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Research article

A novel class of fourth-order derivative-free iterative methods to obtain multiple zeros and their basins of attraction

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Abstract: This article proposed a novel fourth-order class based on weight functions to locate multiple roots numerically, which did not require the evaluation of derivatives at any stage of computation. For particular instances of a multiplicity of zeros, the theoretical convergence behavior of the proposed family has been proven to be symmetrical. This inspired us to show the general results which endorsed the convergence order of the suggested scheme. In addition, some special cases were introduced by using different weight functions. The basins of attraction of the proposed techniques for various parametric values in the complex plane were showcased to verify the stability and convergence features. Finally, we have included a range of problems like Planck's radiation law, the Van der Waals equation, the trajectory of an electron, and a few academic problems. Numerical analyses were performed and compared with other existing algorithms to verify the efficacy and applicability of the proposed techniques.

Keywords: multiple roots; nonlinear equations; derivative-free methods; basins of attractions; iterative methods
Mathematics Subject Classification: 65H05, 37N30, 49M15

1. Introduction

The main goal of this research is to compute the solution to the nonlinear equation

$$\Theta(x) = 0, \tag{1.1}$$

where the function $\Theta : \mathcal{D} \subseteq \mathbb{C} \to \mathbb{C}$ is holomorphic on the domain $\mathcal{D}$ consists of multiple root (η) with multiplicity (m), which implies

$$\Theta^k(\eta) = \begin{cases} 0, & k = 0, 1, 2, 3, \ldots, m-1, \\ \neq 0, & k = m. \end{cases} \tag{1.2}$$

Analytical techniques to locate multiple zeros of a nonlinear function $f(x)$ are almost nonexistent. Consequently, we search for iterative algorithms to obtain approximate solutions. Thus, the main objective of this study is to acquire multiple roots using derivative-free approaches.

The most popular and straightforward approach for calculating multiple roots of Eq (1.1) is the modified Newton's technique [18], defined by

$$x_{t+1} = x_t - m\frac{\Theta(x_t)}{\Theta'(x_t)}, \ t \in \mathbb{N}_0. \tag{1.3}$$

Here, $\mathbb{N}_0$ denotes the collection of all natural numbers, including zero. The aforementioned technique (1.3) requires that the first-order derivative should be calculated at each stage to exhibit quadratic convergence in case of multiple roots. Nonetheless, a plethora of high-order techniques have been presented and investigated in research papers (see Arora et al. [2], Cordero et al. [6], Neta et al. [13], Petkovic et al. [15], Proinov and Ivanov [16], Shengguo et al. [21], Soleymani et al. [24], and Zafar et al. [36]). These iterative techniques can be classified broadly into two categories: (i) with derivatives and (ii) free from derivatives. Evaluating a first or second-order derivative is necessary for the first class of methods. Generally speaking, these methods do not provide the desired results over the non-smooth functions [5]. In order to overcome this fact, a few researchers have analyzed and developed some iterative algorithms without derivatives for multiple roots.

To get rid of derivatives, Traub-Steffensen [26] employed the following approximation

$$\Theta'(x_t) \simeq \frac{\Theta(x_t + \gamma\Theta(x_t)) - \Theta(x_t)}{\gamma\Theta(x_t)}, \ \gamma \in \mathbb{R}\backslash\{0\}, \tag{1.4}$$

where the derivative is replaced with the first-order divided difference approximation (1.4) in the modified Newton method (1.3), and one arrives at

$$x_{t+1} = x_t - m\frac{\Theta(x_t)}{\Theta[u_t, x_t]}, \ \ t \in \mathbb{N}_0, \tag{1.5}$$

known as the modified Traub-Steffensen method (denoted by TM_1), where $\Theta[u_t, x_t] = \frac{\Theta(u_t)-\Theta(x_t)}{u_t - x_t}$ is the first-order divided difference and $u_t = x_t + \gamma\Theta(x_t)$. By maintaining second-order convergence, the approach (1.5) improves upon the modified Newton's method significantly and is derivative-free.

Several versions of the modified Newton's approach have been developed and examined in the literature to approximate multiple zeros of nonlinear functions. Using derivatives, many researchers, such as Cordero et al. [7] and Zafar et al. [35] extended the modified Newton's technique (1.3) for multiple roots. Furthermore, recent and some derivative-free higher-order multipoint techniques have been discussed in [9, 10, 17, 19, 20, 23]. These methods belong to a class of iterative solvers which require the knowledge of the multiplicity. However, the exact value of this multiplicity might only sometimes be available in practice. In such circumstances, a very close approximation of multiplicity can be calculated by using the subsequent forms given by

(i) Traub [26] approximation formula:

$$m \approx \frac{\log|\Theta(x)|}{\log\left|\frac{\Theta(x)}{\Theta'(x)}\right|},$$

as x approaches the multiple root of Θ in close proximity;

(ii) Lagouanelle [11] approximate formula:

$$m \approx \frac{\Theta'(x)^2}{\Theta'(x)^2 - \Theta(x)\Theta''(x)},$$

as x approaches the multiple root of Θ in close proximity.

Moreover, an alternative procedure $h(x) := \frac{\Theta(x)}{\Theta'(x)}$ suggested by Traub [26] can be used on any method that uses multiplicity m. Soleymani et al. [22,25] further implemented this transformation for handling multiple roots in the absence of multiplicity through iterative techniques. In addition to this, such methods are not only restricted to finding the solution of nonlinear equations but can also be used especially in the areas of fluid dynamics, biomechanics, aerodynamics, and many other areas [30–34]. For simple roots, these iterative methods can be extended to multidimensional cases in order to consider alternating direction implicit methods [12,28] and the time-fractional telegraph equation [29].

Developing an efficient and competitive fourth-order derivative-free iterative scheme, compared to existing optimal techniques, is quite challenging. Therefore, motivated by this idea and the weight function notion, we have made an attempt to propose an efficient fourth-order derivative-free family that uses four functional evaluations at each iteration. In literature, most of the iterative schemes are limited to computing multiple roots of multiplicity $m \geq 2$. The key innovation of the proposed derivative-free family is that several efficient variants can be developed using different weight functions. A thorough analysis and comparison are performed to study their convergence behavior. The strength of this scheme lies in its effectiveness for both simple and multiple roots. Additionally, the basins of attraction are analyzed in the complex plane to assess their convergence domains across a range of problems.

The rest of the work is arranged: A new derivative-free class of multiple root solvers is proposed in Sect. 2 along with its convergence analysis for $m = 1, 2$, and 3. The generalized error equation of the proposed algorithm for $m \geq 4$ is established in Sect. 3. Some particular instances are mentioned in Sect. 4. In Sect. 5, the basins of attractions are included to demonstrate the stability features of iterative techniques over complex planes. The suggested scheme is analyzed numerically, using various examples to highlight its effectiveness and accuracy in Sect. 6. Lastly, in Sect. 7, some conclusions are given.

2. Iterative scheme

We propose the following derivative-free family:

$$\begin{aligned} z_t &= x_t - m\frac{\Theta(x_t)}{\Theta[\mu_t, x_t]}, \\ x_{t+1} &= z_t - m\frac{\Theta(h_t)}{\Theta[\mu_t, x_t]}\left[\frac{1+\beta s_t}{1+(\beta-2)s_t}\right]Q[s_t, v_t], \end{aligned} \tag{2.1}$$

where $\mu_t = x_t + \gamma\Theta(x_t)$, $h_t = \frac{x_t + z_t}{2}$, $\gamma \in \mathbb{R}\backslash\{0\}$, $v_t = \left(\frac{\Theta(z_t)}{\Theta(\mu_t)}\right)^{\frac{1}{m}}$, $s_t = \left(\frac{\Theta(z_t)}{\Theta(x_t)}\right)^{\frac{1}{m}}$, and β is a free real parameter. Here, the function $Q : \mathbb{C}^2 \to \mathbb{C}$ is analytic in the neighborhood of origin. In addition, $\Theta[\mu_t, x_t]$ denotes a finite difference of order one. Keep in mind that the mappings s_t and v_t are multivalued. As a result, we take into account their primary analytical branches (see Ahlfors [1]). For instance, we consider $s_t = \exp\left[\frac{1}{m}\log\left(\frac{\Theta(z_t)}{\Theta(x_t)}\right)\right]$, where $\log\left(\frac{\Theta(z_t)}{\Theta(x_t)}\right) = \log\left|\frac{\Theta(z_t)}{\Theta(x_t)}\right| + i\arg\left(\frac{\Theta(z_t)}{\Theta(x_t)}\right)$ for $-\pi < \arg\left(\frac{\Theta(z_t)}{\Theta(x_t)}\right) \leq \pi$. Additionally, $s_t = \left|\frac{\Theta(z_t)}{\Theta(x_t)}\right|^{\frac{1}{m}} \exp\left[\frac{1}{m}\arg\left(\frac{\Theta(z_t)}{\Theta(x_t)}\right)\right]$ can be written $O(e_t)$, where the error at t^{th} step is denoted by e_t. The built-in command of the computer algebra system utilized in this study coincides with the convention of the primary argument $\mathrm{Arg}(z)$ for $z \in \mathbb{C}$.

We demonstrate that the suggested approach (2.1) achieves at least fourth-order convergence for all nonzero real values of γ in the following Theorems 1–4. Also, we shall illustrate that for different cases of the multiplicity m, the theoretical convergence findings of the scheme (2.1) are symmetrical. First, we apply the following theorem to verify its result for the case $m = 1$, i.e., simple zeros.

Theorem 1. *Consider a simple zero (η) of nonlinear function $\Theta : \mathcal{D} \subseteq \mathbb{C} \to \mathbb{C}$ such that the neighborhood of required zero η lies in the domain $\mathcal{D}$. Then, for $Q_{00} = 0$, $Q_{01} = 0$, $Q_{10} = 2$, $Q_{02} = 0$, $Q_{20} = -8$, and $Q_{11} = 3$, the proposed family* (2.1) *achieves at least fourth-order convergence with error equation*

$$e_{t+1} = \frac{1}{4}(1+\sigma_1)p_1\Big((5 + 2\sigma_1 + 4\sigma_1^2 + 8\beta(1+\sigma_1)^2)p_1^2 - (5 + 6\sigma_1)p_2\Big)e_t^4 + O(e_t^5),$$

where $\beta \in \mathbb{R}$, $\sigma_1 = \gamma\Theta'(\eta)$, and $Q_{ij} = \dfrac{\partial^{i+j}}{\partial s_t^i \partial v_t^j} Q(s_t, v_t)\,|_{(s_t=0,\ v_t=0)}$, for $i, j = 0, 1,$ and 2.

