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Research article

Investigation of fractional-order pantograph delay differential equations using Sumudu decomposition method

Asrar Saleh Alsulami*, **Mariam Al-Mazmumy, Maryam Ahmed Alyami and Mona Alsulami**

Department of Mathematics and Statistics, Faculty of Science, University of Jeddah, Jeddah 23218, Saudi Arabia

* **Correspondence:** Email: 2200531@uj.edu.sa.

Abstract: This paper combines the Sumudu transform with the Adomian decomposition method to address Caputo-type fractional-order pantograph delay differential equations. It features numerical evaluations that confirm the effectiveness of the proposed methods. The study introduces a powerful computational technique for solving these equations, providing results that establish its efficiency and relevance through comparisons with existing methods. The findings underscore both the efficiency and accuracy of the proposed algorithm.

Keywords: Sumudu transform; Adomian decomposition method; fractional initial-value problems; fractional pantograph equations
Mathematics Subject Classification: 26A33, 39A60, 65Q20

1. Introduction

Fractional calculus is a branch of mathematics that generalizes integration and differentiation to arbitrary orders, leading to the study of fractional differential equations (FDEs), which have gained prominence in recent times. Its origins trace back to the early conjectures of Leibniz in 1695 and in 1730. The literature offers numerous definitions of fractional derivatives, such as Riemann-Liouville, Caputo, Grunwald-Letnikov, Weyl, Marchaud, and Prabhakar, among others (see [1–3] for a historical overview). Fractional calculus finds practical applications across various fields, including economics [4], chaotic systems [5], dynamics of viscoelastic materials [6], electrochemical processes [7], traffic models in fluid dynamics [8], and optics [9]. Fractional derivatives are particularly useful for describing the historical profiles and genetic characteristics of diverse processes. The significance of FDEs has prompted numerous investigations into the existence and uniqueness of valid solutions [2, 10]. Traditional mathematical modeling often assumes that a system's current state is sufficient for description; however, this assumption can be inadequate. In many cases, especially for dynamical systems exhibiting delays common in fields like economics, control systems, thermoelasticity, hydraulic networks, and biology, it becomes necessary to incorporate past states for its accurate modeling. These time-delay systems can be represented by delay differential equations (DDEs), a specific class of differential equations that relate the derivative of a function at a given time to its values at earlier times. DDEs encompass hereditary systems, systems with dead time, and equations with different arguments. They are particularly useful for modeling time-dependent processes, as they integrate past information into ordinary differential equations. Fractional delay differential equations (FDDEs), which incorporate delay terms within the framework of fractional calculus, offer greater flexibility in modeling than standard DDEs. Recently, FDDEs have attracted significant attention from researchers, as even minor delays can substantially impact system behavior. These equations find applications in diverse scientific fields, including control theory, chemistry, economics, finance, and vibration theory [11, 12]. For mathematical insights into FDDEs, we recommend [13] for information on existence and uniqueness and [14, 15] for findings related to the stability of solutions.

Pantograph equations are a specific type of DDEs that incorporate proportional delay terms and are widely used to model various real-life phenomena. The term "pantograph" originates from a device first used in constructing electric locomotives in 1851. The mathematical modeling of this device was established in 1971 [16]. These equations frequently arise in diverse scientific fields, including population dynamics, electrodynamics, dynamical systems, number theory, and quantum mechanics. Both analytical and numerical techniques have been employed historically and currently to derive closed-form and approximate solutions for fractional pantograph equations. For instance, multi-pantograph systems were addressed using the spectral Tau method in [17], while the Genocchi operational matrix in the collocation method was utilized in [18]. Additionally, the classical operational matrix method [19] and Montez-Legendre polynomials [20] have been applied to computationally tackle pantograph equations with fractional-order derivatives.

In the mid-1980s, George Adomian Introduced a valuable technique known as the Adomian decomposition method (ADM) [21], which effectively provides both closed-form and approximate solutions for linear and nonlinear functional equations. The ADM has gained increasing attention for its convergence properties and the stability of its solutions. Numerous studies have explored these aspects, emphasizing the method's robustness without reiterating the same findings (see [22–25]). The ADM has become widely recognized as a powerful tool for solving various FDEs. Applications include the analysis of fractional Bernoulli's equations [26] and the study of heat transfer processes with fractional components [27]. To enhance its accuracy and efficiency, the ADM has undergone several modifications designed to enhance convergence speed and reduce computational time. These advancements have led to significant progress, showing faster convergence of series solutions compared to the standard ADM. The modified ADM has proven to be computationally efficient across various models, making it invaluable for researchers in applied science. Recently, Masood et al. [28] proposed effective modifications to the ADM in addressing initial-boundary value problems for diffusion equations with fractional-order derivatives. Additionally, another efficient modification was applied by [29] to solve nonlinear FDEs. Furthermore, researchers have integrated the ADM with various integral transformations, such as Laplace, Sumudu, Natural, and Elzaki, which have proven to be robust for solving different FDE models [30–32].

In this context, Watugala [33] introduced the Sumudu integral transform as a modification of the well-known Laplace transform. This new approach offers several motivational advantages, including:

- It preserves units, facilitating problem-solving without the need for frequency domain conversion, which is beneficial in physical sciences where dimensions are crucial.
- The Sumudu integral transform is a linear operator, and conserves linear functions; this means that the units and dimensions remain unchanged.
- The Sumudu transform (in contrast with the Laplace transform) accurately handles initial conditions with singularities in differential equations, typical in practical engineering models.
- Calculating the inverse using the Sumudu transform avoids the complexities of contour integration, offering a simpler solution approach.

Furthermore, the authors in [34, 35] introduced several fundamental properties of the Sumudu transformation, which are instrumental in constructing and solving mathematical models. Additionally, the authors in [36, 37] discussed the combination of the Sumudu transform and the ADM to tackle various models of FDEs.

However, this study aims to utilize the coupling of the ADM and the Sumudu transform, referred to as the Sumudu decomposition method (SDM), to find approximate closed-form solutions for FPDDEs. The proposed method effectively addresses the complexity of these problems while maintaining ease of use. Error comparisons demonstrate that the devised SDM is exceptionally precise.

The structure of the manuscript is as follows: Section 2 provides definitions related to fractional calculus and the Sumudu transform. Section 3 discusses the solution method, while Section 4 applies the SDM to various numerical problems, determining approximate solutions for different FPDDEs. Finally, Section 5 presents the conclusions of the study.

2. Main concepts and theorems

This section presents key definitions and properties of fractional calculus. Additionally, it discusses fundamental features of the Sumudu transform (ST).

2.1. Some basic definitions and properties of the fractional calculus

Definition 2.1. *Let $y(x) \in C([a,b])$ and $a < x < b$. The Riemann-Liouville fractional integral operator of the fractional-order $\alpha > 0$ is given by [1, 2, 10]*

$$I_a^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_a^x y(t)(x-t)^{\alpha-1} dt. \tag{2.1}$$

For $\alpha = 0$, this reduces to the identity operator, i.e., $I_a^0 y(x) = y(x)$. Moreover, when $\alpha \in \mathbb{N}$, $I_a^\alpha y(x)$ coincides with the classical integral operator.

Definition 2.2. *Let $y(x) \in C([a,b])$. The Caputo fractional derivative of the fractional-order $\alpha > 0$ is defined as [1, 2, 10]*

$$D_*^\alpha y(x) = \begin{cases} I^{m-\alpha} y^{(m)}(x), & m-1 < \alpha < m, \\ \frac{d^m}{dx^m} y(x), & \alpha = m, \end{cases} \tag{2.2}$$

where $m = \lceil \alpha \rceil$ and $m \in \mathbb{N}$.

