Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(12): 35949–35963.
DOI:10.3934/math.20241704
Received: 19 October 2024
Revised: 11 December 2024
Accepted: 16 December 2024
Published: 25 December 2024

Research article

The L^∞ estimate of the spatial gradient of the solution to a variational inequality problem originates from the financial contract problem with advanced implementation clauses

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Abstract: The present study investigates a class of variational inequality problems under the framework of the parabolic Kirchhoff operator from the financial contract problem. This particular issue stems from the financial contract problem. By utilizing the energy inequality of the obtained solutions, the energy inequality of the solution gradients, and the Caffarelli–Kohn–Nirenberge inequality, an estimation of the infinite norm of the solution gradients is obtained.

Keywords: variational inequality problem; parabolic Kirchhoff operator; L^∞ estimate of the spatial gradient of the solution; energy inequality; Caffarelli–Kohn–Nirenberge inequality; financial contract problem
Mathematics Subject Classification: 35K99, 97M30

1. Introduction

This paper investigates a class of variational inequality problems under the parabolic Kirchhoff operator framework, which originated from the early exercise provision of financial contracts. Specifically, we consider the problem given by

$$\begin{cases} \max\{Lu, u_0 - u\} = 0 \text{ in } \Omega_T, \\ u(\cdot, 0) = u_0 \text{ in } \Omega, \\ u = u_0 \text{ in } \partial\Omega \times (0, T), \end{cases} \tag{1}$$

where the degenerate parabolic Kirchhoff operator satisfies:

$$Lu = \partial_t u - (1 + \|u\|^p_{L^p(\Omega)}) \times \text{div}(|\nabla u|^{p-1} \nabla u). \tag{2}$$

Here, Ω represents a connected, bounded open region on $\mathbb{R}^N$, and $\partial\Omega$ denotes its boundary. T is a positive constant, while $\Omega_T = \Omega \times (0, T)$, $p \geq 2$, and $\|\cdot\|_{L^p(\Omega)}$ are norms on the space $L^p(\Omega)$. The initial value u_0 satisfies:

$$u_0 \in C(\Omega) \cap W_0^{1,p}(\Omega).$$

1.1. Financial background

The issue discussed here is quite common in financial contracts with attached early exercise provisions [1]. Let S_t be the price of a certain stock, and ϕ represent the value of a call option contract linked to the stock's underlying risky asset. If this call option contract is equipped with an early exercise provision, then its value at time t satisfies

$$\begin{cases} \max\{L\phi, (S-K)_+ - \phi\} = 0 \,\mathrm{in}\, \mathrm{R}_+ \times (0,T), \\ \phi(\,\cdot\,, T) = (S-K)_+ \,\mathrm{in}\, \mathrm{R}_+. \end{cases} \tag{3}$$

Here $L\phi = \partial_t\phi + \frac{1}{2}\varepsilon^2 S^2 \partial_{SS}\phi + rS\partial_S\phi - r\phi$, r represents the risk-free interest rate prevailing in the market. Numerous studies have shown that when trading costs exist, the volatility of the risky asset is often related to the sign of $\mathrm{div}(|\nabla u|^{p-1}\nabla u)$ [2]. This serves as the first motivation for our investigation of this type of variational inequality problem. Additionally, in the field of option pricing, $\partial_S\phi$ is referred to as the sensitivity of the option contract's value to changes in the price of the risky asset [3]. It is important to note that stocks are considered risky assets, and their prices, represented by S_t, often fluctuate stochastically with market information. As investors, it is natural to desire a lower sensitivity, or at least one within a certain controllable range. This serves as the reason for studying the estimation of the infinite norm of solution gradients in this variational inequality problem, and it is the second motivation of this paper. Finally, there is evidence to suggest that holding $\partial_S\phi$ shares of the risky asset can effectively match one unit of currency, resulting in a risk-free portfolio [4]. Although this approach requires constant adjustments to the allocation of risky asset shares, it has become an important method for financial practitioners and scholars to construct risk-free investment portfolios. This serves as the third motivation for our investigation of such problems.

1.2. Relevant research developments

The existence of solutions is the cornerstone of research on problems like the variational inequality (1), and there is a relatively large body of literature available. Reference [5] analyzes a class of variational hemivariational inequality problems under the framework of nonlinear evolution operators. By establishing the existence and uniqueness of solutions through the existence of solutions for discretized stationary problems and the convergence of semi-discrete schemes, the paper provides insights into the existence of solutions for variational hemivariational inequality problems. Inspired by fuzzy fractional damping variational inequality problems, reference [6] investigates a class of fuzzy fractional damping variational inequality problems. The existence of solutions for fractional differential variational inequalities is established by introducing mappings and constraint sets, as well as analyzing the continuous dependence of solutions on time. In Banach spaces, reference [7] studies a coupled inequality system composed of a variational-hemivariational inequality and a quasi-hemivariational inequality. By employing topological methods and analyzing the continuous dependence of the maximum operator on parameters, the study obtains results on the existence of solutions for the coupled system.

The estimate of space gradients of solutions to variational inequality problems has been explored in several studies. For instance, reference [8] presents a unified approach to investigate the Besov regularity and optimal estimates of double obstacle variational inequality problems on cylindrical domains, yielding certain results on space gradient estimation. There is also a significant body of literature on gradient estimation in the field of equation problems with structures similar to the operator (2). In reference [9], a discussion is conducted on initial-boundary value problems composed of a combination of local and non-local terms in degenerate parabolic operators. The study obtains nonlinear Calderón–Zygmund-type estimates on the space gradient, which improve upon the L_1 estimate for the space gradient. Reference [10] focuses on obtaining maximal modulus estimates for the ratio between the space gradient of the solution and the solution itself, as well as Hamilton-type space gradient estimation. Similarly, in reference [11], a Souplet–Zhang type space gradient estimation is developed for a nonlinear parabolic equation involving the Witten Laplacian. Compared to reference [10], a more favorable upper bound is obtained for the ratio between the space gradient of the solution and the solution itself. Furthermore, there are additional energy estimates concerning solutions to degenerate parabolic initial-boundary value problems. Interested readers can refer to references [12–15] for further information, as they provide detailed discussions on these topics.

The study of energy estimates for solutions typically relies on energy inequalities derived from weak solutions [16–18]. These inequalities involve energy functionals of the solutions on both sides, which facilitates the construction of recursive inequalities for the energy functionals, allowing for the derivation of energy estimates for the solutions through the properties of the recursive sequences [17,19]. Unfortunately, this approach is not suitable for analyzing energy estimates of the gradients of the solutions, as the energy norm of the solutions gradient only appears on the right-hand side of the energy inequality. This absence prevents the formation of recursive inequalities for the energy functionals associated with the gradient, making it impossible to obtain energy estimates for the gradients. Furthermore, the order of the energy norm on the left-hand side of the recursive inequality generated by the solutions is too high, which limits our ability to apply Sobolev inequalities for amplification. Consequently, this would only yield results that estimate lower-order norms using higher-order energy norms [20], which are generally not valuable for research. This paper proposes to establish a dedicated energy inequality specifically for the gradients of the solutions, aiming to derive energy estimates for these gradients, which serves as a primary motivation for this study.

