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Research article

A new approach to error inequalities: From Euler-Maclaurin bounds to cubically convergent algorithm

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Abstract: In this paper, we aimed to investigate the error inequality of the open method, known as Euler-Maclaurin's inequality, which is similar to Simpson's rule. We intended to explore some novel Maclaurin-like inequalities involving functions having convexity properties. To further accomplish this task, we built an identity and demonstrated new inequalities. With the help of a new auxiliary result and some well-known ones, like Hölder's, the power mean, improved Hölder, improved power mean, convexity, and bounded features of the function, we obtained new bounds for Euler-Maclaurin's inequality. From an applicable perspective, we developed several intriguing applications of our results, which illustrated the relationship between the means of real numbers and the error bounds of quadrature schemes. We also included a graphical breakdown of our outcomes to demonstrate their validity. Additionally, we constructed a new iterative scheme for non-linear equations that is cubically convergent. Afterwards, we provided a comparative study between the proposed algorithm and standard methods. We also discussed the proposed algorithm's impact on the basins of attraction.

Keywords: convex functions; Simspon's rule; Euler-Maclaurin's inequality; Hölder's inequality; iterative scheme
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1. Introduction and preliminaries

The theory of integral inequalities is studied to estimate the various mathematical quantities, and relying on the concept of inequality, numerous concepts of mathematical analysis are developed. One of these concepts, convex functions, is particularly useful in determining notable inequalities such as Young's inequality, Hölder's type inequalities, Jensen's inequality, trapezium's inequality, Ostrowski's type inequalities, Simpson's type inequalities, and so on. It is interesting to note that Jensen's and Hermite-Hadamard inequalities are studied as equivalent definitions of convex mappings and necessary and sufficient conditions to check the concavity of functions, respectively.

Definition 1.1. *Let* $F : [m_1, m_2] \to \mathbb{R}$ *be considered as convex mapping if*

$$F((1-\mathrm{z})\omega + \mathrm{z}\omega_1) \leq (1-\mathrm{z})F(\omega) + \mathrm{z}F(\omega_1), \quad \forall\, \omega, \omega_1 \in [m_1, m_2], \tag{1.1}$$

where $\mathrm{z} \in [0, 1]$.

Let $F : I = [m_1, m_2] \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping, then

$$F\left(\frac{m_1 + m_2}{2}\right) \leq \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} F(\omega)\mathrm{dz} \leq \frac{F(m_1) + F(m_2)}{2}. \tag{1.2}$$

Innovative and creative strategies have been employed to analyze the Hermite-Hadamard inequality. We can determine the boundaries for the remainder in trapezoidal and mid-point rules employed for numerical integration by assessing the inequality from both the right and left sides, respectively. For more details, see [1]. As far as we are aware, Simpson's rule is the most frequently applied three-point approximation integration rule, although it has the disadvantage of being inapplicable to functions that are not differentiable at the domain's endpoints. In the part that follows, we recapture the Simpson's $\frac{1}{8}$ formula and the well-known Simpson's inequality, which is demonstrated as:

$$\int_{m_1}^{m_2} F(\omega)\mathrm{d}\omega \approx \frac{1}{6}\left[F(m_1) + 4F\left(\frac{m_1 + m_2}{2}\right) + F(m_2)\right].$$

Theorem 1.1. *[2] If* $F : [m_1, m_2] \to \mathbb{R}$ *is a four times continuously differentiable on* (m_1, m_2), *and* $\|F^{(4)}\|_\infty = \sup_{\omega\in(m_1,m_2)} |F^{(4)}| < \infty$, *then*

$$\left|\frac{1}{6}\left[F(m_1) + 4F\left(\frac{m_1 + m_2}{2}\right) + F(m_2)\right] - \frac{1}{m_2 - m_1}\int_{m_1}^{m_2} F(\omega)\mathrm{d}\omega\right| \leq \frac{1}{2880}\|F^{(4)}\|_\infty (m_2 - m_1)^5.$$

To address the limitation of Simpson's inequality, the Maclaurin method is employed since it does not involve any boundary points in its quadrature rules. We present the Maclaurin's inequality, which is stated as:

Theorem 1.2. *Let* $F : [m_1, m_2] \to \mathbb{R}$ *be a four times continuously differentiable on* (m_1, m_2), *and* $\|F^{(4)}\|_\infty = \sup_{\omega\in(m_1,m_2)} |F^{(4)}| < \infty$, *then*

$$\left|\frac{1}{8}\left(3F\left(\frac{5m_1 + m_2}{6}\right) + 2F\left(\frac{m_1 + m_2}{2}\right) + 3F\left(\frac{m_1 + 5m_2}{6}\right)\right) - \frac{1}{m_2 - m_1}\int_{m_1}^{m_2} F(\omega)\mathrm{d}\omega\right|$$
$$\leq \frac{7(m_2 - m_1)^2}{51840}\|F^{(4)}\|_\infty.$$

In 2009, Alomari et al. [3] explored the general form of Simpson's-like inequalities utilizing s-convexity of functions. Budak et al. [4] presented some fresh improvements regarding Simpson's type inequalities by making use of newly proposed quantum operators. Yang et al. [5] obtained new representations of Simpson's-like inequalities using (s, m) convex mappings. Noor et al. [6] presented the inequalities of the trapezium type, utilizing the definition of harmonic convexity for two dimensions. Set et al. [7] obtained some Ostrowski-like inequalities by making use of s-convexity and fractional concepts. In [8], the authors discussed the Dragomir-Agarwal type inequalities incorporated with the generalized convexity and (p, q) calculus. In 2020, Chu et al. [9] came up with a new type of n-polynomial strongly convex mappings and worked out some new, stronger Simpson's type inequalities using Katugampola fractional integral operators. Dragomir and Rassias [10] published a monograph on Ostrowski's type inequalities and their utilities. In [11] applied a unified technique to extract the bounds for various error inequalities incorporated with monotone mappings. Ujevic [12] discussed the sharp inequalities of Simpsons's and Ostrowski's type. For more details, see [13–15].

In 2013, Alomari and Dragomir [16] introduced a new unified kernel and developed the error bounds of several Newton-Cotes formulas, including Euler-Maclaurin's inequality. Meftah et al. [17] came up with error bounds for Euler-Maclaurin's method using the idea of generalized convexity in the fractal domain. They then used the results in numerical integration to show that they were correct. In 2023, Hezenci and his fellows [18] investigated the fractional forms of Euler-Maclaurin-like inequalities associated with convex functions. Additionally, simulations verify the outcomes. In [19], the authors utilized the q approach to obtain more general and improved Euler-Maclaurin's type inequalities. In the continuation, Peng and Du [20] implemented the concepts of multiplicative calculus and established the new error estimates regarding Maclaurin's formula. In 2013, for the first time, Alomari [21] derived the error estimates for Milne's formula for first-order differentiable mappings. This article paved the way to investigate these kinds of methods. Budak et al. [22] investigated the error estimates of open methods involving convex functions and functions having the property of bounded variation and their applications. Also, Bin-Mohsin et al. [23] used quantum mechanics and the Jensen-Mercer inequality to look at new inequalities of Milne's type and used graphs to show that their results were correct.

Research on Euler-Maclaurin's inequality has motivated us to derive more accurate upper bounds for under-consideration inequality and its applications, particularly a novel iterative algorithm related to the open method of Maclaurin's rule. We determine Euler-Maclaurin's type inequalities via a new identity established for first-order differentiable and convex mappings, bridging some elementary results from the theory of inequalities. Our obtained estimates will yield more accurate results than those found in other research. Moreover, we visualize and verify our primary outcomes through graphical representations and numerical examples. To increase reliability, we present novel applications of theoretical means, numerical integration, and a new iterative method to solve non-linear problems.

2. Major results

In the following part of the investigation, we construct new error boundaries for first order differentiable convex by leveraging the elementary concepts of inequalities. First, we prove a new differentiable identity concerning Maclaurin's procedure.

Lemma 2.1. *Let $F : I = [m_1, m_2] \to \mathbb{R}$ be a differential mapping on I^o with $m_1 < m_2$ and $F \in L[m_1, m_2]$, then the following equality hold*

$$\frac{1}{8}\left[3F\left(\frac{5m_1+m_2}{6}\right)+2F\left(\frac{m_1+m_2}{2}\right)+3F\left(\frac{m_1+5m_2}{6}\right)\right]-\frac{1}{m_2-m_1}\int_{m_1}^{m_2}F(\omega)\mathrm{d}\omega \tag{2.1}$$
$$=(m_2-m_1)\left[\int_0^{\frac{1}{6}}\mathrm{z}F'((1-\mathrm{z})m_1+\mathrm{z}m_2)\mathrm{dz}+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\mathrm{z}-\frac{3}{8}\right)F'((1-\mathrm{z})m_1+\mathrm{z}m_2)\mathrm{dz}\right.$$
$$\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\mathrm{z}-\frac{5}{8}\right)F'((1-\mathrm{z})m_1+\mathrm{z}m_2)\mathrm{dz}+\int_{\frac{5}{6}}^{1}(\mathrm{z}-1)F'((1-\mathrm{z})m_1+\mathrm{z}m_2)\mathrm{dz}\right].$$

Proof. Let

$$I_1=\int_0^{\frac{1}{6}}\mathrm{z}F'((1-\mathrm{z})m_1+\mathrm{z}m_2)\mathrm{dz}$$
$$I_2=\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\mathrm{z}-\frac{3}{8}\right)F'((1-\mathrm{z})m_1+\mathrm{z}m_2)\mathrm{dz}$$
$$I_3=\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\mathrm{z}-\frac{5}{8}\right)F'((1-\mathrm{z})m_1+\mathrm{z}m_2)\mathrm{dz}$$
$$I_4=\int_{\frac{5}{6}}^{1}(\mathrm{z}-1)F'((1-\mathrm{z})m_1+\mathrm{z}m_2)\mathrm{dz}.$$

Implementing the integration by parts, we have

$$I_1=\mathrm{z}\left(\frac{F((1-\mathrm{z})m_1+\mathrm{z}m_2)}{m_2-m_1}\right)\Bigg|_0^{\frac{1}{6}}-\int_0^{\frac{1}{6}}\frac{F((1-\mathrm{z})m_1+\mathrm{z}m_2)}{m_2-m_1}\mathrm{dz}$$
$$=\frac{1}{m_2-m_1}\left[\frac{1}{6}F\left(\frac{5m_1+m_2}{6}\right)-\int_0^{\frac{1}{6}}F((1-\mathrm{z})m_1+\mathrm{z}m_2)\mathrm{dz}\right]$$
$$=\frac{1}{m_2-m_1}\left[\frac{1}{6}F\left(\frac{5m_1+m_2}{6}\right)-\frac{1}{m_2-m_1}\int_{m_1}^{\frac{5m_1+m_2}{6}}F(\omega)\mathrm{d}\omega\right]. \tag{2.2}$$

