}, & \text{if } \Gamma^{-1}(\rho) \neq \phi, \\ 0, & \text{if } \Gamma^{-1}(\rho) = \phi, \end{cases}$$

where $\rho \in \mathfrak{H}$. Also, $\Gamma^{-1}(\mu_2)(\delta) = \mu_2(\Gamma(\delta))$, $\Gamma^{-1}(\gamma_2)(\delta) = \gamma_2(\Gamma(\delta))$, and $\Gamma^{-1}(\zeta_2)(\delta) = \zeta_2(\Gamma(\delta))$.

3. Main results

3.1. Neutrosophic modules

Let us now present the notion of a neutrosophic module defined over a neutrosophic ring and module.
First, we define a neutrosophic module over a neutrosophic ring. Consider M is a module over a ring R, and M' is a module over a neutrosophic sub-ring $\mathfrak{R}$.

Definition 3.1. *An NS $\mathfrak{M}_{M'} = \{< \ell, \mu_{M'}(\ell), \gamma_{M'}(\ell), \zeta_{M'}(\ell) >: \ell \in M'\}$ over M' is say a neutrosophic sub-module if the next axioms are met:*

$$(i) \begin{cases} \mu_{M'}(\ell + u) \geq \min(\mu_{M'}(\ell), \mu_{M'}(u)), \\ \gamma_{M'}(\ell + u) \leq \max(\gamma_{M'}(\ell), \gamma_{M'}(u)), \\ \zeta_{M'}(\ell + u) \leq \max(\zeta_{M'}(\ell), \zeta_{M'}(u)). \end{cases}$$

$$(ii) \begin{cases} \mu_{M'}(\lambda\ell) \geq \min(\mu_{M'}(\lambda), \mu_{M'}(\ell)), \\ \gamma_{M'}(\lambda\ell) \leq \max(\gamma_{M'}(\lambda), \gamma_{M'}(\ell)), \\ \zeta_{M'}(\lambda\ell) \leq \max(\zeta_{M'}(\lambda), \zeta_{M'}(\ell)). \end{cases}$$

$$(iii)\begin{cases}\mu_{M'}(0)=1,\\ \gamma_{M'}(0)=0,\\ \zeta_{M'}(0)=0,\end{cases}$$

where $u \in M', \mu_{M'}(\ell), \gamma_{M'}(\ell), \zeta_{M'}(\ell) : M' \to [0,1]$ *and* $\lambda \in \Re$.

Now, we introduce the neutrosophic module over module.

Definition 3.2. *An NS* $\mathfrak{M}_M = \{< \ell, \mu_M(\ell), \gamma_M(\ell), \zeta_M(\ell) >: \ell \in M\}$ *over* M *is say a neutrosophic R-sub-module if the next axioms are met:*

$$(i)\begin{cases}\mu_M(\ell+u)\geq \min(\mu_M(\ell),\mu_M(u)),\\ \gamma_M(\ell+u)\leq \max(\gamma_M(\ell),\gamma_M(u)),\\ \zeta_M(\ell+u)\leq \max(\zeta_M(\ell),\zeta_M(u)).\end{cases}$$

$$(ii)\begin{cases}\mu_M(\lambda\ell)\geq \min(\mu_M(\lambda),\mu_M(\ell)),\\ \gamma_M(\lambda\ell)\leq \max(\gamma_M(\lambda),\gamma_M(\ell)),\\ \zeta_M(\lambda\ell)\leq \max(\zeta_M(\lambda),\zeta_M(\ell)).\end{cases}$$

$$(iii)\begin{cases}\mu_M(0)=1,\\ \gamma_M(0)=0,\\ \zeta_M(0)=0,\end{cases}$$

where $u \in M$, $\mu_M(\ell), \gamma_M(\ell), \zeta_M(\ell) : M \to [0,1]$ *and* $\lambda \in R$.

Example 3.3. *Presume* $R = \mathbb{Z}$ *is a ring and* $M = \mathbb{Z}$ *over itself. Then, define a neutrosophic subset* $\mathfrak{M}_M = \{< \varrho, \mu(\varrho), \gamma(\varrho), \zeta(\varrho) >: \varrho \in \mathbb{Z}\}$ *by:*

$$\mu(\varrho)=\begin{cases}1 & \text{if } \varrho=0,\\ 0.2 & \text{if } \varrho\neq 0 \text{ is even },\\ 0.3 & \text{if } \varrho \text{ is odd.}\end{cases}$$

$$\gamma(\varrho)=\begin{cases}0 & \text{if } \varrho=0,\\ 0.5 & \text{if } \varrho\neq 0 \text{ is even },\\ 0.8 & \text{if } \varrho \text{ is odd.}\end{cases}$$

$$\zeta(\varrho)=\begin{cases}0 & \text{if } \varrho=0,\\ 0.4 & \text{if } \varrho\neq 0 \text{ is even },\\ 0.7 & \text{if } \varrho \text{ is odd.}\end{cases}$$

Thus, $\mathfrak{M}_M$ *is a neutrosophic module.*

The following assertions describe the characteristics of the system of condition (i) and (ii) for different classes of rings.

Proposition 3.4. *Let* R *be a ring with identity, then* $\mu_M(\lambda\ell) = \mu_M(\ell)$, $\gamma_M(\lambda\ell) = \gamma_M(\ell)$, *and* $\zeta_M(\lambda\ell) = \zeta_M(\ell)$.