This study focuses on the estimation of the space gradient's infinity norm for solutions to variational inequality problems under the framework of parabolic Kirchhoff operators. First, by utilizing the time smoothing operator, we establish energy inequalities for the solution and the gradient of the solution. Second, with the aid of these energy inequalities, we construct a sequence that converges to zero, thus proving an upper bound on the space gradient of the solution in the infinity norm. The contributions of this study are as follows: 1) We construct specialized energy inequalities for the space gradient, which lead to improved estimation results; 2) By employing the Caffarelli–Kohn–Nirenberg inequality, we construct a convergent sequence for the space gradient of the solution. By proving its convergence to zero, we obtain an estimation of the infinity norm of the space gradient for solutions to the variational inequality.

2. Preliminaries

In this section, we introduce some formula symbols and present accompanying useful results. We define the time smoothing operator u_h for u, which satisfies:

$$u_h(x,t) = \frac{1}{h}\int_0^t \exp\{\frac{s-t}{h}\}u(x,t)\mathrm{d}s. \tag{4}$$

Additionally, we cite the following results without proof, which can be found in references [16,21].
Lemma 2.1. For any $u \in L^p(\Omega)$, we have

$$\partial_t u_h = \frac{1}{h}(u - u_h),\ \|u_h\|_{L^p(\Omega)} \leq \|u_h\|_{L^p(\Omega)},$$

and the operator u_h converges to u in the $L^p(\Omega)$ norm, i.e.,

$$\|u_h - u\|_{L^p(\Omega)} \to 0 \,\text{as}\, h \to 0.$$

Lemma 2.2. If u^k converges to u in the $L^p(\Omega)$ norm, then u_h^k converges to u_h in the $L^p(\Omega)$ norm. Furthermore, we have

$$\|\partial_t u_h^k - \partial_t u_h\|_{L^p(\Omega)} \to 0,\ \|\nabla u_h^k - \nabla u_h\|_{L^p(\Omega)} \to 0 \,\text{as}\, h \to 0.$$

In order to facilitate the estimation of the infinite upper bound of the gradient of the distinguished solution, we make use of the Caffarelli–Kohn–Nirenberg inequality. In reference [17], specific parameters are assigned, leading to the following results.
Lemma 2.3. There exists a non-negative constant C_{C-K-N}, which only depend on n and p, such that

$$\int\int_{\Omega_T} |v|^{p\frac{(N+q)}{N}}\mathrm{d}x\mathrm{d}t \leq C_{C-K-N}\left(\int\int_{\Omega_T}|\nabla v|^p\mathrm{d}x\mathrm{d}t\right)\left(ess\sup_{t\in(0,T)}\int_{\Omega}|\nabla v|^q\mathrm{d}x\right)^{\frac{p}{N}}.$$

Lemma 2.4. Suppose a sequence $\{X_n, n = 0, 1, 2, \cdots\}$ satisfies: $X_{n+1} \leq Cb^n X_n^{1+\alpha}$, where C, b, and α are non-negative constants. If $X_0 \leq C^{-1/\alpha}b^{-1/\alpha^2}$, then

$$X_n \to 0 \,\text{as}\, n \to \infty.$$

Lemma 2.5. If a sequence $\{X_n, n = 0, 1, 2, \cdots\}$ satisfies $X_{n+1} \leq Cb^n X_n^{1+\alpha}$, where C, b, and α are non-negative constants, $\alpha \in (0,\ 1)$, then we have

$$X_0 \leq \left(\frac{2C}{b^{1-\alpha^{-1}}}\right)^{\alpha^{-1}}.$$

3. Energy inequality

In this section, we analyze the energy inequality of the solution u and its gradient ∇u for the variational inequality (1). We first examine the energy inequality of u. However, before that, we present a result regarding the overall boundedness of the variational inequality (1), which can be found in reference [18]

$$u \le |u_0|_\infty \,\text{in}\, \Omega_T,\ \nabla u \in L^p(\Omega_T). \tag{5}$$

Here, $|u_0|_\infty$ denotes the supremum of u_0 over the domain Ω, which is defined as follows:

$$|u_0|_\infty = \sup_{x\in\Omega} u_0(x).$$

By utilizing the set of maximal monotone maps specified in [18], namely

$$G = \{u | u(x) = 0, x > 0; u(x) \in [-M_0, 0], \quad x = 0\},$$

we present the following weak solution, where M_0 is a positive constant.

Definition 3.1. A pair (u, ξ) is considered a generalized solution to the variational inequality (1) if (u, ξ) fulfills the condition expressed in $u \in L^\infty(0, T, W^{1,p}(\Omega))$, $\partial_t u \in L^\infty(0, T, L^2(\Omega))$, and $\xi \in G$ for any $(x, t) \in \Omega_T$,

(a) $u(x, t) \ge u_0(x)$, $u(x, 0) = u_0(x)$ for any $(x, t) \in \Omega_T$,

(b) for every test function $\varphi \in C^1(\bar{\Omega}_T)$ and $t \in [0, T]$, the equality

$$\int\int_{\Omega_t} \partial_t u \cdot \varphi + (1 + \|u\|^p_{L^p(\Omega)})|\nabla u|^{p-2}\nabla u \nabla\varphi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_t} \xi \cdot \varphi \mathrm{d}x\mathrm{d}t$$

holds.