Additionally, several crucial properties of the fractional differential and integral operators based on the previously mentioned definitions are outlined as follows [1, 2, 10]:

(1)
$$I_a^{\alpha} I_a^{\beta} y(x) = I_a^{\alpha+\beta} y(x), \quad \alpha, \beta \geq 0; \tag{2.3}$$

(2)
$$I_a^{\alpha} D_*^{\alpha} y(x) = y(x) - \sum_{k=0}^{m-1} y^{(k)}(0^+) \frac{x^k}{\Gamma(k+1)}, \text{ where } m-1 < \alpha \leq m; \tag{2.4}$$

(3)
$$I_a^{\alpha} x^n = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} x^{n+\alpha}, \text{ where } x > 0,\ m-1 < \alpha \leq m,\ n > -1; \tag{2.5}$$

(4)
$$D_*^{\alpha} C = 0, \text{ where } C \text{ is a real constant}; \tag{2.6}$$

(5)
$$D_*^{\alpha} x^n = \begin{cases} 0, & n < m-1, \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & n \geq m-1. \end{cases} \tag{2.7}$$

Moreover, since this study focuses on solving Caputo-type FPDDEs, the Caputo fractional derivative is chosen. This derivative allows for the specification of additional initial conditions, ensuring a unique and precise solution for the FDEs. It is widely recognized for its capacity to define initial conditions based on the integer-order derivatives of the functions involved in the models under consideration. Additionally, it is worth noting that numerous studies have thoroughly explored the geometric interpretation of fractional derivatives [38].

2.2. Some basic definitions and properties of the Sumudu transform

Definition 2.3. *The ST over the following set [33, 34]:*

$$A = \{y(x) \mid \exists\, M, \nu_1, \nu_2 > 0,\ |y(x)| < Me^{|x|/\nu_i},\ if\ \ x \in (-1)^i \times [0, \infty)\}, \tag{2.8}$$

of functions is defined by

$$\mathbb{S}[y(x)] = \int_0^{\infty} y(ux) e^{-x} dx = G(u), \quad u \in (-\nu_1, \nu_2), \tag{2.9}$$

where u is a ST parameter, real or complex, and independent of x. In addition, the formula for inverting the ST is expressed as follows:

$$\mathbb{S}^{-1}[G(u)] = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} \frac{e^{ux}}{u} G\left(\frac{1}{u}\right) du = y(x). \tag{2.10}$$

Definition 2.4. *The ST of the Caputo derivative is expressed as [33, 34]*

$$\mathbb{S}[D^{\alpha} y(x)] = u^{-\alpha} \mathbb{S}[y(x)] - \sum_{k=0}^{m-1} u^{-\alpha+k} y^{(k)}(0), \ \ m-1 < \alpha \leq m. \tag{2.11}$$

Furthermore, to illustrate how the adopted integral transform operates, Table 1 presents the Sumudu transform of several elementary functions [33, 34]. Additionally, the authors in [34] demonstrate the equivalence of the Laplace and Sumudu transformations. However, one may prove more convenient for computations than the other, particularly when addressing FDEs. Consequently, this study selects the Sumudu transform for solving the FPDDEs.

Table 1. ST of some functions.

$y(x)$	$\mathbb{S}\{y(x)\} = Y(u)$
1	1
e^{ax}	$\frac{1}{1-au}$
$\frac{x^{n-1}}{\Gamma(n)}$	u^{n-1} for $n > 0$
$\cos ax$	$\frac{1}{1+a^2u^2}$
$\sin ax$	$\frac{au}{1+a^2u^2}$

3. Treatment of fractional pantograph delay differential equations via Sumudu decomposition method

To give the outline of the deployed SDM on the class of FPDDEs, the current section makes consideration of the generalized initial-value problem for FDDE with delay terms q_1 and q_2 as follows:

$$D_*^\alpha y(x) = F(x^\alpha, y(x), y(q_1 x), y(q_2 x)) + g(x), \tag{3.1}$$

$$y^i(0) = y_i, \quad i = 1, \ldots, m-1, \tag{3.2}$$

where $x \in I = [0, T]$, $m - 1 < \alpha \leq m$, $0 < q_1, q_2 < 1$, F is an arbitrary continuous nonlinear function, $g(x)$ is the given source function, $y_i \in \mathbb{R}$, while D_*^α is the Caputo fractional derivative of order α.

To solve the problem expressed in (3.1) and (3.2) via the deployed method, SDM starts by operating the ST on both sides of the governing equation to obtain the following:

$$\mathbb{S}[D_*^\alpha y(x)] = \mathbb{S}[F(x^\alpha, y(x), y(q_1 x), y(q_2 x))] + \mathbb{S}[g(x)]. \tag{3.3}$$

Further, on using (2.11) together with the prescribed initial conditions, one obtains

$$u^{-\alpha}\mathbb{S}[y(x)] - \sum_{i=1}^{m-1} u^{i-\alpha} y^{(i)}(0) = \mathbb{S}[F(x^\alpha, y(x), y(q_1 x), y(q_2 x))] + \mathbb{S}[g(x)], \tag{3.4}$$

or alternatively

$$\mathbb{S}[y(x)] - \sum_{i=1}^{m-1} u^i y_i = \frac{1}{u^{-\alpha}}\mathbb{S}[F(x^\alpha, y(x), y(q_1 x), y(q_2 x))] + \frac{1}{u^{-\alpha}}\mathbb{S}[g(x)]. \tag{3.5}$$

Next, upon applying the inverse ST on both sides of the latter equation, one then obtains

$$y(x) - \mathbb{S}^{-1}\Big[\sum_{i=1}^{m-1} u^i y_i\Big] = \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[F(x^\alpha, y(x), y(q_1 x), y(q_2 x))\Big]\Big] + \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[g(x)\Big]\Big]. \tag{3.6}$$

Further, the standard ADM procedure proceeds by expressing the solution $y(x)$ by the following infinite series:

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{3.7}$$

while the nonlinear term $F(x^\alpha, y(x), y(q_1 x), y(q_2 x))$ is expressed through the summation series of Adomian polynomials A_n as follows:

$$F(x^\alpha, y(x), y(q_1 x), y(q_2 x)) = \sum_{n=0}^{\infty} A_n, \tag{3.8}$$

where A_n are the Adomian polynomials, unequivocally expressed as follows:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\Big[N(\sum_{i=0}^{\infty} \lambda^i y_i)\Big]_{\lambda=0}, \quad n = 0, 1, 2, \ldots \tag{3.9}$$

Therefore, with the above-decomposed series, Eq (3.6) is thus re-expressed as follows:

$$\sum_{n=0}^{\infty} y_n(x) = \mathbb{S}^{-1}\Big[\sum_{i=1}^{m-1} u^i y_i\Big] + \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[\sum_{n=0}^{\infty} A_n\Big]\Big] + \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[g(x)\Big]\Big], \tag{3.10}$$

upon which the formal recursive relation is acquired from the standard ADM procedure as follows:

$$\begin{cases} y_0(x) = \mathbb{S}^{-1}\Big[\sum_{i=1}^{m-1} u^i y_i\Big] + \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}[g(x)]\Big] = p(x), \\ y_n(x) = \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}[A_{n-1}]\Big], \ n \geq 1. \end{cases} \tag{3.11}$$

Notably, it is crucial to note that the standard ADM involves defining the zeroth component $y_0(x)$ based on the function $p(x)$ derived from the source term and the prescribed initial conditions. However, in this context, Wazwaz [39] proposed an extension of the standard ADM by further decomposing the zeroth component $y_0(x)$ into two components as follows:

$$y_0(x) = p_0(x) + p_1(x),$$

where the first part is associated with the initial term, and the second part is attached to the subsequent iterate. Thus, under this assumption, the new modified recursive algorithm from (3.11) takes the following form:

$$\begin{cases} y_0(x) = p_0(x), \\ y_1(x) = \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}[A_0]\Big] + p_1(x), \\ y_n(x) = \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}[A_{n-1}]\Big], \ n \geq 2. \end{cases} \tag{3.12}$$

Certainly, the closed-form solution determined in Eq (3.12) (of course after taking the net sum of the components, $y(x) = \sum_{n=0}^{\infty} y_n(x)$) converges more rapidly than the standard ADM, as shown in [39]. In addition, the modified scheme by Wazwaz heavily relies on the selection of the right $p_0(x)$ and $p_1(x)$ functions; this dependency will be demonstrated in the next application section. In Algorithm 1, we systematically outline our method for solving FPDDSs.

