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Research article

Multi attribute decision-making algorithms using Hamacher Choquet-integral operators with complex intuitionistic fuzzy information

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Abstract: The Choquet integral is a fuzzy measure that serves as an effective aggregation operator for combining a limited number of components into a single set. In 1978, Hamacher introduced the Hamacher t-norm and t-conorm, an expanded version of algebraic t-norms. In this article, we present the aggregation operators for the Choquet integral that utilize the Hamacher t-norms to handle the theory of complex intuitionistic fuzzy values. These operators include the complex intuitionistic fuzzy Hamacher Choquet integral averaging and geometric operators. Additionally, an analysis is conducted on the attributes and special situations of the suggested methodologies. In addition, a novel approach is presented, utilizing newly developed operators for solving multi-attribute decision-making issues with complex intuitionistic fuzzy values. The operational stages of this strategy are thoroughly presented. Finally, we conducted a comprehensive comparison between the proposed methodology and existing approaches, using illustrative examples to validate the effectiveness and demonstrate the advantages of the proposed methods.

Keywords: Choquet-integral; Hamacher t-norms and t-conorms; complex intuitionistic fuzzy sets; average aggregation operators; geometric aggregation operators; decision-making methods
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1. Introduction

In multi-attribute decision-making (MADM) scenarios, particularly in environments that involve complex intuitionistic fuzzy (CIF) values, it is crucial to account for both the interaction between criteria and the logical conjunction of values. The combination of the Choquet Integral, often known as the C-I and the Hamacher t-norm, enables this dual handling of information. The Choquet Integral is known for its ability to model the interaction among criteria through fuzzy measures, making it an effective tool for criteria that influence each other. However, by itself, it does not provide the flexibility needed to define conjunctions of fuzzy values, especially when dealing with varying levels of uncertainty. The Hamacher t-norm complements the Choquet Integral by offering a generalized approach to algebraic conjunctions, which can be adjusted through its parameter to reflect different logical relationships between attributes.

By fusing the Hamacher t-norm with the Choquet Integral, the newly developed operator provides a more robust aggregation framework, particularly suited to MADM problems that involve CIF values. The advantage of this fusion lies in the ability to not only model interactions (via the Choquet integral) but also control the degree of conjunction (via the Hamacher t-norm), resulting in greater flexibility and accuracy in decision-making. This new operator stands apart from other operators due to its ability to handle more complex forms of uncertainty and interaction, offering improved performance in decision-making tasks compared to operators that rely on either Hamacher t-norms or Choquet integral alone. In addition, Wang et al. [2] conducted research on the extraction of knowledge using fuzzy rule-based methods. Moreover, Roman-Flores et al. [3] investigated how to solve several fuzzy differential equations. Dehghan et al. [4] developed the solution for fuzzy linear systems of equations. In 1994, Heiden and Brickmann [5] investigated the process of dividing protein surfaces into segments using fuzzy logic. In addition, Atanassov [6] made modifications to the FS and developed a new theory known as intuitionistic FS (IFS). The IFS consists of the degrees of both truth and falsity, whereas the FS is a specific instance of the IFS. Several applications have been described in the following manner: Liu et al. [7] introduced the linguistic IFS, Xie et al. [8] discussed data quality for IFS, Liu et al. [9] explored internet human decision for IFST, Garg et al. [10] investigated the cubic IFS, Wang et al. [11] evaluated the probabilistic dominance relation for IFS, and Ecer [12] presented the MAIRCA for IFS. Zhang et al. [13] used the fuzzy proportional-integral-derivative for packaging gas distribution system. Garg et al. [14] Schweizer and Sklar developed the prioritized operators for IFS, whereas Mahmood et al. [15] suggested the power operators for intuitionistic hesitation. Both FS and IFS have focused solely on the amplitude term, neglecting the phase. As a result, a large amount of data has been lost throughout the decision-making process. It has been observed that the inclusion of the phase term offers numerous advantages, particularly in situations involving two-dimensional data. For example, when a potential buyer, referred to as "A", visits a car showroom to purchase a car, the owner provides two types of data for each vehicle: The name of the car (representing the real part) and the production data of the car (representing the imaginary part). It is important to note that the FS is unable to evaluate this type of data. Thus, Ramot et al. [16] developed the complex FS (CFS) to represent the truth grade using complex numbers. The real and imaginary components of the complex number lie within the unit interval. In addition, Liu et al. [17] developed the distance measure for CFS, whereas Mahmood et al. [18] assessed the interdependence of complex fuzzy neighborhood operators. In addition, the role of falsity grades in CFS is absent. Instead, Alkouri et al. [19] derived the complex IFS (CIFS). The theory of CIFS has garnered significant attention from various scholars, resulting in numerous applications. For example, Mahmood et al. [20] introduced the Aczel-Alsina power operators, Garg et al. [21] explored the trigonometric operators, Azeem et al. [22] investigated the

Einstein operators, Garg et al. [23] examined the geometric operators, and Ali et al. [24] presented the prioritized operators.

Prior to investigating the primary subject of the proposed work, it is important to examine the fundamental concepts that are highly beneficial for the proposed work. As a result, our initial focus was on Hamacher's [25] 1975 theory of the t-norm and t-conorm. Hamacher norms are a modified approach to algebraic norms. In addition, Choquet [26] conducted an examination of the Choquet integral in 1953. Several researchers have applied the Hamacher norms, Choquet integral, or both in their work. Huang [27] created the Hamacher operator for IFS, Akram et al. [28] created the Hamacher operator for CIFS, Xu [29] described the Choquet integral for weighted IFS, Wang et al. [30] created the Choquet integral based on averaging operators for IFS, Tan et al. [31,32] examined the Choquet integral operators for IFS and induced IFS, and Mahmood et al. [33] created the Hamacher Choquet-integral operators for IFS. It has been observed that no one has suggested the theory of Choquet integral for CIFS, and the theory of Hamacher Choquet-integral operators for CIFS has also not been obtained. In 2023, Mahmood et al. [34], introduced Aczel Alsina aggregation operators. Ejegwa et al. [35] present the applications of emergency management and pattern recognition using intuitionistic fuzzy similarity operators. In 2023, Akram et al. [36] defined a new decision model by combination of CIF with Hamacher aggregation operators. In 2023, Al-Qubati [37], presented Choquet integral aggregation operators with TOPSIS technique and using the Hamacher norm for complex intuitionistic fuzzy set. Some bibliometric analysis on decision-making analysis is given by the various authors and are summarized in Ref. [38–41]. Our main focus of this study is to assess the provided information, including

(1) To introduce the Hamacher operating rules for the CIF values.
(2) To obtain the CIFHC-IA operator, CIFHC-IOA operator, CIFHC-IG operator and CIFHC-IOG operator.
(3) Additionally, an analysis is conducted on the attributes and special situations of the suggested approaches.
(4) The operational phases for MADM issues with CIF values were shown in detail to introduce a novel method based on the created operators.
(5) Finally, a comparison study using the shown cases is provided between the proposed and current methodologies to demonstrate the superiority and validity of the developed approaches.

This article is sectioned as follows: Section 2 has a comprehensive discussion of CIFSs, fuzzy measures, Choquet integral, Hamacher t-norm, and Hamacher t-conorm, all of which are basic concepts related to a specific set $\mathbb{X}$. In Section 3, we introduce the Hamacher operational laws, namely the CIFHC-IA operator, CIFHC-IOA operator, CIFHC-IG operator, and CIFHC-IOG operator. Additionally, an analysis of the attributes and special situations of the suggested methodologies is conducted. In Section 4, a novel approach is presented utilizing newly devised operators for Multiple Attribute Decision Making (MADM) issues with CIF values. The procedural stages were thoroughly illustrated. In Section 5, we provide a comparative examination of the proposed and current methodologies, using illustrative examples to demonstrate the superiority and validity of the derived approaches. The final and conclusive observations are presented in Section 6.

2. Preliminaries

In this section, we present the basic concepts of CIFSs, fuzzy measures, Choquet integral, Hamacher t-norm, and Hamacher t-conorm for a certain set $\mathbb{X}$.

Definition 1. [19] A CIFS E_{ci} is structured in the following manner:

$$E_{ci} = \left\{\left(\mathfrak{W}_{\mathfrak{Y}}(\mathfrak{b}), \mathfrak{N}_{\mathfrak{Y}}(\mathfrak{b})\right) : \mathfrak{b} \in \mathbb{X}\right\} \tag{1}$$

where $\mathfrak{W}_{\mathfrak{Y}}(\mathfrak{b}) = \left(\mathfrak{W}_{\mathfrak{Y}}^{R}(\mathfrak{b}), \mathfrak{W}_{\mathfrak{Y}}^{I}(\mathfrak{b})\right)$ and $\mathfrak{N}_{\mathfrak{Y}a}(\mathfrak{b}) = \left(\mathfrak{N}_{\mathfrak{Y}}^{R}(\mathfrak{b}), \mathfrak{N}_{\mathfrak{Y}}^{I}(\mathfrak{b})\right)$ are memberships, and non-membership is represented by a complicated integer with two significant attributes, such as $0 \le \mathfrak{W}_{\mathfrak{Y}}^{R}(\mathfrak{b}) + \mathfrak{N}_{\mathfrak{Y}}^{R}(\mathfrak{b}) \le 1$ and $0 \le \mathfrak{W}_{\mathfrak{Y}}^{I}(\mathfrak{b}) + \mathfrak{N}_{\mathfrak{Y}}^{I}(\mathfrak{b}) \le 1$. Additionally, the computed structure $\mathfrak{K}_r(\mathfrak{b}) = \left(\mathfrak{K}_r^{R}(\mathfrak{b}), \mathfrak{K}_r^{I}(\mathfrak{b})\right) = \left(1 - \left(\mathfrak{W}_{\mathfrak{Y}}^{R}(\mathfrak{b}) + \mathfrak{N}_{\mathfrak{Y}}^{I}(\mathfrak{b})\right), 1 - \left(\mathfrak{W}_{\mathfrak{Y}}^{R}(\mathfrak{b}) + \mathfrak{N}_{\mathfrak{Y}}^{I}(\mathfrak{b})\right)\right)$ represents the value of neutral information with the simple form of CIF number (CIFN), such as $E_{ci}^{*} = \left(\left(\mathfrak{W}_{\mathfrak{Y}_*}^{R}, \mathfrak{W}_{\mathfrak{Y}_*}^{I}\right), \left(\mathfrak{N}_{\mathfrak{Y}_*}^{R}. \mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)\right), * = 1,2, \dots, \imath$.