3.1. Energy inequality for u

Given $\varepsilon > 1$, let us choose a test function $\phi = u^m \psi(x)^{m+1}\eta(t)$, where $\psi \in C^\infty(\Omega)$ and $\eta \in C^\infty((0, T))$, and they also satisfy:

$$0 \le \psi \le 1 \,\text{in}\, \Omega, 0 \le \eta \le 1 \,\text{in}\, (0, T). \tag{6}$$

Note that, by utilizing the norm convergence result of Lemma 2.2 for L^p, and with the aid of Hölder's inequality and (6), it is straightforward to obtain the following as $h \to 0$,

$$\int_\Omega \partial_t u_h \phi \mathrm{d}x \to \int_\Omega \partial_t u \phi \mathrm{d}x, \tag{7}$$

$$\int_\Omega (|\nabla u|^{p-2}\nabla u)_h \nabla\phi \mathrm{d}x \to \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi \mathrm{d}x, \tag{8}$$

$$\int_\Omega (1 + \|u\|^p_{L^p(\Omega)})(|\nabla u|^{p-2}\nabla u)_h \nabla\phi \mathrm{d}x \to \int_\Omega (1 + \|u\|^p_{L^p(\Omega)})|\nabla u|^{p-2}\nabla u \nabla\phi \mathrm{d}x. \tag{9}$$

Therefore, when h is sufficiently small, we have

$$\int\int_{\Omega\times(t_1,t_2)} \partial_t u_h \phi \mathrm{d}x\mathrm{d}t + \int\int_{\Omega\times(t_1,t_2)} (1 + \|u\|^p_{L^p(\Omega)})(|\nabla u|^{p-2}\nabla u)_h \nabla\phi \mathrm{d}x\mathrm{d}t \le 0. \tag{10}$$

We begin by analyzing $\int\int_{\Omega\times(t_1,t_2)} \partial_t u_h \phi \mathrm{d}x\mathrm{d}t$. Through the process of integration by parts, it is evident that we can obtain:

$$\begin{aligned} &\int\int_{\Omega\times(t_1,t_2)} \partial_t u_h \phi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega\times(t_1,t_2)} \partial_t u_h u^m \psi(x)^{m+1}\eta(t)\mathrm{d}x\mathrm{d}t \\ &= \int\int_{\Omega\times(t_1,t_2)} \partial_t u_h (u^m - u_h^m)\psi(x)^{m+1}\eta(t)\mathrm{d}x\mathrm{d}t + \int\int_{\Omega\times(t_1,t_2)} \partial_t u_h u_h^m \psi(x)^{m+1}\eta(t)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{11}$$

Note that $\partial_t u_h = \frac{u-u_h}{h}$, combined with the fact that u^m is an increasing function with respect to u, we can conclude that

$$\partial_t u_h(u^m - u_h^m) \geq 0.$$

Take note of $\psi \in C^\infty(\Omega)$, $\eta \in C^\infty((0,T))$, and in conjunction with Lemma 2.2, we can deduce

$$\int\int_{\Omega\times(t_1,t_2)} \partial_t u_h(u^m - u_h^m)\psi(x)^{m+1}\eta(t)\mathrm{d}x\mathrm{d}t \searrow +0 \,\text{as}\, h \to 0. \tag{12}$$

Next, let us analyze $\int\int_\Omega \partial_t u_h^{m+1}\psi(x)^{m+1}\eta(t)\mathrm{d}x\mathrm{d}t$. By performing integration by parts, we can obtain

$$\begin{aligned}&\int\int_{\Omega\times(t_1,t_2)} \partial_t u_h^{m+1}\psi(x)^{m+1}\eta(t)\mathrm{d}x\mathrm{d}t\\ &= \int\int_\Omega u_h(x,t_2)^{m+1}\psi(x)^{m+1}\eta(t_2)\mathrm{d}x - \int\int_{\Omega\times(t_1,t_2)} u_h^{m+1}\psi(x)^{m+1}\partial_t\eta(t)\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{13}$$

By utilizing Lemmas 2.1 and 2.2, combined with the result (5), we have

$$\int_\Omega u_h(x,t_2)^{m+1}\psi(x)^{m+1}\eta(t_2)\mathrm{d}x \to \int_\Omega u(x,t_2)^{m+1}\psi(x)^{m+1}\eta(t_2)\mathrm{d}x \,\text{as}\, h \to 0, \tag{14}$$

$$\int\int_{\Omega\times(t_1,t_2)} u_h^{m+1}\psi(x)^{m+1}\partial_t\eta(t)\mathrm{d}x\mathrm{d}t \to \int\int_{\Omega\times(t_1,t_2)} u^{m+1}\psi(x)^{m+1}\partial_t\eta(t)\mathrm{d}x\mathrm{d}t\text{as}\, h \to 0. \tag{15}$$

For ease of description, let us define

$$I_0 = \frac{1}{m+1}\int_\Omega u(x,t_2)^{m+1}\psi(x)^{m+1}\eta(t_2)\mathrm{d}x - \frac{1}{m+1}\int\int_{\Omega\times(t_1,t_2)} u^{m+1}\psi(x)^{m+1}\partial_t\eta(t)\mathrm{d}x\mathrm{d}t.$$

Consequently, we have

$$\lim_{h\to 0} I_h = I_0. \tag{16}$$

Now, let us analyze $L_0 = \int\int_{\Omega\times(t_1,t_2)} (1 + \|u\|^p_{L^p(\Omega)})|\nabla u|^{p-2}\nabla u\nabla\phi\mathrm{d}x\mathrm{d}t$. From (9), it is easy to derive

$$L_h = \int\int_{\Omega\times(t_1,t_2)} (1 + \|u\|^p_{L^p(\Omega)})(|\nabla u|^{p-2}\nabla u)_h\nabla\phi\mathrm{d}x\mathrm{d}t \to L_0 \,\text{as}\, h \to 0. \tag{17}$$

Please note that from (6), we can obtain $0 \leq \psi(x) \leq 1$, which in turn leads to $\psi(x)^{m+1} \leq \psi(x)^m$. By utilizing the integration by parts, we can deduce:

$$\begin{aligned}L_0 &= \int\int_{\Omega\times(t_1,t_2)} (1 + \|u\|^p_{L^p(\Omega)})|\nabla u|^{p-2}\nabla u\nabla(u^m\psi(x)^{m+1}\eta(t))\mathrm{d}x\mathrm{d}t\\ &\leq m\int\int_{\Omega\times(t_1,t_2)} (1 + \|u\|^p_{L^p(\Omega)})u^{m-1}|\nabla u|^p\psi(x)^{m+1}\eta(t)\mathrm{d}x\mathrm{d}t\\ &\quad + (m+1)\int\int_{\Omega\times(t_1,t_2)} (1 + \|u\|^p_{L^p(\Omega)})u^m\psi(x)^m\eta(t)|\nabla u|^{p-1}|\nabla\psi(x)|\mathrm{d}x\mathrm{d}t\\ &\leq m\int\int_{\Omega\times(t_1,t_2)} (1 + \|u\|^p_{L^p(\Omega)})u^{m-1}|\nabla u|^p\psi(x)^m\eta(t)\mathrm{d}x\mathrm{d}t\\ &\quad + (m+1)\int\int_{\Omega\times(t_1,t_2)} (1 + \|u\|^p_{L^p(\Omega)})u^m\psi(x)^m\eta(t)|\nabla u|^{p-1}|\nabla\psi(x)|\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{18}$$

By selecting the parameters $\frac{p-1}{p}$ and $\frac{1}{p}$, and utilizing the weighted Hölder's inequality and Young's inequality, we can obtain:

$$\begin{aligned}&\int\int_{\Omega\times(t_1,t_2)} (1 + \|u\|^p_{L^p(\Omega)})u^m\psi(x)^m\eta(t)|\nabla u|^{p-1}|\nabla\psi(x)|\mathrm{d}x\mathrm{d}t\\ &\leq \frac{p-1}{p}\int\int_{\Omega\times(t_1,t_2)} (1 + \|u\|^p_{L^p(\Omega)})u^{m-1}\psi(x)^m\eta(t)|\nabla u|^p\mathrm{d}x\mathrm{d}t\\ &\quad + \frac{1}{p}\Big(\frac{m+1}{m}\Big)^{p-1}\int\int_{\Omega\times(t_1,t_2)} (1 + \|u\|^p_{L^p(\Omega)})u^{p+m-1}\psi(x)^m\eta(t)|\nabla\psi(x)|^p\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{19}$$

By combining Eqs (17)–(19) and simplifying, we obtain the following result.