Proof. Assume the error at the t^{th} iteration is $e_t = x_t - \eta$ and expands the functions $\Theta(x_t)$ and $\Theta(\mu_t)$ using Taylor's series around $x = \eta$ such that $\Theta(\eta) = 0$ and $\Theta'(\eta) \neq 0$; then one gets

$$\Theta(x_t) = \Theta'(\eta)e_t\Big(1 + p_1e_t + p_2e_t^2 + p_3e_t^3 + p_4e_t^4 + O(e_t^5)\Big), \tag{2.2}$$

and

$$\begin{aligned}\Theta(\mu_t) = \Theta'(\eta)e_t[&1 + \sigma_1 + (p_1 + 3\sigma_1p_1 + \sigma_1^2p_1)e_t + (2\sigma_1p_1^2 + 2\sigma_1^2p_1^2 + p_2 + 4\sigma_1p_2 \\ &+ 3\sigma_1^2p_2 + \sigma_1^3p_2)e_t^2 + (\sigma_1^2p_1^3 + 5\sigma_1p_1p_2 + 8\sigma_1^2p_1p_2 + 3\sigma_1^3p_1p_2 + p_3 \\ &+ 5\sigma_1p_3 + 6\sigma_1^2p_3 + 4\sigma_1^3p_3 + \sigma_1^4p_3)e_t^3] + O(e_t^5),\end{aligned} \tag{2.3}$$

respectively. Here,

$$p_k = \frac{1!}{(1+k)!}\frac{\Theta^{(1+k)}(\eta)}{\Theta'(\eta)},\ \forall\ k = 1, 2, \ldots.$$

To get the error of approximation at the first sub-step of family (2.1), substitute the Eqs (2.2) and (2.3), and one can have

$$\begin{aligned}z_t - \eta = (1+\sigma_1)p_1e_t^2 + &\Big[-(2 + 2\sigma_1 + \sigma_1^2)p_1^2 + (2 + 3\sigma_1 + \sigma_1^2)p_2\Big]e_t^3 + [(4 + 5\sigma_1 + 3\sigma_1^2 \\ &+ \sigma_1^3)p_1^3 - (7 + 10\sigma_1 + 7\sigma_1^2 + 2\sigma_1^3)p_1p_2 + (3 + 6\sigma_1 + 4\sigma_1^2 + \sigma_1^3)p_3]e_t^4 + O(e_t^5).\end{aligned} \tag{2.4}$$

Utilizing Eq (2.4) and the Taylor series of the function $\Theta(z_t)$, one arrives at

$$\begin{aligned}\Theta(z_t) = \Theta'(\eta)e_t[&(1+\sigma_1)p_1e_t + (-(2 + 2\sigma_1 + \sigma_1^2)p_1^2 + (2 + 3\sigma_1 + \sigma_1^2)p_2)e_t^2 + ((5 + 7\sigma_1 \\ &+ 4\sigma_1^2 + \sigma_1^3)p_1^3 - (7 + 10\sigma_1 + 7\sigma_1^2 + 2\sigma_1^3)p_1p_2 + (3 + 6\sigma_1 + 4\sigma_1^2 + \sigma_1^3)p_3)e_t^3] \\ &+ O(e_t^5).\end{aligned} \tag{2.5}$$

By substituting the functions $\Theta(x_t)$, $\Theta(\mu_t)$ and $\Theta(z_t)$ from Eqs (2.2)–(2.5), we get the following expression for s_t and v_t:

$$\begin{aligned} s_t = \left(\frac{\Theta(z_t)}{\Theta(x_t)}\right) &= (1+\sigma_1)p_1e_t + \left(-(-3+3\sigma_1+\sigma_1^2)p_1^2 + (2+3\sigma_1+\sigma_1^2)p_2\right)e_t^2 \\ &+((8+10\sigma_1+5\sigma_1^2+\sigma_1^3)p_1^3 - 2(5+7\sigma_1+4\sigma_1^2+\sigma_1^3)p_1p_2 \\ &+(3+6\sigma_1+4\sigma_1^2+\sigma_1^3)p_3)e_t^3 + (-(20+30\sigma_1+20\sigma_1^2+7\sigma_1^3 \\ &+\sigma_1^4)p_1^4 + (37+60\sigma_1+44\sigma_1^2+17\sigma_1^3+3\sigma_1^4)p_1^2p_2 - (8+15\sigma_1 \\ &+13\sigma_1^2+6\sigma_1^3+\sigma_1^4)p_2^4 - (14+25\sigma_1^2+20\sigma_1^2+9\sigma_1^3+2\sigma_1^4)p_1p_3 \\ &+(4+10\sigma_1+10\sigma_1^2+5\sigma_1^3+\sigma_1^4)p_4)e_t^4 + O(e_t^5), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} v_t = \left(\frac{\Theta(z_t)}{\Theta(\mu_t)}\right) &= p_1e_t + \left(-(3+2\sigma_1)p_1^2 + (2+\sigma_1)p_2\right)e_t^2 + ((8+8\sigma_1+3\sigma_1^2)p_1^3 \\ &-(10+11\sigma_1+4\sigma_1^2)p_1p_2 + (3+3\sigma_1+\sigma_1^2)p_3)e_t^3 + (-(20 \\ &+26\sigma_1+15\sigma_1^2+4\sigma_1^3)p_1^4 + (37+52\sigma_1+33\sigma_1^2+9\sigma_1^3)p_1^2p_2 - (8 \\ &+13\sigma_1+9\sigma_1^2+2\sigma_1^3)p_2^2 - (14+21\sigma_1+14\sigma_1^2+4\sigma_1^3)p_1p_3 + (4 \\ &+6\sigma_1+4\sigma_1^2+\sigma_1^3)p_4)e_t^4 + O(e_t^5). \end{aligned} \tag{2.7}$$

From the Eqs (2.6) and (2.7), we conclude that s_t and v_t are of order e_t.

As we know, the Taylor series expansion of a multivariable function $f(x,y)$ about the point $(0,0)$ up to second-order terms can be written as

$$f(x,y) = f_{00} + \frac{1}{1!}\left(f_{10}x + f_{01}y\right) + \frac{1}{2!}\left(f_{20}x^2 + 2f_{11}xy + f_{02}y^2\right), \tag{2.8}$$

where $f_{ij} = \dfrac{\partial^{i+j}}{\partial x^i \partial y^j}(f(x,y))|_{(x=0,y=0)}$, for $i,j \in \{0,1,2\}$. Now, replacing $x = s_t$ and $y = v_t$ in (2.8) and expanding the weight function $Q(s_t, v_t)$ about the point $(0,0)$, we obtain

$$Q(s_t,v_t) = Q_{00} + (Q_{10}s_t + Q_{01}v_t) + \frac{1}{2}\left(Q_{20}s_t^2 + 2Q_{11}s_tv_t + Q_{02}v_t^2\right), \tag{2.9}$$

where $Q_{ij} = \dfrac{\partial^{i+j}}{\partial s_t^i \partial v_t^j} Q(s_t, v_t)|_{(s_t=0,\ v_t=0)}$, for $i,j \in \{0,1,2\}$.

Finally, the required expressions (2.2)–(2.9) are substituted in the final sub-step of family (2.1), one gets the following relation:

$$\begin{aligned} e_{t+1} = &-\frac{Q_{00}}{2}e_t - \frac{1}{4}\Big(Q_{00}(3+4\sigma_1) + 2(Q_{01} + (-2+Q_{10})(1+\sigma_1)\Big)p_1e_t^2 + \frac{1}{8}\Big(2(-8 \\ &+3Q_{01} - Q_{02} + 3Q_{10} - 2Q_{11} - Q_{20} - 8\sigma_1 - Q_{10}\sigma_1 - 2Q_{11}\sigma_1 - 2Q_{20}\sigma_1 - 4\sigma_1^2 \\ &-2Q_{10}\sigma_1^2 - Q_{20}\sigma_1^2 + Q_{00}(6+\sigma_1-2\sigma_1^2+4\beta(1+\sigma_1)^2))p_1^2 - (Q_{00}(13+24\sigma_1 \\ &+8\sigma_1^2) + 4(2+\sigma_1)(Q_{01} + (-2+Q_{10})(1+\sigma_1)))p_2\Big)e_t^3 + \varphi_1 e_t^4 + O(e_t^5), \end{aligned} \tag{2.10}$$

where $\varphi_1 = \varphi_1(\gamma, \beta, p_1, p_2, p_3, p_4, Q_{00}, Q_{10}, Q_{01}, Q_{20}, Q_{11}, Q_{02})$.

The main aim here is to achieve the highest possible convergence order, which can be obtained by equating the coefficients (e_t^i, i=1,2,3) equal to zero. Therefore, one obtains the following conditions on the weight function:

$$Q_{00} = 0,\ Q_{10} = 2,\ Q_{01} = 0,\ Q_{20} = -8,\ Q_{02} = 0,\ Q_{11} = 3. \tag{2.11}$$

By substituting conditions of (2.11) in Eq (2.10), it yields the final error equation

$$e_{t+1} = \frac{1}{4}(1+\sigma_1)p_1\left((5+2\sigma_1+4\sigma_1^2+8\beta(1+\sigma_1)^2)p_1^2-(5+6\sigma_1)p_2\right)e_t^4+O(e_t^5), \tag{2.12}$$

where $\sigma_1 = \gamma\Theta'(\eta)$. Hence, equation (2.12) leads to the conclusion of at least fourth-order convergence of scheme (2.1) for simple zero. □

The next theorem provides the convergence analysis of proposed scheme (2.1) for $m = 2$.

Theorem 2. *Now, consider the same hypothesis as of Theorem 1 with multiplicity two, then the family* (2.1) *achieves at least fourth-order convergence, provided* $Q_{00} = 0$, $Q_{10} = 4 - Q_{01}$, $Q_{01} = 4$, *and* $Q_{02} = -8 - 2Q_{11} - Q_{20}$, *where* $\{|Q_{11}|, |Q_{20}|\} < \infty$ *satisfying the following error equation*

$$\begin{aligned} e_{t+1} = &\frac{1}{128}(\sigma_2+2c_1)(\sigma_2^2(-4+4\beta-Q_{11}-Q_{20})+2\sigma_2(-10+8\beta-Q_{11}-Q_{20})c_1 \\ &+4(5+4\beta)c_1^2-24c_2)e_t^4+O(e_t^5), \end{aligned}$$

where $\beta \in \mathbb{R}$, $\sigma_2 = \gamma\Theta''(\eta)$, *and* $Q_{ij} = \dfrac{\partial^{i+j}}{\partial s_t^i \partial v_t^j}Q(s_t, v_t)\ |_{(s_t=0,\ v_t=0)}$ *for* $i, j = 0, 1,$ *and* 2.