Similarly, we obtain

$$I_2=\left(\mathrm{z}-\frac{3}{8}\right)\frac{F((1-\mathrm{z})m_1+\mathrm{z}m_2)}{m_2-m_1}\Bigg|_{\frac{1}{6}}^{\frac{1}{2}}-\int_{\frac{1}{6}}^{\frac{1}{2}}\frac{F((1-\mathrm{z})m_1+\mathrm{z}m_2)}{m_2-m_1}\mathrm{dz}$$
$$=\frac{1}{m_2-m_1}\left[\frac{1}{8}F\left(\frac{m_1+m_2}{2}\right)+\frac{5}{24}F\left(\frac{5m_1+m_2}{6}\right)-\frac{1}{m_2-m_1}\int_{\frac{5m_1+m_2}{6}}^{\frac{m_1+m_2}{2}}F(\omega)\mathrm{d}\omega\right], \tag{2.3}$$
$$I_3=\left(\mathrm{z}-\frac{5}{8}\right)\frac{F((1-\mathrm{z})m_1+\mathrm{z}m_2)}{m_2-m_1}\Bigg|_{\frac{1}{2}}^{\frac{5}{6}}-\int_{\frac{1}{2}}^{\frac{5}{6}}\frac{F((1-\mathrm{z})m_1+\mathrm{z}m_2)}{m_2-m_1}\mathrm{dz}$$
$$=\frac{1}{m_2-m_1}\left[\frac{1}{8}F\left(\frac{m_1+m_2}{2}\right)+\frac{5}{24}F\left(\frac{m_1+5m_2}{6}\right)-\frac{1}{m_2-m_1}\int_{\frac{m_1+m_2}{2}}^{\frac{m_1+5m_2}{6}}F(\omega)\mathrm{d}\omega\right], \tag{2.4}$$

$$I_4 = (\mathrm{z}-1)\frac{F((1-\mathrm{z})m_1+\mathrm{z}m_2)}{m_2-m_1}\bigg|_{\frac{5}{6}}^{1} - \int_{\frac{5}{6}}^{1}\frac{F((1-\mathrm{z})m_1+\mathrm{z}m_2)}{m_2-m_1}\mathrm{dz}$$
$$= \frac{1}{m_2-m_1}\left[\frac{1}{6}F\left(\frac{m_1+5m_2}{6}\right) - \frac{1}{m_2-m_1}\int_{\frac{m_1+5m_2}{6}}^{1}F(\omega)\mathrm{d}\omega\right]. \tag{2.5}$$

Summing (2.2–2.5) and then taking the product of the obtained results by (m_2-m_1), we get (2.1). □

Theorem 2.1. *Assume that all of the requirements of Lemma 2.1 are fulfilled. If $|F'|$ is a convex mapping on $[m_1, m_2]$, then*

$$\left|\frac{1}{8}\left[3F\left(\frac{5m_1+m_2}{6}\right)+2F\left(\frac{m_1+m_2}{2}\right)+3F\left(\frac{m_1+5m_2}{6}\right)\right]-\frac{1}{m_2-m_1}\int_{m_1}^{m_2}F(\omega)\mathrm{d}\omega\right| \tag{2.6}$$
$$\leq \frac{25(m_2-m_1)}{576}(|F'(m_1)|+|F'(m_2)|).$$

Proof. Considering Lemma 2.1 and implementing the modulus characteristic and the convexity of $|F'|$, we have

$$\left|\frac{1}{8}\left[3F\left(\frac{5m_1+m_2}{6}\right)+2F\left(\frac{m_1+m_2}{2}\right)+3F\left(\frac{m_1+5m_2}{6}\right)\right]-\frac{1}{m_2-m_1}\int_{m_1}^{m_2}F(\omega)\mathrm{d}\omega\right|$$
$$\leq (m_2-m_1)\left[\int_0^{\frac{1}{6}}\mathrm{z}|F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|\mathrm{dz}+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\mathrm{z}-\frac{3}{8}\right||F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|\mathrm{dz}\right.$$
$$\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\mathrm{z}-\frac{5}{8}\right||F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|\mathrm{dz}+\int_{\frac{5}{6}}^{1}|\mathrm{z}-1||F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|\mathrm{dz}\right]$$
$$\leq (m_2-m_1)\left[\int_0^{\frac{1}{6}}\mathrm{z}((1-\mathrm{z})|F'(m_1)|+\mathrm{z}|F'(m_2)|)\mathrm{dz}+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\mathrm{z}-\frac{3}{8}\right|((1-\mathrm{z})|F'(m_1)|+\mathrm{z}|F'(m_2)|)\mathrm{dz}\right.$$
$$\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\mathrm{z}-\frac{5}{8}\right|((1-\mathrm{z})|F'(m_1)|+\mathrm{z}|F'(m_2)|)\mathrm{dz}+\int_{\frac{5}{6}}^{1}|\mathrm{z}-1|((1-\mathrm{z})|F'(m_1)|+\mathrm{z}|F'(m_2)|)\mathrm{dz}\right]$$
$$= (m_2-m_1)\left[\int_0^{\frac{1}{6}}\mathrm{z}((1-\mathrm{z})|F'(m_1)|+\mathrm{z}|F'(m_2)|)\mathrm{dz}\right.$$
$$+\left(\int_{\frac{1}{6}}^{\frac{3}{8}}\left(\frac{3}{8}-\mathrm{z}\right)+\int_{\frac{3}{8}}^{\frac{1}{2}}\left(\mathrm{z}-\frac{3}{8}\right)\right)((1-\mathrm{z})|F'(m_1)|+\mathrm{z}|F'(m_2)|)\mathrm{dz}$$
$$+\left(\int_{\frac{1}{2}}^{\frac{5}{8}}\left(\frac{5}{8}-\mathrm{z}\right)+\int_{\frac{5}{8}}^{\frac{5}{6}}\left(\mathrm{z}-\frac{5}{8}\right)\right)((1-\mathrm{z})|F'(m_1)|+\mathrm{z}|F'(m_2)|)\mathrm{dz}$$
$$\left.+\int_{\frac{5}{6}}^{1}|\mathrm{z}-1|((1-\mathrm{z})|F'(m_1)|+\mathrm{z}|F'(m_2)|)\mathrm{dz}\right]$$
$$= \frac{25(m_2-m_1)}{576}(|F'(m_1)|+|F'(m_2)|).$$

The proof is completed. □

We now give a graphical illustration of the above mentioned result.

Example 2.1. *Suppose all the properties of Theorem 2.1 are met, and considering the mapping* $F(z) = \frac{\theta}{r+2\theta} z^{\frac{r}{\theta}+2}$ *defined on* $\mathbb{R}^+$ *with* $r \geq 1$, $\theta > 1$, $m_1 = 1$ *and* $m_2 = 3$ *be a convex function. Then,*

$$\left| \frac{\theta\left(3\left(\frac{4}{3}\right)^{\frac{r+2\theta}{\theta}} + 2(2^{\frac{r+2\theta}{\theta}}) + 3\left(\frac{8}{3}\right)^{\frac{r+2\theta}{\theta}}\right)}{8(2\theta+r)} - \frac{\theta^2\left(3^{\frac{r+3\theta}{\theta}} - 1\right)}{2((r+2\theta)(r+3\theta))} \right|$$
$$\leq \frac{25}{288}\left(3^{\frac{r+\theta}{\theta}} + 1\right).$$

Under similar assumptions with $\alpha = 1 = r = \theta$, *Theorem 3.2 proved in [17] and Theorem 2.1 provide the following bounds for Maclaurin's inequality:* $0 < 0.7525$ *and* $0 < 0.8681$, *respectively. Our results provide better estimations compared to results previously established.*

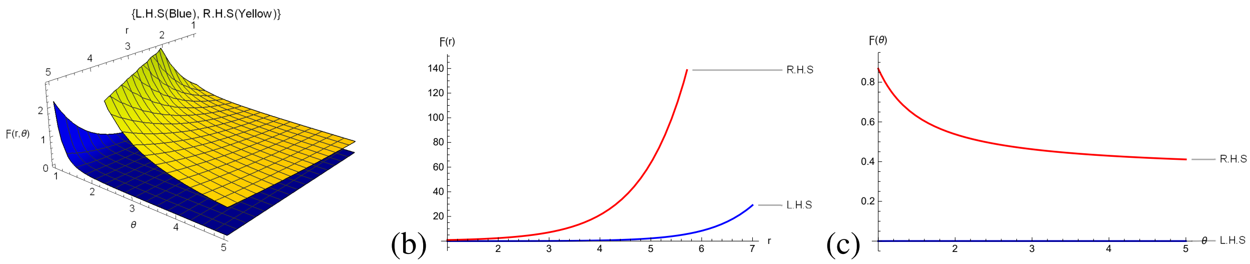

Figure 1. Graphical visuals of left and right sides of Theorem 2.1.

- *For Figure 1(a)–(c), we vary* $r \in [1, 5]$ *and* $\theta \in [1, 5]$ *to illustrate the comparison between the left and right sides of Theorem 2.1. From these visuals, one can easily observe that the left side is strictly less than the right hand side of Theorem 2.1, which confirms the accuracy of under-consideration of the result.*

Theorem 2.2. *Assume that all of the requirements of Lemma 2.1 are fulfilled. If* $|F'|^q$ *is a convex mapping on* $[m_1, m_2]$, $q > 1$, *then*

$$\left| \frac{1}{8}\left[3F\left(\frac{5m_1+m_2}{6}\right) + 2F\left(\frac{m_1+m_2}{2}\right) + 3F\left(\frac{m_1+5m_2}{6}\right)\right] - \frac{1}{m_2-m_1}\int_{m_1}^{m_2} F(\omega)\mathrm{d}\omega \right|$$
$$\leq (m_2-m_1)\left[\left(\frac{1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{11}{72}|F'(m_1)|^q + \frac{1}{72}|F'(m_2)|^q\right)^{\frac{1}{q}}\right.$$
$$+ \left(\frac{1}{p+1}\left(\left(\frac{5}{24}\right)^{p+1} + \left(\frac{1}{8}\right)^{p+1}\right)\right)^{\frac{1}{p}}\left(\frac{2}{9}|F'(m_1)|^q + \frac{1}{9}|F'(m_2)^q\right)^{\frac{1}{q}}$$
$$+ \left(\frac{1}{p+1}\left(\left(\frac{1}{8}\right)^{p+1} + \left(\frac{5}{24}\right)^{p+1}\right)\right)^{\frac{1}{p}}\left(\frac{1}{9}|F'(m_1)|^q + \frac{2}{9}|F'(m_2)^q\right)^{\frac{1}{q}}$$
$$\left. + \left(\frac{1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{72}|F'(m_1)|^q + \frac{11}{72}|F'(m_2)^q\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Taking $p > 1$. Considering Lemma 2.1, and making the utility of notable Hölder's integral inequality and the convexity of $|F'|^q$, we achieve