Proof. Assume that $\mathfrak{M}_M$ is a neutrosophic sub-module; then we have

$$\begin{aligned}
\mu_M(\ell) = \mu_M((1-\lambda)\ell + \lambda\ell) &\geq \min(\mu_M((1-\lambda)\ell), \mu_M(\lambda\ell)) \\
&\geq \min(\min(\mu_M(\ell), \mu_M(-\lambda\ell)), \mu_M(\lambda\ell)) \\
&\geq \min(\min(\mu_M(\ell), \mu_M(-\lambda\ell)), \mu_M(\ell)) \\
&\geq \min(\mu_M(\ell), \mu_M(\lambda\ell)). \\
\gamma_M(\ell) = \gamma_M((1-\lambda)\ell + \lambda\ell) &\leq \max(\gamma_M((1-\lambda)\ell), \gamma_M(\lambda\ell)) \\
&\leq \max(\max(\gamma_M(\ell), \gamma_M(-\lambda\ell)), \gamma_M(\lambda\ell)) \\
&\leq \max(\max(\gamma_M(\ell), \gamma_M(-\lambda\ell)), \gamma_M(\ell)) \\
&\leq \max(\gamma_M(\ell), \gamma_M(\lambda\ell)). \\
\zeta_M(\ell) = \zeta_M((1-\lambda)\ell + \lambda\ell) &\leq \max(\zeta_M((1-\lambda)\ell), \zeta_M(\lambda\ell)) \\
&\leq \max(\max(\zeta_M(\ell), \zeta_M(-\lambda\ell)), \zeta_M(\lambda\ell)) \\
&\leq \max(\max(\zeta_M(\ell), \zeta_M(-\lambda\ell)), \zeta_M(\ell)) \\
&\leq \max(\zeta_M(\ell), \zeta_M(\lambda\ell)).
\end{aligned}$$

From the above and Definition 3.2, (*ii*) we obtain $\mu_M(\lambda\ell) = \mu_M(\ell)$, $\gamma_M(\lambda\ell) = \gamma_M(\ell)$, and $\zeta_M(\lambda\ell) = \zeta_M(\ell)$. □

Proposition 3.5. *Let R be a field and $0 \neq \lambda \in R$, then $\mu_M(\lambda\ell) = \mu_M(\ell)$, $\gamma_M(\lambda\ell) = \gamma_M(\ell)$, and $\zeta_M(\lambda\ell) = \zeta_M(\ell)$.*

Proof. Assume that $0 \neq \lambda \in R$ and R is a field, then

$$\begin{aligned}
\mu_M(\lambda\ell) \geq \mu_M(\ell) = \mu_M(\frac{1}{\lambda}\lambda\ell) \geq \mu_M(\lambda\ell), \\
\gamma_M(\lambda\ell) \leq \gamma_M(\ell) = \gamma_M(\frac{1}{\lambda}\lambda\ell) \leq \gamma_M(\lambda\ell), \\
\zeta_M(\lambda\ell) \leq \zeta_M(\ell) = \zeta_M(\frac{1}{\lambda}\lambda\ell) \leq \zeta_M(\lambda\ell).
\end{aligned}$$

From the above and Definition 3.2, (*ii*) we follow that $\mu_M(\lambda\ell) = \mu_M(\ell)$, $\gamma_M(\lambda\ell) = \gamma_M(\ell)$, and $\zeta_M(\lambda\ell) = \zeta_M(\ell)$. □

3.2. Properties of neutrosophic modules

Proposition 3.6. *A neutrosophic R-sub-module $\mathfrak{M}_M$, then $\mathbb{M}_1 = \{\gimel : \gimel \in M, \mu_M(\gimel) = 1, \gamma_M(\gimel) = \zeta_M(\gimel) = 0\}$ is an R-sub-module of the module M; also $\mathfrak{M}_{\mathbb{M}_1}$ is a neutrosophic R-sub-module.*

Proof. Suppose that $\gimel, \ell \in \mathbb{M}_1$ and $\lambda \in R$, then

$$\begin{aligned}
\mu_M(\gimel + \ell) \geq \min(\mu_M(\gimel), \mu_M(\ell)) = 1, \\
\gamma_M(\gimel + \ell) \leq \max(\gamma_M(\gimel), \gamma_M(\ell)) = 0, \\
\zeta_M(\gimel + \ell) \leq \max(\zeta_M(\gimel), \zeta_M(\ell)) = 0,
\end{aligned}$$

so $\mu_M(\gimel+\ell)=1$ and $\gamma_M(\gimel+\ell)=\zeta_M(\gimel+\ell)=0$, thus $\gimel+\ell\in\mathbb{M}_1$. Since

$$\begin{aligned}\mu_M(\lambda\gimel)&\geq\mu_M(\gimel)=1,\\ \gamma_M(\lambda\gimel)&\leq\gamma_M(\gimel)=0,\\ \zeta_M(\lambda\gimel)&\leq\zeta_M(\gimel)=0,\end{aligned}$$

thus we obtain $\mu_M(\lambda\gimel)=1$, and $\gamma_M(\lambda\gimel)=\zeta_M(\lambda\gimel)=0$. This follows that $\lambda\gimel\in\mathbb{M}_1$. Finally, since $\mu_M(0)=1$, $\gamma_M(0)=0$, and $\zeta_M(0)=0$. Therefore $0\in\mathbb{M}_1$. So $\mathbb{M}_1$ is an R-sub-module of the module M. The last part of the proposition's statement is self-evident. □

Proposition 3.7. *Let R be a ring with unity and $\mathfrak{M}_M$ be a neutrosophic R-sub-module, then $\mathfrak{M}_M$ is a neutrosophic sub-group of M.*