Proof. Suppose the error at the t^{th} iteration is $e_t = x_t - \eta$ and expands the functions $\Theta(x_t)$ and $\Theta(\mu_t)$ using Taylor's series around $x = \eta$ such that $\Theta(\eta) = \Theta'(\eta) = 0$ and $\Theta''(\eta) \neq 0$; then one gets

$$\Theta(x_t) = \frac{\Theta''(\eta)}{2!}e_t^2\left(1+c_1e_t+c_2e_t^2+c_3e_t^3+c_4e_t^4+O(e_t^5)\right), \tag{2.13}$$

and

$$\begin{aligned} \Theta(\mu_t) = &\frac{\Theta''(\eta)}{2!}e_t^2\Big[1+(\sigma_2+c_1)e_t+\frac{1}{4}(\sigma_2^2+10\sigma_2c_1+4c_2)e_t^2+\frac{1}{4}(5\sigma_2^2c_1+6\sigma_2c_1^2 \\ &+12\sigma_2c_2+4c_3)e_t^3+O(e_t^4)\Big], \end{aligned} \tag{2.14}$$

respectively. Here,

$$c_k = \frac{2!}{(2+k)!}\frac{\Theta^{(2+k)}(\eta)}{\Theta''(\eta)},\ \forall\ k = 1, 2, \ldots$$

and $\sigma_2 = \gamma\Theta''(\eta)$. To get the error of approximation at the first sub-step of scheme (2.1), substitute the Eqs (2.13) and (2.14), and one can have

$$\begin{aligned} z_t - \eta = &\frac{1}{4}(\sigma_2+2c_1)e_t^2-\frac{1}{16}\left[\sigma_2^2-8\sigma_2c_1+12c_1^2-16c_2\right]e_t^3+\frac{1}{64}\Big[\sigma_2^3-10c_1(\sigma_2^2+16c_2) \\ &-20\sigma_2c_1^2+64\sigma_2c_2+72c_1^3+96c_3\Big]e_t^4+O(e_t^5). \end{aligned} \tag{2.15}$$

Employing Eq (2.15) and Taylor's expansion of the function $\Theta(z_t)$, one obtains

$$\Theta(z_t) = \frac{\Theta''(\eta)}{2!} e_t^2 \Big[\frac{1}{16}(\sigma_2 + 2c_1)^2 e_t^2 - \frac{1}{32}(\sigma_2 + 2c_1)(\sigma_2^2 - 8\sigma_2 c_1 + 12c_1^2 - 16c_2) e_t^3 + O(e_t^4) \Big]. \tag{2.16}$$

By adopting Eqs (2.13), (2.14), and (2.16) in order to obtain the order of s_t and v_t, one arrives at

$$\begin{aligned} s_t = \left(\frac{\Theta(z_t)}{\Theta(x_t)} \right)^{\frac{1}{m}} = & \frac{1}{4}(\sigma_2 + 2c_1) e_t - \frac{1}{16}(\sigma_2^2 - 6\sigma_2 c_1 + 16c_1^2 - 16c_2) e_t^2 \\ & + \frac{1}{64}(\sigma_2^3 - 6\sigma_2^2 c_1 - 22\sigma_2 c_1^2 + 56\sigma_2 c_2 + 116c_1^3 \\ & - 208c_1 c_2 + 96c_3) e_t^3 + O(e_t^4), \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} v_t = \left(\frac{\Theta(z_t)}{\Theta(\mu_t)} \right)^{\frac{1}{m}} = & \frac{1}{4}(\sigma_2 + 2c_1) e_t + \frac{1}{16}(-3\sigma_2^2 + 2\sigma_2 c_1 - 16c_1^2 + 16c_2) e_t^2 \\ & + \frac{1}{64}(7\sigma_2^3 - 22\sigma_2^2 c_1 - 14\sigma_2 c_1^2 + 24\sigma_2 c_2 + 116c_1^3 \\ & - 208c_1 c_2 + 96c_3) e_t^3 + O(e_t^4). \end{aligned} \tag{2.18}$$

From the Eqs (2.17) and (2.18), we conclude that s_t and v_t are of order e_t.

We can expand the weight function $Q(s_t, v_t)$ about the point $(0, 0)$ using Taylor's series up to second-order terms only, as given by

$$Q(s_t, v_t) = Q_{00} + \frac{1}{1!}(Q_{10} s_t + Q_{01} v_t) + \frac{1}{2!}\left(Q_{20} s_t^2 + 2Q_{11} s_t v_t + Q_{02} v_t^2 \right). \tag{2.19}$$

where $Q_{ij} = \dfrac{\partial^{i+j}}{\partial s_t^i \partial v_t^j} Q(s_t, v_t)|_{(s_t=0,\ v_t=0)}$, for $i, j \in \{0, 1, 2\}$.

By substituting the Eqs (2.13)–(2.19) in the family (2.1), the relation we have obtained is:

$$\begin{aligned} e_{t+1} = & -\frac{Q_{00}}{4} e_t + \frac{1}{16}\big(-\sigma_2(-4 + 3Q_{00} + Q_{01} + Q_{10}) - 2(-4 + 2Q_{00} + Q_{01} + Q_{10}) c_1 \big) e_t^2 \\ & + \frac{1}{128}\Big(\sigma_2^2(-8 + 4(1-\beta)Q_{00} - Q_{02} - 4Q_{10} - 2Q_{11} - Q_{20}) + 4\sigma_2(16 + 2(-9 \\ & + 2\beta)Q_{00} - 6Q_{01} - Q_{02} - 8Q_{10} - 2Q_{11} - Q_{20}) c_1 + 4(-24 + 4(3+\beta)Q_{00} \\ & + 4Q_{01} - Q_{02} + 4Q_{10} - 2Q_{11} - Q_{20}) c_1^2 - 8(9Q_{00} + 4(-4 + Q_{01} + Q_{10})) c_2 \Big) e_t^3 \\ & + \varphi_2 e_t^4 + O(e_t^5), \end{aligned} \tag{2.20}$$

where $\varphi_2 = \varphi_2(\gamma, \beta, c_1, c_2, c_3, Q_{00}, Q_{10}, Q_{01}, Q_{20}, Q_{11}, Q_{02})$.

Now, to achieve a higher convergence order, we equate the coefficients (e_t^i, i=1,2,3) of the above error equation equal to zero. Therefore, one gets

$$Q_{00} = 0,\ Q_{10} = 4 - Q_{01},\ Q_{01} = 4,\ Q_{02} = -8 - 2Q_{11} - Q_{20}. \tag{2.21}$$

On substituting the conditions of (2.21) in Eq (2.20), it yields

$$e_{t+1} = \frac{1}{128}(\sigma_2 + 2c_1)(\sigma_2^2(-4 + 4\beta - Q_{11} - Q_{20}) + 2\sigma_2(-10 + 8\beta - Q_{11} - Q_{20})c_1 + 4(5 + 4\beta)c_1^2 - 24c_2)e_t^4 + O(e_t^5),$$

where $\sigma_2 = \gamma\Theta''(\eta)$. For multiplicity two, the family (2.1) thus reaches at least fourth-order convergence. □

We shall prove the convergence results of the suggested technique for $m = 3$ in the next theorem.

Theorem 3. *Now, consider the same hypothesis as of Theorem 1 with multiplicity* $m = 3$, *and the scheme* (2.1) *achieves at least fourth-order convergence, provided* $Q_{00} = 0$, $Q_{01} = 8 - Q_{10}$, *and* $Q_{02} = -24 - 2Q_{11} - Q_{20}$, *where* $\beta \in \mathbb{R}$, $\sigma_3 = \gamma\Theta'''(\eta)$, *and* $\{|Q_{10}|, |Q_{11}|, |Q_{20}|\} < \infty$. *The error equation becomes*

$$e_{t+1} = \frac{1}{432}b_1\left(-3\sigma_3(12 + Q_{10}) + 4(17 + 8\beta)b_1^2 - 84b_2\right)e_t^4 + O(e_t^5),$$

Proof. Assume the error at the t^{th} iteration is $e_t = x_t - \eta$ and expands the functions $\Theta(x_t)$ and $\Theta(\mu_t)$ using Taylor's series around $x = \eta$ such that $\Theta(\eta) = \Theta'(\eta) = \Theta''(\eta) = 0$ and $\Theta'''(\eta) \neq 0$; then one gets

$$\Theta(x_t) = \frac{\Theta'''(\eta)}{3!}e_t^2(1 + b_1e_t + b_2e_t^2 + b_3e_t^3 + b_4e_t^4 + O(e_t^5)), \tag{2.22}$$

and

$$\Theta(\mu_t) = \frac{\Theta'''(\eta)}{3!}e_t^3\Big[1 + b_1e_t + \frac{1}{2}(\sigma_3 + 2b_2)e_t^2 + \Big(\frac{7}{6}\sigma_3b_1 + b_3\Big)e_t^3 + \frac{1}{12}(\sigma_3^2 + 8\sigma_2b_1^2 + 16\sigma_3b_2 + 12b_4)e_t^4 + O(e_t^5)\Big], \tag{2.23}$$

where the error constants are denoted by

$$b_j = \frac{3!}{(3+j)!}\frac{\Theta^{(3+j)}(\eta)}{\Theta^{(3)}(\eta)},\ j = 1, 2, \ldots.$$

The error of approximation at the first sub-step of family (2.1) is obtained by substituting the Eqs (2.22) and (2.23), and one can have

$$z_t - \eta = \frac{b_1}{3}e_t^2 + \frac{1}{18}(3\sigma_3 - 8b_1^2 + 12b_2)e_t^3 + \Big(\frac{1}{9}b_1(2\sigma_3 - 13b_2) + \frac{16b_1^3}{27} + b_3\Big)e_t^4 + O(e_t^5). \tag{2.24}$$

Using Eq (2.24) and Taylor's expansion of the function $\Theta(z_t)$ brings us to

$$\Theta(z_t) = \frac{\Theta'''(\eta)}{3!}e_t^3\left[\frac{b_1^3}{27}e_t^3 + \frac{1}{54}b_1^2(3\sigma_3 - 8b_1^2 + 12b_2)e_t^4 + O(e_t^5)\right]. \tag{2.25}$$

By adopting Eqs (2.22), (2.23) and (2.25), one arrives at

$$s_t = \left(\frac{\Theta(z_t)}{\Theta(x_t)}\right)^{\frac{1}{3}} = \frac{b_1}{3}e_t + \frac{1}{18}(3\sigma_3 - 10b_1^2 + 12b_2)e_t^2 + \frac{1}{54}(9\sigma_3b_1 + 46b_1^3 - 96b_2b_1 + 54b_3)e_t^3 + O(e_t^4), \tag{2.26}$$

and

$$v_t = \left(\frac{\Theta(z_t)}{\Theta(\mu_t)}\right)^{\frac{1}{3}} = \frac{b_1}{3}e_t + \frac{1}{18}(3\sigma_3 - 10b_1^2 + 12b_2)e_t^2 + \frac{1}{27}(3\sigma_3 b_1 + 23b_1^3 - 48b_2b_1 + 27b_3)e_t^3 + O(e_t^4). \tag{2.27}$$

From the Eqs (2.26) and (2.27), we conclude that s_t and v_t are of order e_t.