$$\begin{aligned}
&\left|\frac{1}{8}\left[3F\left(\frac{5m_1+m_2}{6}\right)+2F\left(\frac{m_1+m_2}{2}\right)+3F\left(\frac{m_1+5m_2}{6}\right)\right]-\frac{1}{m_2-m_1}\int_{m_1}^{m_2}F(\omega)\mathrm{d}\omega\right|\\
&\leq (m_2-m_1)\left[\int_0^{\frac{1}{6}}\mathrm{z}|F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|\mathrm{dz}+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\mathrm{z}-\frac{3}{8}\right||F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|\mathrm{dz}\right.\\
&\quad\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\mathrm{z}-\frac{5}{8}\right||F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|\mathrm{dz}+\int_{\frac{5}{6}}^{1}|\mathrm{z}-1||F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|\mathrm{dz}\right]\\
&\leq (m_2-m_1)\left[\left(\int_0^{\frac{1}{6}}\mathrm{z}^p\mathrm{dz}\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{6}}|F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|^q\mathrm{dz}\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\mathrm{z}-\frac{3}{8}\right|^p\mathrm{dz}\right)^{\frac{1}{p}}\left(\int_{\frac{1}{6}}^{\frac{1}{2}}|F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|^q\mathrm{dz}\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\mathrm{z}-\frac{5}{8}\right|^p\mathrm{dz}\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{5}{6}}|F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|^q\mathrm{dz}\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{5}{6}}^{1}|\mathrm{z}-1|^p\mathrm{dz}\right)^{\frac{1}{p}}\left(\int_{\frac{5}{6}}^{1}|F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|^q\mathrm{dz}\right)^{\frac{1}{q}}\right]\\
&\leq (m_2-m_1)\left[\left(\int_0^{\frac{1}{6}}\mathrm{z}^p\mathrm{dz}\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{6}}((1-\mathrm{z})|F'(m_1)|^q+\mathrm{z}|F'(m_2)^q)\mathrm{dz}\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\mathrm{z}-\frac{3}{8}\right|^p\mathrm{dz}\right)^{\frac{1}{p}}\left(\int_{\frac{1}{6}}^{\frac{1}{2}}((1-\mathrm{z})|F'(m_1)|^q+\mathrm{z}|F'(m_2)^q)\mathrm{dz}\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\mathrm{z}-\frac{5}{8}\right|^p\mathrm{dz}\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{5}{6}}((1-\mathrm{z})|F'(m_1)|^q+\mathrm{z}|F'(m_2)^q)\mathrm{dz}\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{5}{6}}^{1}|\mathrm{z}-1|^p\mathrm{dz}\right)^{\frac{1}{p}}\left(\int_{\frac{5}{6}}^{1}((1-\mathrm{z})|F'(m_1)|^q+\mathrm{z}|F'(m_2)^q)\mathrm{dz}\right)^{\frac{1}{q}}\right]\\
&=(m_2-m_1)\left[\left(\frac{1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{11}{72}|F'(m_1)|^q+\frac{1}{72}|F'(m_2)|^q\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\frac{1}{p+1}\left(\left(\frac{5}{24}\right)^{p+1}+\left(\frac{1}{8}\right)^{p+1}\right)\right)^{\frac{1}{p}}\left(\frac{2}{9}|F'(m_1)|^q+\frac{1}{9}|F'(m_2)^q\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{1}{p+1}\left(\left(\frac{1}{8}\right)^{p+1}+\left(\frac{5}{24}\right)^{p+1}\right)\right)^{\frac{1}{p}}\left(\frac{1}{9}|F'(m_1)|^q+\frac{2}{9}|F'(m_2)^q\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\frac{1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{72}|F'(m_1)|^q+\frac{11}{72}|F'(m_2)^q\right)^{\frac{1}{q}}\right],
\end{aligned}$$

which ends the proof. □

Below is the graphical illustration of Theorem 2.2.

Example 2.2. *Suppose all the properties of Theorem 2.2 are met, and considering the mapping* $F(z) = \frac{\theta}{r+2\theta} z^{\frac{r}{\theta}+2}$ *defined on* $\mathbb{R}^+$ *with* $r \geq 1$, $\theta > 1$, $m_1 = 1$ *and* $m_2 = 3$ *be a convex function. Then,*

$$\left| \frac{\theta\left(3\left(\frac{4}{3}\right)^{\frac{r+2\theta}{\theta}} + 2(2^{\frac{r+2\theta}{\theta}}) + 3\left(\frac{8}{3}\right)^{\frac{r+2\theta}{\theta}}\right)}{8(r+2\theta)} - \frac{\theta^2\left(3^{\frac{r+3\theta}{\theta}} - 1\right)}{2((r+2\theta)(r+3\theta))} \right|$$
$$\leq 2\left[\sqrt{\frac{1}{648}}\left(\sqrt{\frac{11}{72} + \frac{1}{72}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right) + \sqrt{\frac{19}{184}}\left(\sqrt{\frac{2}{9} + \frac{1}{9}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right) \right.$$
$$\left. + \sqrt{\frac{19}{184}}\left(\sqrt{\frac{1}{9} + \frac{2}{9}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right) + \sqrt{\frac{1}{648}}\left(\sqrt{\frac{1}{72} + \frac{11}{72}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right) \right].$$

- *For Figure 2(a)–(c), we vary* $r \in [1, 5]$ *and* $\theta \in [1, 5]$ *to illustrate the comparison between the left and right sides of Theorem 2.2. From these visuals, one can easily observe that the left side is strictly less than the right side of Theorem 2.2, which confirms the accuracy of under-consideration of the result.*

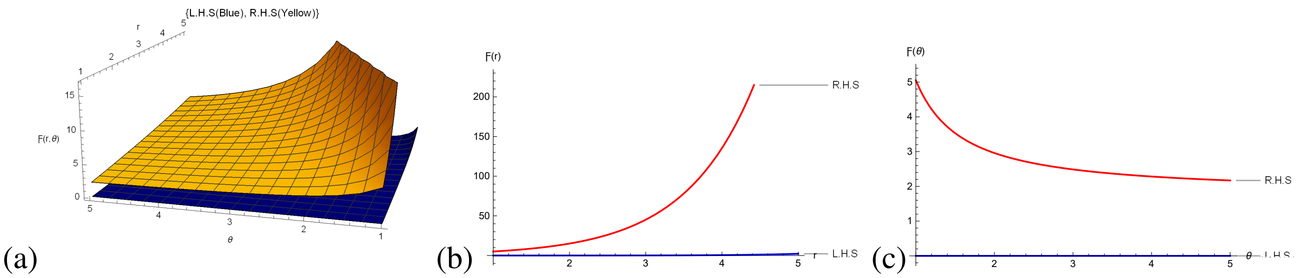

Figure 2. Graphical visuals of left and right sides of Theorem 2.2.

Theorem 2.3. *Assume that all of the requirements of Lemma 2.1 are fulfilled. If* $|F'|^q$ *is a convex mapping on* $[m_1, m_2]$, $q \geq 1$, *then*

$$\left| \frac{1}{8}\left[3F\left(\frac{5m_1+m_2}{6}\right) + 2F\left(\frac{m_1+m_2}{2}\right) + 3F\left(\frac{m_1+5m_2}{6}\right)\right] - \frac{1}{m_2-m_1}\int_{m_1}^{m_2} F(\omega)d\omega \right|$$
$$\leq (m_2-m_1)\left[\left(\frac{1}{72}\right)^{1-\frac{1}{q}}\left(\frac{1}{81}|F'(m_1)|^q + \frac{1}{648}|F'(m_2)|^q\right)^{\frac{1}{q}} + \left(\frac{17}{576}\right)^{1-\frac{1}{q}}\left(\frac{863}{41472}|F'(m_1)|^q + \frac{361}{41472}|F'(m_2)^q\right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\frac{17}{576}\right)^{1-\frac{1}{q}}\left(\frac{361}{41472}|F'(m_1)|^q + \frac{863}{41472}|F'(m_2)^q\right)^{\frac{1}{q}} + \left(\frac{1}{72}\right)^{1-\frac{1}{q}}\left(\frac{1}{648}|F'(m_1)|^q + \frac{1}{81}|F'(m_2)^q\right)^{\frac{1}{q}}\right].$$

Proof. Considering Lemma 2.1, implementing the power-mean's inequality and the convexity of $|F'|^q$, we get

$$\left| \frac{1}{8}\left[3F\left(\frac{5m_1+m_2}{6}\right) + 2F\left(\frac{m_1+m_2}{2}\right) + 3F\left(\frac{m_1+5m_2}{6}\right)\right] - \frac{1}{m_2-m_1}\int_{m_1}^{m_2} F(\omega)d\omega \right|$$

$$\leq (m_2 - m_1)\left[\int_0^{\frac{1}{6}} z|F'((1-z)m_1 + zm_2)|dz + \int_{\frac{1}{6}}^{\frac{1}{2}} \left|z - \frac{3}{8}\right| |F'((1-z)m_1 + zm_2)|dz \right.$$
$$\left. + \int_{\frac{1}{2}}^{\frac{5}{6}} \left|z - \frac{5}{8}\right| |F'((1-z)m_1 + zm_2)|dz + \int_{\frac{5}{6}}^{1} |z-1||F'((1-z)m_1 + zm_2)|dz\right]$$
$$\leq (m_2 - m_1)\left[\left(\int_0^{\frac{1}{6}} zdz\right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{6}} z((1-z)|F'(m_1)|^q + z|F'(m_2)^q)dz\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left|z - \frac{3}{8}\right| dz\right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left|z - \frac{3}{8}\right| ((1-z)|F'(m_1)|^q + z|F'(m_2)^q)dz\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}} \left|z - \frac{5}{8}\right| dz\right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^{\frac{5}{6}} \left|z - \frac{5}{8}\right| ((1-z)|F'(m_1)|^q + z|F'(m_2)^q)dz\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{5}{6}}^{1} |z-1|dz\right)^{1-\frac{1}{q}} \left(\int_{\frac{5}{6}}^{1} |z-1|((1-z)|F'(m_1)|^q + z|F'(m_2)^q)dz\right)^{\frac{1}{q}}\right]$$
$$= (m_2 - m_1)\left[\left(\frac{1}{72}\right)^{1-\frac{1}{q}} \left(\frac{1}{81}|F'(m_1)|^q + \frac{1}{648}|F'(m_2)|^q\right)^{\frac{1}{q}} + \left(\frac{17}{576}\right)^{1-\frac{1}{q}} \left(\frac{863}{41472}|F'(m_1)|^q + \frac{361}{41472}|F'(m_2)^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{17}{576}\right)^{1-\frac{1}{q}} \left(\frac{361}{41472}|F'(m_1)|^q + \frac{863}{41472}|F'(m_2)^q\right)^{\frac{1}{q}} + \left(\frac{1}{72}\right)^{1-\frac{1}{q}} \left(\frac{1}{648}|F'(m_1)|^q + \frac{1}{81}|F'(m_2)^q\right)^{\frac{1}{q}}\right].$$

Hence, we acquire our required result. □

Here is the visual analysis of Theorem 2.3.