Definition 2. [20] For a CIFN $E_{ci}^{*} = \left(\left(\mathfrak{W}_{\mathfrak{Y}_*}^{R}, \mathfrak{W}_{\mathfrak{Y}_*}^{I}\right), \left(\mathfrak{N}_{\mathfrak{Y}_*}^{R}. \mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)\right), * = 1$. The focus has been on reviewing the concept of score and accuracy function, such as

$$E_{S-ci}^{*} = \frac{1}{2}\left(\mathfrak{W}_{\mathfrak{Y}_*}^{R} - \mathfrak{N}_{\mathfrak{Y}_*}^{R} + \mathfrak{W}_{\mathfrak{Y}_*}^{I} - \mathfrak{N}_{\mathfrak{Y}_*}^{I}\right) \in [-1,1] \tag{2}$$

$$E_{H-ci}^{*} = \frac{1}{2}\left(\mathfrak{W}_{\mathfrak{Y}_*}^{R} + \mathfrak{N}_{\mathfrak{Y}_*}^{R} + \mathfrak{W}_{\mathfrak{Y}_*}^{I} + \mathfrak{N}_{\mathfrak{Y}_*}^{I}\right) \in [\Xi, 1]. \tag{3}$$

Here, some rules for the data in Eqs (2) and (3) are also explained, such as

(1) When $E_{S-ci}^{1} > E_{S-ci}^{2}$, then $E_{ci}^{1} > E_{ci}^{2}$.

(2) When $E_{S-ci}^{1} < E_{S-ci}^{2}$, then $E_{ci}^{1} < E_{ci}^{2}$.

(3) When $E_{S-ci}^{1} = E_{S-ci}^{2}$, then

  1) When $E_{H-ci}^{1} > E_{H-ci}^{2}$, then $E_{ci}^{1} > E_{ci}^{2}$.

  2) When $E_{H-ci}^{1} < E_{H-ci}^{2}$, then $E_{ci}^{1} < E_{ci}^{2}$.

Definition 3. [25] The fundamental concept of Hamacher t-norm and Hamacher t-conorm has been re-examined for any combination of positive values.

$$\eth \oplus \mathfrak{o} = \frac{\eth\mathfrak{o}}{\varrho + (1-\varrho)(\eth + \mathfrak{o} - \eth\mathfrak{o})}, \tag{4}$$

$$\eth \otimes \mathfrak{o} = \frac{\eth + \mathfrak{o} - \eth\mathfrak{o} - (1-\varrho)\eth\mathfrak{o}}{1-(1-\varrho)\eth\mathfrak{o}}. \tag{5}$$

In addition, by varying the parameter ϱ, several types of t-norms can be obtained. For example, by substituting $\varrho = 1$ into Eqs (4) and (5), the following result is obtained:

$$\eth \oplus \mathfrak{o} = \eth\mathfrak{o}, \tag{6}$$

$$\eth \otimes \mathfrak{o} = \eth + \mathfrak{o} - \eth\mathfrak{o}. \tag{7}$$

The data in Eqs (6) and (7) represent the mathematical form of algebraic t-norms. When $\varrho = 2$ is substituted into Eqs (4) and (5), the following result is obtained:

$$\eth \oplus \mathfrak{o} = \frac{\eth\mathfrak{o}}{1+(1-\eth)(1-\mathfrak{o})}, \tag{8}$$

$$\eth \otimes \mathfrak{o} = \frac{\eth + \mathfrak{o}}{1+\eth\mathfrak{o}}. \tag{9}$$

The data in Eqs (8) and (9) are stated the mathematical shape of Einstein t-norms.

Definition 4. [26] The major idea of Choquet integral based on the fuzzy measure is described below:

$$\int \not{f} dE = \sum_{*=1}^{\daleth} \left(E\left(\bar{\bar{E}}_{\Xi(*)}\right) - E\left(\bar{\bar{E}}_{\Xi(*-1)}\right) \right) \not{f}_{\Xi(*)} \tag{10}$$

$$E\left(\bar{\bar{E}}\right) = E\left(\coprod_{*=1}^{\daleth} \mathfrak{b}_*\right) = \begin{cases} \frac{1}{\blacksquare}\left[\prod_{*=1}^{\daleth}\left(1+\gamma E(\mathfrak{b}_*)\right)-1\right] & \blacksquare \neq \Xi \\ \sum_{\mathfrak{b}_* \in A} E(\mathfrak{b}_*) & \blacksquare = \Xi \end{cases} \tag{11}$$

where $\Xi(*)$ represents permutations for $(1,2,\dots,\daleth)$ with of $\not{f}_{\Xi(1)} \geq \not{f}_{\Xi(2)} \geq \cdots \geq \not{f}_{\Xi(\daleth)}$ and $\bar{\bar{E}} = \theta$, $\bar{\bar{E}}_{\Xi(*)} = \left\{E'_{\Xi(1)}, E'_{\Xi(2)}, \dots, E'_{\Xi(*)}\right\}$.

3. CIF Hamacher C-I operators

In section, we present the original concept of Hamacher operational laws for CIFNs. Subsequently, the theory of four operators CIFHC-IA, CIFHC-IOA, CIFHC-IG, and CIFHC-IOG, is assessed using these operational laws. The Hamacher parameter plays a crucial role in the decision-making analysis by influencing the aggregation process in fuzzy logic or multi-criteria decision analysis (MCDA). Specifically, it controls the degree of interaction between the criteria being evaluated. A higher value of the Hamacher parameter tends to emphasize the more influential criteria, leading to a stronger impact of the most significant factors on the final decision. Conversely, a lower value of the Hamacher parameter tends to distribute influence more evenly among the criteria, resulting in a more balanced aggregation.

Moreover, some desirable properties and important results are also examined in this section for the collection of CIFNs $E_{ci}^{*} = \left(\left(\mathfrak{W}_{\mathfrak{Y}_*}^{R}, \mathfrak{W}_{\mathfrak{Y}_*}^{I}\right), \left(\mathfrak{N}_{\mathfrak{Y}_*}^{R}. \mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)\right), *= 1,2,\dots,\daleth$. Here, the Hamacher operations for CIFNs have been examined or derived, such as

$E_{ci}^{1} \oplus E_{ci}^{2} \oplus \dots \oplus E_{ci}^{\daleth} =$

$$\left(\begin{array}{c}\left(\frac{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)-\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)}, \frac{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)-\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)}\right), \\ \left(\frac{\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)-\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}-\mathfrak{N}_{\mathfrak{Y}a_*}^{R}\right)}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)}, \frac{\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)-\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}-\mathfrak{N}_{\mathfrak{Y}a_*}^{I}\right)}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)}\right)\end{array}\right) \tag{12}$$

$E_{ci}^{1} \otimes E_{ci}^{2} \otimes \dots \otimes E_{ci}^{\daleth} =$

$$\left(\begin{array}{c}\left(\frac{\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)-\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)}, \frac{\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)-\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}-\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)}\right), \\ \left(\frac{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)-\prod_{*=1}^{\daleth}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)}, \frac{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)-\prod_{*=1}^{\daleth}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)}\right)\end{array}\right) \tag{13}$$

$$\blacksquare E_{ci}^{1} = \left(\begin{array}{c}\left(\frac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\blacksquare}-\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\blacksquare}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\blacksquare}-(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\blacksquare}}, \frac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\blacksquare}-\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\blacksquare}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\blacksquare}-(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\blacksquare}}\right), \\ \left(\frac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\blacksquare}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{R}-\mathfrak{N}_{\mathfrak{Y}_1}^{R}\right)^{\blacksquare}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\blacksquare}-(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\blacksquare}}, \frac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\blacksquare}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{I}-\mathfrak{N}_{\mathfrak{Y}_1}^{I}\right)^{\blacksquare}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\blacksquare}-(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\blacksquare}}\right)\end{array}\right) \tag{14}$$

$$E_{ci}^{1\blacksquare} = \left(\begin{array}{c}\left(\frac{\varrho\left(1-\mathfrak{N}_{\mathfrak{Y}_1}^R\right)^{\blacksquare}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^R-\mathfrak{N}_{\mathfrak{Y}_1}^R\right)^{\blacksquare}}{\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_1}^R\right)^{\blacksquare}-(\varrho-1)\left(1-\mathfrak{N}_{\mathfrak{Y}_1}^R\right)^{\blacksquare}}, \frac{\varrho\left(1-\mathfrak{N}_{\mathfrak{Y}_1}^I\right)^{\blacksquare}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^I-\mathfrak{N}_{\mathfrak{Y}_1}^I\right)^{\blacksquare}}{\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_1}^I\right)^{\blacksquare}-(\varrho-1)\left(1-\mathfrak{N}_{\mathfrak{Y}_1}^I\right)^{\blacksquare}}\right), \\ \left(\frac{\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_1}^R\right)^{\blacksquare}-\left(1-\mathfrak{N}_{\mathfrak{Y}_1}^R\right)^{\blacksquare}}{\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_1}^R\right)^{\blacksquare}-(\varrho-1)\left(1-\mathfrak{N}_{\mathfrak{Y}_1}^R\right)^{\blacksquare}}, \frac{\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_1}^I\right)^{\blacksquare}-\left(1-\mathfrak{N}_{\mathfrak{Y}_1}^I\right)^{\blacksquare}}{\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_1}^I\right)^{\blacksquare}-(\varrho-1)\left(1-\mathfrak{N}_{\mathfrak{Y}_1}^I\right)^{\blacksquare}}\right)\end{array}\right). \tag{15}$$