We can expand the weight function $Q(s_t, v_t)$ about the point $(0, 0)$ using Taylor's series up to second-order terms only, as given by

$$Q(s_t, v_t) = Q_{00} + \frac{1}{1!}\left(Q_{10}s_t + Q_{01}v_t\right) + \frac{1}{2!}\left(Q_{20}s_t^2 + 2Q_{11}s_tv_t + Q_{02}v_t^2\right), \tag{2.28}$$

where $Q_{ij} = \dfrac{\partial^{i+j}}{\partial s_t^i \partial v_t^j} Q(s_t, v_t)|_{(s_t=0,\ v_t=0)}$, for $i, j \in \{0, 1, 2\}$.

Now, substitute the Eqs (2.22)–(2.28) in the scheme (2.1), and one gets

$$\begin{aligned} e_{t+1} =& -\frac{Q_{00}}{8}e_t - \frac{1}{48}(5Q_{00} + 2(-8 + Q_{01} + Q_{10}))b_1e_t^2 + \frac{1}{288}\Big(2(-64 + 4(5+\beta)Q_{00} \\ &+ 5Q_{01} - Q_{02} + 5Q_{10} - 2Q_{11} - Q_{20})b_1^2 - 3(2\sigma_3(-8 + 4Q_{00} + Q_{01} + Q_{10}) \\ &+ (23Q_{00} + 8(-8 + Q_{10} + Q_{01}))b_2)e_t^3 + \varphi_3e_t^4 + O(e_t^5), \end{aligned} \tag{2.29}$$

where $\varphi_3 = \varphi_3(\gamma, \beta, b_1, b_2, b_3, Q_{00}, Q_{10}, Q_{01}, Q_{20}, Q_{11}, Q_{02})$.

To achieve at least fourth-order convergence, we equate the coefficients (e_t^i, i=1,2,3) of the above error equation equal to zero. Therefore, one gets

$$Q_{00} = 0,\ Q_{01} = 8 - Q_{10},\ Q_{02} = -24 - 2Q_{11} - Q_{20}. \tag{2.30}$$

Using Eq (2.30) in Eq (2.29), one can have

$$e_{t+1} = \frac{1}{432}b_1\left(-3\sigma_3(12 + Q_{10}) + 4(17 + 8\beta)b_1^2 - 84b_2\right)e_t^4 + O(e_t^5), \tag{2.31}$$

where $\sigma_3 = \gamma\Theta'''(\eta)$. Thus, for multiplicity three, the proposed approach (2.1) achieves at least fourth-order convergence. □

3. Generic form of error equation

The generic form of the error equation for the derivative-free scheme (2.1) will now be presented when $m \geq 4$.

Theorem 4. *Consider the same hypothesis as of Theorem 1 with multiplicity* $m \geq 4$*, then the scheme* (2.1) *achieves at least fourth-order convergence, provided* $Q_{00} = 0$, $Q_{01} = 2^m - Q_{10}$, *and* $Q_{02} = -(2^m m + 2Q_{11} + Q_{20})$*, where* $\{|Q_{10}|,\ |Q_{11}|,\ |Q_{20}|\} < \infty$*, satisfies the following error equation*

$$e_{t+1} = \frac{1}{4m^3}\left[((1 + m)^2 + 1 + 8\beta)w_1^3 - m(m + 4)w_1w_2\right]e_t^4 + O(e_t^5).$$

Proof. Assume the error at the t^{th} iteration is $e_t = x_t - \eta$ and the error constants $w_k = \frac{m!}{(m+j)!}\frac{\Theta^{(m+j)}(\eta)}{\Theta^{(m)}(\eta)}$, $j = 1, 2, \ldots$. Now, expand the functions $\Theta(x_t)$ and $\Theta(\mu_t)$ using Taylor's series around $x = \eta$ such that it follows expression (1.2), and one gets

$$\Theta(x_t) = \frac{\Theta^{(m)}(\eta)}{m!} e_t^m (1 + w_1 e_t + w_2 e_t^2 + w_3 e_t^3 + w_4 e_t^4 + O(e_t^5)), \tag{3.1}$$

and

$$\Theta(\mu_t) = \frac{\Theta^{(m)}(\eta)}{m!} e_t^m \left[1 + \sum_{i=0}^{2} \delta_i e_t^{i+1} + O(e_t^4)\right], \tag{3.2}$$

respectively.
Here $\delta_i = \delta_i(m, \gamma, w_1, w_2, w_3, w_4, \Theta^{(m)}(\eta))$. For example, the first few coefficients can be expressed simply as $\delta_0 = w_1, \delta_1 = w_2$, and

$$\delta_2 = \begin{cases} \frac{1}{6}\left(\gamma\Theta^{(4)}(\eta) + 6w_3\right) & m = 4, \\ w_3 & m \geq 5. \end{cases}$$

Now, utilizing the Eqs (3.1) and (3.2) in the proposed scheme (2.1), we get

$$\begin{aligned} e_{z_t} = z_t - \eta =& \frac{w_1}{m} e_t^2 + \frac{1}{m^2}(2mw_2 - (1+m)w_1^2)e_t^3 + \frac{1}{m^3}(\delta_1 + m^2 w_3 + (1+m)^2 w_1^3 \\ &- m(3m+4)w_2 w_1)e_t^4 + O(e_t^5), \end{aligned} \tag{3.3}$$

where

$$\delta_1 = \begin{cases} m\gamma\Theta^{(4)}(\eta) & m = 4, \\ 0 & m \geq 5. \end{cases}$$

On operating Eq (3.3) and the Taylor series of the function $\Theta(z_t)$, one can have

$$\Theta(z_t) = \frac{\Theta^{(m)}(\eta)}{m!} e_{z_t}^m (1 + w_1 e_{z_t} + w_2 e_{z_t}^2 + w_3 e_{z_t}^3 + w_4 e_{z_t}^4 + O(e_t^5)). \tag{3.4}$$

From the Eqs (3.1), (3.2), and (3.4), we obtain

$$\begin{aligned} s_t = \left(\frac{\Theta(z_t)}{\Theta(x_t)}\right)^{\frac{1}{m}} =& \frac{w_1}{m} e_t + \left(\frac{2}{m} w_2 - \frac{(m+2)}{m^2} w_1^2\right) e_t^2 + \frac{1}{2m^3}(\delta_2 + (2m^2 + 7m \\ &+ 7)w_1^3 - 2m(3m+7)w_1 w_2 + 6m^2 w_3)e_t^3 + O(e_t^4), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} v_t = \left(\frac{\Theta(z_t)}{\Theta(\mu_t)}\right)^{\frac{1}{m}} =& \frac{w_1}{m} e_t + \frac{1}{m^2}\left(2mw_2 - (m+2)w_1^2\right) e_t^2 + \frac{1}{2m^3}(\delta_2 + (2m^2 + 7m \\ &+ 7)w_1^3 - 2m(3m+7)w_1 w_2 + 6m^2 w_3)e_t^3 + O(e_t^4), \end{aligned} \tag{3.6}$$

where

$$\delta_2 = \begin{cases} 2m\gamma\Theta^{(4)}(\eta) & m = 4, \\ 0 & m \geq 5. \end{cases}$$

Expanding the weight function $Q(s_t, v_t)$ about the point $(0, 0)$ using Taylor's series up to second-order terms as:

$$Q(s_t, v_t) = Q_{00} + s_t Q_{10} + v_t Q_{01} + \frac{1}{2}s_t^2 Q_{20} + Q_{11} s_t v_t + \frac{1}{2}v_t^2 Q_{02}. \tag{3.7}$$

On substituting the Eqs (3.1)–(3.7) in the technique (2.1), one arrives at the relation:

$$\begin{aligned} e_{t+1} =& -\frac{Q_{00}}{2^m}e_t - \frac{1}{m2^{m+1}}((2+m)Q_{00} + 2(-2^{m+1} + Q_{01} + Q_{10}))w_1 e_t^2 + \frac{1}{m^2 2^{m+2}} \\ &\times \Big(2(-(1+m)2^{m+1} + ((1+m)(2+m) + 4\beta)Q_{00} + (m+2)(Q_{01} + Q_{10}) \\ &- Q_{02} - 2Q_{11} - Q_{20})w_1^2 - m((4(m+2) + m)Q_{00} + 8(-2^m + Q_{01} \\ &+ Q_{10}))w_2\Big)e_t^3 + \varphi_3 e_t^4 + O(e_t^5), \end{aligned} \tag{3.8}$$

where $\varphi_3 = \varphi_3(\gamma, \beta, w_1, w_2, w_3, w_4, Q_{00}, Q_{10}, Q_{01}, Q_{20}, Q_{11}, Q_{02})$.

In a similar fashion, as in previous theorems, to achieve higher convergence order, we equate the coefficients (e_t^i, i=1,2,3) equal to zero. Therefore, one gets

$$Q_{00} = 0,\ Q_{01} = 2^m - Q_{10},\ Q_{02} = -(2^m m + 2Q_{11} + Q_{20}). \tag{3.9}$$

By using Eq (3.9) in Eq. (3.8), the error equation becomes

$$e_{t+1} = \frac{1}{4m^3}\left[((m+1)^2 + 1 + 8\beta)w_1^3 - m(m+4)w_1 w_2\right]e_t^4 + O(e_t^5). \tag{3.10}$$

The scheme (2.1) for nonzero real values of γ exhibits at least fourth-order convergence when $m \geq 4$, as shown by Eq. (3.10). □

Remark 1. *Subsequently, the proposed family* (2.1) *for the respective weight function can be expressed as:*

$$\begin{aligned} z_t &= x_t - m\frac{\Theta(x_t)}{\Theta[\mu_t, x_t]}, \\ x_{t+1} &= z_t - m\frac{\Theta(h_t)}{\Theta[\mu_t, x_t]}\left[\frac{1+\beta s_t}{1+(\beta-2)s_t}\right]\Big((Q_{10}s_t + (2^m - Q_{10})v_t) + \frac{1}{2}\left(Q_{20}s_t^2 + 2Q_{11}s_t v_t\right. \\ &\quad \left.-(2^m m + 2Q_{11} + Q_{20})v_t^2\right)\Big), \end{aligned} \tag{3.11}$$

where β is the free parameter, and the values of Q_{10}, Q_{20}, and Q_{11} can be taken from Table 1 for different multiplicities. It is essential to note that the family (3.11) *satisfies all of the criteria stated in the preceding Theorems 1–4 and has a simple body structure. However, the choice of weight function is not restricted to polynomial form; it can also be of rational form, which will be discussed in the next section.*

Remark 2. *The proposed scheme* (2.1) *is based on the modified Newton's method, which is well-known for its simplicity and good local convergence properties. However, a good convergence of Newton's method can only be expected when the initial guess is adequately chosen. The significance of the choice of initial approximations becomes even more important if higher-order iteratives are applied due to their sensitivity to perturbations. If the initial approximation is not sufficiently close to the desired root, these methods may exhibit slow convergence at the beginning of the iterative process, leading to decreased computational efficiency.*

Remark 3. *The computational complexity of an iterative method can be determined by the Ostrowski approach [14], given by*

$$E = \rho^{1/C}, \tag{3.12}$$

where ρ denotes the order of convergence, C denotes the computational cost of an iterative method that includes a total number of functional evaluations and its derivatives per iteration for solving scalar nonlinear equations, and E denotes the efficiency index. Here, the computational cost of the proposed scheme (2.1) *is four, i.e., it requires $\Theta(x_t)$, $\Theta(\mu_t)$, $\Theta(h_t)$, and $\Theta(z_t)$ functional evaluations to attain at least fourth-order convergence. Therefore, the efficiency index of the proposed scheme becomes* $4^{1/4}$.