Theorem 3.1. Let u be a solution to the variational inequality (1). For any given $t_1, t_2 \in (0, T)$, $m > 0$, and $p \geq 2$, if $t_1 < t_2$ holds, then there exists a non-negative constant C that depends only on m and p, such that

$$\begin{aligned}&\int_\Omega u(x,t_2)^{m+1}\psi(x)^{m+1}\eta(t_2)\mathrm{d}x\\ &\leq \frac{1}{p}(m+1)^{p+1}m^{1-p}\int\int_{\Omega\times(t_1,t_2)}(1+\|u\|^p_{L^p(\Omega)})u^{p+m-1}\psi(x)^\varepsilon\eta(t)|\nabla\psi(x)|^p\mathrm{d}x\mathrm{d}t\\ &\quad+\int\int_{\Omega\times(t_1,t_2)}(1+\|u\|^p_{L^p(\Omega)})u^{m+1}\psi(x)^{m+1}|\partial_t\eta(t)|\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{20}$$

and

$$\begin{aligned}&\int\int_{\Omega\times(t_1,t_2)}u^{m-1}|\nabla u|^p\psi(x)^m\eta(t)\mathrm{d}x\mathrm{d}t\\ &\leq (m+1)^p m^{-p}\int\int_{\Omega\times(t_1,t_2)}(1+\|u\|^p_{L^p(\Omega)})u^{p+m-1}\psi(x)^\varepsilon\eta(t)|\nabla\psi(x)|^p\mathrm{d}x\mathrm{d}t\\ &\quad+\frac{p}{m(m+1)}\int\int_{\Omega\times(t_1,t_2)}(1+\|u\|^p_{L^p(\Omega)})u^{m+1}\psi(x)^{m+1}|\partial_t\eta(t)|\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{21}$$

3.2. Energy inequality for ∇u

We examine the energy inequality regarding the gradient ∇u. For ease of discussion, let us define:

$$\mathrm{O}(\rho,\theta)=\mathrm{O}(\rho,\theta|(x_0,t_0))=\Theta_\rho\times\Xi_\theta=\{x\,||x-x_0|<\rho\}\times(t_0-\theta,t_0), \tag{22}$$

where (x_0, t_0) is a point located within the interior of Ω_T. Furthermore, let us set an undetermined constant $\delta \in (0, 1)$ such that

$$\rho_n=\sigma\rho+\frac{1-\sigma}{2^n}\rho,\theta_n=\sigma\theta+\frac{1-\sigma}{2^n}\theta,\mathrm{O}_n=\mathrm{O}(\rho_n,\theta_n)=\Theta_{\rho_n}\times\Xi_{\theta_n}. \tag{23}$$

It is worth noting that $\mathrm{O}_0 = \mathrm{O}(\rho, \theta)$, $\mathrm{O}_\infty = \mathrm{O}(\sigma\rho, \sigma\theta)$. Additionally, we need the following cylindrical region:

$$\tilde{\mathrm{O}}_n=\mathrm{O}(\tilde{\rho}_n,\tilde{\theta}_n)=\Theta_{\tilde{\rho}_n}\times\Xi_{\tilde{\theta}_n}, \tag{24}$$

where

$$\tilde{\rho}_n=\frac{1}{2}(\rho_n+\rho_{n+1}),\tilde{\theta}_n=\frac{1}{2}(\theta_n+\theta_{n+1}). \tag{25}$$

Clearly, within these cylindrical regions,

$$\mathrm{O}_{n+1}\subset\tilde{\mathrm{O}}_n\subset\mathrm{O}_n. \tag{26}$$

Building upon the foundations of O_n and $\tilde{\mathrm{O}}_n$, we provide more detailed definitions for ψ and η. We assume that ψ_n is a truncation factor on $\Theta_{\rho_{n+1}}$, satisfying not only the conditions regarding ψ as stated in (7), but also ψ_n on the boundary of $\Theta_{\rho_{n+1}}$ being 0, as well as

$$\psi_n(x)=1\,\mathrm{in}\,\Theta_{\rho_n},|\nabla\psi_n|\leq\frac{2^{n+2}}{(1-\sigma)\rho}. \tag{27}$$

Furthermore, let us assume that η_n is a truncation function on $\Xi_{\theta_{n+1}}$, which not only satisfies (7), but also η_n at $t_0 - \theta_n$ being 0, as well as

$$\eta_n(x)=1\,\mathrm{in}\,\Xi_{\theta_n},|\nabla\eta_n|\leq\frac{2^{n+2}}{(1-\sigma)\theta}. \tag{28}$$

From (1), we know that when $Lu < 0$, then $u = u_0$. According to the assumption of u_0, it is clear that

$$\nabla u \in L^{\infty}(\{(x,t)|u = u_0\}). \tag{29}$$

Next, we analyze the case of $Lu = 0$ on Ω_T. Let $v = |\nabla u|$, and set

$$\varphi = p(v - k_{n+1})_+^{p-1} \times \psi_n^p \eta_n^p, \zeta_n = I_{\{(x,t)\in \mathrm{O}_n | v \geq k_{n+1}\}}, \tag{30}$$

and $k_n = k - \frac{1}{2^n}k$, where k is a non-negative undetermined constant. Multiply both sides of $\nabla Lu = 0$ by φ and integrate over O_n, we have

$$\int\int_{\mathrm{O}_n} \partial_t \nabla u \varphi \mathrm{d}x\mathrm{d}t + \int\int_{\mathrm{O}_n} (1 + \|u\|^p_{L^p(\Omega)}) \times \mathrm{div}(v^{p-2}\nabla u)\nabla\varphi \mathrm{d}x\mathrm{d}t = 0. \tag{31}$$