Definition 5. For the finite collection of CIFNs, the theory of the CIFHC-IA operator is stated as follows:

$$\int E_{ci}dE = CIFHC - IA(E_{ci}^1, E_{ci}^2, \dots, E_{ci}^{\daleth})$$
$$= \left(E\left(\bar{\bar{E}}_{\Xi(1)}\right) - E\left(\bar{\bar{E}}_{\Xi(\Xi)}\right)\right)E_{ci}^1 \oplus \left(E\left(\bar{\bar{E}}_{\Xi(2)}\right) - E\left(\bar{\bar{E}}_{\Xi(1)}\right)\right)E_{ci}^2 \oplus \dots$$
$$\oplus \left(E\left(\bar{\bar{E}}_{\Xi(\daleth)}\right) - E\left(\bar{\bar{E}}_{\Xi(\daleth-1)}\right)\right)E_{ci}^{\daleth} = \sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right) - E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)E_{ci}^{*}$$

$$= \oplus_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right) - E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)E_{ci}^{*}. \tag{16}$$

Theorem 1. Considering the data in Eq (16), it is proven that the aggregated value is again a CIFN, such as:

$$CIFHC - IA(E_{ci}^1, E_{ci}^2, \dots, E_{ci}^{\daleth})$$
$$= \left(\begin{array}{c}\left(\begin{array}{c}\frac{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^R\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)} - \prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^R\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^R\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)} + (\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^R\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}, \\ \frac{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^I\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)} - \prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^I\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^I\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)} + (\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^I\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}\end{array}\right), \\ \left(\begin{array}{c}\frac{\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^R\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)} - \varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^R-\mathfrak{N}_{\mathfrak{Y}_*}^R\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^R\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)} + (\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^R\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}, \\ \frac{\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^I\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)} - \varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^I-\mathfrak{N}_{\mathfrak{Y}_*}^I\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^I\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)} + (\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^I\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}\end{array}\right)\end{array}\right). \tag{17}$$

Proof. To demonstrate the validity of the facts presented in Eq (17), the method of mathematical induction was utilized. Specifically, when $\daleth = 2$, the following is true:

$$\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(\Xi)}\right)\right)E_{ci}^{1}$$

$$=\left(\begin{array}{c}\left(\frac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}-\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}},\frac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}-\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}}\right),\\ \left(\frac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{R}-\mathfrak{N}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}},\frac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{I}-\mathfrak{N}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(1-1)}\right)\right)}}\right)\end{array}\right)$$

$$\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(1)}\right)\right)E_{ci}^{2}$$

$$=\left(\begin{array}{c}\left(\frac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}-\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}},\frac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}-\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}}\right),\\ \left(\frac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{R}-\mathfrak{N}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}},\frac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{I}-\mathfrak{N}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(2-1)}\right)\right)}}\right)\end{array}\right).$$

Thus,

$$=\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(\Xi)}\right)\right)E_{ci}^{1}\oplus\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(1)}\right)\right)E_{ci}^{2}$$

$$
= \left(\begin{array}{c} \left(\left(\begin{array}{c} \dfrac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}-\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}}, \\ \dfrac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}-\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}} \end{array} \right), \left(\begin{array}{c} \dfrac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{R}-\mathfrak{N}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}}, \\ \dfrac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{I}-\mathfrak{N}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_1}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(1-1)}\right)\right)}} \end{array} \right) \right) \\ \oplus \left(\left(\begin{array}{c} \dfrac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}-\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}}, \\ \dfrac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}-\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}} \end{array} \right), \left(\begin{array}{c} \dfrac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{R}-\mathfrak{N}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}}, \\ \dfrac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{I}-\mathfrak{N}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_2}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(2-1)}\right)\right)}} \end{array} \right) \right) \end{array} \right)
$$

$$
= \left(\begin{array}{c} \left(\begin{array}{c} \dfrac{\prod_{*=1}^{2}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{2}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{2}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{2}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}, \\ \dfrac{\prod_{*=1}^{2}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{2}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{2}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{2}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}} \end{array} \right), \\ \left(\begin{array}{c} \dfrac{\varrho\prod_{*=1}^{2}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{2}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{2}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{2}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}, \\ \dfrac{\varrho\prod_{*=1}^{2}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{2}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}-\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{2}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{2}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}} \end{array} \right) \end{array} \right).
$$

The data in Eq (17) is successfully reliable for $\imath = 2$, moreover, it is assumed that they also hold for $\imath = k$, such as

$$CIFHC-IA\left(E_{ci}^{1},E_{ci}^{2},\dots,E_{ci}^{k}\right)$$

$$=\left(\begin{array}{c}\left(\dfrac{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}},\dfrac{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}\right),\\ \left(\dfrac{\varrho\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}},\dfrac{\varrho\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}-\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}\right)\end{array}\right).$$

Thus, finally, it is proven for $\imath = k+1$, such as

$$\begin{aligned}&CIFHC-IA\left(E_{ci}^{1},E_{ci}^{2},\dots,E_{ci}^{\imath}\right)\\&=\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(\Xi)}\right)\right)E_{ci}^{1}\oplus\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(1)}\right)\right)E_{ci}^{2}\oplus\dots\\&\oplus\left(E\left(\bar{\bar{E}}_{\Xi(k)}\right)-E\left(\bar{\bar{E}}_{\Xi(k-1)}\right)\right)E_{ci}^{k}\oplus\left(E\left(\bar{\bar{E}}_{\Xi(k+1)}\right)-E\left(\bar{\bar{E}}_{\Xi(k+1-1)}\right)\right)E_{ci}^{k+1}\\&=\sum_{*=1}^{k}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)E_{ci}^{*}\oplus\left(E\left(\bar{\bar{E}}_{\Xi(k+1)}\right)-E\left(\bar{\bar{E}}_{\Xi(k+1-1)}\right)\right)E_{ci}^{k+1}\\&=\oplus_{*=1}^{k}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)E_{ci}^{*}\oplus\left(E\left(\bar{\bar{E}}_{\Xi(k+1)}\right)-E\left(\bar{\bar{E}}_{\Xi(k+1-1)}\right)\right)E_{ci}^{k+1}\end{aligned}$$

$$=\left(\begin{array}{c}\left(\dfrac{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}},\dfrac{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}\right),\\ \left(\dfrac{\varrho\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}},\dfrac{\varrho\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}-\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}\right)\end{array}\right)$$

$$\oplus\left(E\left(\bar{\bar{E}}_{\Xi(k+1)}\right)-E\left(\bar{\bar{E}}_{\Xi(k+1-1)}\right)\right)E_{ci}^{k+1}$$

$$
= \left(\begin{pmatrix}\begin{pmatrix}\dfrac{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}, \\ \dfrac{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}\end{pmatrix}, \\ \begin{pmatrix}\dfrac{\varrho\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}, \\ \dfrac{\varrho\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}-\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}\end{pmatrix}\end{pmatrix}\right)
$$

$$
\oplus \left(\begin{pmatrix}\begin{pmatrix}\dfrac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}-\left(1-\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}}, \\ \dfrac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}-\left(1-\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}}\end{pmatrix}, \\ \begin{pmatrix}\dfrac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{R}-\mathfrak{N}_{\mathfrak{Y}_{k+1}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}}, \\ \dfrac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{I}-\mathfrak{N}_{\mathfrak{Y}_{k+1}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}_{k+1}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(k+1)}\right)-E\left(\overline{\overline{E}}_{\Xi(k+1-1)}\right)\right)}}\end{pmatrix}\end{pmatrix}\right)
$$

$$
= \left(\begin{pmatrix}\begin{pmatrix}\dfrac{\prod_{*=1}^{k+1}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{k+1}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k+!}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k+1}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}, \\ \dfrac{\prod_{*=1}^{k+1}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{k+1}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k+1}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k+1}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}\end{pmatrix}, \\ \begin{pmatrix}\dfrac{\varrho\prod_{*=1}^{k+1}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{k+1}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k+1}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k+1}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}, \\ \dfrac{\varrho\prod_{*=1}^{k+1}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{k+1}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}-\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{k+1}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{k+1}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}\end{pmatrix}\end{pmatrix}\right).
$$

The data in Eq (17) is effectively computed for positive values of ı.

To assess or streamline the operators, the fundamental characteristics of the devised theory, including idempotency, monotonicity, and boundedness, were outlined.

Property 1. Several characteristics have been defined for the finite collection of CIFNs, including:

If $E^{*}_{ci} = E_{ci} = \left(\left(\mathfrak{W}^{R}_{\mathfrak{Y}}, \mathfrak{W}^{I}_{\mathfrak{Y}}\right), \left(\mathfrak{N}^{R}_{\mathfrak{Y}}, \mathfrak{N}^{I}_{\mathfrak{Y}}\right)\right), *= 1,2, \dots, \daleth$, thus

$$CIFHC - IA(E^{1}_{ci}, E^{2}_{ci}, \dots, E^{\daleth}_{ci}) = E_{ci}. \tag{18}$$

(1) If $E^{*}_{ci} = \left(\left(\mathfrak{W}^{R}_{\mathfrak{Y}_*}, \mathfrak{W}^{I}_{\mathfrak{Y}_*}\right), \left(\mathfrak{N}^{R}_{\mathfrak{Y}_*}, \mathfrak{N}^{I}_{\mathfrak{Y}_*}\right)\right) \leq E^{**}_{ci} = \left(\left({\mathfrak{W}^{*}_{\mathfrak{Y}_*}}^{R}, {\mathfrak{W}^{*}_{\mathfrak{Y}_*}}^{I}\right), \left({\mathfrak{N}^{*}_{\mathfrak{Y}_*}}^{R}, {\mathfrak{N}^{*}_{\mathfrak{Y}_*}}^{I}\right)\right)$, thus

$$CIFHC - IA(E^{1}_{ci}, E^{2}_{ci}, \dots, E^{\daleth}_{ci}) \leq CIFHC - IA\left({E^{*}}^{1}_{ci}, {E^{*}}^{2}_{ci}, \dots, {E^{*}}^{\daleth}_{ci}\right). \tag{19}$$

(2) If $E^{-}_{ci} = \left(\left(\min_{*} \mathfrak{W}^{R}_{\mathfrak{Y}_*}, \min_{*} \mathfrak{W}^{I}_{\mathfrak{Y}_*}\right), \left(\max_{*} \mathfrak{N}^{R}_{\mathfrak{Y}_*}, \max_{*} \mathfrak{N}^{I}_{\mathfrak{Y}_*}\right)\right)$ and

$E^{+}_{ci} = \left(\left(\max_{*} \mathfrak{W}^{R}_{\mathfrak{Y}_*}, \max_{*} \mathfrak{W}^{I}_{\mathfrak{Y}_*}\right), \left(\min_{*} \mathfrak{N}^{R}_{\mathfrak{Y}_*}, \min_{*} \mathfrak{N}^{I}_{\mathfrak{Y}_*}\right)\right)$, thus

$$E^{-}_{ci} \leq CIFHC - IA(E^{1}_{ci}, E^{2}_{ci}, \dots, E^{\daleth}_{ci}) \leq E^{+}_{ci}. \tag{20}$$

Proof. The mathematical proof of all information is stated below.