Remark 4. *It can be noticed that the results for multiplicity $m = 1, 2, \geq 3$, the presented scheme* (2.1) *achieves the fourth-order convergence with a number of restrictions on Q_{ij} and are* $6, 4, 3$, *respectively. Also, for $m \geq 3$, their corresponding error equations satisfy the common conditions, as given in Table 1.*

Table 1. Number of restrictions on weight function Q.

Multiplicity (m)	**Number of Restrictions**	**Conditions**
1	6	$Q_{00} = 0,\ Q_{01} = 0,\ Q_{10} = 2,\ Q_{02} = 0,\ Q_{20} = -8,\ Q_{11} = 3$
2	4	$Q_{00} = 0,\ Q_{10} = 0,\ Q_{01} = 4,\ Q_{02} = -8 - 2Q_{11} - Q_{20}$
≥ 3	3	$Q_{00} = 0,\ Q_{01} = 2^m - Q_{10},\ Q_{02} = -(2^m m + 2Q_{11} + Q_{20})$

Furthermore, for multiplicity $m \geq 1$, these conditions can be generalized as $Q_{00} = 0,\ Q_{01} = (m-1)2^m,\ Q_{10} = 2^m - Q_{01},\ Q_{02} = -(2^m m + 2Q_{11} + Q_{20}),\ Q_{20} = -(m+1)2^{(m+1)}$, *and* $Q_{11} = (m+2)2^{(m-1)}$.

Remark 5. *It can be seen that the final error equation* (3.10) *(for $m \geq 4$) does not contain the parameters γ in the e_t^4 coefficient. Although they exist in the coefficient of e_t^5, we do not state it as we already attain the convergence order four. Moreover, the role of parameters β and γ can be observed in equations* (2.12), (2.22) *and* (2.31) *for $m = 1$, $m = 2$ and $m = 3$, respectively, showing how the parametric values will enhance the convergence order of the proposed iterative algorithm.*

4. Special cases: Weight function analysis

We can devise several new iterative methods by choosing different weight functions $Q(s_t, v_t)$ of the scheme (2.1) that satisfies the conditions of Theorems 1–4. Some specific cases of the suggested scheme are given below:

1. Let us first consider the following weight function in the form of polynomial

$$Q(s_t, v_t) = k_1 + k_2 s_t + k_3 v_t + k_4 s_t^2 + k_5 s_t v_t. \tag{4.1}$$

under the generalized conditions $Q_{00} = 0,\ Q_{01} = (m-1)2^m,\ Q_{10} = 2^m - Q_{01},\ Q_{20} = -(m+1)2^{(m+1)}$, and $Q_{11} = 2^{(m-1)}(m+2)$. Here, $Q_{ij} = \frac{\partial^{i+j}}{\partial s_t^i \partial v_t^j} Q(s_t, v_t)|_{(s_t=0,\ v_t=0)}$. On substituting, we obtain a system of five linear equations and therefore one obtains the k_i's values for $i = 1, 2, 3, 4, 5$ by

$$k_1 = 0,\ k_2 = 2^m(2-m),\ k_3 = 2^m(m-1),\ k_4 = -2^m(m+1),\ \text{and } k_5 = 2^{m-1}(m+2). \tag{4.2}$$

As a result, the first weight function $Q(s_t, v_t) = Q_1(s_t, v_t)$ becomes

$$Q_1(s_t, v_t) = 2^{m-1}(2(m-1)v_t - 2(1+m)s_t^2 + ms_t(v_t - 2) + 2s_t(2 + v_t)).$$

2. Now, there are two ways to get a rational weight function. The first one is to directly apply the generalized conditions obtained above on the following rational weight function

$$Q(s_t, v_t) = \frac{k_1 + k_2 s_t + k_3 v_t + k_4 s_t^2 + k_5 v_t^2 + k_6 s_t v_t}{-1 + 2s_t - 2v_t}, \tag{4.3}$$

and one obtains the weight function

$$Q(s_t, v_t) = \frac{2^{m-1}(7(m-2))s_t v_t - 2^m(m-5)s_t^2 + 2^m(m-2)s_t - 2^{m+1}(m-1)v_t^2 - 2^m(m-1)v_t}{2s_t - 2v_t - 1}.$$

This approach is the basic one that most of the researchers do. However, in this study, we optimize the k_i's values of the weight function (4.3) by applying it directly to the proposed scheme, and the optimized values obtained here are given by

$$\begin{aligned} k_1 = 0,\ k_2 = 2^m(m-2),\ k_3 = 2^m(1-m),\ k_4 = \left(1 - m^m + 2^{2+m}\right), \\ k_5 = m^m - 1,\ k_6 = -2^{m-1}(7m). \end{aligned} \tag{4.4}$$

On subsituting the values, our second weight function $Q(s_t, v_t) = Q_2(s_t, v_t)$ becomes

$$Q_2(s_t, v_t) = \frac{((1 + 2^{2+m}m - m^m)s_t^2 + v_t(2^m - 2^m m - v_t + m^m v_t) - 2^{m-1}s_t(4 + m(-2 + 7v_t)))}{(-1 + 2s_t - 2v_t)}.$$

3. Let us consider another rational weight function of the form

$$Q(s_t, v_t) = \frac{k_1 + k_2 s_t + k_3 v_t + k_4 s_t^2 + k_5 v_t^2 + k_6 s_t v_t}{(-1 + s_t - v_t)^4}, \tag{4.5}$$

and using the same hypothesis as discussed in generating the second weight function, the k_i's values becomes

$$\begin{aligned} k_1 = 0,\ k_2 = 2^m(2-m), k_3 = 2^m(m-1),\ k_4 = m - 12 - 2^{m-1}m + m^2 - m^m, \\ k_5 = m^m - m^2,\ k_6 = 12 - m. \end{aligned} \tag{4.6}$$

Upon solving, we obtain our third weight function $Q(s_t, v_t) = Q_3(s_t, v_t)$ by

$$Q_3(s_t, v_t) = \frac{(2^m(2-m)s_t + (m - 12 - 2^{m-1}m + m^2 - m^m)s_t^2 + 2^m(m-1)v_t + (12-m)s_t v_t + (m^m - m^2)v_t^2)}{(-1 + s_t - v_t)^4}.$$

Therefore, by taking these weight functions in our scheme (2.1), the corresponding methods take the following forms:
Method 1 (LM_1):

$$\begin{cases} z_t = x_t - m\dfrac{\Theta(x_t)}{\Theta[\mu_t, x_t]}, \\ x_{t+1} = z_t - m\dfrac{\Theta(h_t)}{\Theta[\mu_t, x_t]}\left[\dfrac{1+\beta s_t}{1+(\beta-2)s_t}\right]\left(2^{m-1}(2(m-1)v_t - 2(1+m)s_t^2 + ms_t(v_t-2) + 2s_t(v_t+2))\right). \end{cases}$$

Method 2 (LM_2):

$$\begin{cases} z_t = x_t - m\dfrac{\Theta(x_t)}{\Theta[\mu_t, x_t]}, \\ x_{t+1} = z_t - m\dfrac{\Theta(h_t)}{\Theta[\mu_t, x_t]}\left[\dfrac{1+\beta s_t}{1+(\beta-2)s_t}\right] \\ \qquad \times \dfrac{((1+2^{2+m}m - m^m)s_t^2 + v_t(2^m - 2^m m - v_t + m^m v_t) - 2^{m-1}s_t(4+m(-2+7v_t)))}{(-1+2s_t-2v_t)}. \end{cases}$$

Method 3 (LM_3):

$$\begin{cases} z_t = x_t - m\dfrac{\Theta(x_t)}{\Theta[\mu_t, x_t]}, \\ x_{t+1} = z_t - m\dfrac{\Theta(h_t)}{\Theta[\mu_t, x_t]}\left[\dfrac{1+\beta s_t}{1+(\beta-2)s_t}\right] \\ \qquad \times \dfrac{(2^m(2-m)s_t + (m-12-2^{m-1}m+m^2-m^m)s_t^2 + 2^m(m-1)v_t + (12-m)s_t v_t + (m^m-m^2)v_t^2)}{(-1+s_t-v_t)^4}. \end{cases}$$

It is important to note that discontinuities or singularities may occur when the denominators of the weight function Q_1, Q_2, and Q_3 approaches zero. We compared the weight functions for $m = 1$ and $m = 2$, as illustrated in Figures 1–2. Each weight function displays distinct behavior due to its unique numerator and denominator. Additionally, functions that include terms like 2^m or m^m may dominate certain regions because of their scaling differences. We have examined several real-world and standard academic examples to evaluate the effectiveness of iterative approaches using different weight functions.

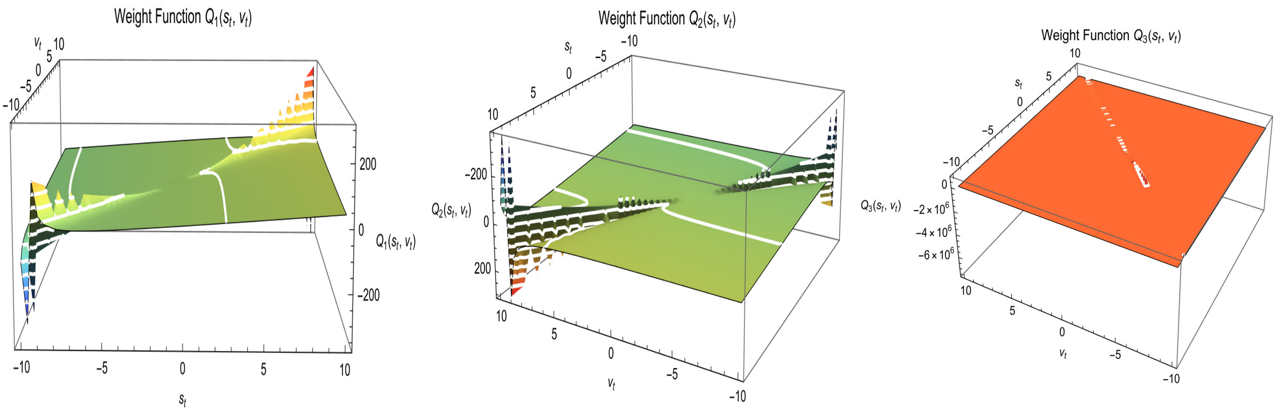

Figure 1. Numerical comparisons of weight functions for $m = 1$.