Algorithm 1. Solution procedure for FPDDEs using SDM

Input: Define the FPDDE with given fractional derivative, nonlinear terms, delay terms, and initial conditions.
Step 1: Apply the Sumudu transform (ST) to the given equation, incorporating the initial conditions.
Step 2: Use the inverse ST to reformulate the problem, expressing the solution in terms of a series and Adomian polynomials.
Step 3: Decompose the solution and nonlinear terms into infinite series. Represent the nonlinear terms using Adomian polynomials.
Step 4: Derive recursive relations for the solution components using inverse ST.
Step 5: For Wazwaz's modified approach, decompose the initial component into two parts for improved convergence.
Step 6: Iteratively compute the solution components until convergence is achieved.
Output: Construct the approximate solution as the sum of all computed components.

4. Applications

This section presents a numerical demonstration of the derived Sumudu decomposition method (SDM) on several initial-value problems (IVPs) for fractional pantograph delay differential equations (FPDDEs). Additionally, graphical illustrations will be provided to compare the obtained approximate solutions with available exact solutions. Furthermore, comparison tables will be included to evaluate the effectiveness of the deployed SDM against other methods found in the literature. All calculations are performed, and the plots are generated by using Maple 22 with 8 digits precision.

Example 4.1. Consider the IVP for FPDDE as follows [40–42]:

$$\begin{cases} D_*^{1/2}y(x) = -y(x) + y(\frac{x}{2}) + \frac{7}{8}x^3 + \frac{16}{5\Gamma(\frac{1}{2})}x^{\frac{5}{2}}, \\ y(0) = 0. \end{cases} \tag{4.1}$$

To solve (4.1) by SDM, we apply ST on both sides of the equation, which when using (2.11) gives

$$\mathbb{S}[y(x)] = \frac{1}{u^{-1/2}}\mathbb{S}[-y(x)] + \frac{1}{u^{-1/2}}\mathbb{S}\Big[y(\frac{x}{2})\Big] + \frac{1}{u^{-1/2}}\mathbb{S}\Big[\frac{7}{8}x^3 + \frac{16}{5\Gamma(\frac{1}{2})}x^{\frac{5}{2}}\Big]. \tag{4.2}$$

Now, applying the inverse ST on both sides of the above equation, one gets

$$y(x) = \mathbb{S}^{-1}\Big[\frac{1}{u^{-1/2}}\mathbb{S}[-y(x)]\Big] + \mathbb{S}^{-1}\Big[\frac{1}{u^{-1/2}}\mathbb{S}\Big[y(\frac{x}{2})\Big]\Big] + \mathbb{S}^{-1}\Big[\frac{1}{u^{-1/2}}\mathbb{S}\Big[\frac{7}{8}x^3 + \frac{16}{5\Gamma(\frac{1}{2})}x^{\frac{5}{2}}\Big]\Big], \tag{4.3}$$

or equally expressed after deploying the standard ADM procedure on the latter equation as follows:

$$\sum_{n=0}^{\infty} y_n(x) = \mathbb{S}^{-1}\Big[\frac{1}{u^{-1/2}}\mathbb{S}\Big[-\sum_{n=0}^{\infty} y_n(x)\Big]\Big] + \mathbb{S}^{-1}\Big[\frac{1}{u^{-1/2}}\mathbb{S}\Big[\sum_{n=0}^{\infty} y_n(\frac{x}{2})\Big]\Big] + \mathbb{S}^{-1}\Big[\frac{1}{u^{-1/2}}\mathbb{S}\Big[\frac{7}{8}x^3 + \frac{16}{5\Gamma(\frac{1}{2})}x^{\frac{5}{2}}\Big]\Big]. \tag{4.4}$$

Moreover, the formal recursive relationship for the governing fractional IVP is obtained as follows:

$$\begin{cases} y_0(x) = \mathbb{S}^{-1}\left[\frac{1}{u^{-1/2}}\mathbb{S}\left[\frac{7}{8}x^3 + \frac{16}{5\Gamma(\frac{1}{2})}x^{\frac{5}{2}}\right]\right] = x^3 + \frac{4}{5\sqrt{\pi}}x^{\frac{7}{2}}, \\ y_n(x) = \mathbb{S}^{-1}\left[\frac{1}{u^{-1/2}}\mathbb{S}\left[-y_{n-1}(x)\right]\right] + \mathbb{S}^{-1}\left[\frac{1}{u^{-1/2}}\mathbb{S}\left[y_{n-1}(\frac{x}{2})\right]\right], \ n \geq 1. \end{cases} \tag{4.5}$$

In addition, upon using the modified recursive algorithm by Wazwaz in (3.12), the governing model via the modified SDM yields the following recursive algorithm as follows:

$$\begin{cases} y_0(x) = x^3, \\ y_1(x) = \mathbb{S}^{-1}\left[\frac{1}{u^{-1/2}}\mathbb{S}\left[-x^3\right]\right] + \mathbb{S}^{-1}\left[\frac{1}{u^{-1/2}}\mathbb{S}\left[\frac{x^3}{8}\right]\right] + \frac{4}{5\sqrt{\pi}}x^{\frac{7}{2}} \\ \qquad = -\frac{4}{5\sqrt{\pi}}x^{\frac{7}{2}} + \frac{4}{5\sqrt{\pi}}x^{\frac{7}{2}} \\ \qquad = 0, \\ y_n(x) = 0, \ \ n \geq 2, \end{cases} \tag{4.6}$$

upon which the exact solution is arrived at after taking the net sum of the above solution components as follows:

$$y(x) = x^3. \tag{4.7}$$

In this context, the authors in [40–42] employed various methods to tackle the same model in (4.1), using the regenerative kernel method (RKM), the piecewise Picard iteration method (PPIM), and the generalized Legendre polynomial configuration method (GLPCM), respectively. Each of these authors obtained approximate solutions by calculating nine terms, while our algorithm achieves an exact solution using only two terms. This advantage leads to more effective convergence to the exact solution compared to numerical methods, significantly reducing computational complexity. Additionally, the study compares the resulting errors from these methods with those produced by the devised SDM, as detailed in Table 2.