(1) If $E^{*}_{ci} = E_{ci} = \left(\left(\mathfrak{W}^{R}_{\mathfrak{Y}}, \mathfrak{W}^{I}_{\mathfrak{Y}}\right), \left(\mathfrak{N}^{R}_{\mathfrak{Y}}, \mathfrak{N}^{I}_{\mathfrak{Y}}\right)\right), *= 1,2, \dots, \daleth$, thus, using the data in Eq (17), it follows that

$$CoIFHC - IA(E^{1}_{ci}, E^{2}_{ci}, \dots, E^{\daleth}_{ci})$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \frac{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}^{R}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)} - \prod_{*=1}^{\daleth}\left(1-\mathfrak{W}^{R}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}^{R}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)} + (\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}^{R}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}, \\ \frac{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}^{I}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)} - \prod_{*=1}^{\daleth}\left(1-\mathfrak{W}^{I}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}^{I}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)} + (\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}^{I}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}} \end{array}\right), \\ \left(\begin{array}{c} \frac{\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}^{R}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)} - \varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}^{R}_{\mathfrak{Y}}-\mathfrak{N}^{R}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}^{R}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)} + (\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}^{R}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}, \\ \frac{\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}^{I}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)} - \varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}^{I}_{\mathfrak{Y}}-\mathfrak{N}^{I}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}^{I}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)} + (\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}^{I}_{\mathfrak{Y}}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}} \end{array}\right) \end{array}\right)$$

$$= \left(\begin{pmatrix} \frac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}}^{R}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\left(1-\mathfrak{W}_{\mathfrak{Y}}^{R}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}}^{R}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}}^{R}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}, \\ \frac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}}^{I}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\left(1-\mathfrak{W}_{\mathfrak{Y}}^{I}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}}^{I}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}}^{I}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}} \end{pmatrix}, \\ \begin{pmatrix} \frac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}}^{R}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}}^{R}-\mathfrak{N}_{\mathfrak{Y}}^{R}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}}^{R}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}}^{R}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}, \\ \frac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}}^{I}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}}^{I}-\mathfrak{N}_{\mathfrak{Y}}^{I}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}}^{I}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}}^{I}\right)^{\sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}} \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} \frac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}}^{R}\right)-\left(1-\mathfrak{W}_{\mathfrak{Y}}^{R}\right)}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}}^{R}\right)+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}}^{R}\right)}, \frac{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}}^{I}\right)-\left(1-\mathfrak{W}_{\mathfrak{Y}}^{I}\right)}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}}^{I}\right)+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}}^{I}\right)} \end{pmatrix}, \\ \begin{pmatrix} \frac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}}^{R}\right)-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}}^{R}-\mathfrak{N}_{\mathfrak{Y}}^{R}\right)}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}}^{R}\right)+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}}^{R}\right)}, \frac{\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}}^{I}\right)-\varrho\left(1-\mathfrak{W}_{\mathfrak{Y}}^{I}-\mathfrak{N}_{\mathfrak{Y}}^{I}\right)}{\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}}^{I}\right)+(\varrho-1)\left(1-\mathfrak{W}_{\mathfrak{Y}}^{I}\right)} \end{pmatrix} \right), \sum_{*=1}^{\daleth}\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right) - E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right) = 1$$

$$= \left(\begin{pmatrix} \frac{1+\varrho\mathfrak{W}_{\mathfrak{Y}}^{R}-\mathfrak{W}_{\mathfrak{Y}}^{R}-1+\mathfrak{W}_{\mathfrak{Y}}^{R}}{1+\varrho\mathfrak{W}_{\mathfrak{Y}}^{R}-\mathfrak{W}_{\mathfrak{Y}}^{R}+\varrho-\varrho\mathfrak{W}_{\mathfrak{Y}}^{R}-1+\mathfrak{W}_{\mathfrak{Y}}^{R}}, \frac{1+\varrho\mathfrak{W}_{\mathfrak{Y}}^{I}-\mathfrak{W}_{\mathfrak{Y}}^{I}-1+\mathfrak{W}_{\mathfrak{Y}}^{I}}{1+\varrho\mathfrak{W}_{\mathfrak{Y}}^{I}-\mathfrak{W}_{\mathfrak{Y}}^{I}+\varrho-\varrho\mathfrak{W}_{\mathfrak{Y}}^{I}-1+\mathfrak{W}_{\mathfrak{Y}}^{I}} \end{pmatrix}, \\ \begin{pmatrix} \frac{\varrho-\varrho\mathfrak{W}_{\mathfrak{Y}}^{R}-\varrho 1+\varrho\mathfrak{W}_{\mathfrak{Y}}^{R}+\varrho\mathfrak{N}_{\mathfrak{Y}}^{R}}{1+\varrho\mathfrak{W}_{\mathfrak{Y}}^{R}-\mathfrak{W}_{\mathfrak{Y}}^{R}+\varrho-\varrho\mathfrak{W}_{\mathfrak{Y}}^{R}-1+\mathfrak{W}_{\mathfrak{Y}}^{R}}, \frac{\varrho-\varrho\mathfrak{W}_{\mathfrak{Y}}^{I}-\varrho 1+\varrho\mathfrak{W}_{\mathfrak{Y}}^{I}+\varrho\mathfrak{N}_{\mathfrak{Y}}^{I}}{1+\varrho\mathfrak{W}_{\mathfrak{Y}}^{I}-\mathfrak{W}_{\mathfrak{Y}}^{I}+\varrho-\varrho\mathfrak{W}_{\mathfrak{Y}}^{I}-1+\mathfrak{W}_{\mathfrak{Y}}^{I}} \end{pmatrix} \right)$$

$$= \left(\left(\mathfrak{W}_{\mathfrak{Y}}^{R}, \mathfrak{W}_{\mathfrak{Y}}^{I}\right), \left(\mathfrak{N}_{\mathfrak{Y}}^{R}, \mathfrak{N}_{\mathfrak{Y}}^{I}\right)\right) = E_{ci}.$$

(1) If $E_{ci}^{*} = \left(\left(\mathfrak{W}_{\mathfrak{Y}_*}^{R}, \mathfrak{W}_{\mathfrak{Y}_*}^{I}\right), \left(\mathfrak{N}_{\mathfrak{Y}_*}^{R}, \mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)\right) \leq E^{**}_{ci} = \left(\left(\mathfrak{W}^{*}_{\mathfrak{Y}_*}{}^{R}, \mathfrak{W}^{*}_{\mathfrak{Y}_*}{}^{I}\right), \left(\mathfrak{N}^{*}_{\mathfrak{Y}_*}{}^{R}, \mathfrak{N}^{*}_{\mathfrak{Y}_*}{}^{I}\right)\right)$, that is $\mathfrak{W}_{\mathfrak{Y}_*}^{R} \leq \mathfrak{W}^{*}_{\mathfrak{Y}_*}{}^{R}, \mathfrak{W}_{\mathfrak{Y}_*}^{I} \leq \mathfrak{W}^{*}_{\mathfrak{Y}_*}{}^{I}$ and $\mathfrak{N}_{\mathfrak{Y}_*}^{R} \geq \mathfrak{N}^{*}_{\mathfrak{Y}_*}{}^{R}, \mathfrak{N}_{\mathfrak{Y}_*}^{I} \geq \mathfrak{N}^{*}_{\mathfrak{Y}_*}{}^{I}$, thus

$$\mathfrak{W}_{\mathfrak{Y}_*}^{R} \leq \mathfrak{W}^{*}_{\mathfrak{Y}_*}{}^{R} \Rightarrow 1-\mathfrak{W}_{\mathfrak{Y}_*}^{R} \geq 1-\mathfrak{W}^{*}_{\mathfrak{Y}_*}{}^{R} \Rightarrow \prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)} \geq \prod_{*=1}^{\daleth}\left(1-\mathfrak{W}^{*}_{\mathfrak{Y}_*}{}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}$$

$$\Rightarrow \prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)} - \prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)} \leq \prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}^{*}_{\mathfrak{Y}_*}{}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)} - \prod_{*=1}^{\daleth}\left(1-\mathfrak{W}^{*}_{\mathfrak{Y}_*}{}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}$$

$$\Rightarrow \frac{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}-\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}}$$
$$\leq \frac{\prod_{*=1}^{\daleth}\left(1+(\varrho-1){\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}-\prod_{*=1}^{\daleth}\left(1-{\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1){\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-{\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}}.$$

Moreover, it is derived from the imaginary parts, such as

$$\mathfrak{W}_{\mathfrak{Y}_*}^{I} \leq {\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{I}$$
$$\Rightarrow \frac{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}-\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}}$$
$$\leq \frac{\prod_{*=1}^{\daleth}\left(1+(\varrho-1){\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{I}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}-\prod_{*=1}^{\daleth}\left(1-{\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{I}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1){\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{I}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-{\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{I}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}}.$$

Further, for falsity information, the following holds.