Regarding $\int\int_{\mathrm{O}_n} \partial_t \nabla u \varphi \mathrm{d}x\mathrm{d}t$, applying the fundamental principle of differential expansion, we can obtain:

$$\begin{aligned} &\int\int_{\mathrm{O}_n} \partial_t \nabla u \times \varphi \mathrm{d}x\mathrm{d}t = p\int\int_{\mathrm{O}_n} \partial_t v(v - k_{n+1})^{p-1} \times \psi_n^p\eta_n^p \mathrm{d}x\mathrm{d}t \\ &= \int\int_{\mathrm{O}_n} (v - k_{n+1})^p \times \psi_n^p\eta_n^p \mathrm{d}x - p\int\int_{\mathrm{O}_n} (v - k_{n+1})^p \times \psi_n^p \eta_n^{p-1}\partial_t\eta_n \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{32}$$

Next, let us analyze the second term on the left-hand side of (31), using the fundamental principle of differential expansion. It is easy to observe that

$$\begin{aligned} &\int\int_{\mathrm{O}_n} (1 + \|u\|^p_{L^p(\Omega)})\mathrm{div}(v^{p-2}\nabla u)\nabla\varphi \mathrm{d}x\mathrm{d}t \\ &= p(p-1)^2 \int\int_{\mathrm{O}_n} (1 + \|u\|^p_{L^p(\Omega)})v^{p-2}|\Delta u|^2(v - k_n)_+^{p-2} \times \psi_n^p\eta_n^p \mathrm{d}x\mathrm{d}t \\ &\quad +(p-1)p^2 \int\int_{\mathrm{O}_n} (1 + \|u\|^p_{L^p(\Omega)})v^{p-2}|\Delta u|(v - k_n)_+^{p-1} \times \psi_n^{p-1}\eta_n^p\nabla\psi_n \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{33}$$

Please take note that if '=' in (31) is changed to '≥', it would not be conducive to constructing an energy inequality, which in turn hinders our search for a lower bound for

$$\int\int_{\mathrm{O}_n} (1 + \|u\|^p_{L^p(\Omega)})v^{p-2}|\Delta u|^2(v - k_n)_+^{p-2} \times \psi_n^p\eta_n^p \mathrm{d}x\mathrm{d}t.$$

Furthermore, due to the fact that for any $n = 1, 2, 3, \cdots$, we have

$$v \geq k_0 = \frac{1}{2}k \text{ in } \{(x,t) \in \mathrm{O}_n | v \geq k_{n+1}\}, \tag{34}$$

it follows that

$$\begin{aligned} &\int\int_{\mathrm{O}_n} (1 + \|u\|^p_{L^p(\Omega)})v^{p-2}|\Delta u|^2(v - k_n)_+^{p-2} \times \psi_n^p\eta_n^p \mathrm{d}x\mathrm{d}t \\ &\geq p\Big(\frac{k}{2}\Big)^{p-2} \int\int_{\mathrm{O}_n} (1 + \|u\|^p_{L^p(\Omega)})|(v - k_n)_+^{\frac{1}{2}p-1}\Delta u|^2 \times \psi_n^p\eta_n^p \mathrm{d}x\mathrm{d}t \\ &= \frac{1}{p}\Big(\frac{k}{2}\Big)^{p-2} \int\int_{\mathrm{O}_n} (1 + \|u\|^p_{L^p(\Omega)})|\nabla(v - k_n)_+^{\frac{1}{2}p}|^2 \times \psi_n^p\eta_n^p \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{35}$$

Be aware that $(v - k_n)_+ \leq v$ on $\{(x,t) \in \mathrm{O}_n | v \geq k_{n+1}\}$, thus resulting in

$$\begin{aligned} &(p-1)p^2 \left|\int\int_{\mathrm{O}_n} (1 + \|u\|^p_{L^p(\Omega)})v^{p-2}|\Delta u|^2(v - k_n)_+^{p-1} \times \psi_n^{p-1}\eta_n^p\nabla\psi_n \mathrm{d}x\mathrm{d}t\right| \\ &\leq (p-1)p^2 \int\int_{\mathrm{O}_n} (1 + \|u\|^p_{L^p(\Omega)})v^{\frac{3}{2}p-2}|\nabla(v - k_n)_+^{\frac{1}{2}p}| \times \psi_n^{p-1}\eta_n^p|\nabla\psi_n| \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{36}$$

We aim to construct an upper bound for

$$\int\int_{O_n}(1+\|u\|^p_{L^p(\Omega)})v^{\frac{3}{2}p-2}|\nabla(v-k_n)_+^{\frac{1}{2}p}|\times\psi_n^{p-1}\eta_n^p|\nabla\psi_n|\mathrm{d}x\mathrm{d}t,$$

in terms of $\int\int_{O_n}(1+\|u\|^p_{L^p(\Omega)})|\nabla(v-k_n)_+^{\frac{1}{2}p}|^2\times\psi_n^p\eta_n^p\mathrm{d}x\mathrm{d}t$, so that we can subsequently apply the Hölder and Young inequalities to obtain:

$$\begin{aligned}&(p-1)p^2\int\int_{O_n}(1+\|u\|^p_{L^p(\Omega)})v^{\frac{3}{2}p-2}|\nabla(v-k_n)_+^{\frac{1}{2}p}|\times\psi_n^{p-1}\eta_n^p|\nabla\psi_n|\mathrm{d}x\mathrm{d}t\\&\le\frac{(p-1)^2}{2p}\left(\frac{k}{2}\right)^{p-2}\int\int_{O_n}(1+\|u\|^p_{L^p(\Omega)})|\nabla(v-k_n)_+^{\frac{1}{2}p}|^2\times\psi_n^p\eta_n^p\mathrm{d}x\mathrm{d}t\\&\quad+\frac{p^2}{2(p-1)(k/2)^{p-2}}\int\int_{O_n}(1+\|u\|^p_{L^p(\Omega)})v^{3p-4}\times\psi_n^{p-2}|\nabla\psi_n|^2\eta_n^p\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{37}$$

By combining formulas (31)–(33) and (35)–(37), we obtain the following result.