Table 2. Comparison of absolute errors at various x_i values for the fractional IVP (4.1).

x	**RKM** [40]	**PPIM** [41]	**GLPCM** [42]	**Present method (SDM)**
0.1	1.475×10^{-7}	1.303×10^{-13}	6.328×10^{-15}	0
0.2	4.492×10^{-6}	3.731×10^{-11}	6.212×10^{-15}	0
0.3	2.361×10^{-5}	1.171×10^{-9}	5.648×10^{-15}	0
0.4	8.239×10^{-5}	1.351×10^{-8}	2.914×10^{-15}	0
0.5	2.107×10^{-4}	2.107×10^{-4}	5.065×10^{-15}	0
0.6	4.403×10^{-4}	4.403×10^{-4}	4.219×10^{-15}	0
0.7	8.037×10^{-4}	8.037×10^{-4}	4.163×10^{-15}	0
0.8	1.333×10^{-3}	1.333×10^{-3}	3.220×10^{-15}	0
0.9	2.064×10^{-3}	2.064×10^{-3}	2.665×10^{-15}	0
1	1.303×10^{-3}	1.303×10^{-3}	2.331×10^{-15}	0

Example 4.2. Consider the nonlinear IVP for FPDDE as follows [42–45]:

$$\begin{cases} D_*^{\alpha}y(x) = 1 - 2y^2\left(\frac{x}{2}\right), \ \ 0 < \alpha \leq 1, \\ y(0) = 0. \end{cases} \tag{4.8}$$

To solve (4.8) by SDM, we apply ST on both sides of the equation to accordingly obtain

$$\mathbb{S}[y(x)] = \frac{1}{u^{-\alpha}}\mathbb{S}[1] - \frac{1}{u^{-\alpha}}\mathbb{S}\Big[2y^2\Big(\frac{x}{2}\Big)\Big]. \tag{4.9}$$

Next, upon applying the inverse ST on the latter equation, one obtains

$$y(x) = \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}[1]\Big] - \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[2y^2\Big(\frac{x}{2}\Big)\Big]\Big], \tag{4.10}$$

such that when the ADM is deployed reveals

$$\sum_{n=0}^{\infty} y_n(x) = \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}[1]\Big] - \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[2\sum_{n=0}^{\infty} A_n\Big(\frac{x}{2}\Big)\Big]\Big], \tag{4.11}$$

where A_n are the Adomian polynomials corresponding to the nonlinear terms $y^2(\frac{x}{2})$ defined as follows:

$$\begin{aligned} A_0 &= y_0\Big(\frac{x}{2}\Big)^2, \\ A_1 &= 2y_0\Big(\frac{x}{2}\Big)y_1\Big(\frac{x}{2}\Big), \\ A_2 &= y_1\Big(\frac{x}{2}\Big)^2 + 2y_0\Big(\frac{x}{2}\Big)y_2\Big(\frac{x}{2}\Big), \\ &\vdots \end{aligned} \tag{4.12}$$

Moreover, the formal recursive relationship for the governing fractional IVP is obtained as follows:

$$\begin{cases} y_0(x) = \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}[1]\Big], \\ y_n(x) = -\mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[2A_{n-1}\Big(\frac{x}{2}\Big)\Big]\Big], \ n \geq 1. \end{cases} \tag{4.13}$$

Accordingly, the recursive scheme is expressed more plainly as follows:

$$\begin{aligned} y_0(x) &= \frac{1}{\Gamma(\alpha+1)}x^{\alpha}, \\ y_1(x) &= -\mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[2A_0\Big(\frac{x}{2}\Big)\Big]\Big] \\ &= -\mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[2\frac{x^{2\alpha}}{4^{\alpha}\Gamma^2(\alpha+1)}\Big]\Big] \\ &= -\frac{2\Gamma(2\alpha+1)}{4^{\alpha}\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}x^{3\alpha}, \\ y_2(x) &= \frac{2^3\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{4^{3\alpha}\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}x^{5\alpha}, \\ &\vdots \end{aligned} \tag{4.14}$$

Hence, on taking the net sum of the latter solution components, one acquires

$$y(x) = \frac{x^{\alpha}}{\Gamma(\alpha+1)} - \frac{2\Gamma(2\alpha+1)}{4^{\alpha}\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}x^{3\alpha} + \frac{2^3\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{4^{3\alpha}\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}x^{5\alpha} + \ldots \tag{4.15}$$

Remarkably, when the fractional-order α takes unity in the above series solution, that is, $\alpha = 1$, one obtains the following compacted closed-form solution:

$$y(x) = x - \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x). \tag{4.16}$$

Similarly, the authors in [42–45] addressed the governing model in (4.8) using the Chebyshev wavelet method (CWM), the Jacobi spectral configuration method (JSCM), the rescaled spectral configuration method (RSCM), and the generalized Legendre polynomial configuration method (GLPCM), respectively. These authors obtained approximate solutions by calculating nine terms, while our algorithm provides an exact solution for the model at $\alpha = 1$. For error analysis, refer to Table 3, which compares the proposed SDM with other competing approaches. Additionally, Figure 1 presents a graphical depiction of the approximate SDM solution and the absolute errors for the governing nonlinear initial-value problem (IVP) for FPDDE (4.8).

Table 3. Comparison of absolute errors at various x_i values for the fractional IVP (4.8).

x	**JSCM [43]**	**CWM [44]**	**RSCM [45]**	**GLPCM** [42]	**Present method (SDM)**
0.125	2.05×10^{-12}	1.71×10^{-12}	1.81×10^{-13}	1.61×10^{-13}	0
0.250	1.61×10^{-12}	2.47×10^{-12}	1.34×10^{-13}	3.14×10^{-14}	0
0.375	3.48×10^{-11}	9.36×10^{-11}	1.50×10^{-13}	5.29×10^{-14}	0
0.500	5.31×10^{-11}	1.78×10^{-11}	1.61×10^{-13}	2.38×10^{-14}	0
0.625	2.07×10^{-11}	1.88×10^{-11}	1.69×10^{-13}	5.35×10^{-14}	0
0.750	2.47×10^{-11}	3.04×10^{-12}	1.30×10^{-13}	6.05×10^{-14}	0
0.875	1.59×10^{-11}	1.64×10^{-12}	1.38×10^{-13}	1.61×10^{-14}	0
1	3.85×10^{-10}	4.51×10^{-10}	8.98×10^{-13}	3.11×10^{-15}	0

Example 4.3. Consider the nonhomogeneous nonlinear IVP for FPDDE as follows [42, 43, 46]:

$$\begin{cases} D_*^{1/2} y(x) = y(\frac{x}{3}) + 3y^2(x) + g(x), \\ y(0) = 0, \end{cases} \tag{4.17}$$

where

$$g(x) = \frac{8}{3\sqrt{\pi}} x^{\frac{3}{2}} - \frac{2}{\sqrt{\pi}} x^{\frac{1}{2}} - x^4 + 2x^3 - \frac{10}{9}x^2 + \frac{1}{3}x.$$

Thus, without much delay, the formal recursive relationship for the governing fractional model is obtained as follows:

$$\begin{cases} y_0(x) = \mathbb{S}^{-1}\left[\frac{1}{u^{-1/2}}\mathbb{S}\left[g(x)\right]\right] = x^2 - x - \frac{256}{315\sqrt{\pi}}x^{\frac{9}{2}} + \frac{64}{35\sqrt{\pi}}x^{\frac{7}{2}} - \frac{32}{27\sqrt{\pi}}x^{\frac{5}{2}} + \frac{4}{9\sqrt{\pi}}x^{\frac{3}{2}} = p(x), \\ y_n(x) = \mathbb{S}^{-1}\left[\frac{1}{u^{-1/2}}\mathbb{S}\left[y_{n-1}(\frac{1}{3}x)\right]\right] + \mathbb{S}^{-1}\left[\frac{1}{u^{-1/2}}\mathbb{S}\left[3A_{n-1}\right]\right], \ n \geq 1, \end{cases} \tag{4.18}$$

where the Adomian polynomials A_n in the above scheme for the nonlinear term $y^2(x)$, take the following forms:

$$\begin{aligned} A_0 &= y_0(x)^2\,, \\ A_1 &= 2y_0(x)\,y_1(x)\,, \\ A_2 &= y_1(x)^2 + 2y_0(x)\,y_2(x)\,, \\ &\vdots \end{aligned} \tag{4.19}$$