Example 2.3. *Suppose all the properties of Theorem 2.3 are met, and considering the mapping* $F(z) = \frac{\theta}{r+2\theta} z^{\frac{r}{\theta}+2}$ *defined on* $\mathbb{R}^+$ *with* $r \geq 1$, $\theta > 1$, $m_1 = 1$ *and* $m_2 = 3$ *be a convex function. Then,*

$$\left|\frac{\theta\left(3\left(\frac{4}{3}\right)^{\frac{r+2\theta}{\theta}} + 2(2^{\frac{r+2\theta}{\theta}}) + 3\left(\frac{8}{3}\right)^{\frac{r+2\theta}{\theta}}\right)}{8(r+2\theta)} - \frac{\theta^2\left(3^{\frac{r+3\theta}{\theta}} - 1\right)}{2((r+2\theta)(r+3\theta))}\right|$$
$$\leq 2\left[\sqrt{\frac{1}{648}}\left(\sqrt{\frac{1}{81} + \frac{1}{648}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right) + \sqrt{\frac{19}{5184}}\left(\sqrt{\frac{863}{41472} + \frac{361}{41472}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)\right.$$
$$\left.+\sqrt{\frac{19}{5184}}\left(\sqrt{\frac{361}{41472} + \frac{863}{41472}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right) + \sqrt{\frac{1}{648}}\left(\sqrt{\frac{1}{648} + \frac{1}{81}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)\right].$$

- *For Figure 3(a)–(c), we vary* $r \in [1,5]$ *and* $\theta \in [1,5]$ *to illustrate the comparison between the left and right sides of Theorem 2.3. From these visuals, one can easily observe that the left side is strictly less than the right side of Theorem 2.3, which confirms the accuracy of under-consideration of the result.*

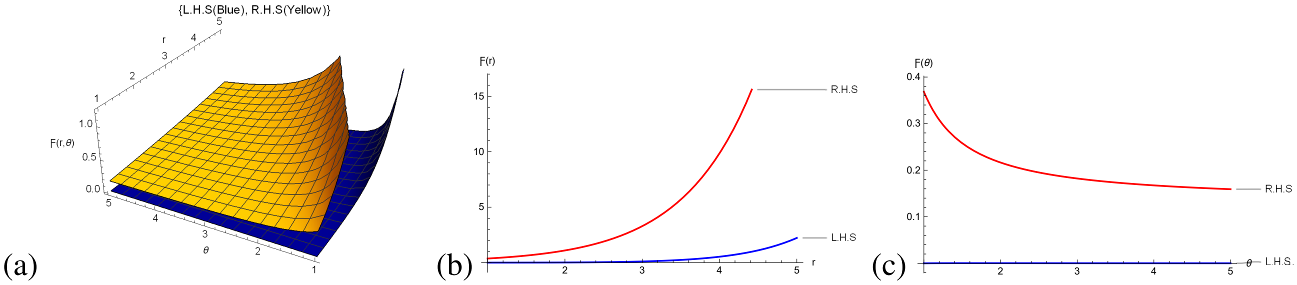

Figure 3. Graphical visuals of left and right sides of Theorem 2.3.

Theorem 2.4. *Assume that all of the requirements of Lemma 2.1 are fulfilled. If $|F'|^q$ is a convex mapping on $[m_1, m_2]$, $q > 1$, then*

$$\left|\frac{1}{8}\left[3F\left(\frac{5m_1+m_2}{6}\right)+2F\left(\frac{m_1+m_2}{2}\right)+3F\left(\frac{m_1+5m_2}{6}\right)\right]-\frac{1}{m_2-m_1}\int_{m_1}^{m_2}F(\omega)\mathrm{d}\omega\right|$$
$$\leq (m_2-m_1)\left\{6\left[\left(\frac{1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{17}{1296}|F'(m_1)|^q+\frac{1}{1296}|F'(m_2)|^q\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\frac{1}{6^{p+2}(p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{81}|F'(m_1)|^q+\frac{1}{648}|F'(m_2)|^q\right)^{\frac{1}{q}}\right]$$
$$+3\left[\left(\frac{3^{p+2}+5^{p+1}(11+8p)}{24^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{13}{324}|F'(m_1)|^q+\frac{5}{324}|F'(m_2)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{5^{p+2}+3^{p+1}(13+8p)}{24^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{11}{324}|F'(m_1)|^q+\frac{7}{324}|F'(m_2)|^q\right)^{\frac{1}{q}}\right]$$
$$+3\left[\left(\frac{5^{p+2}+3^{p+1}(13+8p)}{24^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{7}{324}|F'(m_1)|^q+\frac{11}{324}|F'(m_2)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{3^{p+2}+5^{p+1}(13+8p)}{24^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{5}{324}|F'(m_1)|^q+\frac{13}{324}|F'(m_2)|^q\right)^{\frac{1}{q}}\right]$$
$$+6\left[\left(\frac{1}{6^{p+2}(p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{648}|F'(m_1)|^q+\frac{1}{81}|F'(m_2)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\frac{1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{1296}|F'(m_1)|^q+\frac{17}{1296}|F'(m_2)|^q\right)^{\frac{1}{q}}\right]\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Considering Lemma 2.1, by implementing the improved Hölder's inequality and the convexity of $|F'|^q$, we get

$$\left|\frac{1}{8}\left[3F\left(\frac{5m_1+m_2}{6}\right)+2F\left(\frac{m_1+m_2}{2}\right)+3F\left(\frac{m_1+5m_2}{6}\right)\right]-\frac{1}{m_2-m_1}\int_{m_1}^{m_2}F(\omega)\mathrm{d}\omega\right|$$

$$
\begin{aligned}
&\leq (m_2 - m_1)\left[\int_0^{\frac{1}{6}} z|F'((1-z)m_1 + zm_2)|dz + \int_{\frac{1}{6}}^{\frac{1}{2}} \left|z - \frac{3}{8}\right| |F'((1-z)m_1 + zm_2)|dz\right.\\
&\left. + \int_{\frac{1}{2}}^{\frac{5}{6}} \left|z - \frac{5}{8}\right| |F'((1-z)m_1 + zm_2)|dz + \int_{\frac{5}{6}}^{1} |z-1||F'((1-z)m_1 + zm_2)|dz\right]\\
&\leq (m_2 - m_1)\left\{6\left[\left(\int_0^{\frac{1}{6}} z^p\left(\frac{1}{6} - z\right)dz\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{6}}\left(\frac{1}{6} - z\right)|F'((1-z)m_1 + zm_2)|^q dz\right)^{\frac{1}{q}}\right.\right.\\
&\left. + \left(\int_0^{\frac{1}{6}} z^p(z)dz\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{6}} z|F'((1-z)m_1 + zm_2)|dz\right)^{\frac{1}{q}}\right]\\
&+3\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|z - \frac{3}{8}\right|^p\left(\frac{1}{2} - z\right)dz\right)^{\frac{1}{p}}\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left(\frac{1}{2} - z\right)|F'((1-z)m_1 + zm_2)|^q dz\right)^{\frac{1}{q}}\right.\\
&\left. + \left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|z - \frac{3}{8}\right|^p\left(z - \frac{1}{6}\right)dz\right)^{\frac{1}{p}}\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left(z - \frac{1}{6}\right)|F'((1-z)m_1 + zm_2)|^q dz\right)^{\frac{1}{q}}\right]\\
&+3\left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|z - \frac{5}{8}\right|^p\left(\frac{5}{6} - z\right)dz\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6} - z\right)|F'((1-z)m_1 + zm_2)|^q dz\right)^{\frac{1}{q}}\right.\\
&\left. + \left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|z - \frac{5}{8}\right|^p\left(z - \frac{1}{2}\right)dz\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left(z - \frac{1}{2}\right)|F'((1-z)m_1 + zm_2)|^q dz\right)^{\frac{1}{q}}\right]\\
&+6\left[\left(\int_{\frac{5}{6}}^{1}|z-1|^p(1-z)\,dz\right)^{\frac{1}{p}}\left(\int_{\frac{5}{6}}^{1}(1-z)\,|F'((1-z)m_1 + zm_2)|^q dz\right)^{\frac{1}{q}}\right.\\
&\left.\left. + \left(\int_{\frac{5}{6}}^{1}|z-1|^p\left(z - \frac{5}{6}\right)dz\right)^{\frac{1}{p}}\left(\int_{\frac{5}{6}}^{1}\left(z - \frac{5}{6}\right)|F'((1-z)m_1 + zm_2)|^q dz\right)^{\frac{1}{q}}\right]\right\}\\
&\leq (m_2 - m_1)\left\{6\left[\left(\frac{1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{17}{1296}|F'(m_1)|^q + \frac{1}{1296}|F'(m_2)|^q\right)^{\frac{1}{q}}\right.\right.\\
&\left. + \left(\frac{1}{6^{p+2}(p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{81}|F'(m_1)|^q + \frac{1}{648}|F'(m_2)|^q\right)^{\frac{1}{q}}\right]\\
&+3\left[\left(\frac{3^{p+2} + 5^{p+1}(11+8p)}{24^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{13}{324}|F'(m_1)|^q + \frac{5}{324}|F'(m_2)|^q\right)^{\frac{1}{q}}\right.\\
&\left. + \left(\frac{5^{p+2} + 3^{p+1}(13+8p)}{24^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{11}{324}|F'(m_1)|^q + \frac{7}{324}|F'(m_2)|^q\right)^{\frac{1}{q}}\right]\\
&+3\left[\left(\frac{5^{p+2} + 3^{p+1}(13+8p)}{24^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{7}{324}|F'(m_1)|^q + \frac{11}{324}|F'(m_2)|^q\right)^{\frac{1}{q}}\right.
\end{aligned}
$$

$$+\left(\frac{3^{p+2}+5^{p+1}(13+8p)}{24^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{5}{324}|F'(m_1)|^q+\frac{13}{324}|F'(m_2)|^q\right)^{\frac{1}{q}}\Bigg]$$
$$+6\Bigg[\left(\frac{1}{6^{p+2}(p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{648}|F'(m_1)|^q+\frac{1}{81}|F'(m_2)|^q\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{1296}|F'(m_1)|^q+\frac{17}{1296}|F'(m_2)|^q\right)^{\frac{1}{q}}\Bigg]\Bigg\},$$

which ends the proof. □

Example 2.4. *Suppose all the properties of Theorem 2.4 are met, and considering the mapping* $F(z)=\frac{\theta}{r+2\theta}z^{\frac{r}{\theta}+2}$ *defined on* $\mathbb{R}^+$ *with* $r\geq 1$*,* $\theta>1$*,* $m_1=1$ *and* $m_2=3$ *be a convex function. Then,*

$$\left|\frac{\theta\left(3\left(\frac{4}{3}\right)^{\frac{r+2\theta}{\theta}}+2(2^{\frac{r+2\theta}{\theta}})+3\left(\frac{8}{3}\right)^{\frac{r+2\theta}{\theta}}\right)}{8(r+2\theta)}-\frac{\theta^2\left(3^{\frac{r+3\theta}{\theta}}-1\right)}{2((r+2\theta)(r+3\theta))}\right|$$
$$\leq 2\Bigg\{6\left[\sqrt{\frac{1}{15552}}\left(\sqrt{\frac{17}{1296}+\frac{1}{1296}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)+\sqrt{\frac{1}{5184}}\left(\sqrt{\frac{1}{81}+\frac{1}{648}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)\right]$$
$$+3\left[\sqrt{\frac{1}{1152}}\left(\sqrt{\frac{13}{324}+\frac{5}{324}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)+\sqrt{\frac{677}{1990656}}\left(\sqrt{\frac{11}{324}+\frac{7}{324}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)\right]$$
$$+3\left[\sqrt{\frac{677}{1990656}}\left(\sqrt{\frac{7}{324}+\frac{11}{324}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)+\sqrt{\frac{1}{1152}}\left(\sqrt{\frac{5}{324}+\frac{13}{324}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)\right]$$
$$+6\left[\sqrt{\frac{1}{5184}}\left(\sqrt{\frac{1}{648}+\frac{1}{81}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)+\sqrt{\frac{1}{15552}}\left(\sqrt{\frac{1}{1296}+\frac{17}{1296}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)\right]\Bigg\}.$$