$$\mathfrak{N}_{\mathfrak{Y}_*}^{R} \geq {\mathfrak{N}_{\mathfrak{Y}_*}^{*}}^{R} \Rightarrow 1-\mathfrak{N}_{\mathfrak{Y}_*}^{R} \leq 1-{\mathfrak{N}_{\mathfrak{Y}_*}^{*}}^{R} \Rightarrow 1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}-\mathfrak{N}_{\mathfrak{Y}_*}^{R} \geq 1-{\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{R}-{\mathfrak{N}_{\mathfrak{Y}_*}^{*}}^{R}$$
$$\Rightarrow \prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)} \geq \prod_{*=1}^{\daleth}\left(1-{\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{R}-{\mathfrak{N}_{\mathfrak{Y}_*}^{*}}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}$$
$$\Rightarrow -\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)} \leq -\varrho\prod_{*=1}^{\daleth}\left(1-{\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{R}-{\mathfrak{N}_{\mathfrak{Y}_*}^{*}}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}$$
$$\Rightarrow \varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}-\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}$$
$$\geq \varrho\prod_{*=1}^{\daleth}\left(1-{\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}$$
$$-\varrho\prod_{*=1}^{\daleth}\left(1-{\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{R}-{\mathfrak{N}_{\mathfrak{Y}_*}^{*}}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}$$
$$\Rightarrow \frac{\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}-\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}}$$
$$\geq \frac{\varrho\prod_{*=1}^{\daleth}\left(1-{\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}-\varrho\prod_{*=1}^{\daleth}\left(1-{\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{R}-{\mathfrak{N}_{\mathfrak{Y}_*}^{*}}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1){\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-{\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{R}\right)^{\left(E(\overline{\overline{E}}_{\Xi(*)})-E(\overline{\overline{E}}_{\Xi(*-1)})\right)}}.$$

Additionally, imaginary parts have been evaluated, such as

$$\mathfrak{N}^{I}_{\mathfrak{Y}_*} \geq {\mathfrak{N}^*_{\mathfrak{Y}_*}}^{I}$$

$$\Rightarrow \frac{\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}^{I}_{\mathfrak{Y}_*}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}^{I}_{\mathfrak{Y}_*}-\mathfrak{N}^{I}_{\mathfrak{Y}_*}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1)\mathfrak{W}^{I}_{\mathfrak{Y}_*}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-\mathfrak{W}^{I}_{\mathfrak{Y}_*}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}$$

$$\geq \frac{\varrho\prod_{*=1}^{\daleth}\left(1-{\mathfrak{W}^*_{\mathfrak{Y}_*}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{\daleth}\left(1-{\mathfrak{W}^*_{\mathfrak{Y}_*}}^{I}-{\mathfrak{N}^*_{\mathfrak{Y}_*}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\daleth}\left(1+(\varrho-1){\mathfrak{W}^*_{\mathfrak{Y}_*}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{\daleth}\left(1-{\mathfrak{W}^*_{\mathfrak{Y}_*}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}.$$

Thus, by including the information above with the data provided in Eqs (2) and (3), the following is obtained.

$$CIFHC-IA\left(E^{1}_{ci},E^{2}_{ci},\dots,E^{\daleth}_{ci}\right)\leq CIFHC-IA\left({E^*}^{1}_{ci},{E^*}^{2}_{ci},\dots,{E^*}^{\daleth}_{ci}\right).$$

(2) If $E^{-}_{ci}=\left(\left(\min_{*}\mathfrak{W}^{R}_{\mathfrak{Y}_*},\min_{*}\mathfrak{W}^{I}_{\mathfrak{Y}_*}\right),\left(\max_{*}\mathfrak{N}^{R}_{\mathfrak{Y}_*},\max_{*}\mathfrak{N}^{I}_{\mathfrak{Y}_*}\right)\right)$ and

$E^{+}_{ci}=\left(\left(\max_{*}\mathfrak{W}^{R}_{\mathfrak{Y}_*},\max_{*}\mathfrak{W}^{I}_{\mathfrak{Y}_*}\right),\left(\min_{*}\mathfrak{N}^{R}_{\mathfrak{Y}_*},\min_{*}\mathfrak{N}^{I}_{\mathfrak{Y}_*}\right)\right)$, thus, by considering the above two proofs, it can be concluded that

$$CoIFHC-IA\left(E^{1}_{ci},E^{2}_{ci},\dots,E^{\daleth}_{ci}\right)\leq CoIFHC-IA\left({E^+}^{1}_{ci},{E^+}^{2}_{ci},\dots,{E^+}^{\daleth}_{ci}\right)=E^{+}_{ci}$$

$$CoIFHC-IA\left(E^{1}_{ci},E^{2}_{ci},\dots,E^{\daleth}_{ci}\right)\geq CoIFHC-IA\left({E^-}^{1}_{ci},{E^-}^{2}_{ci},\dots,{E^-}^{\daleth}_{ci}\right)=E^{-}_{ci}.$$

Thus,

$$E^{-}_{ci}\leq CoIFHC-IA\left(E^{1}_{ci},E^{2}_{ci},\dots,E^{\daleth}_{ci}\right)\leq E^{+}_{ci}.$$

Definition 6. The theory of the CIFHC-IOA operator for the finite collection of CIFNs has been presented.

$$\begin{aligned}\int E_{ci}dE &= CIFHC-IOA\left(E^{1}_{ci},E^{2}_{ci},\dots,E^{\daleth}_{ci}\right)\\ &=\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(\Xi)}\right)\right)E^{\Xi(1)}_{ci}\oplus\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(1)}\right)\right)E^{\Xi(2)}_{ci}\oplus\dots\\ &\oplus\left(E\left(\overline{\overline{E}}_{\Xi(\daleth)}\right)-E\left(\overline{\overline{E}}_{\Xi(\daleth-1)}\right)\right)E^{\Xi(\daleth)}_{ci}=\sum_{*=1}^{\daleth}\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)E^{\Xi(*)}_{ci}\\ &=\oplus_{*=1}^{\daleth}\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)E^{\Xi(*)}_{ci}.\end{aligned} \tag{21}$$

Observed that $\Xi(*)\leq\Xi(*-1)$ for the collection of finite permutation $*=1,2,\dots,\daleth$.

Theorem 2. When examining the data in Eq (21), it is demonstrated that the combined value conforms to the CIFN manner, as follows:

$CIFHC-IOA(E_{ci}^{1},E_{ci}^{2},\dots,E_{ci}^{\imath})=$

$$\left(\begin{array}{c}\left(\begin{array}{c}\frac{\prod_{*=1}^{\imath}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{\imath}\left(1-\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\imath}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{\imath}\left(1-\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}},\\ \frac{\prod_{*=1}^{\imath}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{\imath}\left(1-\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\imath}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{\imath}\left(1-\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}\end{array}\right),\\ \left(\begin{array}{c}\frac{\varrho\prod_{*=1}^{\imath}\left(1-\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{\imath}\left(1-\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{R}-\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\imath}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{\imath}\left(1-\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}},\\ \frac{\varrho\prod_{*=1}^{\imath}\left(1-\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{\imath}\left(1-\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{I}-\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\imath}\left(1+(\varrho-1)\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{\imath}\left(1-\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}}\end{array}\right)\end{array}\right). \tag{22}$$

To assess or streamline the operators above, the fundamental characteristics of the devised theory, including idempotency, monotonicity, and boundedness, were defined.

Property 2. For the finite collection of CoIFNs, some properties have been stated, such as:

(1) If $E_{ci}^{*}=E_{ci}=\left(\left(\mathfrak{W}_{\mathfrak{Y}}^{R},\mathfrak{W}_{\mathfrak{Y}}^{I}\right),\left(\mathfrak{N}_{\mathfrak{Y}}^{R},\mathfrak{N}_{\mathfrak{Y}}^{I}\right)\right), *=1,2,\dots,\imath$, thus

$$CIFHC-IOA(E_{ci}^{1},E_{ci}^{2},\dots,E_{ci}^{\imath})=E_{ci}. \tag{23}$$

(2) If $E_{ci}^{*}=\left(\left(\mathfrak{W}_{\mathfrak{Y}_*}^{R},\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right),\left(\mathfrak{N}_{\mathfrak{Y}_*}^{R},\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)\right)\leq {E^{*}}_{ci}^{*}=\left(\left({\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{R},{\mathfrak{W}_{\mathfrak{Y}_*}^{*}}^{I}\right),\left({\mathfrak{N}_{\mathfrak{Y}_*}^{*}}^{R},{\mathfrak{N}_{\mathfrak{Y}_*}^{*}}^{I}\right)\right)$, thus

$$CIFHC-IOA(E_{ci}^{1},E_{ci}^{2},\dots,E_{ci}^{\imath})\leq CoIFHC-IOA\left({E^{*}}_{ci}^{1},{E^{*}}_{ci}^{2},\dots,{E^{*}}_{ci}^{\imath}\right). \tag{24}$$

(3) If $E_{ci}^{-}=\left(\left(\min_{*}\mathfrak{W}_{\mathfrak{Y}_*}^{R},\min_{*}\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right),\left(\max_{*}\mathfrak{N}_{\mathfrak{Y}_*}^{R},\max_{*}\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)\right)$ and

$E_{ci}^{+}=\left(\left(\max_{*}\mathfrak{W}_{\mathfrak{Y}_*}^{R},\max_{*}\mathfrak{W}_{\mathfrak{Y}_*}^{I}\right),\left(\min_{*}\mathfrak{N}_{\mathfrak{Y}_*}^{R},\min_{*}\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)\right)$, thus

$$E_{ci}^{-}\leq CIFHC-IOA(E_{ci}^{1},E_{ci}^{2},\dots,E_{ci}^{\imath})\leq E_{ci}^{+}. \tag{25}$$

Definition 7. The theory of the CIFHC-IG operator for the finite collection of CIFNs has been presented.

$$\begin{aligned}\int E_{ci}dE&=CIFHC-IG(E_{ci}^{1},E_{ci}^{2},\dots,E_{ci}^{\imath})\\ &=E_{ci}^{1\left(E\left(\overline{\overline{E}}_{\Xi(1)}\right)-E\left(\overline{\overline{E}}_{\Xi(\Xi)}\right)\right)}\otimes E_{ci}^{2\left(E\left(\overline{\overline{E}}_{\Xi(2)}\right)-E\left(\overline{\overline{E}}_{\Xi(1)}\right)\right)}\otimes\dots\otimes E_{ci}^{\imath\left(E\left(\overline{\overline{E}}_{\Xi(\imath)}\right)-E\left(\overline{\overline{E}}_{\Xi(\imath-1)}\right)\right)}\\ &=\prod_{*=1}^{\imath}E_{ci}^{*\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}=\otimes_{*=1}^{\imath}E_{ci}^{*\left(E\left(\overline{\overline{E}}_{\Xi(*)}\right)-E\left(\overline{\overline{E}}_{\Xi(*-1)}\right)\right)}\end{aligned} \tag{26}$$

Theorem 3. To analyze the data in Eq (26), it is demonstrated that the combined value corresponds to the CIFN manner, as follows:

$CIFHC-IG(E_{ci}^{1},E_{ci}^{2},\dots,E_{ci}^{\imath})=$

$$\left(\begin{array}{c}\left(\dfrac{\varrho\prod_{*=1}^{\gimel}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{\gimel}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{R}-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\gimel}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{\gimel}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}},\dfrac{\varrho\prod_{*=1}^{\gimel}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{\gimel}\left(1-\mathfrak{W}_{\mathfrak{Y}_*}^{I}-\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\gimel}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{\gimel}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}\right),\\ \left(\dfrac{\prod_{*=1}^{\gimel}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{\gimel}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\gimel}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{\gimel}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}},\dfrac{\prod_{*=1}^{\gimel}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{\gimel}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\gimel}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{\gimel}\left(1-\mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}\right)\end{array}\right). \tag{27}$$

To assess or streamline the above operators, the fundamental characteristics of the proposed theory, including idempotency, monotonicity, and boundedness, were described.