Proof. Suppose that $\gimel\in M$, then

$$\begin{aligned}\mu_M(-\gimel)&=\mu_M((-1).\gimel)\geq\gamma_M(\gimel),\\ \gamma_M(-\gimel)&=\gamma_M((-1).\gimel)\leq\gamma_M(\gimel),\\ \zeta_M(-\gimel)&=\zeta_M((-1).\gimel)\leq\zeta_M(\gimel),\end{aligned}$$

and since $\mathfrak{M}_M$ is a neutrosophic R-sub-module, this leads to $\mathfrak{M}_M$ being a neutrosophic sub-group of M. □

Proposition 3.8. *A neutrosophic R-sub-module $\mathfrak{M}_M$, then $\mathbb{M}_\vartheta=\{\gimel:\gimel\in M,\mu_M(\gimel)\geq\vartheta,\gamma_M(\gimel)\leq\vartheta,\zeta_M(\gimel)\leq\vartheta\}$ is an R-sub-module of the module M also $\mathfrak{M}_{\mathbb{M}_\vartheta}$ is a neutrosophic R-sub-module, where $0\leq\vartheta\leq1$.*

Proof. Assume that $\gimel,\kappa\in M$ and $\lambda\in R$, then

$$(i)\begin{cases}\mu_M(\gimel+\kappa)\geq\min(\mu_M(\gimel),\mu_M(\kappa))=\vartheta,\\ \gamma_M(\gimel+\kappa)\leq\max(\gamma_M(\gimel),\gamma_M(\kappa))=\vartheta,\\ \zeta_M(\gimel+\kappa)\leq\max(\zeta_M(\gimel),\zeta_M(\kappa))=\vartheta,\end{cases}$$

$$(ii)\begin{cases}\mu_M(\lambda\gimel)\geq\min(\mu_M(\lambda),\mu_M(\gimel))=\mu_M(\gimel)\geq\vartheta,\\ \gamma_M(\lambda\gimel)\leq\max(\gamma_M(\lambda),\gamma_M(\gimel))=\gamma_M(\gimel)\leq\vartheta,\\ \zeta_M(\lambda\gimel)\leq\max(\zeta_M(\lambda),\zeta_M(\gimel))=\zeta_M(\gimel)\leq\vartheta,\end{cases}$$

$$(iii)\begin{cases}\mu_M(0)=1\geq\vartheta,\\ \gamma_M(0)=0\leq\vartheta,\\ \zeta_M(0)=0\leq\vartheta.\end{cases}$$

Therefore, $\gimel+\kappa\in\mathbb{M}_\vartheta$, $\lambda\gimel\in\mathbb{M}_\vartheta$ and $0\in\mathbb{M}_\vartheta$. □

Here we suppose that $N\subseteq M$ and M is an R-module; then we define a neutrosophic subset on N as follows: $\mathfrak{M}_N=\{<\gimel,\mu_N(\gimel),\gamma_N(\gimel),\zeta_N(\gimel)>:\gimel\in N\}$ and $\mu_N,\gamma_N,\zeta_N:N\longrightarrow[0,1]$.

Proposition 3.9. *$\mathfrak{M}_M$ is a neutrosophic R-sub-module iff N is a sub-module of M.*

Proof. Suppose that $\mathfrak{M}_M$ is a neutrosophic R-sub-module, then for any $\eta,\kappa\in N$ and $\lambda\in R$, we obtain

$$(i)\begin{cases}\mu_M(\eta+\kappa)\geq\min(\mu_M(\eta),\mu_M(\kappa))=1,\\ \gamma_M(\eta+\kappa)\leq\max(\gamma_M(\eta),\gamma_M(\kappa))=0,\\ \zeta_M(\eta+\kappa)\leq\max(\zeta_M(\eta),\zeta_M(\kappa))=0,\end{cases}$$

$$(ii)\begin{cases}\mu_M(\lambda\eta) \geq \min(\mu_M(\lambda),\mu_M(\eta)) = \mu_M(\eta) = 1,\\ \gamma_M(\lambda\eta) \leq \max(\gamma_M(\lambda),\gamma_M(\eta)) = \gamma_M(\eta) = 0,\\ \zeta_M(\lambda\eta) \leq \max(\zeta_M(\lambda),\zeta_M(\eta)) = \zeta_M(\eta) = 0,\end{cases}$$

$$(iii)\begin{cases}\mu_M(0) = 1,\\ \gamma_M(0) = 0,\\ \zeta_M(0) = 0.\end{cases}$$

So $\eta+\kappa \in N$, $\lambda\eta \in N$ and $0 \in N$. Therefore, N is sub-module of M.

The other direction, assume that N is a sub-module of M. Now, we show some cases:

Case 1. For any $0,\eta,\kappa \in N$ and $\lambda \in R$, we obtain:

$$(i)\begin{cases}\min(\mu_M(\eta),\mu_M(\kappa)) = 1 \leq 1 = \mu_M(\eta+\kappa),\\ \max(\gamma_M(\eta),\gamma_M(\kappa)) = 0 \geq 0 = \gamma_M(\eta+\kappa),\\ \max(\zeta_M(\eta),\zeta_M(\kappa)) = 0 \geq 0 = \zeta_M(\eta+\kappa),\end{cases}$$

$$(ii)\begin{cases}\mu_M(\lambda\eta) = 1 \geq \min(\mu_M(\lambda),\mu_M(\eta)) = \mu_M(\eta),\\ \gamma_M(\lambda\eta) = 0 \leq \max(\gamma_M(\lambda),\gamma_M(\eta)) = \gamma_M(\eta),\\ \zeta_M(\lambda\eta) = 0 \leq \max(\zeta_M(\lambda),\zeta_M(\eta)) = \zeta_M(\eta),\end{cases}$$

$$(iii)\begin{cases}\mu_M(0) = 1,\\ \gamma_M(0) = 0,\\ \zeta_M(0) = 0.\end{cases}$$