Accordingly, when considering the Wazwaz's modification for ADM (3.12) by choosing the functions $p_0(x)$ and $p_1(x)$ as follows:

$$\begin{aligned} p_0(x) &= x^2 - x, \\ p_1(x) &= -\frac{256}{315\sqrt{\pi}}x^{\frac{9}{2}} + \frac{64}{35\sqrt{\pi}}x^{\frac{7}{2}} - \frac{32}{27\sqrt{\pi}}x^{\frac{5}{2}} + \frac{4}{9\sqrt{\pi}}x^{\frac{3}{2}}, \end{aligned} \tag{4.20}$$

the modified SDM algorithm then yields the following scheme:

$$\begin{cases} y_0(x) = p_0(x) = x^2 - x, \\ y_1(x) = \mathbb{S}^{-1}\left[\frac{1}{u^{-1/2}}\mathbb{S}\left[y_0(x)\right]\right] + \mathbb{S}^{-1}\left[\frac{1}{u^{-1/2}}\mathbb{S}\left[3A_0(x)\right]\right] + p_1(x) \\ \quad\quad = \mathbb{S}^{-1}\left[\frac{1}{u^{-1/2}}\mathbb{S}\left[x^2 - x\right]\right] + \mathbb{S}^{-1}\left[\frac{1}{u^{-1/2}}\mathbb{S}\left[3(x^2 - x)^2\right]\right] - \frac{256}{315\sqrt{\pi}}x^{\frac{9}{2}} + \frac{64}{35\sqrt{\pi}}x^{\frac{7}{2}} - \frac{32}{27\sqrt{\pi}}x^{\frac{5}{2}} + \frac{4}{9\sqrt{\pi}}x^{\frac{3}{2}} \\ \quad\quad = \frac{256}{315\sqrt{\pi}}x^{\frac{9}{2}} - \frac{64}{35\sqrt{\pi}}x^{\frac{7}{2}} + \frac{32}{27\sqrt{\pi}}x^{\frac{5}{2}} - \frac{4}{9\sqrt{\pi}}x^{\frac{3}{2}} - \frac{256}{315\sqrt{\pi}}x^{\frac{9}{2}} + \frac{64}{35\sqrt{\pi}}x^{\frac{7}{2}} - \frac{32}{27\sqrt{\pi}}x^{\frac{5}{2}} + \frac{4}{9\sqrt{\pi}}x^{\frac{3}{2}} \\ \quad\quad = 0, \\ y_n(x) = 0,\ n \geq 2, \end{cases} \tag{4.21}$$

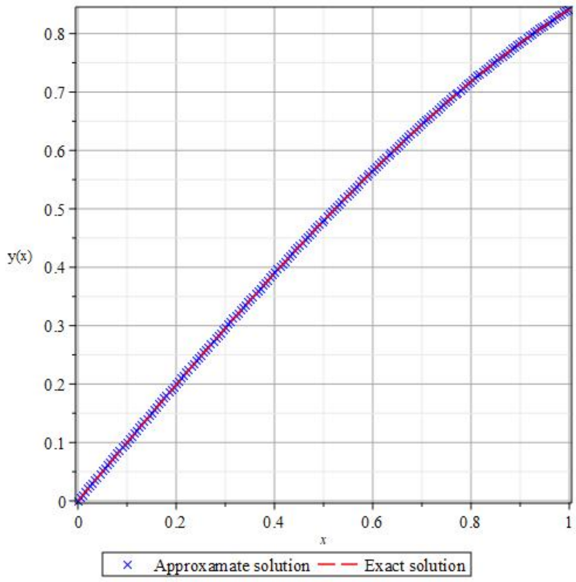
which then sums to yield the resulting exact solution as follows:

$$y(x) = x^2 - x. \tag{4.22}$$

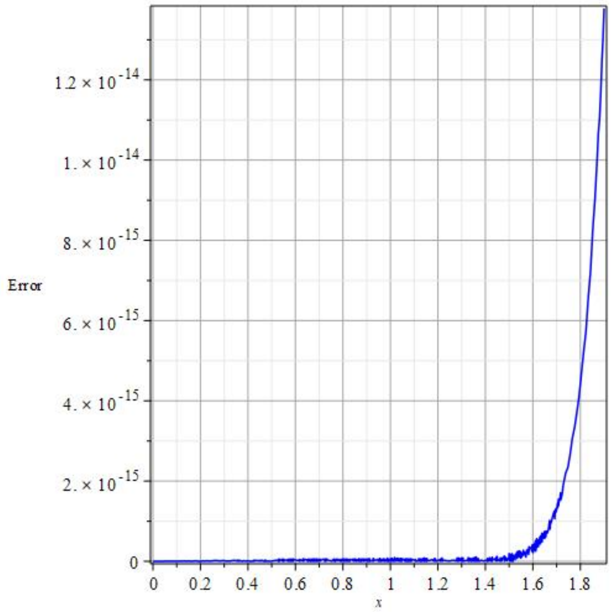
Consequently, the references [42, 43, 46] also addressed the governing model in (4.17) using the Jacobi spectral configuration method (JSCM), the Jacobi spectral Galerkin method (JSGM), and the generalized Legendre polynomial spectral method (GLPSM), respectively. These authors obtained approximate solutions by calculating varying numbers of terms, while our algorithm provides an exact solution using only two terms. This advantage leads to more effective convergence to the exact solution compared to numerical methods, significantly reducing computational complexity. For a comparative analysis of the aforementioned methods with respect to different error norms, see Table 4.

Table 4. Comparison of maximum errors for different methods for the fractional IVP (4.17).

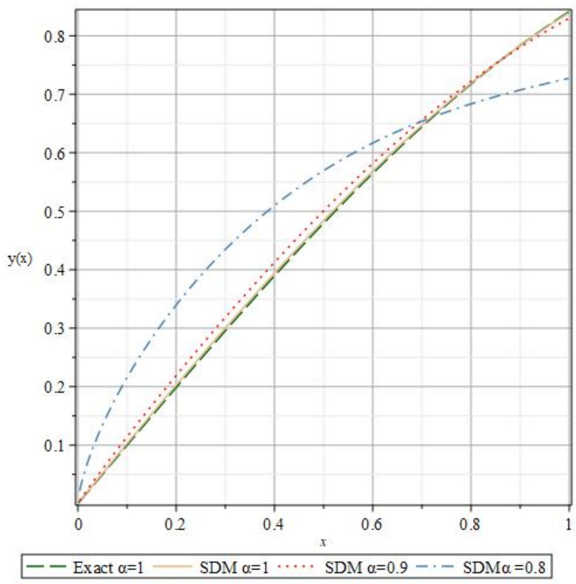
Maximum Error	$n = 6$	$n = 8$	$n = 10$
JSCM [43] (L^2)	3.6822×10^{-6}	1.20720×10^{-8}	1.3049×10^{-11}
JSCM [43] (L^∞)	1.0832×10^{-5}	2.7460×10^{-8}	3.9471×10^{-11}
JSGM [46] (L^2)	2.3632×10^{-6}	1.0032×10^{-8}	1.1006×10^{-11}
JSGM [46] (L^∞)	1.2851×10^{-5}	1.4350×10^{-8}	2.9356×10^{-11}
GLPCM [42]	4.5103×10^{-16}	4.7184×10^{-16}	5.3429×10^{-16}