Theorem 3.2. Assuming $v=|\nabla u|$, for any $n=1,2,3,\cdots$, we have

$$\begin{aligned}&\sup_{t\in\Xi_{\theta_n}}\int_{\Theta_n}(v-k_{n+1})^p\times\psi_n^p\eta_n^p\mathrm{d}x+\frac{(p-1)^2}{2p}\left(\frac{k}{2}\right)^{p-2}\int\int_{O_n}|\nabla(v-k_n)_+^{\frac{1}{2}p}|^2\times\psi_n^p\eta_n^p\mathrm{d}x\mathrm{d}t\\&\le p\int\int_{O_n}(1+\|u\|^p_{L^p(\Omega)})(v-k_{n+1})^p\times\psi_n^p\eta_n^{p-1}|\partial_t\eta_n|\mathrm{d}x\mathrm{d}t\\&\quad+\frac{p^2}{2(p-1)(k/2)^{p-2}}\int\int_{O_n}(1+\|u\|^p_{L^p(\Omega)})v^{3p-4}\times\psi_n^{p-2}|\nabla\psi_n|^2\eta_n^p\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{38}$$

4. Infinite norm estimation of $|\nabla u|$

This section examines the infinite norm estimation of the spatial gradients of solutions near the point (x_0,t_0). By utilizing Theorem 3.2, it is straightforward to obtain

$$\begin{aligned}&\sup_{t\in\Xi_{\theta_n}}\int_{\Theta_n}(v-k_{n+1})^p\times\psi_n^p\eta_n^p\mathrm{d}x+\frac{(p-1)^2}{2p}\left(\frac{k}{2}\right)^{p-2}\int\int_{O_n}|\nabla(v-k_n)_+^{\frac{1}{2}p}|^2\times\psi_n^p\eta_n^p\mathrm{d}x\mathrm{d}t\\&\le p\frac{2^{n+2}}{(1-\sigma)\theta}\int\int_{O_n}(1+\|u\|^p_{L^p(\Omega)})v^p\times\psi_n^p\eta_n^{p-1}I_{(v-k_{n+1})_+>0}\mathrm{d}x\mathrm{d}t\\&\quad+\frac{p^2}{2(p-1)(k/2)^{p-2}}\frac{2^{2n+4}}{(1-\sigma)^2\rho^2}\int\int_{O_n}(1+\|u\|^p_{L^p(\Omega)})v^{3p-4}\times\psi_n^{p-2}\eta_n^pI_{(v-k_{n+1})_+>0}\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{39}$$

Now, we simplify the recursive relation in (39) in order to utilize Lemma 2.1, defining

$$X_n=\int\int_{O_n}(v-k_n)_+^p\mathrm{d}x\mathrm{d}t.$$

Note that when $I_{(v-k_{n+1})_+>0}=1$, we have $v>k_{n+1}$ and $(v-k_{n+1})_+>\frac{k}{2^{n+1}}$, thus resulting in

$$\int\int_{O_n}I_{(v-k_{n+1})_+>0}\mathrm{d}x\mathrm{d}t\le\frac{2^{(n+1)p}}{k^p}\int\int_{O_n}(v-k_n)_+^p\mathrm{d}x\mathrm{d}t=2^{(n+1)p}k^{-p}X_n.\tag{40}$$

Lemma 4.1. For any $n=1,2,3,\cdots$, we can obtain:

$$\int\int_{O_n}v^pI_{(v-k_{n+1})_+>0}\mathrm{d}x\mathrm{d}t\ge2^{np+1}\int\int_{O_n}(v-k_n)_+^p\mathrm{d}x\mathrm{d}t.$$

Proof: Note that $\{(x,t)|(v-k_{n+1})_+>0\}\subset \mathrm{O}_n$, hence, resulting in

$$\int\int_{\mathrm{O}_n}(v-k_n)_+^p\mathrm{d}x\mathrm{d}t\geq\int\int_{\mathrm{O}_n}(v-k_n)_+^pI_{(v-k_{n+1})_+>0}\mathrm{d}x\mathrm{d}t. \tag{41}$$

By further utilizing $k_n=k_{n+1}\frac{2^{n+1}-2}{2^{n+1}-1}$, as well as $v\geq k_{n+1}$ on $\{(x,t)|(v-k_{n+1})_+>0\}\subset \mathrm{O}_n$, we can obtain:

$$\int\int_{\mathrm{O}_n}(v-k_n)_+^p\mathrm{d}x\mathrm{d}t\geq\int\int_{\mathrm{O}_n}v^p\left(1-\frac{2^{n+1}-2}{2^{n+1}-1}\right)_+^pI_{(v-k_{n+1})_+>0}\mathrm{d}x\mathrm{d}t. \tag{42}$$

Finally, to simplify the result, amplifying $2^{n+1}-1$ to 2^{n+1}, we can obtain:

$$\int\int_{\mathrm{O}_n}(v-k_n)_+^p\mathrm{d}x\mathrm{d}t\geq\frac{1}{2^{np+1}}\int\int_{\mathrm{O}_n}v^pI_{(v-k_{n+1})_+>0}\mathrm{d}x\mathrm{d}t.$$

Next, we seek the upper bound of X_{n+1} in order to establish a recursive relation with X_n in (39). Note that $\mathrm{O}_n\supset \mathrm{O}_{n+1}$ and $\zeta_n=1$ in O_{n+1}, thus by using the Hölder's inequality, we have

$$\begin{aligned}X_{n+1}&\leq\int\int_{\mathrm{O}_n}|(v-k_{n+1})_+^{p/2}\times\psi_n\eta_n|^2\mathrm{d}x\mathrm{d}t\\&\leq\left(\int\int_{\mathrm{O}_n}|(v-k_{n+1})_+^{p/2}\times\psi_n\eta_n|^{2\frac{N+2}{N}}\mathrm{d}x\mathrm{d}t\right)^{\frac{N}{N+2}}\times\left(\int\int_{\mathrm{O}_n}I_{(v-k_{n+1})_+>0}\mathrm{d}x\mathrm{d}t\right)^{\frac{2}{N+2}}.\end{aligned} \tag{43}$$

By combining the aforementioned estimation results with (40), it is straightforward to obtain:

$$\begin{aligned}X_{n+1}&\leq\int\int_{\mathrm{O}_n}|(v-k_{n+1})_+^{p/2}\times\psi_n\eta_n|^2\mathrm{d}x\mathrm{d}t\\&\leq\left(\int\int_{\mathrm{O}_n}|(v-k_{n+1})_+^{p/2}\times\psi_n\eta_n|^{2\frac{N+2}{N}}\mathrm{d}x\mathrm{d}t\right)^{\frac{N}{N+2}}4^{np}k^{-p\frac{2}{N+2}}X_n^{\frac{2}{N+2}}.\end{aligned} \tag{44}$$

Due to the high exponent of $\int\int_{\mathrm{O}_n}|(v-k_{n+1})_+^{p/2}\times\psi_n\eta_n|^{2\frac{N+2}{N}}\mathrm{d}x\mathrm{d}t$, it becomes challenging to control it using the norms in the above results. Therefore, we employ the Caffarelli–Kohn–Nirenberge inequality to reduce the order of the norms, resulting in

$$\begin{aligned}&\int\int_{\mathrm{O}_n}|(v-k_{n+1})_+^{p/2}\times\psi_n\eta_n|^{2\frac{N+2}{N}}\mathrm{d}x\mathrm{d}t\\&\leq C_{C-K-N}\left(\int\int_{\mathrm{O}_n}|\nabla(v-k_{n+1})_+^{p/2}\times\psi_n\eta_n|^2\mathrm{d}x\mathrm{d}t\right)\left(ess\sup_{t\in\Xi_n}\int_{\Theta_n}|(v-k_{n+1})_+^{p/2}\times\psi_n\eta_n|^2\mathrm{d}x\right)^{\frac{2}{N}}.\end{aligned} \tag{45}$$