Case 2. For any $0,\eta \in N,\kappa \notin N$ and $\lambda \in R$, we obtain:

$$(i)\begin{cases}\min(\mu_M(\eta),\mu_M(\kappa)) = \min(1,0) = 0 \leq \mu_M(\eta+\kappa),\\ \max(\gamma_M(\eta),\gamma_M(\kappa)) = \max(1,0) = 1 \geq 0 = \gamma_M(\eta+\kappa),\\ \max(\zeta_M(\eta),\zeta_M(\kappa)) = \max(1,0) = 1 \geq 0 = \zeta_M(\eta+\kappa),\end{cases}$$

$$(ii)\begin{cases}\mu_M(\lambda\eta) = 1 \geq \min(\mu_M(\lambda),\mu_M(\eta)) = \mu_M(\eta),\\ \gamma_M(\lambda\eta) = 0 \leq \max(\gamma_M(\lambda),\gamma_M(\eta)) = \gamma_M(\eta),\\ \zeta_M(\lambda\eta) = 0 \leq \max(\zeta_M(\lambda),\zeta_M(\eta)) = \zeta_M(\eta),\end{cases}$$

$$(iii)\begin{cases}\mu_M(0) = 1,\\ \gamma_M(0) = 0,\\ \zeta_M(0) = 0.\end{cases}$$

Case 3. For any $0,\eta \notin N,\kappa \in N$ and $\lambda \in R$, we obtain:

$$(i)\begin{cases}\min(\mu_M(\eta),\mu_M(\kappa)) = \min(0,1) = 0 \leq \mu_M(\eta+\kappa),\\ \max(\gamma_M(\eta),\gamma_M(\kappa)) = \max(0,1) = 1 \geq 0 = \gamma_M(\eta+\kappa),\\ \max(\zeta_M(\eta),\zeta_M(\kappa)) = \max(0,1) = 1 \geq 0 = \zeta_M(\eta+\kappa),\end{cases}$$

$$(ii)\begin{cases}\mu_M(\eta) = 0 \leq \mu_M(\lambda\eta),\\ \gamma_M(\eta) = 1 \geq \gamma_M(\lambda\eta),\\ \zeta_M(\eta) = 1 \geq \zeta_M(\lambda\eta)\end{cases}$$

$$(iii)\begin{cases}\mu_M(0) = 1,\\ \gamma_M(0) = 0,\\ \zeta_M(0) = 0.\end{cases}$$

Case 4. For any $0,\eta \notin N,\kappa \notin N$ and $\lambda \in R$, we obtain:

$$(i)\begin{cases} \min(\mu_M(\eta),\mu_M(\kappa)) = \min(0,0) = 0 \le \mu_M(\eta+\kappa), \\ \max(\gamma_M(\eta),\gamma_M(\kappa)) = \max(0,0) = 0 \ge 0 = \gamma_M(\eta+\kappa), \\ \max(\zeta_M(\eta),\zeta_M(\kappa)) = \max(0,0) = 0 \ge 0 = \zeta_M(\eta+\kappa), \end{cases}$$

$$(ii)\begin{cases} \mu_M(\eta) = 0 \le \mu_M(\lambda\eta), \\ \gamma_M(\eta) = 1 \ge \gamma_M(\lambda\eta), \\ \zeta_M(\eta) = 1 \ge \zeta_M(\lambda\eta) \end{cases}$$

$$(iii)\begin{cases} \mu_M(0) = 1, \\ \gamma_M(0) = 0, \\ \zeta_M(0) = 0. \end{cases}$$

Thus, $\mathfrak{M}_M$ is a neutrosophic R-sub-module. □

In what follows, the set of all neutrosophic R-sub-modules of $\mathfrak{M}_M$ is denoted by **NSM(R)**.

Proposition 3.10. *Let $\mathfrak{M}_M, \mathfrak{P}_M \in$**NSM(R)**, then $\mathfrak{M}_M \cap_1 \mathfrak{P}_M \in$**NSM(R)**.*

Proof. Assume that $\gimel, \kappa \in M$ and $\lambda \in R$, then

$$(i)\begin{cases} (\mu_M \vee \mu'_M)(\gimel+\kappa) = \mu_M(\gimel+\kappa) \vee \mu'_M(\gimel+\kappa) \\ \qquad \ge \min(\mu_M(\gimel),\mu_M(\kappa)) \vee \min(\mu'_M(\gimel),\mu'_M(\kappa)) \\ \qquad = \min((\mu_M \vee \mu'_M)(\gimel),(\mu_M \vee \mu'_M)(\kappa)), \\ (\gamma_M \wedge \gamma'_M)(\gimel+\kappa) = \gamma_M(\gimel+\kappa) \wedge \gamma'_M(\gimel+\kappa) \\ \qquad \le \max(\gamma_M(\gimel),\gamma_M(\kappa)) \wedge \max(\gamma'_M(\gimel),\gamma'_M(\kappa)) \\ \qquad = \max((\gamma_M \wedge \gamma'_M)(\gimel),(\gamma_M \wedge \gamma'_M)(\kappa)), \\ (\zeta_M \wedge \zeta'_M)(\gimel+\kappa) = \zeta_M(\gimel+\kappa) \wedge \zeta'_M(\gimel+\kappa) \\ \qquad \le \max(\zeta_M(\gimel),\zeta_M(\kappa)) \wedge \max(\zeta'_M(\gimel),\zeta'_M(\kappa)) \\ \qquad = \max((\zeta_M \wedge \zeta'_M)(\gimel),(\zeta_M \wedge \zeta'_M)(\kappa)), \end{cases}$$