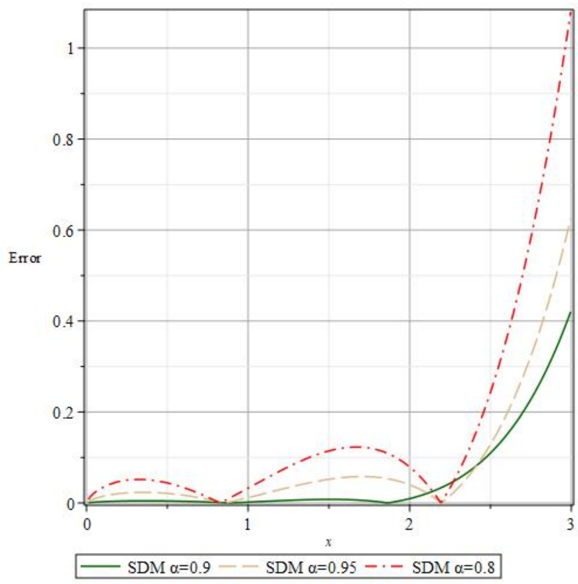
(a) Comparison of the approximate and exact solutions



(b) Absolute error curve for SDM solution, $\alpha = 1$



(c) SDM solutions for various values of α



(d) Absolute error curve for various values of α

Figure 1. SDM solutions and error curves for the fractional IVP (4.8).

Example 4.4. Consider the IVP for FPDDE of the following form:

$$\begin{cases} D_*^{\alpha} y(x) = 1 + 2y(\frac{x}{2}) - y(x), \\ y(0) = 0, \end{cases} \tag{4.23}$$

where the fractional-order α assumes a general value over the interval $0 < \alpha \leq 1$.

Subsequently, upon applying the ST on the fractional IVP in via (2.11), one obtains

$$\mathbb{S}[y(x)] = \frac{1}{u^{-\alpha}} + \frac{1}{u^{-\alpha}}\mathbb{S}\Big[2y(\frac{x}{2})\Big] - \frac{1}{u^{-\alpha}}\mathbb{S}\Big[y(x)\Big]. \tag{4.24}$$

Now, on applying the inverse ST on the latter equation, one obtains

$$y(x) = \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\Big] + \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[2y(\frac{x}{2})\Big]\Big] - \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[y(x)\Big]\Big], \tag{4.25}$$

such that when the ADM is deployed, one obtains

$$\sum_{n=0}^{\infty} y_n(x) = \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\Big] + \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[2\sum_{n=0}^{\infty} y_n(\frac{x}{2})\Big]\Big] - \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[\sum_{n=0}^{\infty} y_n(x)\Big]\Big]. \tag{4.26}$$

Accordingly, the formal recursive relation for the governing fractional IVP is obtained as follows:

$$\begin{cases} y_0(x) = \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\Big], \\ y_n(x) = \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[2y_{n-1}(\frac{x}{2})\Big]\Big] - \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[y_{n-1}(x)\Big]\Big], \ n \geq 1, \end{cases} \tag{4.27}$$

or more openly from the above recursive relation as follows:

$$\begin{aligned} y_0(x) &= \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\Big] = \frac{x^{\alpha}}{\Gamma(\alpha+1)}, \\ y_1(x) &= \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[2y_0(\frac{x}{2})\Big]\Big] - \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[y_0(x)\Big]\Big] = \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\Big(\frac{1}{2^{\alpha-1}} - 1\Big), \\ y_2(x) &= \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[2y_1(\frac{x}{2})\Big]\Big] - \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[y_1(x)\Big]\Big] \\ &= \frac{x^{3\alpha}}{\Gamma(3\alpha+1)}\Big(\frac{1}{2^{\alpha-1}} - 1\Big)\Big(\frac{1}{2^{\alpha-1}} - 1\Big), \\ y_3(x) &= \frac{x^{4\alpha}}{\Gamma(4\alpha+1)}\Big(\frac{1}{2^{\alpha-1}} - 1\Big)\Big(\frac{1}{2^{\alpha-1}} - 1\Big)\Big(\frac{1}{2^{\alpha-1}} - 1\Big), \\ &\ \vdots \end{aligned} \tag{4.28}$$

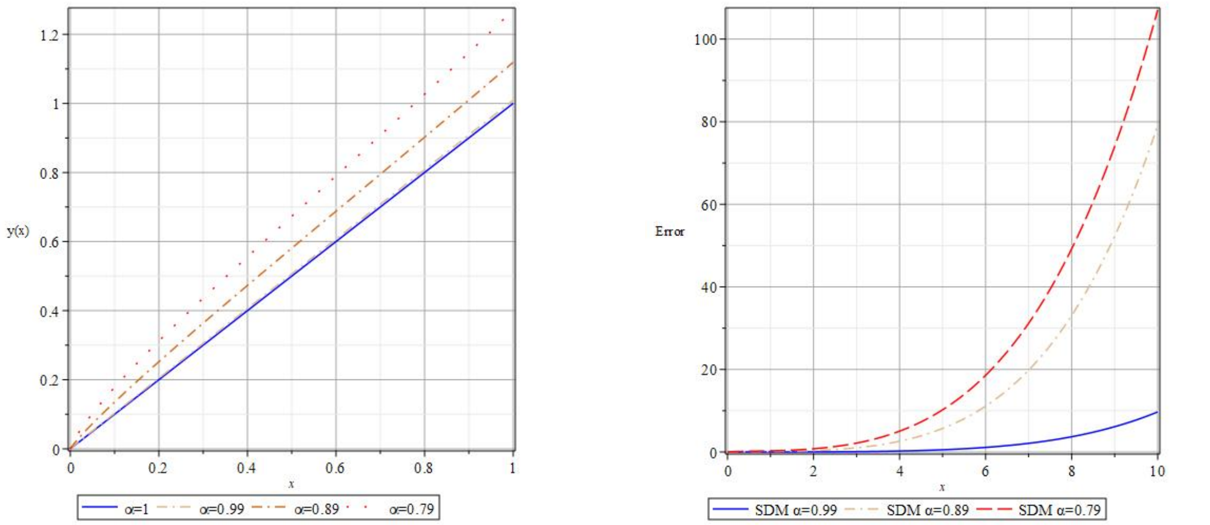
Hence, on taking the net sum of the above solution components, one obtains

$$y(x) = \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=1}^{\infty}\Big(\prod_{l=1}^{i}\frac{1}{2^{l\alpha-1}} - 1\Big)\frac{x^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)}. \tag{4.29}$$

Additionally, when the fractional-order α assumes full integer-order, that is, when $\alpha = 1$, the second term in the above series solution vanishes, which then leads to the acquisition of the exact solution of the integer-order model as follows:

$$y(x) = x. \tag{4.30}$$

Moreover, see Figure 2 for the pictorial representation of the approximate SDM solution, together with the resulting absolute error for various values of the fractional-order α for the governing IVP in (4.23).



(a) SDM solution for various values of $\alpha = 1, 0.99, 0.89, 0.79$, respectively.

(b) Absolute error for various values of $\alpha = 0.99, 0.89, 0.79$, respectively.

Figure 2. SDM solution and absolute error of the fractional IVP (4.23) for various values of α.