Property 3. To assess or streamline the above operators, the fundamental characteristics of the proposed theory, including idempotency, monotonicity, and boundedness, were described.

(1) If $E_{ci}^{*} = E_{ci} = \left(\left(\mathfrak{W}_{\mathfrak{Y}}^{R}, \mathfrak{W}_{\mathfrak{Y}}^{I}\right), \left(\mathfrak{N}_{\mathfrak{Y}}^{R}, \mathfrak{N}_{\mathfrak{Y}}^{I}\right)\right), *= 1,2,\dots,\gimel$, thus

$$CIFHC - IG(E_{ci}^{1}, E_{ci}^{2}, \dots, E_{ci}^{\gimel}) = E_{ci}. \tag{28}$$

(2) If $E_{ci}^{*} = \left(\left(\mathfrak{W}_{\mathfrak{Y}_*}^{R}, \mathfrak{W}_{\mathfrak{Y}_*}^{I}\right), \left(\mathfrak{N}_{\mathfrak{Y}_*}^{R}, \mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)\right) \leq E^{*}{}_{ci}^{*} = \left(\left(\mathfrak{W}_{\mathfrak{Y}_*}^{*}{}^{R}, \mathfrak{W}_{\mathfrak{Y}_*}^{*}{}^{I}\right), \left(\mathfrak{N}_{\mathfrak{Y}_*}^{*}{}^{R}, \mathfrak{N}_{\mathfrak{Y}_*}^{*}{}^{I}\right)\right)$, thus

$$CIFHC - IG(E_{ci}^{1}, E_{ci}^{2}, \dots, E_{ci}^{\gimel}) \leq CIFHC - IG\left(E^{*}{}_{ci}^{1}, E^{*}{}_{ci}^{2}, \dots, E^{*}{}_{ci}^{\gimel}\right). \tag{29}$$

(3) If $E_{ci}^{-} = \left(\left(\min_{*} \mathfrak{W}_{\mathfrak{Y}_*}^{R}, \min_{*} \mathfrak{W}_{\mathfrak{Y}_*}^{I}\right), \left(\max_{*} \mathfrak{N}_{\mathfrak{Y}_*}^{R}, \max_{*} \mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)\right)$ and

$$E_{ci}^{+} = \left(\left(\max_{*} \mathfrak{W}_{\mathfrak{Y}_*}^{R}, \max_{*} \mathfrak{W}_{\mathfrak{Y}_*}^{I}\right), \left(\min_{*} \mathfrak{N}_{\mathfrak{Y}_*}^{R}, \min_{*} \mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)\right), \text{ thus}$$

$$E_{ci}^{-} \leq CIFHC - IG(E_{ci}^{1}, E_{ci}^{2}, \dots, E_{ci}^{\gimel}) \leq E_{ci}^{+}. \tag{30}$$

Definition 8. The theory of the CIFHC-IOG operator for the finite set of CIFNs has been presented.

$$\int E_{ci} dE = CIFHC - IOG(E_{ci}^{1}, E_{ci}^{2}, \dots, E_{ci}^{\gimel}) = E_{ci}^{\Xi(1)^{\left(E\left(\bar{\bar{E}}_{\Xi(1)}\right)-E\left(\bar{\bar{E}}_{\Xi(\Xi)}\right)\right)}} \otimes E_{ci}^{\Xi(2)^{\left(E\left(\bar{\bar{E}}_{\Xi(2)}\right)-E\left(\bar{\bar{E}}_{\Xi(1)}\right)\right)}} \otimes \dots \otimes E_{ci}^{\Xi(\gimel)^{\left(E\left(\bar{\bar{E}}_{\Xi(\gimel)}\right)-E\left(\bar{\bar{E}}_{\Xi(\gimel-1)}\right)\right)}} = \prod_{*=1}^{\gimel} E_{ci}^{\Xi(*)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}} = \otimes_{*=1}^{\gimel} E_{ci}^{\Xi(*)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}. \tag{31}$$

Observed that $\Xi(*) \leq \Xi(* - 1)$ for the finite collection of permutation $*= 1,2,\dots,\gimel$.

Theorem 4. To analyze the data in Eq (31), it is demonstrated that the combined value corresponds to the CIFN manner, as follows:

$CIFHC - IG(E_{ci}^{1}, E_{ci}^{2}, \dots, E_{ci}^{\gimel}) =$

$$\left(\begin{array}{c}\left(\frac{\varrho\prod_{*=1}^{\imath}\left(1-\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{\imath}\left(1-\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{R}-\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\imath}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{\imath}\left(1-\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}},\right.\\ \left.\frac{\varrho\prod_{*=1}^{\imath}\left(1-\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\varrho\prod_{*=1}^{\imath}\left(1-\mathfrak{W}_{\mathfrak{Y}_{\Xi(*)}}^{I}-\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\imath}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{\imath}\left(1-\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}\right),\\ \left(\frac{\prod_{*=1}^{\imath}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{\imath}\left(1-\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\imath}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{\imath}\left(1-\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{R}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}},\right.\\ \left.\frac{\prod_{*=1}^{\imath}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}-\prod_{*=1}^{\imath}\left(1-\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}{\prod_{*=1}^{\imath}\left(1+(\varrho-1)\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}+(\varrho-1)\prod_{*=1}^{\imath}\left(1-\mathfrak{N}_{\mathfrak{Y}_{\Xi(*)}}^{I}\right)^{\left(E\left(\bar{\bar{E}}_{\Xi(*)}\right)-E\left(\bar{\bar{E}}_{\Xi(*-1)}\right)\right)}}\right)\end{array}\right). \tag{32}$$

The fundamental characteristics of the devised theory, including idempotency, monotonicity, and boundedness, were outlined to assess or streamline the operators.

Property 4. Considering the finite collection of CIFNs, some characteristics have been mentioned, including:

(1) If $E_{ci}^{*}=E_{ci}=\left(\left(\mathfrak{W}_{\mathfrak{Y}}^{R},\mathfrak{W}_{\mathfrak{Y}}^{I}\right),\left(\mathfrak{N}_{\mathfrak{Y}}^{R},\mathfrak{N}_{\mathfrak{Y}}^{I}\right)\right), *=1,2,\dots,\imath$, thus

$$CIFHC-IOG\left(E_{ci}^{1},E_{ci}^{2},\dots,E_{ci}^{\imath}\right)=E_{ci}. \tag{33}$$

(2) If $E_{ci}^{*}=\left(\left(\mathfrak{W}_{\mathfrak{Y}_{*}}^{R},\mathfrak{W}_{\mathfrak{Y}_{*}}^{I}\right),\left(\mathfrak{N}_{\mathfrak{Y}_{*}}^{R},\mathfrak{N}_{\mathfrak{Y}_{*}}^{I}\right)\right)\le {E^{*}}_{ci}^{*}=\left(\left({\mathfrak{W}_{\mathfrak{Y}_{*}}^{*}}^{R},{\mathfrak{W}_{\mathfrak{Y}_{*}}^{*}}^{I}\right),\left({\mathfrak{N}_{\mathfrak{Y}_{*}}^{*}}^{R},{\mathfrak{N}_{\mathfrak{Y}_{*}}^{*}}^{I}\right)\right)$, thus

$$CIFHC-IOG\left(E_{ci}^{1},E_{ci}^{2},\dots,E_{ci}^{\imath}\right)\le CIFHC-IOG\left({E^{*}}_{ci}^{1},{E^{*}}_{ci}^{2},\dots,{E^{*}}_{ci}^{\imath}\right). \tag{34}$$

(3) If $E_{ci}^{-}=\left(\left(\min_{*}\mathfrak{W}_{\mathfrak{Y}_{*}}^{R},\min_{*}\mathfrak{W}_{\mathfrak{Y}_{*}}^{I}\right),\left(\max_{*}\mathfrak{N}_{\mathfrak{Y}_{*}}^{R},\max_{*}\mathfrak{N}_{\mathfrak{Y}_{*}}^{I}\right)\right)$ and

$E_{ci}^{+}=\left(\left(\max_{*}\mathfrak{W}_{\mathfrak{Y}_{*}}^{R},\max_{*}\mathfrak{W}_{\mathfrak{Y}_{*}}^{I}\right),\left(\min_{*}\mathfrak{N}_{\mathfrak{Y}_{*}}^{R},\min_{*}\mathfrak{N}_{\mathfrak{Y}_{*}}^{I}\right)\right)$, thus

$$E_{ci}^{-}\le CIFHC-IOG\left(E_{ci}^{1},E_{ci}^{2},\dots,E_{ci}^{\imath}\right)\le E_{ci}^{+}. \tag{35}$$

4. Problem of Multiple Attribute Decision Making (MADM) for derived approaches

In this section, we present a novel Multi-Attribute Decision-Making (MADM) approach that leverages Hamacher Choquet-Integral operators to handle Complex Intuitionistic Fuzzy (CIF) information. This method is designed to provide a robust framework for decision-making problems where interdependencies among attributes and complex uncertainty need to be addressed. The proposed method integrates CIF sets, Hamacher aggregation operators, and the Choquet Integral to capture both individual attribute evaluations and the interactions between them. The approach follows a systematic process to evaluate and rank alternatives based on complex-valued intuitionistic fuzzy data. Here, we discuss the use of the Multiple Attribute Decision Making (MADM) technique using innovative techniques, namely the CIFHC-IA operator and CIFHC-IG operator.