$$(ii)\begin{cases} (\mu_M \vee \mu'_M)(\lambda\gimel) = \mu_M(\lambda\gimel) \vee \mu'_M(\lambda\gimel) \\ \qquad \ge \mu_M(\gimel) \vee \mu'_M(\gimel) \\ \qquad = (\mu_M \vee \mu'_M)(\gimel), \\ (\gamma_M \wedge \gamma'_M)(\lambda\gimel) = \gamma_M(\lambda\gimel) \wedge \gamma'_M(\lambda\gimel) \\ \qquad \le \gamma_M(\gimel) \wedge \gamma'_M(\gimel) \\ \qquad = (\gamma_M \wedge \gamma'_M)(\gimel), \\ (\zeta_M \wedge \zeta'_M)(\lambda\gimel) = \zeta_M(\lambda\gimel) \wedge \zeta'_M(\lambda\gimel) \\ \qquad \le \zeta_M(\gimel) \wedge \zeta'_M(\gimel) \\ \qquad = (\zeta_M \wedge \zeta'_M)(\gimel), \end{cases}$$

$$(iii)\begin{cases} (\mu_M \vee \mu'_M)(0) = 1, \\ (\gamma_M \wedge \gamma'_M)(0) = 0, \\ (\zeta_M \wedge \zeta'_M)(0) = 0. \end{cases}$$

□

Example 3.11. *Let $R = Z_2$ be a ring; then we have a module $M = Z_2$. Define NS $\mathfrak{M} = \{< 0,1,0,0 >, < 1,0.3,0.4,0.5 >\}$ and $\mathfrak{B} = \{< 0,1,0,0 >, < 1,0.2,0.6,0.7 >\}$ over M. It is clear that $\mathfrak{M}, \mathfrak{B} \in$**NSM(R)**. Also, $\mathfrak{M} \cap_1 \mathfrak{B} = \{< 0,1,0,0 >, < 1,0.3,0.6,0.7 >\} \in$**NSM(R)**.*

Now, we show the generalization of Proposition 3.10.

Corollary 3.12. *Let $\mathfrak{M}_{i_M}$ ∈**NSM(R)** with $i = 1, 2, \ldots, n$, then $\cap_{i_1} \mathfrak{M}_{i_M}$ ∈**NSM(R)**.*

Next, we introduce the definition of direct product of **NSM**(R).

Definition 3.13. *Let $\mathfrak{M}_{i_M}$ ∈**NSM(R)** with $i = 1, 2, \ldots, n$, then the direct product of $\mathfrak{M}_{i_M}$ is defined as $\mathfrak{M}_M = \prod_{i=1}^{n} \mathfrak{M}_{i_M}$ with*

$$\mu_M(\gimel_1, \gimel_2, \ldots, \gimel_n) = (\prod_{i=1}^{n} \mu_{i_M})(\gimel_1, \gimel_2, \ldots, \gimel_n) = \min(\mu_{1_M}(\gimel_1), \mu_{2_M}(\gimel_2), \ldots, \mu_{n_M}(\gimel_n)),$$

$$\gamma_M(\gimel_1, \gimel_2, \ldots, \gimel_n) = (\prod_{i=1}^{n} \gamma_{i_M}(\gimel_1, \gimel_2, \ldots, \gimel_n) = \max(\gamma_{1_M}(\gimel_1), \gamma_{2_M}(\gimel_2), \ldots, \gamma_{n_M}(\gimel_n)),$$

$$\zeta_M(\gimel_1, \gimel_2, \ldots, \gimel_n) = (\prod_{i=1}^{n} \zeta_{i_M}(\gimel_1, \gimel_2, \ldots, \gimel_n) = \max(\zeta_{1_M}(\gimel_1), \zeta_{2_M}(\gimel_2), \ldots, \zeta_{n_M}(\gimel_n)).$$

The set of R-modules is denoted by **M**(R). Also, $M = \prod_{i=1}^{n} M_i$ is a direct product where M_i ∈**M**(R).

Theorem 3.14. $\mathfrak{M}_M = \prod_{i=1}^{n} \mathfrak{M}_{i_M}$ *is a neutrosophic R-sub-module.*

Proof. Suppose that $\gimel, \kappa \in M$ and $\lambda \in R$, where $\gimel = (\gimel_1, \gimel_2, \ldots, \gimel_n)$ and $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$. Then