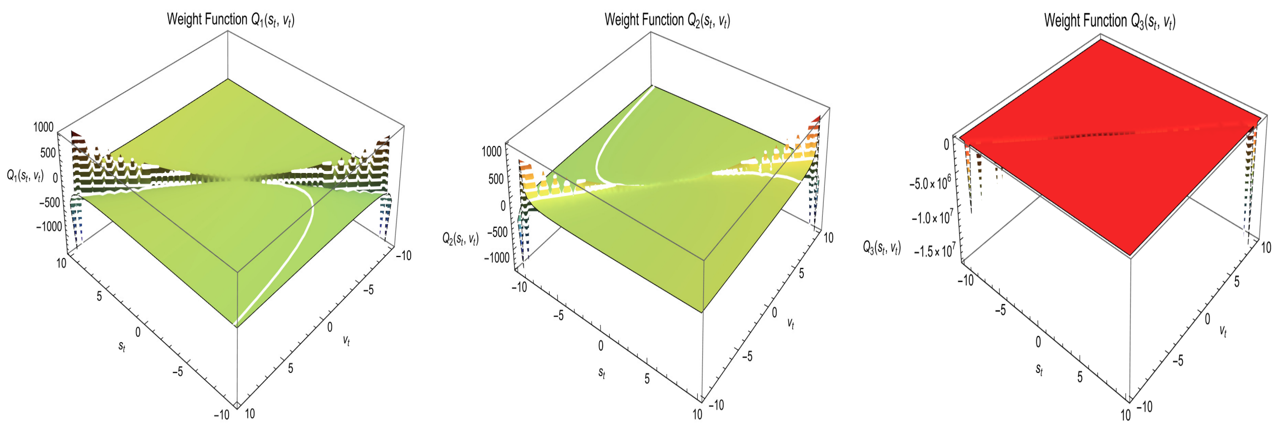

Figure 2. Numerical comparisons of weight functions for $m = 2$.

5. Basins of attraction

Here, we use the technique of some complex polynomials $p(z)$ to analyze the basins of attraction of the iterative schemes and highlight the key information about their convergence and stability. Attraction basins are typically thought of as visual geometrical tools for comparing iterative methods that describe the behavior of a particular scheme at several initial points. The following contents offer a succinct summary of some fundamental concepts related to the visual tool; further details can be found in Varona [27].

Let the rational map $\psi : \mathbb{C} \to \mathbb{C}$ be defined on the Riemann sphere. The orbit is defined for any given point $z_0 \in \mathbb{C}$ by

$$\{z_0, \psi(z_0), \psi^2(z_0), \ldots, \psi^n(z_0), \ldots\}.$$

If $\psi^q(z_0) = z_0$, then a point $z_0 \in \mathbb{C}$ is typically referred to as a periodic point with period q (q is the least integer). Consequently, a fixed point is a periodic point when $q = 1$. If $|\psi'(z_0)| < 1$, then z_0 is an attracting complex point; if $|\psi'(z_0)| > 1$, it repels; if $|\psi'(z_0)| = 1$, it is neutral; and if $|\psi'(z_0)| = 0$, it is super attracting. Think of α^* as the rational function's attractive fixed point. Regarding each attractive fixed point α^*, the collection of initial points z_0 whose orbits tend to α^* constitutes the attraction basins (represented by $D(\alpha^*)$) given as

$$D(\alpha^*) = \{z_0 \in \mathbb{C} : \psi^n(z_0) \to \alpha^*, \text{ as } n \to \infty\}.$$

The Fatou set collects all points whose orbits are getting closer to the fixed point α^*. The Julia set (J_s) is its complementary set. The lines connecting the basins are verified from J_s recognized by the closure of the set where fixed points repel. Thus, when we employ a method of iteration on a polynomial, the procedure of attraction basins allows us to select those initial points that are convergent to the required root. Additionally, we can see those starting regions that are unfit for the iterative techniques. We choose initial point z_0 in the domain D_1, where $D_1 = [-3, 3] \times [-3, 3] \subseteq \mathbb{C} \times \mathbb{C}$ denotes the rectangular domain with 512×512 mesh points so that it contains each root of $p(z) = 0$; otherwise, we enlarge the domain to enclose the desired root. The convergence of a scheme starting at a point $z_0 \in D_1$ to the zero of a function $p(z)$ is not guaranteed. To draw the attraction basins, we set the tolerance 10^{-3} in the stopping criterion for the convergence, limited to 25 iterations. The scheme, including the findings demonstrating non-convergence w.r.t the iteration formula begun from z_0, will be rejected if the tolerance has not been reached in the expected amount of iterations. We allocate a single color to each z_0 throughout the formation of the basins. These attraction basins with the designated color are formed if the iterative scheme converges for the initial point z_0. If not, the scheme will be colored blue since it deviates from the predicted number of iterations.

For comparison purposes, we consider several existing optimal fourth-order variants of Newton-like techniques for calculating the multiple zeros of a function. The schemes are expressed as follows:

Zafar et al. [35] method:

$$\begin{cases} z_t = x_t - m\dfrac{\Theta(x_t)}{\Theta'(x_t)}, \\ x_{t+1} = z_t - m\dfrac{u_t(4u_t + 1)}{(u_t + 1)^2}\dfrac{\Theta(x_t)}{\Theta'(x_t)}, \end{cases} \tag{5.1}$$

and

$$\begin{cases} z_t = x_t - m\dfrac{2\Theta(x_t)}{2\Theta'(x_t) + m\Theta(x_t)}, \\ x_{t+1} = z_t - mu_t\left(1 + 2u_t + \dfrac{11}{2}u_t^2\right)\dfrac{\Theta(x_t)}{\Theta'(x_t) + m\Theta(x_t)}, \end{cases} \tag{5.2}$$

where $u_t = \left(\dfrac{\Theta(z_t)}{\Theta(x_t)}\right)^{\frac{1}{m}}$. We have denoted the methods (5.1) and (5.2) by FM_1 and FM_2, respectively.

Sharma et al. [19] method:

$$\begin{cases} w_t = u_t - m\dfrac{\Theta(u_t)}{\Theta[v_t, u_t]}, \\ u_{t+1} = w_t - \left(\dfrac{s_t - y_t + my_t - m^2 s_t y_t + 2ms_t y_t}{-ms_t + s_t^2 + 1}\right)\dfrac{\Theta(u_t)}{\Theta[v_t, u_t]}, \end{cases} \tag{5.3}$$

and

$$\begin{cases} w_t = u_t - m\dfrac{\Theta(u_t)}{\Theta[v_t, u_t]}, \\ \\ u_{t+1} = w_t + \left(\dfrac{s_t + ms_t^2 - (m-1)y_t(my_t - 1)}{my_t - 1}\right)\dfrac{\Theta(u_t)}{\Theta[v_t, u_t]}, \end{cases} \tag{5.4}$$

where $v_t = u_t + \beta\Theta(u_t)$, $s_t = \left(\dfrac{\Theta(w_t)}{\Theta(u_t)}\right)^{\frac{1}{m}}$, $y_t = \left(\frac{\Theta(w_t)}{\Theta(v_t)}\right)^{\frac{1}{m}}$, and $\beta = 1/2$. The methods in expressions (5.3) and (5.4) are denoted by SM_1, and SM_2, respectively.

Sharma et al. [20] method:

$$\begin{cases} z_t = u_t - m\dfrac{\Theta(u_t)}{\Theta[v_t, u_t]}, \\ u_{t+1} = z_t - \dfrac{mh_t(1 + 3h_t)}{2}\left(\dfrac{1}{y_t} + 1\right)\dfrac{\Theta(u_t)}{\Theta[v_t, u_t]}, \end{cases} \tag{5.5}$$

where $v_t = u_t + \beta\Theta(u_t)$, $x_t = \left(\dfrac{\Theta(z_t)}{\Theta(u_t)}\right)^{\frac{1}{m}}$, $y_t = \left(\dfrac{\Theta(v_t)}{\Theta(u_t)}\right)^{\frac{1}{m}}$, $h_t = \dfrac{x_t}{1 + x_t}$, and $\beta = 0.5$. The method (5.5) is denoted by MM_1.

Three polynomials with multiple zeros are employed to study the complex dynamics of the scheme (2.1). For testing, we fix $\beta = -\frac{25}{26}$ in the proposed methods LM_1, LM_2, and LM_3. The test problems considered here are as follows:

Problem 1. *Consider a polynomial $p_1(z) = (z^2 + 5z + 6)^2$ that consists of two zeros $\{-3, -2\}$ of multiplicity $m = 2$. For $\gamma = 0.01, 10^{-4}, 10^{-6}$, the polynomial's basins are shown in the Figures 3–5, respectively. The orange color is assigned to the initial approximations whenever it converges to the roots of an Eq $p_1(z) = 0$ while drawing its basins of attractions.*

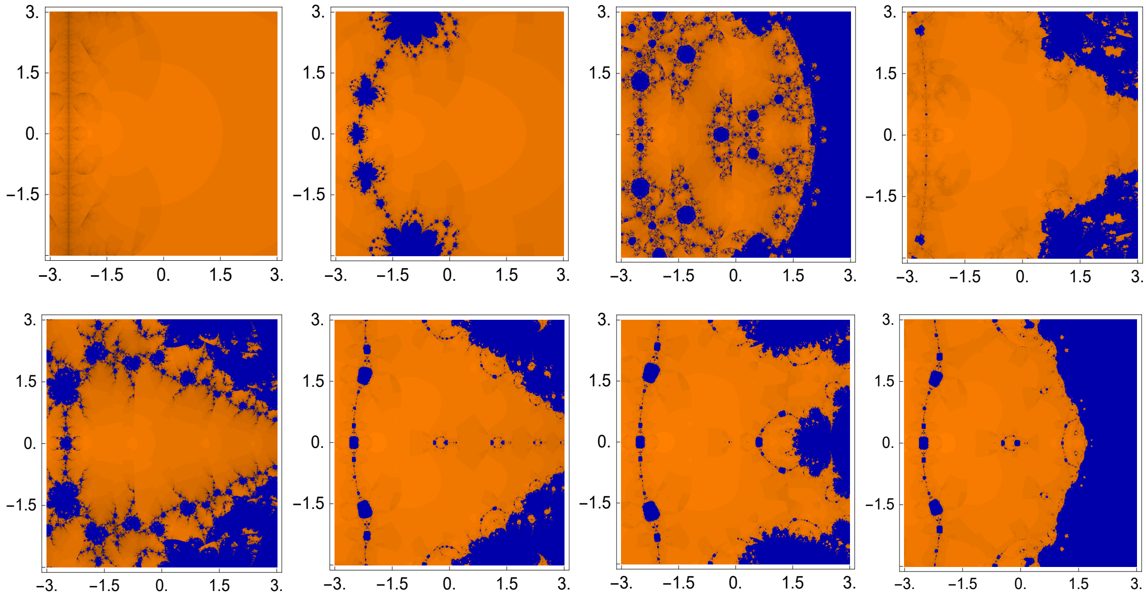

Figure 3. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for $\gamma = 10^{-2}$ in Problem 1.