The use of Hamacher Choquet-Integral operators provides a robust mechanism to aggregate CIF information while accounting for interdependencies among attributes. The operators inherently incorporate uncertainty by combining uncertain data (CIF values) in a way that respects their fuzzy and complex nature, thus preserving the integrity of the uncertainty during the aggregation process. The fuzzy measure used in the Choquet Integral captures the interaction between attributes, which often carries inherent uncertainty. By modeling these interactions explicitly, our approach ensures that uncertainties related to attribute interdependencies are properly accounted for in the decision-making process.

For this, it is assumed that the finite family of alternatives $E_{ci}^{1}, E_{ci}^{2}, \dots, E_{ci}^{\imath}$ and their attributes $E_{A}^{1}, E_{A}^{2}, \dots, E_{A}^{m}$ giving by a finite number of experts $(E_1, E_2, \dots, E_g) g = 1,2, \dots, n$. To assigned a CIF number to each attribute in every option and arrange them in a matrix format, such as $[r_{ij}]_{m\times n}$, where each term in the constructed matrix is defined in the shape of CIF values, looked at here $\mathfrak{W}_{\mathfrak{Y}}(\mathfrak{b}) = \left(\mathfrak{W}_{\mathfrak{Y}}^{R}(\mathfrak{b}), \mathfrak{W}_{\mathfrak{Y}}^{I}(\mathfrak{b})\right)$ and $\mathfrak{N}_{\mathfrak{Y}a}(\mathfrak{b}) = \left(\mathfrak{N}_{\mathfrak{Y}}^{R}(\mathfrak{b}), \mathfrak{N}_{\mathfrak{Y}}^{I}(\mathfrak{b})\right)$, the value of membership and non-membership is demonstrated with the representation of a complex number with two significant attributes, such as $\Xi \leq \mathfrak{W}_{\mathfrak{Y}}^{R}(\mathfrak{b}) + \mathfrak{N}_{\mathfrak{Y}}^{R}(\mathfrak{b}) \leq 1$ and $\Xi \leq \mathfrak{W}_{\mathfrak{Y}}^{I}(\mathfrak{b}) + \mathfrak{N}_{\mathfrak{Y}}^{I}(\mathfrak{b}) \leq 1$. Additionally, the computed structure $\mathfrak{K}_{r}(\mathfrak{b}) = \left(\mathfrak{K}_{r}^{R}(\mathfrak{b}), \mathfrak{K}_{r}^{I}(\mathfrak{b})\right) = \left(1 - \left(\mathfrak{W}_{\mathfrak{Y}}^{R}(\mathfrak{b}) + \mathfrak{N}_{\mathfrak{Y}}^{I}(\mathfrak{b})\right), 1 - \left(\mathfrak{W}_{\mathfrak{Y}}^{R}(\mathfrak{b}) + \mathfrak{N}_{\mathfrak{Y}}^{I}(\mathfrak{b})\right)\right)$ represents the value of neutral information with the simple form of CIFN, such as $E_{ci}^{*} = \left(\left(\mathfrak{W}_{\mathfrak{Y}_*}^{R}, \mathfrak{W}_{\mathfrak{Y}_*}^{I}\right), \left(\mathfrak{N}_{\mathfrak{Y}_*}^{R}. \mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)\right), * = 1,2, \dots, \imath$. Thus, the following methodology or strategy has been introduced for assessing previous issues, including:

Step 1. Given the CIFNs information, our focus is to represent it in matrix form and subsequently normalize it. This is particularly applicable when the information is provided in the form of cost type.

$$Z = \begin{cases} \left(\left(\mathfrak{W}_{\mathfrak{Y}_*}^{R}, \mathfrak{W}_{\mathfrak{Y}_*}^{I}\right), \left(\mathfrak{N}_{\mathfrak{Y}_*}^{R}. \mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)\right) & \text{for benefit attributes} \\ \left(\left(\mathfrak{N}_{\mathfrak{Y}_*}^{R}. \mathfrak{N}_{\mathfrak{Y}_*}^{I}\right), \left(\mathfrak{W}_{\mathfrak{Y}_*}^{R}, \mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)\right) & \text{for cost attributes} \end{cases}.$$

However, when it comes to benefits, normalization is optional.

Step 2. To address issues with group decision-making, the CIFHC-IA operator is applied to combine the group data into a single matrix.

Step 3. Furthermore, the data is aggregated using the CIFHC-IA operator and CIFHC-IG operator and converted into individual numbers.

Step 4. Evaluate the score values of the aggregated information using the data provided in Eqs (2) and (3).

Step 5. Rank all the options in order based on the values of score information and evaluate the top-ranked option.

To streamline the approach above, our attention is directed towards assessing real-world instances in the presence of the operators above to demonstrate the efficiency of the generated theory.

A. Illustrative example

In this instance, several varieties of glass are selected for use in the windows. Four experts are tasked with collecting data on four different types of glass for this purpose. The four experts E_1, E_2, E_3, E_4 have arranged the data for the following four best glasses $E_{ci}^{1}, E_{ci}^{2}, E_{ci}^{3}, E_{ci}^{4}$, such as:

E_{ci}^{1}: **Heat-strengthened glass:** This kind of glass converts heartiness into coolness, they are treated for strength or energy capability.

E_{ci}^{2}: **Float glass:** This type of glass is fundamental and innovative for contemporary windows.

E_{ci}^{3}: Laminated glass: That is highly durable and resistant to breakage.

E_{ci}^{4}: **Tinted glass:** This type of glass absorbs solar heat, aiding in room cooling.

A set of characteristics is presented for each glass type to help professionals easily select the optimal option from four distinct types, for instance, E_A^1: Climate, E_A^2: Location, E_A^3: Energy efficiency, and E_A^4: Safety and security. Given the facts provided, our objective is to choose the optimal option. To achieve this, the decision-making process described above is applied to assess the desired outcomes.

Step 1. Based on the CIFNs provided in Tables 1–4, our focus is to represent the data in matrix form and subsequently normalize it. This normalization process is applicable when the information is given in the form of cost type.

$$Z = \begin{cases} \left(\left(\mathfrak{W}_{\mathfrak{Y}_*}^{R}, \mathfrak{W}_{\mathfrak{Y}_*}^{I}\right), \left(\mathfrak{N}_{\mathfrak{Y}_*}^{R} . \mathfrak{N}_{\mathfrak{Y}_*}^{I}\right)\right) & \text{for benefit attributes} \\ \left(\left(\mathfrak{N}_{\mathfrak{Y}_*}^{R} . \mathfrak{N}_{\mathfrak{Y}_*}^{I}\right), \left(\mathfrak{W}_{\mathfrak{Y}_*}^{R}, \mathfrak{W}_{\mathfrak{Y}_*}^{I}\right)\right) & \text{for cost attributes} \end{cases}.$$

Normalization is not required because the data consists of benefit types.

Table 1. CIF decision matrix by E_1.

	E_A^1	E_A^2	E_A^3	E_A^4
E_{ci}^1	$((0.5,0.3),(0.3,0.4))$	$((0.51,0.31),(0.31,0.41))$	$((0.52,0.32),(0.32,0.42))$	$((0.53,0.33),(0.33,0.43))$
E_{ci}^2	$((0.4,0.4),(0.3,0.3))$	$((0.41,0.41),(0.31,0.31))$	$((0.42,0.42),(0.32,0.32))$	$((0.43,0.43),(0.33,0.33))$
E_{ci}^3	$((0.3,0.6),(0.2,0.2))$	$((0.31,0.61),(0.21,0.21))$	$((0.32,0.62),(0.22,0.22))$	$((0.33,0.63),(0.23,0.23))$
E_{ci}^4	$((0.5,0.2),(0.3,0.3))$	$((0.51,0.21),(0.31,0.31))$	$((0.52,0.22),(0.32,0.32))$	$((0.53,0.23),(0.33,0.33))$

Table 2. CIF decision matrix by E_2.

	E_A^1	E_A^2	E_A^3	E_A^4
E_{ci}^1	$((0.8,0.7),(0.1,0.2))$	$((0.81,0.71),(0.11,0.21))$	$((0.82,0.72),(0.12,0.22))$	$((0.83,0.73),(0.13,0.23))$
E_{ci}^2	$((0.7,0.6),(0.2,0.3))$	$((0.71,0.61),(0.21,0.31))$	$((0.72,0.62),(0.22,0.32))$	$((0.73,0.63),(0.23,0.33))$
E_{ci}^3	$((0.6,0.5),(0.3,0.4))$	$((0.61,0.51),(0.31,0.41))$	$((0.62,0.52),(0.32,0.42))$	$((0.63,0.53),(0.33,0.43))$
E_{ci}^4	$((0.5,0.4),(0.4,0.4))$	$((0.51,0.41),(0.41,0.41))$	$((0.52,0.42),(0.42,0.42))$	$((0.53,0.43),(0.43,0.43))$

Table 3. CIF decision matrix by E_3.

	E_A^1	E_A^2	E_A^3	E_A^4
E_{ci}^1	$((0.3,0.4),(0.3,0.4))$	$((0.31,0.41),(0.31,0.41))$	$((0.32,0.42),(0.32,0.42))$	$((0.33,0.43),(0.33,0.43))$
E_{ci}^2	$((0.2,0.3),(0.3,0.4))$	$((0.21,0.31),(0.31,0.41))$	$((0.22,0.32),(0.32,0.42))$	$((0.23,0.33),(0.33,0.43))$
E_{ci}^3	$((0.3,0.4),(0.3,0.4))$	$((0.31,0.41),(0.31,0.41))$	$((0.32,0.42),(0.32,0.42))$	$((0.33,0.43),(0.33,0.43))$
E_{ci}^4	$((0.2,0.3),(0.3,0.4))$	$((0.21,0.31),(0.31,0.41))$	$((0.22,0.32),(0.32,0.42))$	$((0.23,0.33),(0.33,0.43))$

Table 4. CIF decision matrix by E_4.