$$(i) \begin{cases} \mu_M(\gimel + \kappa) = \mu_M(\gimel_1 + \kappa_1, \gimel_2 + \kappa_2, \ldots, \gimel_n + \kappa_n) \\ \quad = \min(\mu_{1_M}(\gimel_1 + \kappa_1), \mu_{2_M}(\gimel_2 + \kappa_2), \ldots, \mu_{n_M}(\gimel_n + \kappa_n)) \\ \quad \geq \min(\min(\mu_{1_M}(\gimel_1), \mu_{1_M}(\kappa_1)), \min(\mu_{2_M}(\gimel_2), \mu_{2_M}(\kappa_2)), \ldots, \min(\mu_{n_M}(\gimel_n), \mu_{n_M}(\kappa_n))) \\ \quad = \min(\min(\mu_{1_M}(\gimel_1), \mu_{2_M}(\gimel_2), \ldots, \mu_{n_M}(\gimel_n)), \min(\mu_{1_M}(\kappa_1), \mu_{2_M}(\kappa_2), \ldots, \mu_{n_M}(\kappa_n))) \\ \quad = \min(\mu_M(\gimel), \mu_M(\kappa)), \\ \gamma_M(\gimel + \kappa) = \gamma_M(\gimel_1 + \kappa_1, \gimel_2 + \kappa_2, \ldots, \gimel_n + \kappa_n) \\ \quad = \max(\gamma_{1_M}(\gimel_1 + \kappa_1), \gamma_{2_M}(\gimel_2 + \kappa_2), \ldots, \gamma_{n_M}(\gimel_n + \kappa_n)) \\ \quad \leq \max(\max(\gamma_{1_M}(\gimel_1), \gamma_{1_M}(\kappa_1)), \max(\gamma_{2_M}(\gimel_2), \gamma_{2_M}(\kappa_2)), \ldots, \max(\gamma_{n_M}(\gimel_n), \gamma_{n_M}(\kappa_n))) \\ \quad = \max(\max(\gamma_{1_M}(\gimel_1), \gamma_{2_M}(\gimel_2), \ldots, \gamma_{n_M}(\gimel_n)), \max(\mu_{1_M}(\kappa_1), \gamma_{2_M}(\kappa_2), \ldots, \gamma_{n_M}(\kappa_n))) \\ \quad = \max(\gamma_M(\gimel), \gamma_M(\kappa)), \\ \zeta_M(\gimel + \kappa) = \zeta_M(\gimel_1 + \kappa_1, \gimel_2 + \kappa_2, \ldots, \gimel_n + \kappa_n) \\ \quad = \max(\zeta_{1_M}(\gimel_1 + \kappa_1), \zeta_{2_M}(\gimel_2 + \kappa_2), \ldots, \zeta_{n_M}(\gimel_n + \kappa_n)) \\ \quad \leq \max(\max(\zeta_{1_M}(\gimel_1), \zeta_{1_M}(\kappa_1)), \max(\zeta_{2_M}(\gimel_2), \zeta_{2_M}(\kappa_2)), \ldots, \max(\zeta_{n_M}(\gimel_n), \zeta_{n_M}(\kappa_n))) \\ \quad = \max(\max(\zeta_{1_M}(\gimel_1), \zeta_{2_M}(\gimel_2), \ldots, \zeta_{n_M}(\gimel_n)), \max(\mu_{1_M}(\kappa_1), \zeta_{2_M}(\kappa_2), \ldots, \zeta_{n_M}(\kappa_n))) \\ \quad = \max(\zeta_M(\gimel), \zeta_M(\kappa)), \end{cases}$$

□

Proof. (*ii*)
$$\begin{cases} \mu_M(\lambda\gimel) = \mu_M(\lambda\gimel_1, \lambda\gimel_2, \ldots, \lambda\gimel_n) \\ \qquad\qquad = \min(\mu_{1_M}(\lambda\gimel_1), \mu_{2_M}(\lambda\gimel_2), \ldots, \mu_{n_M}(\lambda\gimel_n)) \\ \qquad\qquad \geq \min(\mu_{1_M}(\gimel_1), \mu_{2_M}(\gimel_2), \ldots, \mu_{n_M}(\gimel_n)) \\ \qquad\qquad = \mu_M(\gimel), \\ \gamma_M(\lambda\gimel) = \gamma_M(\lambda\gimel_1, \lambda\gimel_2, \ldots, \lambda\gimel_n) \\ \qquad\qquad = \max(\gamma_{1_M}(\lambda\gimel_1), \gamma_{2_M}(\lambda\gimel_2), \ldots, \gamma_{n_M}(\lambda\gimel_n)) \\ \qquad\qquad \leq \max(\gamma_{1_M}(\gimel_1), \gamma_{2_M}(\gimel_2), \ldots, \gamma_{n_M}(\gimel_n)) \\ \qquad\qquad = \gamma_M(\gimel), \\ \zeta_M(\lambda\gimel) = \zeta_M(\lambda\gimel_1, \lambda\gimel_2, \ldots, \lambda\gimel_n) \\ \qquad\qquad = \max(\zeta_{1_M}(\lambda\gimel_1), \zeta_{2_M}(\lambda\gimel_2), \ldots, \zeta_{n_M}(\lambda\gimel_n)) \\ \qquad\qquad \leq \max(\zeta_{1_M}(\gimel_1), \zeta_{2_M}(\gimel_2), \ldots, \zeta_{n_M}(\gimel_n)) \\ \qquad\qquad = \zeta_M(\gimel), \end{cases}$$

(*iii*)
$$\begin{cases} \mu_M(0) = \mu_M(0,0,\ldots,0) = \min(\mu_{1_M}(0), \mu_{2_M}(0), \ldots, \mu_{n_M}(0)) = \min(1,1,\ldots,1) = 1, \\ \gamma_M(0) = \gamma_M(0,0,\ldots,0) = \max(\gamma_{1_M}(0), \gamma_{2_M}(0), \ldots, \gamma_{n_M}(0)) = \max(0,0,\ldots,0) = 0, \\ \zeta_M(0) = \zeta_M(0,0,\ldots,0) = \max(\zeta_{1_M}(0), \zeta_{2_M}(0), \ldots, \zeta_{n_M}(0)) = \max(0,0,\ldots,0) = 0. \end{cases}$$
□