By further utilizing (39), it is evident that

$$ess\sup_{t\in\Xi_n}\int_{\Theta_n}(v-k_{n+1})^p\times\psi_n\eta_n\mathrm{d}x\leq \mathrm{H}, \tag{46}$$

$$\int\int_{\mathrm{O}_n}|\nabla(v-k_n)_+^{\frac{1}{2}p}|^2\times\psi_n\eta_n\mathrm{d}x\mathrm{d}t\leq\frac{2p^2}{(p-1)^2}\left(\frac{k}{2}\right)^{2-p}\times \mathrm{H}, \tag{47}$$

where

$$\begin{aligned}\mathrm{H}=&\frac{2^{n+2}p}{(1-\sigma)\theta}\int\int_{\mathrm{O}_n}(1+\|u\|_{L^p(\Omega)}^p)\times v^pI_{(v-k_{n+1})_+>0}\mathrm{d}x\mathrm{d}t\\&+\frac{p^4}{2(k/2)^{p-2}}\frac{2^{2n+4}}{(1-\sigma)^2\rho^2}\int\int_{\mathrm{O}_n}(1+\|u\|_{L^p(\Omega)}^p)\times v^{3p-4}I_{(v-k_{n+1})_+>0}\mathrm{d}x\mathrm{d}t.\end{aligned}$$

By substituting (46) and (47) into (45), we obtain an estimation for $\int\int_{O_n}|(v-k_{n+1})_+^{p/2}\times\psi_n\eta_n|^{2\frac{N+2}{N}}\mathrm{d}x\mathrm{d}t$, denoted as:

$$\int\int_{O_n}|(v-k_{n+1})_+^{p/2}\times\psi_n\eta_n|^{2\frac{N+2}{N}}\mathrm{d}x\mathrm{d}t\le C_{C-K-N}\frac{2p^2}{(p-1)^2}\left(\frac{k}{2}\right)^{2-p}\mathrm{H}^{1+\frac{2}{N}}. \tag{48}$$

By further utilizing the Hölder's inequality, we can obtain:

$$\begin{aligned}&\int\int_{O_n}(1+\|u\|^p_{L^p(\Omega)})v^{3p-4}I_{(v-k_{n+1})_+>0}\mathrm{d}x\mathrm{d}t\\&\le\Big(\int\int_{O_n}(1+\|u\|^p_{L^p(\Omega)})v^{(p-2)(N+2)+p}\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{N+2}}\Big(\int\int_{O_n}(1+\|u\|^p_{L^p(\Omega)})v^pI_{(v-k_{n+1})_+>0}\mathrm{d}x\mathrm{d}t\Big)^{\frac{N+1}{N+2}}.\end{aligned} \tag{49}$$

Using the energy inequality (Lemma 3.1), we obtain:

$$\int\int_{O_n}(1+\|u\|^p_{L^p(\Omega)})v^{(p-2)(N+2)+p}\mathrm{d}x\mathrm{d}t\le C\int_{\Xi_{\theta_n}}(1+\|u\|^p_{L^p(\Omega)})\int_{\Theta_{\rho n}}u^{(p-2)(N+2)+p}\mathrm{d}x\mathrm{d}t.$$

Furthermore, from (6), we can derive

$$\int\int_{Q_n}(1+\|u\|^p_{L^p(\Omega)})v^{3p-4}I_{(v-k_{n+1})_+>0}\mathrm{d}x\mathrm{d}t\le C(p,|u_0|_\infty)2^{np+1}X_n^{\frac{N+1}{N+2}}. \tag{50}$$

By utilizing Lemma 4.1 and (50), we have

$$\begin{aligned}&\int\int_{Q_n}|(v-k_{n+1})_+^{p/2}\times\psi_n\eta_n|^{2\frac{N+2}{N}}\mathrm{d}x\mathrm{d}t\\&\le C(C_{C-K-N},p,\rho,\sigma,\theta)\Big(\frac{k}{2}\Big)^{2-p}\Big(2^{np+1}X_n+C(p,|u_0|_\infty)2^{np+1}X_n^{\frac{N+1}{N+2}}\Big)^{1+\frac{2}{N}}.\end{aligned} \tag{51}$$

Combining (44) and (51), we obtain:

$$X_{n+1}\le(C_{C-K-N},p,\rho,\sigma,\theta,|u_0|_\infty)k^{-\frac{p}{N+2}}4^{np}X_n^{1+\frac{2}{N+2}}.$$

By further restricting $k>1$, we have

$$X_{n+1}\le(C_{C-K-N},p,\rho,\sigma,\theta,|u_0|_\infty)4^{np}X_n^{1+\frac{2}{N+2}}.$$

Finally, by applying Lemmas 2.4 and 2.5, we can obtain:

$$X_0\le(C_{C-K-N},p,\rho,\sigma,\theta,|u_0|_\infty)^{-N/2}4^{-N^2p/4}. \tag{52}$$

It is worth noting that in the above equation, we can choose $\sigma=0.5$, thereby obtaining the main result of this paper.

Theorem 4.1. For any $O(\rho,\theta)\subset\Omega_T$, there exist non-negative constants that depend only on p, ρ, θ, and $|u_0|_\infty$, such that the solution u of the variational inequality (1) satisfies:

$$\nabla u\in L^\infty(O(\rho,\theta)).$$

Note that Ω_T is a bounded and open set on R_{n+1}, and we can find a finite number of points (let us assume there are m points)

$$(x_1,t_1),\ (x_2,t_2),\ \cdots,\ (x_m,t_m)$$

on Ω_T to construct the set

$$O(\rho,\theta|(x_1,t_1)),\ O(\rho,\theta|(x_2,t_2)),\ \cdots,\ O(\rho,\theta|(x_m,t_m)).$$