- *For Figure 4(a)–(c), we vary* $r\in[1,5]$ *and* $\theta\in[1,5]$ *to illustrate the comparison between the left and right hand side of Theorem 2.4. From these visuals, one can easily observe that the left side is strictly less than the right side of Theorem 2.4, which confirms the accuracy of under-consideration of the result.*

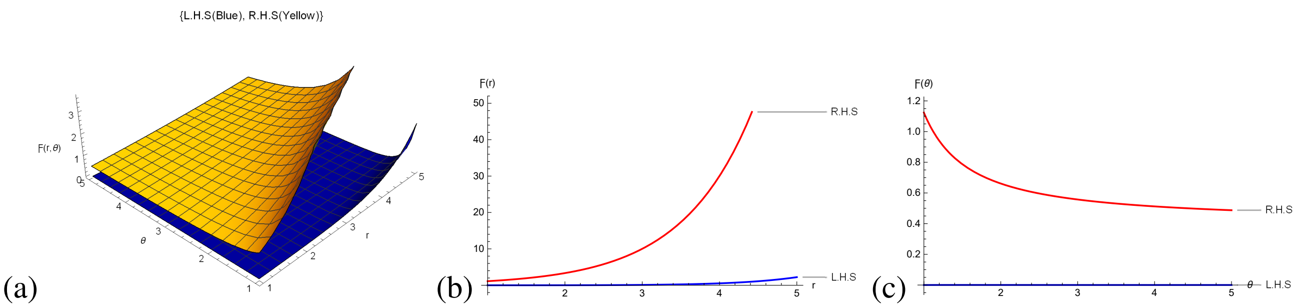

Figure 4. Graphical visuals of left and right sides of Theorem 2.4.

Theorem 2.5. *Assume that all of the requirements of Lemma 2.1 are fulfilled. If* $|F'| \leq M$, $M > 0$ *is convex function on* $[m_1, m_2]$, *then*

$$\left|\frac{1}{8}\left[3F\left(\frac{5m_1+m_2}{6}\right)+2F\left(\frac{m_1+m_2}{2}\right)+3F\left(\frac{m_1+5m_2}{6}\right)\right]-\frac{1}{m_2-m_1}\int_{m_1}^{m_2}F(\omega)\mathrm{d}\omega\right|$$
$$\leq \frac{25M(m_2-m_1)}{288}.$$

Proof. Considering Lemma 2.1 and the convexity of $|F'|$, we have

$$\left|\frac{1}{8}\left[3F\left(\frac{5m_1+m_2}{6}\right)+2F\left(\frac{m_1+m_2}{2}\right)+3F\left(\frac{m_1+5m_2}{6}\right)\right]-\frac{1}{m_2-m_1}\int_{m_1}^{m_2}F(\omega)\mathrm{d}\omega\right|$$
$$\leq (m_2-m_1)\left[\int_0^{\frac{1}{6}} \mathrm{z}|F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|\mathrm{dz}+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\mathrm{z}-\frac{3}{8}\right||F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|\mathrm{dz}\right.$$
$$\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\mathrm{z}-\frac{5}{8}\right||F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|\mathrm{dz}+\int_{\frac{5}{6}}^{1}|\mathrm{z}-1||F'((1-\mathrm{z})m_1+\mathrm{z}m_2)|\mathrm{dz}\right]$$
$$\leq (m_2-m_1)\left[\int_0^{\frac{1}{6}} \mathrm{z}((1-\mathrm{z})|F'(m_1)|+\mathrm{z}|F'(m_2)|)\mathrm{dz}+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\mathrm{z}-\frac{3}{8}\right|((1-\mathrm{z})|F'(m_1)|+\mathrm{z}|F'(m_2)|)\mathrm{dz}\right.$$
$$\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\mathrm{z}-\frac{5}{8}\right|((1-\mathrm{z})|F'(m_1)|+\mathrm{z}|F'(m_2)|)\mathrm{dz}+\int_{\frac{5}{6}}^{1}|\mathrm{z}-1|((1-\mathrm{z})|F'(m_1)|+\mathrm{z}|F'(m_2)|)\mathrm{dz}\right]$$
$$\leq M(m_2-m_1)\left(\int_0^{\frac{1}{6}}\mathrm{zdz}+\left(\int_{\frac{1}{6}}^{\frac{3}{8}}\left(\frac{3}{8}-\mathrm{z}\right)+\int_{\frac{3}{8}}^{\frac{1}{2}}\left(\mathrm{z}-\frac{3}{8}\right)\right)\mathrm{dz}\right.$$
$$\left.+\left[\int_{\frac{1}{2}}^{\frac{5}{8}}\left(\frac{5}{8}-\mathrm{z}\right)+\int_{\frac{5}{8}}^{\frac{5}{6}}\left(\mathrm{z}-\frac{5}{8}\right)\right)\mathrm{dz}+\int_{\frac{5}{6}}^{1}|\mathrm{z}-1|\mathrm{dz}\right]$$
$$=\frac{25M(m_2-m_1)}{288}.$$

Hence, we acquire the required outcome. □

Here is the graphical analysis of Theorem 2.5.

Example 2.5. *Suppose all the properties of Theorem 2.5 are met, and considering the mapping* $F(\mathrm{z}) = \frac{\theta}{r+2\theta}\mathrm{z}^{\frac{r}{\theta}+2}$ *defined on* $\mathbb{R}^+$ *with* $r \geq 1$, $\theta > 1$, $m_1 = 1$ *and* $m_2 = 3$ *be a convex function. Then,*

$$\left|\frac{\theta\left(3\left(\frac{4}{3}\right)^{\frac{r+2\theta}{\theta}}+2(2^{\frac{r+2\theta}{\theta}})+3\left(\frac{8}{3}\right)^{\frac{r+2\theta}{\theta}}\right)}{8(r+2\theta)}-\frac{\theta^2\left(3^{\frac{r+3\theta}{\theta}}-1\right)}{2((r+2\theta)(r+3\theta))}\right|$$
$$\leq \frac{25}{288}\left(1+3^{\frac{r+\theta}{\theta}}\right).$$

- *For Figure 5(a)–(c), we vary* $r \in [1,5]$ *and* $\theta \in [1,5]$ *to illustrate the comparison between the left and right hand side of Theorem 2.5. From these visuals, one can observe that the left side is strictly less than the right side of Theorem 2.5, which confirms the accuracy of under-consideration of the result.*

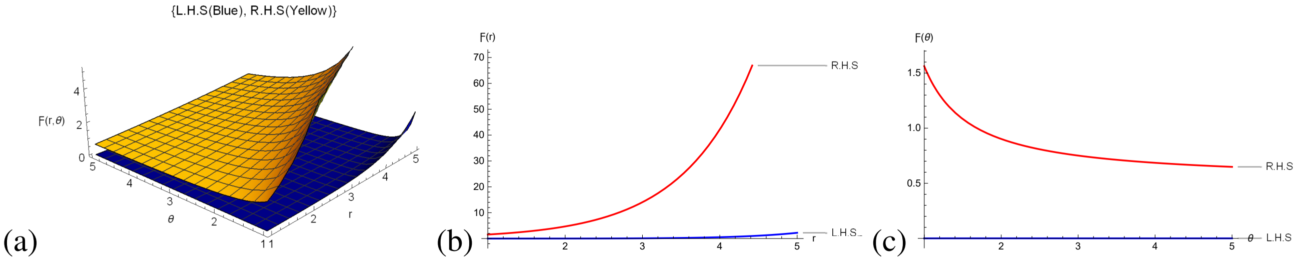

Figure 5. Graphical visuals of left and right sides of Theorem 2.5.

Theorem 2.6. *Assume that all of the requirements of Lemma 2.1 are fulfilled. If $|F'|^q$ is a convex mapping on $[m_1, m_2]$, $q > 1$, then*

$$\begin{aligned}
&\left|\frac{1}{8}\left[3F\left(\frac{5m_1+m_2}{6}\right)+2F\left(\frac{m_1+m_2}{2}\right)+3F\left(\frac{m_1+5m_2}{6}\right)\right]-\frac{1}{m_2-m_1}\int_{m_1}^{m_2}F(\omega)d\omega\right| \\
&\leq (m_2-m_1)\left\{6\left[\left(\frac{1}{1296}\right)^{1-\frac{1}{q}}\left(\frac{11}{15552}|F'(m_1)|^q+\frac{1}{15552}|F'(m_2)|^q\right)^{\frac{1}{q}}\right.\right. \\
&\quad\left.+\left(\frac{1}{648}\right)^{1-\frac{1}{q}}\left(\frac{7}{5184}|F'(m_1)|^q+\frac{1}{5184}|F'(m_2)|^q\right)^{\frac{1}{q}}\right] \\
&\quad+3\left[\left(\frac{251}{41472}\right)^{1-\frac{1}{q}}\left(\frac{3059}{663552}|F'(m_1)|^q+\frac{319}{221184}|F'(m_2)|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{157}{41472}\right)^{1-\frac{1}{q}}\left(\frac{4631}{1990656}|F'(m_1)|^q+\frac{2905}{1990656}|F'(m_2)|^q\right)^{\frac{1}{q}}\right] \\
&\quad+3\left[\left(\frac{157}{41472}\right)^{1-\frac{1}{q}}\left(\frac{2905}{1990656}|F'(m_1)|^q+\frac{4631}{1990656}|F'(m_2)|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{251}{41472}\right)^{1-\frac{1}{q}}\left(\frac{319}{221184}|F'(m_1)|^q+\frac{3059}{663552}|F'(m_2)|^q\right)^{\frac{1}{q}}\right] \\
&\quad+6\left[\left(\frac{1}{648}\right)^{1-\frac{1}{q}}\left(\frac{1}{5184}|F'(m_1)|^q+\frac{7}{5184}|F'(m_2)|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.\left.+\left(\frac{1}{1296}\right)^{1-\frac{1}{q}}\left(\frac{1}{15552}|F'(m_1)|^q+\frac{11}{15552}|F'(m_2)|^q\right)^{\frac{1}{q}}\right]\right\},
\end{aligned}$$

where $q > 1$.