3.3. Homomorphism between neutrosophic R-sub-modules

Proposition 3.15. *Let* Γ *be an epimorphism from M into N R-modules. When* $\mathfrak{M}_M \in$***NSM(R)****, then* $\Gamma(\mathfrak{M}_M) \in$ ***NSN(R)****.*

Proof. Suppose that $\rho_1, \rho_2 \in N$ and $\lambda \in R$, then

(*i*)
$$\begin{cases} \Gamma(\mu_M)(\rho_1+\rho_2) = \sup\{\mu_M(\varrho_1+\varrho_2) : \varrho_1,\varrho_2 \in M, \Gamma(\varrho_1)=\rho_1, \Gamma(\varrho_2)=\rho_2\} \\ \qquad \geq \sup\{\min(\mu_M(\varrho_1), \mu_M(\varrho_2)) : \varrho_1,\varrho_2 \in M, \Gamma(\varrho_1)=\rho_1, \Gamma(\varrho_2)=\rho_2\} \\ \qquad = \min(\sup\{\mu_M(\varrho_1) : \Gamma(\varrho_1)=\rho_1\}, \sup\{\mu_M(\varrho_2) : \Gamma(\varrho_2)=\rho_2\}) \\ \qquad = \min(\Gamma(\mu_M)(\rho_1), \Gamma(\mu_M)(\rho_2)), \\ \Gamma(\gamma_M)(\rho_1+\rho_2) = \inf\{\gamma_M(\varrho_1+\varrho_2) : \varrho_1,\varrho_2 \in M, \Gamma(\varrho_1)=\rho_1, \Gamma(\varrho_2)=\rho_2\} \\ \qquad \leq \inf\{\max(\gamma_M(\varrho_1), \gamma_M(\varrho_2)) : \varrho_1,\varrho_2 \in M, \Gamma(\varrho_1)=\rho_1, \Gamma(\varrho_2)=\rho_2\} \\ \qquad = \max(\inf\{\gamma_M(\varrho_1) : \Gamma(\varrho_1)=\rho_1\}, \inf\{\gamma_M(\varrho_2) : \Gamma(\varrho_2)=\rho_2\}) \\ \qquad = \max(\Gamma(\gamma_M)(\rho_1), \Gamma(\gamma_M)(\rho_2)) \\ \Gamma(\zeta_M)(\rho_1+\rho_2) = \inf\{\zeta_M(\varrho_1+\varrho_2) : \varrho_1,\varrho_2 \in M, \Gamma(\varrho_1)=\rho_1, \Gamma(\varrho_2)=\rho_2\} \\ \qquad \leq \inf\{\max(\zeta_M(\varrho_1), \zeta_M(\varrho_2)) : \varrho_1,\varrho_2 \in M, \Gamma(\varrho_1)=\rho_1, \Gamma(\varrho_2)=\rho_2\} \\ \qquad = \max(\inf\{\zeta_M(\varrho_1) : \Gamma(\varrho_1)=\rho_1\}, \inf\{\zeta_M(\varrho_2) : \Gamma(\varrho_2)=\rho_2\}) \\ \qquad = \max(\Gamma(\zeta_M)(\rho_1), \Gamma(\zeta_M)(\rho_2)), \end{cases}$$
□

$$\textit{Proof. } (ii) \begin{cases} \Gamma(\mu_M)(\lambda\rho_1) = \sup\{\mu_M(\lambda\varrho_1) : \lambda\varrho_1 \in M, \Gamma(\lambda\varrho_1) = \lambda\rho_1\} \\ \qquad\qquad \geq \sup\{\mu_M(\varrho_1) : \varrho_1 \in M, \Gamma(\varrho_1) = \rho_1\} \\ \qquad\qquad = \Gamma(\mu_M)(\rho_1), \\ \Gamma(\gamma_M)(\lambda\rho_1) = \inf\{\gamma_M(\lambda\varrho_1) : \lambda\varrho_1 \in M, \Gamma(\lambda\varrho_1) = \lambda\rho_1\} \\ \qquad\qquad \leq \inf\{\gamma_M(\varrho_1) : \varrho_1 \in M, \Gamma(\varrho_1) = \rho_1\} \\ \qquad\qquad = \Gamma(\gamma_M)(\rho_1), \\ \Gamma(\zeta_M)(\lambda\rho_1) = \inf\{\zeta_M(\lambda\varrho_1) : \lambda\varrho_1 \in M, \Gamma(\lambda\varrho_1) = \lambda\rho_1\} \\ \qquad\qquad \leq \inf\{\zeta_M(\varrho_1) : \varrho_1 \in M, \Gamma(\varrho_1) = \rho_1\} \\ \qquad\qquad = \Gamma(\zeta_M)(\rho_1), \end{cases}$$

$$(iii) \begin{cases} \Gamma(\mu_M)(0) = \sup\{\mu_M(0) : 0 \in M, \Gamma(0) = 0\} = 1, \\ \Gamma(\gamma_M)(0) = \inf\{\gamma_M(0) : 0 \in M, \Gamma(0) = 0\} = 0, \\ \Gamma(\zeta_M)(0) = \inf\{\zeta_M(0) : 0 \in M, \Gamma(0) = 0\} = 0. \end{cases}$$

Therefore, $\Gamma(\mathfrak{M}_M) \in$ **NSN(***R***)**. □

Proposition 3.16. *Let* Γ *be an epimorphism from* M *into* N *R-modules. When* $\mathfrak{P}_N \in$***NSN(R)****, then* $\Gamma^{-1}(\mathfrak{P}_N) \in$ ***NSM(R)****.*