Example 4.5. Consider the inhomogeneous IVP for FPDDE [18, 47]

$$\begin{cases} D_*^{5/2} y(x) = -y(x) - y(x-0.5) + g(x), \quad x \in [0,1], \\ y(0) = y'(0) = y''(0) = 0, \end{cases} \tag{4.31}$$

where

$$g(x) = \frac{\Gamma(4)}{\Gamma(3/2)} x^{1/2} + x^3 + (x-0.5)^3.$$

Accordingly, upon applying the ST and its subsequent inverse transform on (4.31) as previously highlighted, one thus obtains

$$y(x) = \mathbb{S}^{-1}\left[\frac{1}{u^{-5/2}}\mathbb{S}[-y(x)]\right] - \mathbb{S}^{-1}\left[\frac{1}{u^{-5/2}}\mathbb{S}[y(x-0.5)]\right] + \mathbb{S}^{-1}\left[\frac{1}{u^{-5/2}}\mathbb{S}[g(x)]\right], \tag{4.32}$$

such that the standard ADM reveals the resulting recurrent scheme as follows:

$$\begin{cases} y_0(x) = \mathbb{S}^{-1}\left[\frac{1}{u^{-5/2}}\mathbb{S}[g(x)]\right] = p(x), \\ y_n(x) = -\mathbb{S}^{-1}\left[\frac{1}{u^{-5/2}}\mathbb{S}\left[y_{n-1}(x)\right]\right] - \mathbb{S}^{-1}\left[\frac{1}{u^{-5/2}}\mathbb{S}\left[y_{n-1}(x-0.5)\right]\right], \ n \geq 1. \end{cases} \tag{4.33}$$

In the same fashion, when considering the Wazwaz's modification for ADM (3.12) by choosing the functions $p_0(x)$ and $p_1(x)$ as follows:

$$\begin{aligned} p_0(x) &= x^3, \\ p_1(x) &= \frac{\Gamma(4)}{\Gamma(13/2)} x^{11/2} + \mathbb{S}^{-1}\left[u^{5/2}(u-0.5)^3\Gamma(4)\right], \end{aligned} \tag{4.34}$$

the modified SDM scheme is then expressed as follows:

$$\begin{aligned}
y_0(x) &= p_0(x) = x^3,\\
y_1(x) &= -\mathbb{S}^{-1}\left[\frac{1}{u^{-5/2}}\mathbb{S}\left[y_0(x)\right]\right] - \mathbb{S}^{-1}\left[\frac{1}{u^{-5/2}}\mathbb{S}\left[y_0(x-0.5)\right]\right] + p_1(x),\\
&= -\mathbb{S}^{-1}\left[\frac{1}{u^{-5/2}}\mathbb{S}\left[x^3\right]\right] - \mathbb{S}^{-1}\left[\frac{1}{u^{-5/2}}\mathbb{S}\left[(x-0.5)^3\right]\right] + \frac{\Gamma(4)}{\Gamma(13/2)}x^{11/2}\\
&\quad + \mathbb{S}^{-1}\left[u^{5/2}(u-0.5)^3\Gamma(4)\right],\\
&= -\frac{\Gamma(4)}{\Gamma(13/2)}x^{11/2} - \mathbb{S}^{-1}\left[u^{5/2}(u-0.5)^3\Gamma(4)\right] + \frac{\Gamma(4)}{\Gamma(13/2)}x^{11/2}\\
&\quad + \mathbb{S}^{-1}\left[u^{5/2}(u-0.5)^3\Gamma(4)\right],\\
&= 0,\\
y_n(x) &= 0,\ n \geq 2,
\end{aligned} \tag{4.35}$$

such that the closed-form solution of the governing IVP is attained by summing the above solution components as follows:

$$y(x) = x^3. \tag{4.36}$$

Moreover, upon simulating the derived SDM scheme for the examining model, together with some contending computational methods, Table 5 then provides the absolute error contrast between the deployed SDM (with only a few terms, $N = 2$), the results reported in [47], where $N = 22$ using different Laguerre's parameters β through the application of the Laguerre-Gauss collocation scheme (L-GCS), and the approximate solution in [18] using the operational matrix via the Genocchi polynomials collocation method (GPCM), where $N = 4$. Certainly, the results show that the proposed SDM method provides an almost exact solution over the contending approaches.

Table 5. Comparison of the absolute errors at various x_i values for the fractional IVP (4.31).

x	L-GCS [47] $N = 22$			GPCM [18]	Present method (SDM)
	$\beta = 2$	$\beta = 3$	$\beta = 5$	$N = 4$	$N = 2$
0.1	1.030×10^{-4}	1.019×10^{-5}	6.273×10^{-6}	6.17040×10^{-9}	0
0.2	6.510×10^{-4}	6.051×10^{-5}	3.892×10^{-5}	4.93630×10^{-8}	0
0.3	1.740×10^{-3}	1.495×10^{-4}	1.023×10^{-4}	1.66600×10^{-7}	0
0.4	3.283×10^{-3}	2.559×10^{-4}	1.901×10^{-4}	3.94910×10^{-7}	0
0.5	5.138×10^{-3}	3.546×10^{-4}	2.944×10^{-4}	7.71300×10^{-7}	0
0.6	7.175×10^{-3}	4.261×10^{-4}	4.088×10^{-4}	1.33280×10^{-6}	0
0.7	9.303×10^{-3}	4.592×10^{-4}	5.306×10^{-4}	2.11640×10^{-6}	0
0.8	1.147×10^{-2}	4.510×10^{-4}	6.597×10^{-4}	3.15920×10^{-6}	0
0.9	1.367×10^{-2}	4.055×10^{-4}	7.977×10^{-4}	4.49820×10^{-6}	0
1.0	1.589×10^{-2}	3.311×10^{-4}	9.468×10^{-4}	6.17040×10^{-6}	0

Example 4.6. Consider the coupled system of nonhomogeneous IVP for FPDDE as follows [48]:

$$\begin{cases} D_*^\alpha y_1(x) = e^{-x^\alpha} - e^{\frac{x^\alpha}{2}} + y_1(\frac{x}{2}) + y_1(x) - y_2(x), & y_1(0) = 1, \\ D_*^\alpha y_2(x) = e^{x^\alpha} + e^{\frac{x^\alpha}{2}} - y_2(\frac{x}{2}) - y_1(x) - y_2(x), & y_2(0) = 1. \end{cases} \tag{4.37}$$

Accordingly, the deployed SDM posses the recursive scheme for the governing coupled fractional model as follows:

$$\begin{cases} y_{1,0}(x) = 1 + \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}[e^{-x^\alpha} - e^{\frac{x^\alpha}{2}}]\Big], \\ y_{2,0}(x) = 1 + \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}[e^{x^\alpha} + e^{\frac{x^\alpha}{2}}]\Big], \end{cases} \tag{4.38}$$

and $\forall\ n \geq 1$ as follows:

$$\begin{cases} y_{1,n}(x) = \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}[y_{1,n-1}(\frac{x}{2}) + y_{1,n-1}(x) - y_{2,n-1}(x)]\Big], \\ y_{2,n}(x) = -\mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}[y_{2,n-1}(\frac{x}{2}) + y_{1,n-1}(x) + y_{2,n-1}(x)]\Big]. \end{cases} \tag{4.39}$$

Certainly, the current coupled system of fractional IVP extends the corresponding integer-order model examined in [49]. Thus, upon considering the integer-order version in [49], where the fractional-order takes unity, that is, when $\alpha = 1$, the model admits the following exact solution $y_1(x) = e^x$, $y_2(x) = e^{-x}$. What is more, upon simplifying the derived scheme further, one replaces the exponential terms in (4.38) with the first seven terms of their Taylor series expansions to eventually obtain as follows:

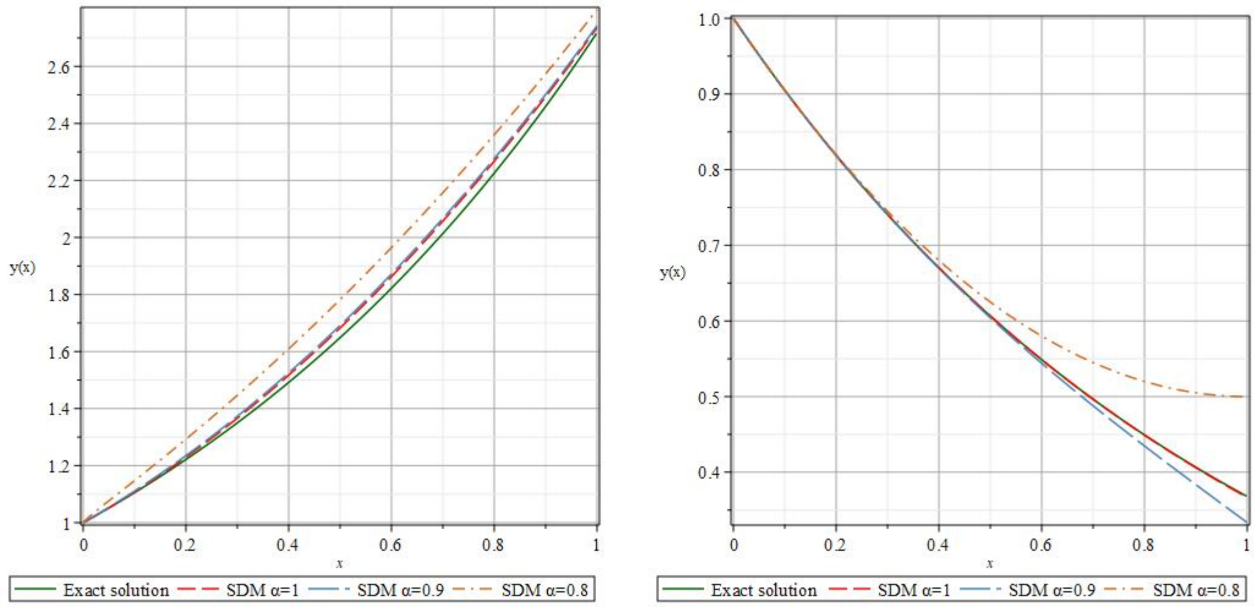
$$\begin{cases} y_{1,0}(x) = 1 + \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[\frac{3x^\alpha}{2} - \frac{3x^{2\alpha}}{8} + \frac{3x^{3\alpha}}{16} - \frac{5x^{4\alpha}}{128} + \frac{11x^{5\alpha}}{1280} - \frac{7x^{6\alpha}}{5120}\Big]\Big], \\ y_{2,0}(x) = 1 - \mathbb{S}^{-1}\Big[\frac{1}{u^{-\alpha}}\mathbb{S}\Big[2 + \frac{x^\alpha}{2} + \frac{5x^{2\alpha}}{8} + \frac{7x^{3\alpha}}{48} + \frac{17x^{4\alpha}}{384} + \frac{31x^{5\alpha}}{3840} + \frac{13x^{6\alpha}}{9216}\Big]\Big], \end{cases} \tag{4.40}$$

that is,

$$\begin{aligned} y_{1,0}(x) &= 1 + \frac{3x^{2\alpha}\Gamma(1+\alpha)}{2\Gamma(1+2\alpha)} - \frac{3x^{3\alpha}\Gamma(1+\alpha)}{8\Gamma(1+3\alpha)} + \frac{3x^{4\alpha}\Gamma(1+3\alpha)}{16\Gamma(1+4\alpha)} - \frac{5x^{4\alpha}\Gamma(1+\alpha)}{128\Gamma(1+128\alpha)} \\ &\quad + \frac{11x^{6\alpha}\Gamma(1+5\alpha)}{1280\Gamma(1+6\alpha)} - \frac{7x^{7\alpha}\Gamma(1+6\alpha)}{5120\Gamma(1+7\alpha)}, \\ y_{2,0}(x) &= 1 - \frac{2x^\alpha}{\Gamma(1+\alpha)} - \frac{x^{2\alpha}\Gamma(1+\alpha)}{2\Gamma(1+2\alpha)} - \frac{5x^{3\alpha}\Gamma(1+2\alpha)}{8\Gamma(1+3\alpha)} - \frac{7x^{4\alpha}\Gamma(1+3\alpha)}{48\Gamma(1+4\alpha)} \\ &\quad - \frac{17x^{5\alpha}\Gamma(1+4\alpha)}{384\Gamma(1+5\alpha)} - \frac{31x^{6\alpha}\Gamma(1+5\alpha)}{3840\Gamma(1+6\alpha)} - \frac{13x^{7\alpha}\Gamma(1+6\alpha)}{9216\Gamma(1+7\alpha)}, \\ y_{1,1}(x) &= \frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{2x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{2^\alpha x^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots - \frac{x^{8\alpha}\Gamma(1+6\alpha)}{23040\Gamma(1+8\alpha)} + \frac{x^{8\alpha}\Gamma(1+6\alpha)}{72^{7\alpha+10} \times 5\Gamma(1+8\alpha)}, \\ y_{2,1}(x) &= \frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{2^\alpha x^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{x^{2\alpha}\Gamma(1+\alpha)}{2\Gamma(1+2\alpha)} - \cdots - \frac{x^{8\alpha}\Gamma(1+6\alpha)}{360\Gamma(1+8\alpha)} - \frac{x^{8\alpha}\Gamma(1+6\alpha)}{132^{7\alpha+10} \times 9\Gamma(1+8\alpha)}, \\ &\ \ \vdots \end{aligned}$$

In addition, Figure 3 depicts the pictorial view of the obtained SDM solution for the fractional-order system (4.37). Furthermore, with the consideration of the corresponding integer-order case, that is, when $\alpha = 1$, the first 6 terms of the solution components by SDM sum to yield the following series solution:

$$\begin{cases} y_1(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots \approx e^x, \\ y_2(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \cdots \approx e^{-x}. \end{cases} \tag{4.41}$$



(a) Exact and SDM solutions of $y_1(x)$ for various values of α.

(b) Exact and SDM solutions of $y_2(x)$ for various values α.

Figure 3. Exact and SDM solutions of the coupled fractional IVP (4.37) for various values α.

5. Conclusions

The current manuscript introduces a novel computational approach for solving the class of FPDDEs by combining the Sumudu transform with the reliable ADM. This approach enables the derivation of approximate or exact solutions through straightforward iterative calculations that are easily implementable in computer programs. The numerical results confirm the precision of the proposed method. Furthermore, the study successfully achieves its goal by obtaining numerical solutions in the form of rapidly converging series with easily calculable components. To validate the accuracy and efficiency of the algorithms adopted via the SDM, we present several important and interesting problems, comparing the obtained solutions both numerically and graphically. The results demonstrate that our method is highly effective, straightforward, and yields approximate solutions that closely match the exact solutions. Additionally, the paper recommends applying this technique to more complex neutral pantograph equations involving fractional derivatives on the right-hand side, utilizing various fractional operators.

Author contributions

M. Al-Mazmumy: Conceptualization; A. S. Alsulami: Formal analysis, Writing–original draft; A. S. Alsulami, M. Al-Mazmumy, M. A. Alyami and M. Alsulami: Methodology, Investigation, Writing–review & editing; A. S. Alsulami and M. Al-Mazmumy: Software. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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