In this case, we have $\cup_{i=1}^m O(\rho,\theta|(x_i,t_i))\supset\Omega_T$, and on each $O(\rho,\theta|(x_i,t_i))$, we also have $\nabla u\in L^\infty(O(\rho,\theta))$. Therefore, by the finite open cover theorem, there exist non-negative constants that depend only on p, N, T, $\dim(\Omega)$, and $|u_0|_\infty$, such that the solution u of the variational inequality (1) satisfies

$$\nabla u\in L^\infty(\Omega_T).$$

5. An example from option pricing

We continue to examine the valuation of call options with early exercise features, as detailed in Model (3). This is commonly referred to as American options in the literature [1–4]. In financial scenarios, to mitigate significant losses, option issuers typically impose barrier clauses on top of Model (3). When the price of the stock underlying the American option exceeds (S_1, S_2), the option becomes void. At this point, the value of the American option is given by:

$$\begin{cases} \max\{L\phi, (S-K)_+ - \phi\} = 0 \text{ in } (S_1, S_2) \times (0, T), \\ \phi(\cdot, T) = (S-K)_+ \text{ in } (S_1, S_2), \\ \phi(S_1 \cdot, t) = \phi(S_2 \cdot, t) = 0 \text{ in } (0, T), \end{cases} \tag{53}$$

where $L\phi = \partial_t \phi + \frac{1}{2}\varepsilon^2 S^2 \partial_{SS}\phi + rS\partial_S \phi - r\phi$, S_1, and S_2 represent the lower and upper price limits of the stock as specified in the option agreement, and $0 < S_1 < S_2$. In contrast, if the option does not include early exercise features, we refer to it as a European option, and its value is given by:

$$\begin{cases} L\Phi = 0 \text{ in } (S_1, S_2) \times (0, T), \\ \Phi(\cdot, T) = (S-K)_+ \text{ in } (S_1, S_2), \\ \Phi(S_1 \cdot, t) = \Phi(S_2 \cdot, t) = 0 \text{ in } (0, T). \end{cases} \tag{54}$$

By applying the transformation $x = \ln S$ and $\tau = T - t$, and letting $M\phi = \partial_\tau\phi - \frac{1}{2}\varepsilon^2\partial_{xx}\phi + (r - \frac{1}{2}\varepsilon^2)\partial_x\phi + r\phi$, the valuation models for the American option and the European option, represented by (53) and (54), can be rewritten as:

$$\begin{cases} \max\{M\phi, (e^x - K)_+ - \phi\} = 0 \text{ in } (\ln S_1, \ln S_2) \times (0, T), \\ \phi(\cdot, T) = (e^x - K)_+ \text{ in } (\ln S_1, \ln S_2), \\ \phi(\ln S_1 \cdot, t) = \phi(\ln S_2 \cdot, t) = 0 \text{ in } (0, T), \end{cases} \tag{55}$$

and

$$\begin{cases} M\Phi = 0 \text{ in } (\ln S_1, \ln S_2) \times (0, T), \\ \Phi(\cdot, 0) = (e^x - K)_+ \text{ in } (\ln S_1, \ln S_2), \\ \Phi(\ln S_1 \cdot, t) = \phi(\ln S_2 \cdot, t) = 0 \text{ in } (0, T), \end{cases} \tag{56}$$

respectively.

Next, we will use the American option model (55) and the European option model (56) to verify the results of Theorem 4.1. Compared to European options, American options include early exercise features, allowing investors greater flexibility in hedging strategies. This means that when stock prices fluctuate, American options tend to exhibit greater stability than European options [1,2], as expressed by

$$\left|\frac{\partial}{\partial x}\phi\right| \le \left|\frac{\partial}{\partial x}\Phi\right|. \tag{57}$$

On the other hand, by choosing $v = \frac{\partial}{\partial x}\Phi$ and taking the partial derivatives with respect to x on both sides of the three equations in (56), we obtain $SS2$.

$$\begin{cases} Mv = 0 \text{ in } (\ln S_1, \ln S_2) \times (0, T), \\ v(\cdot, 0) = e^x I_{e^x \ge K} \text{ in } (\ln S_1, \ln S_2), \\ v(\ln S_1 \cdot, t) = v(\ln S_2 \cdot, t) = 0 \text{ in } (0, T). \end{cases} \tag{58}$$

Note that $MS_2 = rS_2 > 0$, $v(\cdot, 0) \leq S_2$ in $(\ln S_1, \ln S_2)$, and

$$v(\ln S_1 \cdot, t) = v(\ln S_2 \cdot, t) = 0 \leq S_2 \text{ in } (0, T).$$

By the comparison principle, we can conclude that$v \leq S_2$ in $(\ln S_1, \ln S_2) \times (0, T)$. Furthermore, by applying the comparison principle again, we can derive $v \geq -S_2$ in $(\ln S_1, \ln S_2) \times (0, T)$, leading to

$$\left|\frac{\partial}{\partial x}\phi\right| \leq \left|\frac{\partial}{\partial x}\Phi\right| \leq S_2. \tag{59}$$

Clearly, this result is consistent with the conclusion of Theorem 4.1.

6. Conclusions and discussions

This paper investigates the variational inequality initial-boundary value problem for a class of degenerate parabolic Kirchhoff operators, denoted as:

$$Lu = \partial_t u - (1 + \|u\|^p_{L^p(\Omega)}) \times \mathrm{div}(|\nabla u|^{p-1}\nabla u).$$

First, by utilizing the time smoothing operator $u_h(x,t) = \frac{1}{h}\int_0^t \exp\{\frac{s-t}{h}\}u(x,t)\mathrm{d}s$, the $C^\infty(\Omega)$-continuity of functions ψ, and the $C^\infty((0,T))$-continuity of function η, as well as the Hölder's inequality, the energy inequality for the solution u is obtained. Next, based on ψ and η, we construct the spatial truncation function ψ_n and the temporal truncation function η_n, and analyze the energy inequality for the spatial gradient ∇u of the solution. Finally, by utilizing the Caffarelli–Kohn–Nirenberg inequality (Lemma 2.3), the result of a convergent sequence of sets (Lemmas 2.4 and 2.5), and combining the results of the two energy inequalities, we obtain the boundedness of the infinity norm of the spatial gradient ∇u of the solution.

Throughout the completion of this paper, there are some important notes that readers should pay attention to:

1) The construction method adopted in analyzing the energy inequality of the solution's gradient in this paper is quite stringent, and it is no longer feasible to extend the parabolic operator to more complex scenarios, such as degenerate parabolic nonlinear operators, as shown in Eq (30).

2) When analyzing the energy inequality of the solution's gradient, it is required that p must be greater than or equal to 2; otherwise, Eqs (33), (35), and (36) are not valid.

Use of Generative-AI tools declaration

The author declares that no Artificial Intelligence (AI) tools were employed in the creation of this article.

Acknowledgments

The author sincerely thanks the editors and anonymous reviewers for their insightful comments and constructive suggestions, which greatly improved the quality of the paper. The study was supported by the Social Science Projects of Chongqing (Project No. 2017YBGL160).

Conflict of interest

The author declares no conflict of interest.

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