	E_A^1	E_A^2	E_A^3	E_A^4
E_{ci}^1	$((0.6,0.5),(0.3,0.4))$	$((0.61,0.51),(0.31,0.41))$	$((0.62,0.52),(0.32,0.42))$	$((0.63,0.53),(0.33,0.43))$
E_{ci}^2	$((0.5,0.4),(0.4,0.4))$	$((0.51,0.41),(0.41,0.41))$	$((0.52,0.42),(0.42,0.42))$	$((0.53,0.43),(0.43,0.43))$
E_{ci}^3	$((0.2,0.3),(0.3,0.4))$	$((0.21,0.31),(0.31,0.41))$	$((0.22,0.32),(0.32,0.42))$	$((0.23,0.33),(0.33,0.43))$
E_{ci}^4	$((0.3,0.4),(0.3,0.4))$	$((0.31,0.41),(0.31,0.41))$	$((0.32,0.42),(0.32,0.42))$	$((0.33,0.43),(0.33,0.43))$

Step 2. To address issues with group decision-making, the CIFHC-IA operator is used to aggregate the group data into a single matrix, as shown in Table 5

Table 5. Aggregation using the CIFHC-IA operator.

	E_A^1	E_A^2	E_A^3	E_A^4
E_{ci}^1	$\begin{pmatrix}(0.2736,0.18),\\(0.2603,0.2988)\end{pmatrix}$	$\begin{pmatrix}(0.2797,0.185),\\(0.284,0.3219)\end{pmatrix}$	$\begin{pmatrix}(0.2859,0.19),\\(0.3121,0.3496)\end{pmatrix}$	$\begin{pmatrix}(0.2922,0.1951),\\(0.3474,0.3853)\end{pmatrix}$
E_{ci}^2	$\begin{pmatrix}(0.218,0.1864),\\(0.2814,0.2488)\end{pmatrix}$	$\begin{pmatrix}(0.2232,0.1913),\\(0.3046,0.2658)\end{pmatrix}$	$\begin{pmatrix}(0.2285,0.1962),\\(0.3322,0.2851)\end{pmatrix}$	$\begin{pmatrix}(0.2338,0.2011),\\(0.3675,0.3079)\end{pmatrix}$
E_{ci}^3	$\begin{pmatrix}(0.1263,0.2222),\\(0.1555,0.2163)\end{pmatrix}$	$\begin{pmatrix}(0.1308,0.2276),\\(0.168,0.2334)\end{pmatrix}$	$\begin{pmatrix}(0.1353,0.2331),\\(0.1827,0.2529)\end{pmatrix}$	$\begin{pmatrix}(0.1398,0.2386),\\(0.2012,0.2759)\end{pmatrix}$
E_{ci}^4	$\begin{pmatrix}(0.1984,0.1194),\\(0.2479,0.2249)\end{pmatrix}$	$\begin{pmatrix}(0.2034,0.1238),\\(0.2656,0.238)\end{pmatrix}$	$\begin{pmatrix}(0.2084,0.1282),\\(0.2857,0.2522)\end{pmatrix}$	$\begin{pmatrix}(0.2134,0.1326),\\(0.3094,0.2679)\end{pmatrix}$

Step 3. In addition, the data is aggregated using the CIFHC-IA and CIFHC-IG operators to generate individual values, as shown in Table 6.

Table 6. Aggregated decision matrix using the CIFHC-IA operator and CoIFHC-IG operator.

	CoIFHC-IA operator	CoIFHC-IG operator
E_{ci}^1	$((0.2798,0.185),(0.2879,0.3257))$	$((0.2807,0.1856),(0.2811,0.306))$
E_{ci}^2	$((0.2232,0.1913),(0.3083,0.2676))$	$((0.224,0.1916),(0.3074,0.2672))$
E_{ci}^3	$((0.1308,0.2277),(0.1695,0.2353))$	$((0.1309,0.228),(0.1632,0.231))$
E_{ci}^4	$((0.2034,0.1238),(0.2675,0.2388))$	$((0.2038,0.1239),(0.2574,0.2253))$

Step 4. Examine the score values of the aggregate information using the data provided in Eqs (2) and (3), which can be seen in Table 7.

Table 7. Score values in the matrix.

	CoIFHC-IA operator	CoIFHC-IG operator
E_{ci}^1	-0.074	-0.06
E_{ci}^2	-0.081	-0.08
E_{ci}^3	-0.023	-0.018
E_{ci}^4	-0.09	-0.078

Step 5. Arrange all the options in order based on the values of the score information and analyze the highest-scoring option, as shown in Table 8.

Table 8. Assessing facts and depicting the optimal choice.

Methods	Ranking Results	Best one
CIFHC-IA operator	$E_{ci}^3 > E_{ci}^1 > E_{ci}^2 > E_{ci}^4$	E_{ci}^3
CIFHC-IG operator	$E_{ci}^3 > E_{ci}^1 > E_{ci}^4 > E_{ci}^2$	E_{ci}^3

The most optimum solution, as determined by both operators, is E_{ci}^3. In order to streamline the procedure above, our focus is on assessing the comparison between the suggested methodologies and many established procedures of value, to demonstrate the efficiency of the generated theory.

5. Comparisons analysis

In this section, we provide a concise analysis of the contrast between the suggested methods and proposed techniques, considering FS and their expansions. To compare the suggested approaches with the proposed methods, certain established operators based on FS and their variations are required. Therefore, based on the concepts mentioned, a comparison with the proposed techniques is conducted using the data provided in Table 5. The proposed ideas are expressed as follows: The Hamacher operator for IFS was derived by Huang [27], while the Hamacher operator for CIFS was developed by Akram et al. [28]. Xu [29] exposed the Choquet integral for weighted IFS, and Wang [30] derived the Choquet integral based on averaging operators for IFS. Chen [31,32] examined the Choquet integral operators for IFST and induced IFS. Mahmood et al. [33] derived the Hamacher Choquet-integral operators for IFS. Mahmood et al. [20] developed the Aczel-Alsina power operators for CIFS. Garg [23] assessed the geometric operators for CIFS. Mahmood et al. [34] introduced the Aczel-Alsina operators for CIFS. Table 9 presents the comparison based on the data provided in Table 5.

Table 9. Conducting a comparative study of the data presented in Table 5.

Techniques	Scoring System	Ranking Data
Huang [27]	*Not able to evaluate the data*	*Not able to evaluate the data*
Akram et al. [28]	*Not able to evaluate the data*	*Not able to evaluate the data*
Xu [29]	*Not able to evaluate the data*	*Not able to evaluate the data*
Wang [30]	*Not able to evaluate the data*	*Not able to evaluate the data*
Chen [31]	*Not able to evaluate the data*	*Not able to evaluate the data*
Chen [32]	*Not able to evaluate the data*	*Not able to evaluate the data*
Mahmood et al. [20]	$-0.4872, -0.4844, -0.4142, -0.4746$	$E_{ci}^3 > E_{ci}^4 > E_{ci}^2 > E_{ci}^1$
Garg [23]	$-0.484, -0.528, -0.437, -0.503$	$E_{ci}^3 > E_{ci}^1 > E_{ci}^4 > E_{ci}^2$
Mahmood et al. [34]	$0.3709, 0.3674, 0.3759, 0.3331$	$E_{ci}^3 > E_{ci}^1 > E_{ci}^2 > E_{ci}^4$
CIFHC-IA operator	$-0.074, -0.081, -0.023, -0.09$	$E_{ci}^3 > E_{ci}^1 > E_{ci}^2 > E_{ci}^4$
CIFHC-IG operator	$-0.06, -0.08, -0.018, -0.078$	$E_{ci}^3 > E_{ci}^1 > E_{ci}^4 > E_{ci}^2$

Following a thorough evaluation, the ranking information obtained corresponds to the results of Mahmood et al. [20], Garg et al. [23], Mahmood et al. [35], and derived methodologies, indicating that the optimal option is E_{ci}^3. In addition, several proposed methods were unable to analyze the data shown in Table 5 due to their dependency on operators calculated using the IFS, which is a particular case of the suggested information. Therefore, the presented work is very influential and reliable in comparison to the theory mentioned in Ref. [27–32]. In the future, our goal is to enhance it further or establish other operators using the evaluated methodologies.

6. Conclusions

The major contributions of this work are listed below:

First, the Hamacher operational laws are presented in consideration of the CoIF values.

Second, the CIFHC-IA operator, CIFHC-IOA operator, CIFHC-IG operator, and CIFHC-IOG operator were examined, followed by an analysis of the attributes and special instances of the proposed approaches.

Third, a novel approach is introduced using the created operators for Multiple Attribute Decision Making (MADM) issues with CIF values, with the operational procedures thoroughly illustrated.

In conclusion, a comparative analysis is conducted to evaluate the proposed techniques in comparison to the current ones. This analysis is based on the presented cases and aims to demonstrate the superiority and validity of the derived approaches.

In future research work, we will extend the presented approach to dynamic decision-making scenarios where attribute values or criteria weights evolve over time. This could involve developing time-dependent CIF models or incorporating real-time data streams. Also, future work could entail integrating the CIFHC-IA operator with machine learning techniques or optimization algorithms to automatically learn attribute interdependencies and fuzzy measures from data, enhancing decision-making efficiency in large-scale problems. While we provide a general framework, future research could include the proposed method to specific domains such as healthcare, supply chain management, or financial risk assessment, tailoring the method to domain-specific requirements [42,43]. Another promising direction is enhancing the model to better handle group decision-making scenarios where experts may have conflicting opinions, potentially using consensus-building mechanisms.

Author contributions

Harish Garg: Writing–review & editing, writing–original draft, validation, methodology, investigation, formal analysis, conceptualization; Tehreem: Writing–original draft, validation, methodology, formal analysis; Kinza Ayaz: Writing–original draft, investigation; Walid Emam: Formal analysis, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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