Proof. Suppose that $\varrho_1, \varrho_2 \in M$ and $\lambda \in R$, then

$$(i) \begin{cases} \Gamma^{-1}(\mu'_N)(\varrho_1 + \varrho_2) = \mu'_N(\Gamma(\varrho_1 + \varrho_2)) \\ \qquad\qquad = \mu'_N(\Gamma(\varrho_1) + \Gamma(\varrho_2)) \\ \qquad\qquad \geq \min(\mu'_N(\Gamma^{-1}(\varrho_1)), \mu'_N(\Gamma^{-1}(\varrho_2)) \\ \qquad\qquad = \min(\Gamma^{-1}(\mu'_N)(\varrho_1), \Gamma^{-1}(\mu'_N)(\varrho_2)), \\ \Gamma^{-1}(\gamma'_N)(\varrho_1 + \varrho_2) = \gamma'_N(\Gamma(\varrho_1 + \varrho_2)) \\ \qquad\qquad = \gamma'_N(\Gamma(\varrho_1) + \Gamma(\varrho_2)) \\ \qquad\qquad \leq \max(\gamma'_N(\Gamma^{-1}(\varrho_1)), \gamma'_N(\Gamma^{-1}(\varrho_2)) \\ \qquad\qquad = \max(\Gamma^{-1}(\gamma'_N)(\varrho_1), \Gamma^{-1}(\gamma'_N)(\varrho_2)) \\ \Gamma^{-1}(\zeta'_N)(\varrho_1 + \varrho_2) = \zeta'_N(\Gamma(\varrho_1 + \varrho_2)) \\ \qquad\qquad = \zeta'_N(\Gamma(\varrho_1) + \Gamma(\varrho_2)) \\ \qquad\qquad \leq \max(\zeta'_N(\Gamma^{-1}(\varrho_1)), \zeta'_N(\Gamma^{-1}(\varrho_2)) \\ \qquad\qquad = \max(\Gamma^{-1}(\zeta'_N)(\varrho_1), \Gamma^{-1}(\zeta'_N)(\varrho_2)), \end{cases}$$

□

Proof. (*ii*)
$$\begin{cases} \Gamma^{-1}(\mu'_N)(\lambda\varrho_1) = \mu'_N(\Gamma(\lambda\varrho_1)) \\ \qquad = \mu'_N(\lambda\Gamma(\varrho_1)) \\ \qquad \geq \mu'_N(\Gamma(\varrho_1)) \\ \qquad = \Gamma^{-1}(\mu'_N)(\varrho_1), \\ \Gamma^{-1}(\gamma'_N)(\lambda\varrho_1) = \gamma'_N(\Gamma(\lambda\varrho_1)) \\ \qquad = \gamma'_N(\lambda\Gamma(\varrho_1)) \\ \qquad \leq \gamma'_N(\Gamma(\varrho_1)) \\ \qquad = \Gamma^{-1}(\gamma'_N)(\varrho_1), \\ \Gamma^{-1}(\zeta'_N)(\lambda\varrho_1) = \zeta'_N(\Gamma(\lambda\varrho_1)) \\ \qquad = \zeta'_N(\lambda\Gamma(\varrho_1)) \\ \qquad \leq \zeta'_N(\Gamma(\varrho_1)) \\ \qquad = \Gamma^{-1}(\zeta'_N)(\varrho_1), \end{cases}$$

(*iii*)
$$\begin{cases} \Gamma^{-1}(\mu'_N)(0) = \mu'_N(\Gamma(0)) = \mu'_N(0) = 1, \\ \Gamma^{-1}(\gamma'_N)(0) = \gamma'_N(\Gamma(0)) = \gamma'_N(0) = 0, \\ \Gamma^{-1}(\zeta'_N)(0) = \zeta'_N(\Gamma(0)) = \zeta'_N(0) = 0. \end{cases}$$

Therefore $\Gamma^{-1}(\mathfrak{P}_N) \in \mathbf{NSM}(R)$. □

Remark 3.17. We have enhanced the definition of a neutrosophic sub-module by building on the foundation established in [9, 12] and using the methodology applied by the researchers in [4–6]. This revised approach offers significant advantages as it is consistent with the qualitative properties of the components. In particular, the component μ is treated as a measure of positive quality, while γ and ζ are associated with negative qualities. This distinction justifies the consistent application of operations, with γ and ζ being subjected to the same operations, such as max/max and $\leqslant$ / $\leqslant$. By refining the structure in this way, the new definition better reflects the underlying theoretical framework and provides a more coherent and practical perspective on the properties and behavior of neutrosophic sub-modules.

4. Conclusions

This study has significantly extended the theoretical framework of neutrosophic algebra by exploring the structure and properties of neutrosophic modules over rings and their associated systems. By systematically analyzing the fundamental properties of neutrosophic modules, the research has shed light on their behavior in direct product operations and homomorphism and provided a deeper understanding of their algebraic nature.

The results provide a solid foundation for further study extensions and variations of neutrosophic modules. They could open new avenues of research in the field of algebraic structures dealing with uncertainty and indeterminacy, such as the neutrosophic Artinian multiplication module and the neutrosophic Jacobson radical. Moreover, these findings could have wider implications for applied mathematics, as they could improve decision-making methods, artificial intelligence, and system modeling, where dealing with uncertain and inconsistent data is crucial by opening up possibilities for practical applications.

Author contributions

Ali Yahya Hummdi: Writing-review and editing; Amr Elrawy: Conceptualization, formal analysis, investigation, methodology; Ayat A. Temraz: Visualisation, writing-original and editing, draft acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

This work does not have any conflict of interest.

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