Proof. Through Lemma 2.1, implementing the improved power-mean inequality and the convexity of $|F'|^q$, we get

$$\begin{aligned}
&\left|\frac{1}{8}\left[3F\left(\frac{5m_1+m_2}{6}\right)+2F\left(\frac{m_1+m_2}{2}\right)+3F\left(\frac{m_1+5m_2}{6}\right)\right]-\frac{1}{m_2-m_1}\int_{m_1}^{m_2}F(\omega)d\omega\right| \\
&\leq (m_2-m_1)\left[\int_0^{\frac{1}{6}} z|F'((1-z)m_1+zm_2)|dz+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|z-\frac{3}{8}\right||F'((1-z)m_1+zm_2)|dz\right.
\end{aligned}$$

$$+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|z-\frac{5}{8}\right||F'((1-z)m_1+zm_2)|dz+\int_{\frac{5}{6}}^{1}|z-1||F'((1-z)m_1+zm_2)|dz\Bigg]$$

$$\leq (m_2-m_1)\Bigg\{6\Bigg[\left(\int_0^{\frac{1}{6}} z\left(\frac{1}{6}-z\right)dz\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{6}}\left(\frac{1}{6}-z\right)z|F'((1-z)m_1+zm_2)|^q dz\right)^{\frac{1}{q}}$$

$$+\left(\int_0^{\frac{1}{6}} z^2 dz\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{6}} z^2|F'((1-z)m_1+zm_2)|dz\right)^{\frac{1}{q}}\Bigg]$$

$$+3\Bigg[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|z-\frac{3}{8}\right|\left(\frac{1}{2}-z\right)dz\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|z-\frac{3}{8}\right|\left(\frac{1}{2}-z\right)|F'((1-z)m_1+zm_2)|^q dz\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|z-\frac{3}{8}\right|\left(z-\frac{1}{6}\right)dz\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|z-\frac{3}{8}\right|\left(z-\frac{1}{6}\right)|F'((1-z)m_1+zm_2)|^q dz\right)^{\frac{1}{q}}\Bigg]$$

$$+3\Bigg[\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|z-\frac{5}{8}\right|\left(\frac{5}{6}-z\right)dz\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-z\right)\left|z-\frac{5}{8}\right||F'((1-z)m_1+zm_2)|^q dz\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|z-\frac{5}{8}\right|\left(z-\frac{1}{2}\right)dz\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left(z-\frac{1}{2}\right)\left|z-\frac{5}{8}\right||F'((1-z)m_1+zm_2)|^q dz\right)^{\frac{1}{q}}\Bigg]$$

$$+6\Bigg[\left(\int_{\frac{5}{6}}^{1}|z-1|(1-z)dz\right)^{1-\frac{1}{q}}\left(\int_{\frac{5}{6}}^{1}(1-z)|z-1||F'((1-z)m_1+zm_2)|^q dz\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{5}{6}}^{1}|z-1|\left(z-\frac{5}{6}\right)dz\right)^{1-\frac{1}{q}}\left(\int_{\frac{5}{6}}^{1}\left(z-\frac{5}{6}\right)|z-1||F'((1-z)m_1+zm_2)|^q dz\right)^{\frac{1}{q}}\Bigg]\Bigg\}$$

$$\leq (m_2-m_1)\Bigg\{6\Bigg[\left(\frac{1}{1296}\right)^{1-\frac{1}{q}}\left(\frac{11}{15552}|F'(m_1)|^q+\frac{1}{15552}|F'(m_2)|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{648}\right)^{1-\frac{1}{q}}\left(\frac{7}{5184}|F'(m_1)|^q+\frac{1}{5184}|F'(m_2)|^q\right)^{\frac{1}{q}}\Bigg]$$

$$+3\Bigg[\left(\frac{251}{41472}\right)^{1-\frac{1}{q}}\left(\frac{3059}{663552}|F'(m_1)|^q+\frac{319}{221184}|F'(m_2)|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{157}{41472}\right)^{1-\frac{1}{q}}\left(\frac{4631}{1990656}|F'(m_1)|^q+\frac{2905}{1990656}|F'(m_2)|^q\right)^{\frac{1}{q}}\Bigg]$$

$$+3\Bigg[\left(\frac{157}{41472}\right)^{1-\frac{1}{q}}\left(\frac{2905}{1990656}|F'(m_1)|^q+\frac{4631}{1990656}|F'(m_2)|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{251}{41472}\right)^{1-\frac{1}{q}}\left(\frac{319}{221184}|F'(m_1)|^q+\frac{3059}{663552}|F'(m_2)|^q\right)^{\frac{1}{q}}\Bigg]$$

$$+6\left[\left(\frac{1}{648}\right)^{1-\frac{1}{q}}\left(\frac{1}{5184}|F'(m_1)|^q+\frac{7}{5184}|F'(m_2)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\frac{1}{1296}\right)^{1-\frac{1}{q}}\left(\frac{1}{15552}|F'(m_1)|^q+\frac{11}{15552}|F'(m_2)|^q\right)^{\frac{1}{q}}\right]\right\},$$

which ends the proof. □

Example 2.6. *Suppose all the properties of Theorem 2.6 are met, and considering the mapping* $F(z) = \frac{\theta}{r+2\theta}z^{\frac{r}{\theta}+2}$ *defined on* $\mathbb{R}^+$ *with* $r \geq 1$, $\theta > 1$, $m_1 = 1$ *and* $m_2 = 3$ *be a convex function. Then,*

$$\left|\frac{\theta\left(3\left(\frac{4}{3}\right)^{\frac{r+2\theta}{\theta}}+2(2^{\frac{r+2\theta}{\theta}})+3\left(\frac{8}{3}\right)^{\frac{r+2\theta}{\theta}}\right)}{8(r+2\theta)}-\frac{\theta^2\left(3^{\frac{r+3\theta}{\theta}}-1\right)}{2((r+2\theta)(r+3\theta))}\right|$$
$$\leq 2\left\{6\left[\sqrt{\frac{1}{1296}}\left(\sqrt{\frac{11}{15552}+\frac{1}{15552}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)+\sqrt{\frac{1}{648}}\left(\sqrt{\frac{7}{5184}+\frac{1}{5184}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)\right]\right.$$
$$+3\left[\sqrt{\frac{251}{41472}}\left(\sqrt{\frac{3059}{663552}+\frac{319}{221184}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)+\sqrt{\frac{157}{41472}}\left(\sqrt{\frac{4631}{1990656}+\frac{2905}{1990656}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)\right]$$
$$+3\left[\sqrt{\frac{157}{41472}}\left(\sqrt{\frac{2905}{1990656}+\frac{4631}{1990656}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)+\sqrt{\frac{251}{41472}}\left(\sqrt{\frac{319}{221184}+\frac{3059}{663552}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)\right]$$
$$\left.+6\left[\sqrt{\frac{1}{648}}\left(\sqrt{\frac{1}{5184}+\frac{7}{5184}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)+\sqrt{\frac{1}{1296}}\left(\sqrt{\frac{1}{15552}+\frac{11}{15552}\left(3^{\frac{2r+2\theta}{\theta}}\right)}\right)\right]\right\}.$$

- *For Figure 6(a)–(c), we vary* $r \in [1,5]$ *and* $\theta \in [1,5]$ *to illustrate the comparison between the left and right hand side of Theorem 2.6. From these visuals, one observe that the left side is strictly less than the right side of Theorem 2.6, which confirms the accuracy of under-consideration of the result.*

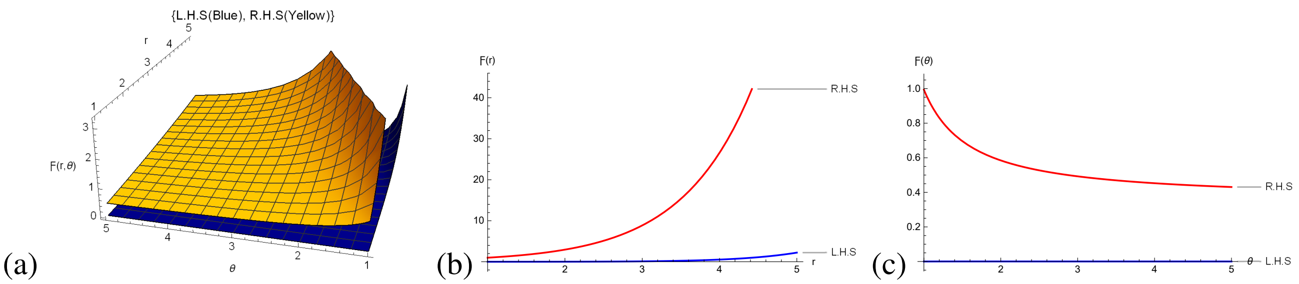

Figure 6. Graphical visuals of left and right sides of Theorem 2.6.

3. Applications

In this section, we discuss applications of our primary innovations. First, we establish a relationship between the means of nonnegative real numbers by taking into account certain outcomes from the previous section. We also present various numerical integration implementations.

3.1. The quadrature formula

Suppose a partition $\mathcal{P}: m_1 = \omega_0 < \omega_1 < ... < \omega_{n-1} < \omega_n = m_2$ is obtained by dividing the interval $[m_1, m_2]$ into n subintervals $[\omega_i, \omega_{i+1}]$ with $i = 0, 1, \ldots, n-1$, then

$$\int_{m_1}^{m_2} F(\omega)dx = T(\omega) + R(\omega).$$

Here,

$$T(\omega) = \frac{m_2 - m_1}{8}\left[3F\left(\frac{5m_1 + m_2}{6}\right) + 2F\left(\frac{m_1 + m_2}{2}\right) + 3F\left(\frac{m_1 + 5m_2}{6}\right)\right],$$

and $R(\omega)$ denotes the error term.