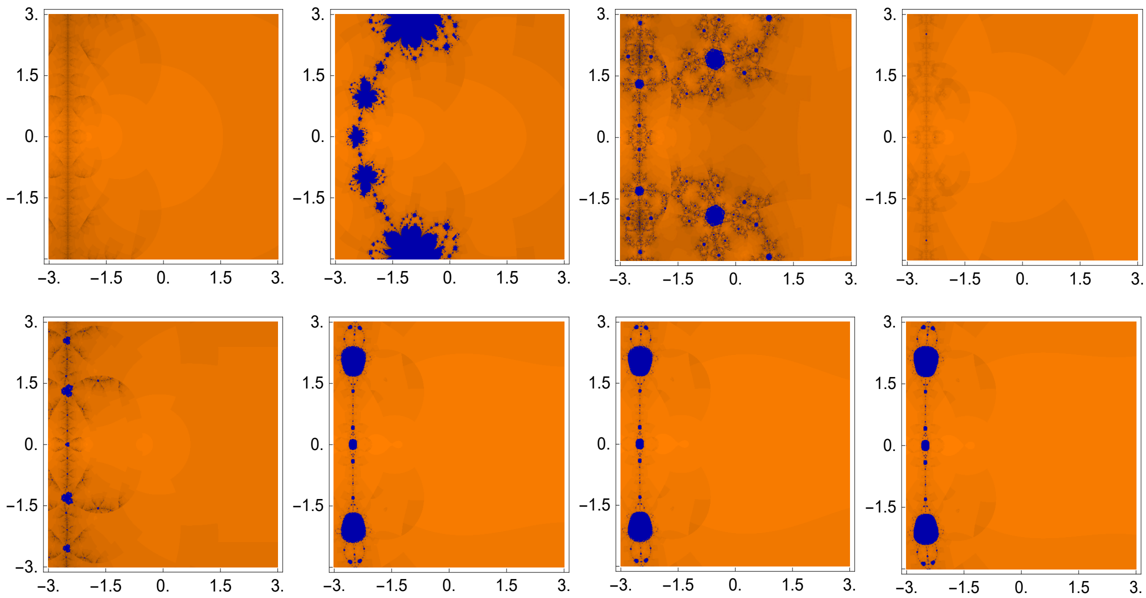

Figure 4. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for $\gamma = 10^{-4}$ in Problem 1.

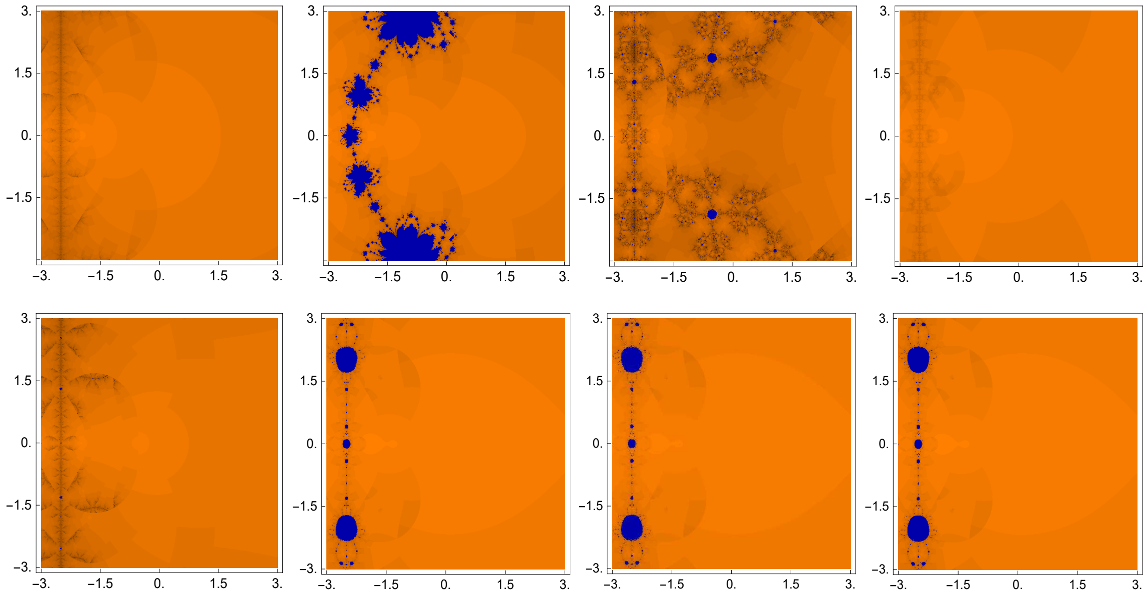

Figure 5. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for $\gamma = 10^{-6}$ in Problem 1.

Problem 2. *For further testing, suppose the polynomial* $p_2(z) = (z^2 + \frac{1}{z} + 2)^3$ *carrying three zeros* $\{0.453, -0.226{+}1.467I, -0.226{-}1.467I\}$ *with multiplicity* $m = 3$. *The basins of attractors are estimated by different schemes for* $p_2(z)$, *represented in Figures 6–8 for the parametric value* γ *as* $0.01, 10^{-4}, 10^{-6}$, *respectively. The corresponding basins are distinguished via a color allotted to them. Particularly, we allocate an orange color to all convergent points approaching to the zeros of a function* $p_2(z)$.

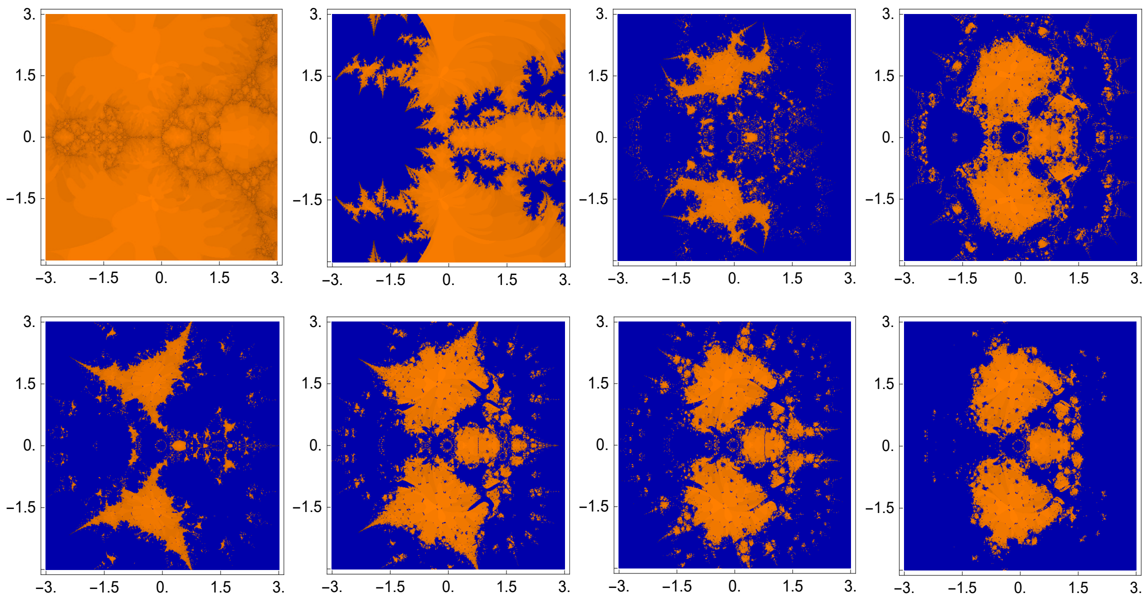

Figure 6. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for $\gamma = 10^{-2}$ in Problem 2.

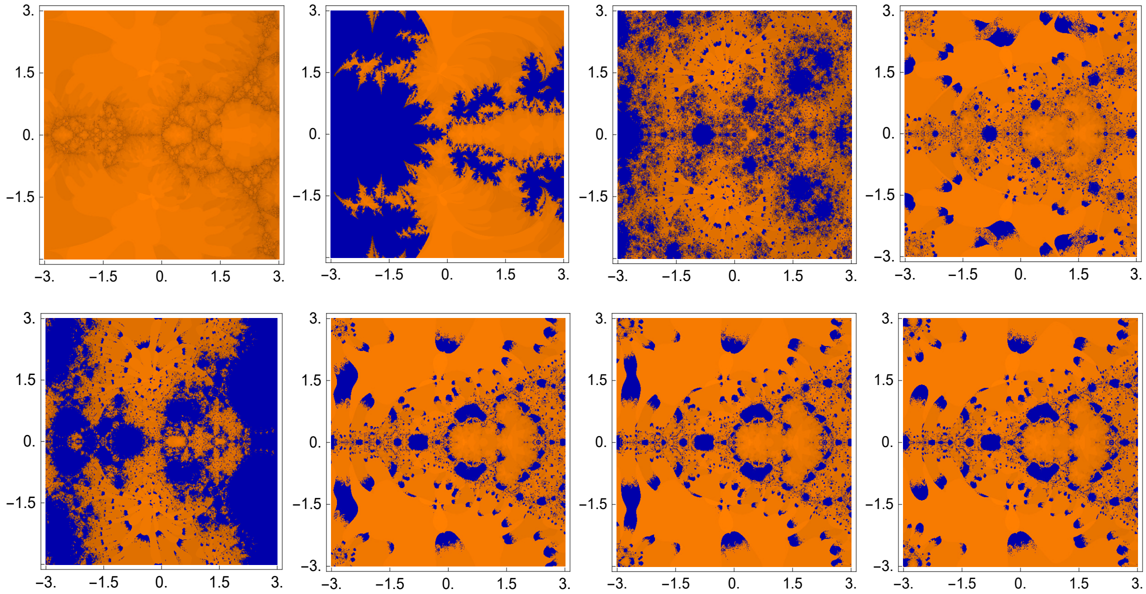

Figure 7. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for $\gamma = 10^{-4}$ in Problem 2.