Proposition 3.1. *All the conditions of Theorem 2.1 are fulfilled, then*

$$|R(\omega)| \leq \sum_{i=0}^{n-1} \frac{25(\omega_{i+1} - \omega_i)}{576}(|F'(\omega_i)| + |F'(\omega_{i+1})|).$$

Proof. The proof is simply attained by applying the sum from $i = 0$ to $n-1$ over subinterval $[\omega_i, \omega_{i+1}]$ in Theorem 2.1. □

Proposition 3.2. *All the conditions of Theorem 2.4 are fulfilled, so*

$$\begin{aligned}
|R(\omega)| \leq \sum_{i=0}^{n-1}(\omega_{i+1} - \omega_i)\Bigg\{ & 6\Bigg[\left(\frac{1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{17}{1296}|F'(\omega_i)|^q + \frac{1}{1296}|F'(\omega_{i+1})|^q\right)^{\frac{1}{q}} \\
& + \left(\frac{1}{6^{p+2}(p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{81}|F'(\omega_i)|^q + \frac{1}{648}|F'(\omega_{i+1})|^q\right)^{\frac{1}{q}}\Bigg] \\
& + 3\Bigg[\left(\frac{3^{p+2} + 5^{p+1}(11 + 8p)}{24^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{13}{324}|F'(\omega_i)|^q + \frac{5}{324}|F'(\omega_{i+1})|^q\right)^{\frac{1}{q}} \\
& + \left(\frac{5^{p+2} + 3^{p+1}(13 + 8p)}{24^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{11}{324}|F'(\omega_i)|^q + \frac{7}{324}|F'(\omega_{i+1})|^q\right)^{\frac{1}{q}}\Bigg] \\
& + 3\Bigg[\left(\frac{5^{p+2} + 3^{p+1}(13 + 8p)}{24^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{7}{324}|F'(\omega_i)|^q + \frac{11}{324}|F'(\omega_{i+1})|^q\right)^{\frac{1}{q}} \\
& + \left(\frac{3^{p+2} + 5^{p+1}(13 + 8p)}{24^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{5}{324}|F'(\omega_i)|^q + \frac{13}{324}|F'(\omega_{i+1})|^q\right)^{\frac{1}{q}}\Bigg] \\
& + 6\Bigg[\left(\frac{1}{6^{p+2}(p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{648}|F'(\omega_i)|^q + \frac{1}{81}|F'(\omega_{i+1})|^q\right)^{\frac{1}{q}} \\
& + \left(\frac{1}{6^{p+2}(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{17}{1296}|F'(\omega_i)|^q + \frac{1}{1296}|F'(\omega_{i+1})|^q\right)^{\frac{1}{q}}\Bigg]\Bigg\}.
\end{aligned}$$

Proof. The proof is simply attained by applying the sum from $i = 0$ to $n-1$ over subinterval $[\omega_i, \omega_{i+1}]$ in Theorem 2.4. □

Proposition 3.3. *All the conditions of Theorem 2.5 are fulfilled, then*

$$|R(\omega)| \leq \sum_{i=0}^{n-1} \frac{25M(\omega_{i+1}-\omega_i)}{288}.$$

Proof. The proof is simply attained by applying the sum from $i = 0$ to $n-1$ over subinterval $[\omega_i, \omega_{i+1}]$ in Theorem 2.5. □

3.2. Applications to means

We recall some notable means for non-negative real numbers.

(1) The arithmetic mean:

$$A(m_1, m_2) = \frac{m_1 + m_2}{2}.$$

(2) The Weighted arithmetic mean:

$$_{w}A(w_1, w_2; m_1, m_2) = \frac{m_1 w_1 + m_2 w_2}{w_1 + w_2}.$$

(3) The log-mean:

$$L_r(m_1, m_2) = \left[\frac{{m_2}^{r+1} - {m_1}^{r+1}}{(r+1)(m_2 - m_1)}\right]^{\frac{1}{r}}; \quad r \in \Re \setminus \{-1, 0\}.$$

Proposition 3.4. *From Theorem 2.2, we get*

$$\begin{aligned}
&\left|\frac{m}{8(r+2\theta)}\left[3A^{\frac{r}{\theta}+2}\left(m_1, m_2, \frac{5}{6}, \frac{1}{6}\right) + 2A^{\frac{r}{\theta}+2}(m_1, m_2) + 3A^{\frac{r}{\theta}+2}\left(m_1, m_2, \frac{1}{6}, \frac{5}{6}\right)\right] - 3L_{\frac{r}{\theta}+2}^{\frac{r}{\theta}+2}(m_1, m_2)\right| \\
&\leq (m_2 - m_1)\left[\left(\frac{1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{36}A\left(11{m_1}^{(\frac{r}{\theta}+1)q}, {m_2}^{(\frac{r}{\theta}+1)q}\right)\right)^{\frac{1}{q}}\right. \\
&\quad + \left(\frac{1}{p+1}\left(\left(\frac{5}{24}\right)^{p+1} + \left(\frac{1}{8}\right)^{p+1}\right)\right)^{\frac{1}{p}}\left(\frac{2}{9}A\left(2{m_1}^{(\frac{r}{\theta}+1)q}, {m_2}^{(\frac{r}{\theta}+1)q}\right)\right)^{\frac{1}{q}} \\
&\quad + \left(\frac{1}{p+1}\left(\left(\frac{1}{8}\right)^{p+1} + \left(\frac{5}{24}\right)^{p+1}\right)\right)^{\frac{1}{p}}\left(\frac{2}{9}A\left({m_1}^{(\frac{r}{\theta}+1)q}, 2{m_2}^{(\frac{r}{\theta}+1)q}\right)\right)^{\frac{1}{q}} \\
&\quad \left. + \left(\frac{1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{36}A\left({m_1}^{(\frac{r}{\theta}+1)q}, 11{m_2}^{(\frac{r}{\theta}+1)q}\right)\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. The statement of claim is directly followed by substitution $F(z) = \frac{\theta}{r+2\theta} z^{\frac{r}{\theta}+2}$ in Theorem 2.2. □

Proposition 3.5. *From Theorem 2.3, we have*

$$\left|\frac{m}{8(r+2\theta)}\left[3A^{\frac{r}{\theta}+2}\left(m_1, m_2, \frac{5}{6}, \frac{1}{6}\right) + 2A^{\frac{r}{\theta}+2}(m_1, m_2) + 3A^{\frac{r}{\theta}+2}\left(m_1, m_2, \frac{1}{6}, \frac{5}{6}\right)\right] - 3L_{\frac{r}{\theta}+2}^{\frac{r}{\theta}+2}(m_1, m_2)\right|$$

$$\leq (m_2 - m_1)\left[\left(\frac{1}{72}\right)^{1-\frac{1}{q}}\left(\frac{1}{81}A\left({m_1}^{(\frac{r}{\theta}+1)q}, 8{m_2}^{(\frac{r}{\theta}+1)q}\right)\right)^{\frac{1}{q}} + \left(\frac{17}{576}\right)^{1-\frac{1}{q}}\left(\frac{1}{20736}A\left(863{m_1}^{(\frac{r}{\theta}+1)q}, 361{m_2}^{(\frac{r}{\theta}+1)q}\right)\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{17}{576}\right)^{1-\frac{1}{q}}\left(\frac{1}{20736}A\left(361{m_1}^{(\frac{r}{\theta}+1)q}, 863{m_2}^{(\frac{r}{\theta}+1)q}\right)\right)^{\frac{1}{q}} + \left(\frac{1}{72}\right)^{1-\frac{1}{q}}\left(\frac{1}{81}A\left(8{m_1}^{(\frac{r}{\theta}+1)q}, {m_2}^{(\frac{r}{\theta}+1)q}\right)\right)^{\frac{1}{q}}\right].$$

Proof. By substituting $F(z) = \frac{\theta}{r+2\theta} z^{\frac{r}{\theta}+2}$ in Theorem 2.3, we attain our desired result. □

3.3. Applications to probability functions

Assume that X be a convex with respect to probability density mapping, $p : [m_1, m_2] \rightarrow [0, 1]$, and cumulative distribution mapping is explored as:

$$Pr(X \leq m_2) = F(m_2) = \int_{m_1}^{m_2} p(z)dz.$$

Utilizing the fact that

$$E(X) = \int_{m_1}^{m_2} zp(z)dz$$
$$E(X) = m_2 - \int_{m_1}^{m_2} p(z)dz.$$

Proposition 3.6. *Through Theorem 2.1, we have*

$$\left|\frac{1}{8}\left[3Pr\left(X \leq \frac{5m_1 + m_2}{6}\right) + 2Pr\left(X \leq \frac{m_1 + m_2}{2}\right) + 3Pr\left(X \leq \frac{m_1 + 5m_2}{6}\right)\right] - \frac{m_2 - E(X)}{m_2 - m_1}\right| \tag{3.1}$$
$$\leq \frac{25(m_2 - m_1)}{576}(|p(m_1)| + |p(m_2)|).$$

Proof. The assertion follows directly by making use of probability density mapping in Theorem 2.1. □

Proposition 3.7. *Through Theorem 2.1, we have*

$$\left|\frac{1}{8}\left[3Pr\left(X \leq \frac{5m_1 + m_2}{6}\right) + 2Pr\left(X \leq \frac{m_1 + m_2}{2}\right) + 3Pr\left(X \leq \frac{m_1 + 5m_2}{6}\right)\right] - \frac{m_2 - E(X)}{m_2 - m_1}\right| \tag{3.2}$$
$$\leq \frac{25M(m_2 - m_1)}{288}. \tag{3.3}$$

Proof. The assertion follows directly by making use of probability density mapping in Theorem 2.5. □

4. Application to numerical scheme

Algorithm 4.1. *Suppose we have a non-linear equation* $F(\omega) = 0$*, then*

$$\omega_{n+1} = \omega_n - \frac{8F(\omega_n)}{3F'\left(\frac{5\omega_n+\omega_{1n}}{6}\right) + 2F'\left(\frac{\omega_n+\omega_{1n}}{2}\right) + 3F'\left(\frac{\omega_n+5\omega_{1n}}{6}\right)}, \tag{4.1}$$

where

$$\omega_{1n} = \omega_n - \frac{F(\omega_n)}{F'(\omega_n)}.$$

Proof. From Theorem 2.1 applying the result over $[\omega_n, \omega]$ for F', we achieve the following result,

$$-F(\omega_n) = \frac{\omega - \omega_n}{8}\left[3F'\left(\frac{5\omega_n + \omega}{6}\right) + 2F'\left(\frac{\omega_n + \omega}{2}\right) + F'\left(\frac{\omega_1 + 5\omega}{6}\right)\right].$$

This implies that

$$\omega = \omega_n - \frac{8F(\omega_n)}{3F'\left(\frac{5\omega_n+\omega}{6}\right) + 2F'\left(\frac{\omega_n+\omega}{2}\right) + F'\left(\frac{\omega_1+5\omega}{6}\right)}.$$

Taking Newton's method as predictor, we obtained our desired Algorithm 4.1. □

4.1. Numerical analysis

In the following subsequent portion, we demonstrate the numerical analysis of the developed scheme 4.1. For this, we consider some physical problems.

(1) The first problem we consider is the Blood Rheology and Fractional Non-Linear Equations Model [24]. Since blood is Casson fluid. To discuss the plug flow of Casson fluids, we take into account the given non-linear fractional equation, where a decrease in flow rate is measured by:

$$F(\omega) = 1 - \frac{16}{7}\sqrt{\omega} + \frac{4}{3}\omega - \frac{1}{21}\omega^4 - G,$$

where reduction in flow rate is measured by $G = 0.4$. If we choose an initial guess of $\omega_0 = 0.1$. Then, we use using the proposed Algorithm 4.1, which gives the required solution $\omega = 0.104698651536548228 12$ in four iterations.

(2) For the second problem, we consider is the Fluid Permeability in Biogels [24]. The link between pressure gradient and fluid velocity in porous media (such as agarose gel or extracellular fibre matrix) is shown by the nonlinear equation below:

$$F(\omega) = \Re_e\omega^3 - 20\kappa(1 - \omega)^2,$$

where $\Re_e = 10\times10^{-9}$ and $\kappa = 0.3655$. Using initial guess of $\omega_0 = 2$, we have noticed that the proposed Algorithm 4.1 reached at destination $\omega = 1.0000369883881891758$ in twenty iterations.

(3) The third problem is [25]:

$$F(\omega) = \frac{\omega}{1 - \omega} - 5\log\left[\frac{0.4(1 - \omega)}{0.4 - 0.5\omega}\right] + 4.45977, \tag{4.2}$$

where ω specifies the conversion of species A in chemical reactor and $\omega \in [0, 1]$, for other value of ω there does not exist any physical meanings. Using Algorithm 4.1, we have searched the solution $\omega = 0.75739624625375387946$ after four iterations. Note that expression (4.2) becomes undefined when $\omega \in [0.8, 1]$. The derivative of (4.2) approaches to 0 when $\omega \in [0, 0.5]$. Thus, we assume the initial approximation to be $\omega_0 = 0.76$ for the current problem.

We now present some more numerical experiments and study the comparison analysis of Algorithm 4.1. For this, we consider the following different types of non-linear equations:

(1) $F(\omega) = \omega^3 + 4\omega^2 - 15$,

(2) $F(\omega) = xe^{\omega^2} - \sin^2 \omega + 3\cos\omega + 5$,

(3) $F(\omega) = 10\omega e^{-\omega^2} - 1$,

(4) $F(\omega) = e^{-\omega} + \cos\omega$.

We compare our proposed method Algorithm 4.1 with well-known techniques such as the Newton method (NM) [26], Abbasbandy's method (AM) [27], Halley's method (HM) [26] and Chun's method (CM) [28]. To determine the approximate root, we employ a tolerance of $\epsilon = 10^{-15}$. The subsequent termination conditions are utilized for computer algorithms:

(1) $|\omega_{n+1} - \omega_n| < \epsilon$,

(2) $|F(\omega_{n+1})| < \epsilon$.

Numerical tests were conducted on an Intel(R) Core(TM) i5 processor with 1.60 GHz and 16GB RAM. Maple 2020 was used for coding, while graphical analysis was carried out using Matlab 2021. After carrying out numerical tests on the software below we present tabular as well as visual illustrations of Algorithm 4.1 for the above mentioned examples in Table 1.

Table 1. Comparison of different methods for various examples.

Methods	ω_0	IT	ω_n	$F(\omega_n)$	δ
NM	2	6	1.6319808055660635175	0	0
AM	2	4	1.6319808055660635175	0	0
HM	2	4	1.6319808055660635175	0	0
CM	2	4	1.6319808055660635175	0	0
ALG	2	4	1.6319808055660635175	0	0
NM	−1	6	−1.2076478271309189270	4.0×10^{-19}	7.58×10^{-17}
AM	−1	5	−1.2076478271309189270	4.0×10^{-19}	0
HM	−1	4	−1.2076478271309189270	4.0×10^{-19}	0
CM	−1	5	−1.2076478271309189270	4.0×10^{-19}	0
ALG	−1	5	−1.2076478271309189270	4.0×10^{-19}	0
NM	1.8	5	1.6796306104284499407	-9×10^{-20}	4.7395×10^{-15}
AM	1.8	4	1.6796306104284499407	-9×10^{-20}	1.0×10^{-19}
HM	1.8	4	1.6796306104284499407	-9×10^{-20}	0
CM	1.8	4	1.6796306104284499407	2.0×10^{-19}	0
ALG	1.8	4	1.6796306104284499407	-9×10^{-20}	0
NM	2	5	1.7461395304080124176	6.0×10^{-20}	1.0×10^{-19}
AM	2	4	1.7461395304080124176	-6×10^{-20}	1.0×10^{-19}
HM	2	4	1.7461395304080124176	6.0×10^{-20}	1.0×10^{-19}
CM	2	3	1.7461395304080124176	-6×10^{-20}	4.63×10^{-17}
ALG	2	4	1.7461395304080124176	6×10^{-20}	1.0×10^{-19}

4.2. Basins of attraction

Here, we briefly describe the Algorithm 4.1 through the basins of attraction. We deploy our proposed Algorithm on $[-2, 2]\times[-2, 2]$ with a 1000×1000 points grid by fixing the tolerance $|F(\omega_n)| < 1 \times 10^{-10}$ and the maximum number of iterations is 50. Along with this, we also present probability distributions of the required iterations for obtaining the basins of attraction. The red line in the plots will indicate the most probable number of iterations. For this purpose, we consider these examples $\omega^2 - 1$, $\omega^3 - 1$, $\omega^5 - 1$, and $\omega^4 + 1$.

Figure 7 contains the basin of attraction of above mentioned examples, while Figure 8 provides us the probability distribution of these examples based on the number of iterations.

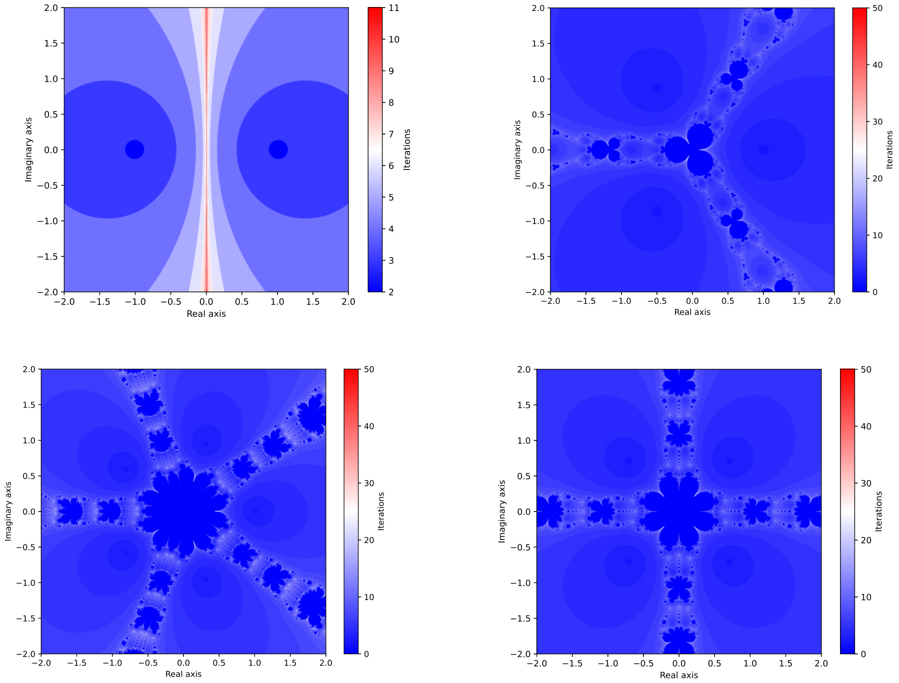

Figure 7. Basins of attraction.

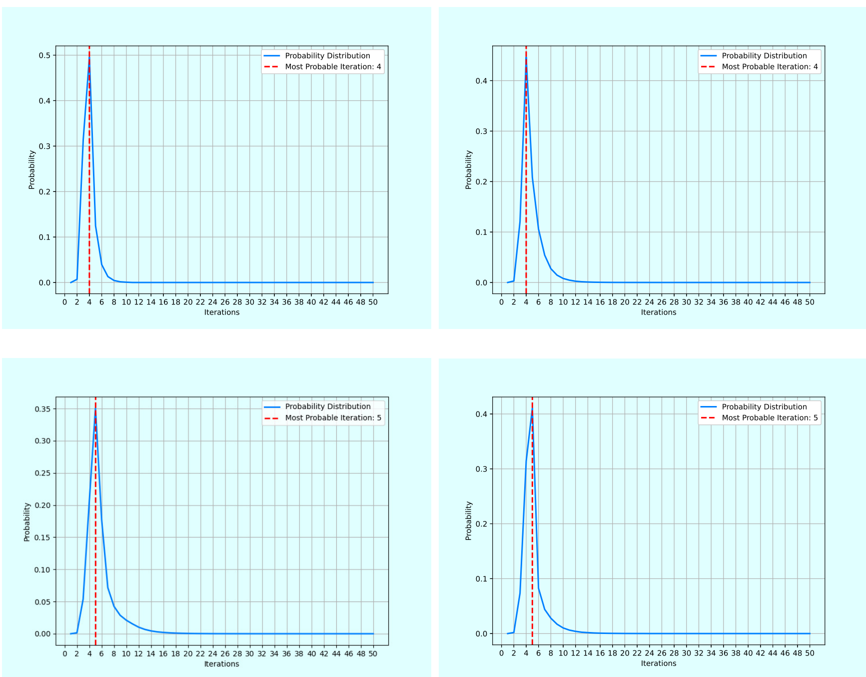

Figure 8. Probability distribution plots.

5. Conclusions

One of the featured aspects of research regarding error analysis is to establish new error bounds for quadrature rules involving different strategies. Integral inequalities are the main source for evaluating the error estimations of numerical integration rules. In this study, we develop new integral inequalities of Euler-Maclaurin's type involving convex mappings. It is worth mentioning that our results provide better upper bounds compared to other results for first-order differentiable mappings. Also, we derive various applications for means, quadrature rules, and novel iterative schemes. We establish a differentiable identity that enables us to achieve several other bounds through various classes of functions, such as strong and uniform convex functions, Breckner convex functions, exponential convex functions, and Godunova-Levin convexity. In the future, we will attempt to establish some tight bounds of this inequality through higher-order differentiable mappings and other functional classes in different frameworks. We hope the methodology and idea of the paper will create new research dimensions.

Author contributions

Miguel Vivas-Cortez: Methodology, software, validation, formal analysis, investigation, writing-review and editing, visualization; Usama Asif: Conceptualization, methodology, software, validation, formal analysis, investigation, writing-original draft preparation, writing-review and editing, visualization; Muhammad Zakria Javed: Conceptualization, methodology, software, validation, formal analysis, investigation, writing-original draft preparation, writing-review and editing, visualization; Muhammad Uzair Awan: Conceptualization, methodology, software, validation, formal analysis, investigation, writing-review and editing, visualization, supervision; Yahya Almalki: Software, validation, formal analysis, investigation, writing-review and editing, visualization; Omar Mutab Alsalami: Methodology, software, validation, formal analysis, investigation, writing-review and editing, visualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interests.

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