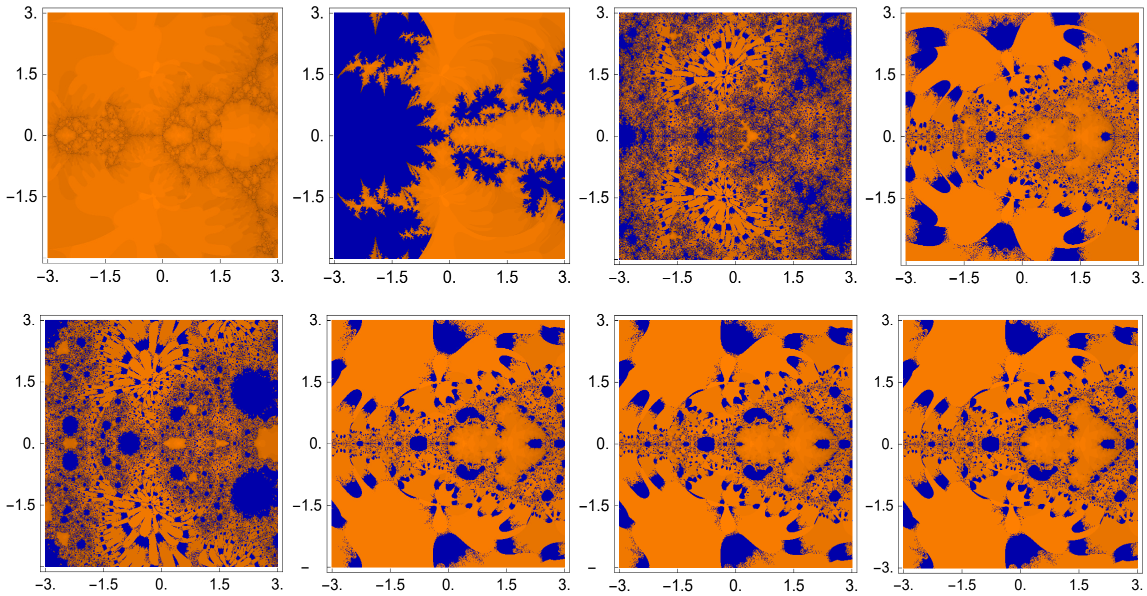

Figure 8. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for $\gamma = 10^{-6}$ in Problem 2.

Problem 3. *Consider a function* $p_3(z) = (z^3 - z)^4$ *having three zeros* $\{-1, 1, 0\}$ *with multiplicity four. Basins of attractions for this problem are illustrated in Figures 9–11 for particular values of* $\gamma = 0.01, 10^{-4}, 10^{-6}$*, respectively. We allot an orange color to the convergent points, whereas a blue color signifies the non-convergent points.*

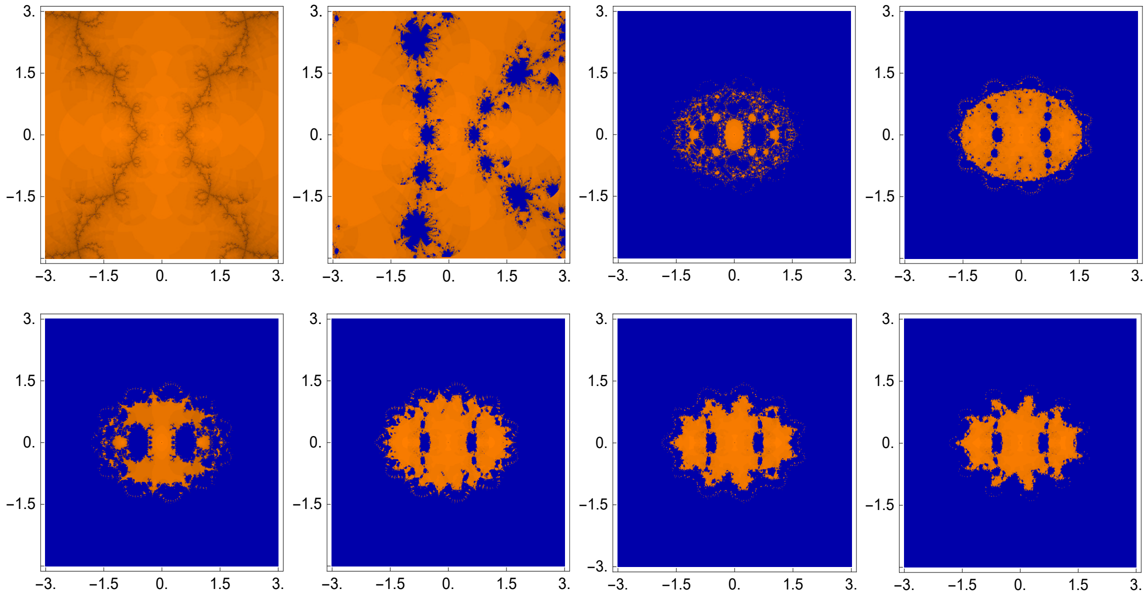

Figure 9. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for $\gamma = 10^{-2}$ in Problem 3.

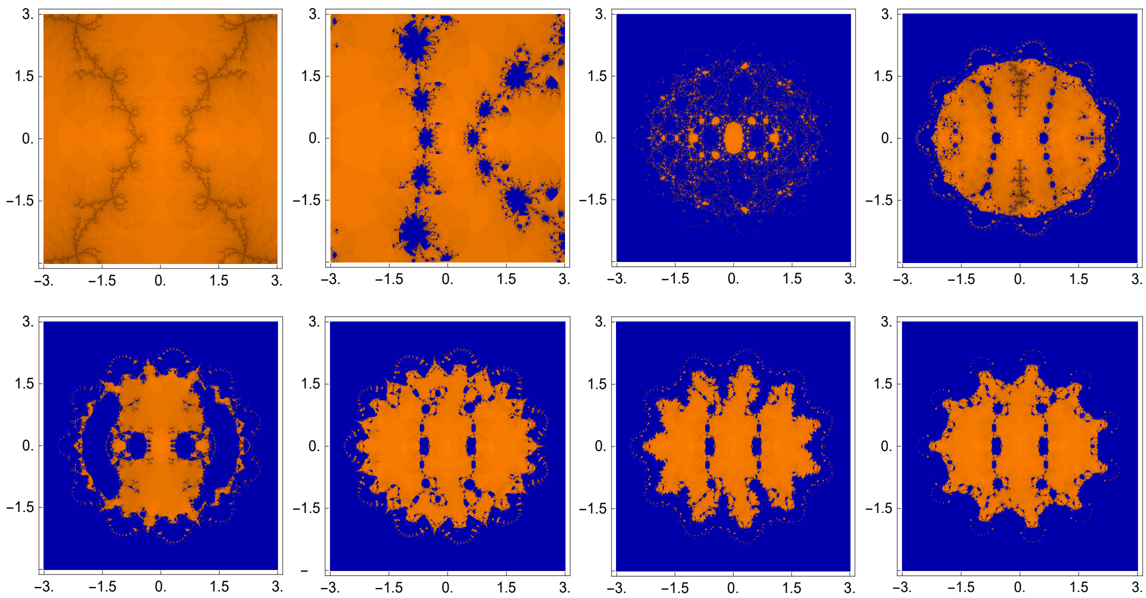

Figure 10. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for $\gamma = 10^{-4}$ in Problem 3.

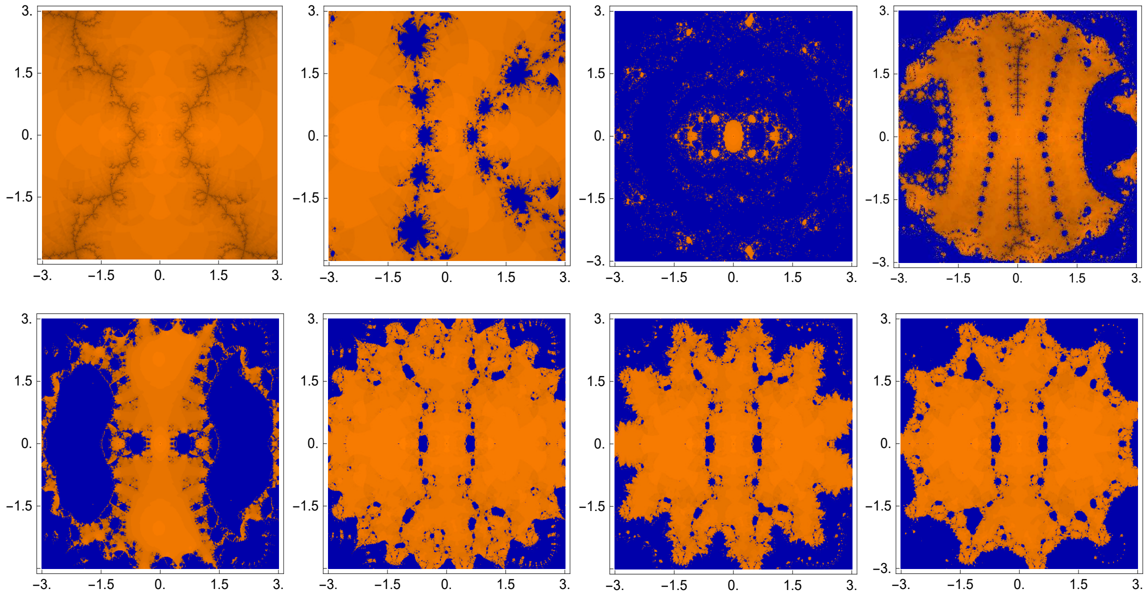

Figure 11. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for $\gamma = 10^{-6}$ in Problem 3.

We present quantitative data on the above three problems in Tables 2–4 by considering the mean value of iterations per point. The mean number of iterations in an algorithm is the total number of iterations needed per point until the root is reached, displayed in the first column (by M/P). If the point is not converging within maximum iterations, the point is regarded as non-convergent. Tables 2–4 display the percentage of non-convergent points (NC(%)) in the second column. The blue zones in the fractal images shown in Figures 3–11 represent these points.

Remember that the non-convergent points, which always contributed with the maximum 25 permitted iterations, determine the value of M/P. Contrastingly, convergent points were typically reached very quickly because we use higher-order multipoint approaches. To reduce round-off errors, we also provide an additional column (M_C/C) that shows the mean value of iterations per convergent point.

It is evident that the convergence behavior suggested family (2.1) is significantly impacted by the estimation of parameter γ. For this reason, we have chosen several values of γ. An increasing size of the basin attractors is shown by the lowering parameter γ. Conversely, the fractals get smaller when the parameter γ is large. Furthermore, when the value of γ drops, the blue areas represent the divergence zones, which are likewise getting smaller. In conclusion, we find that the convergence of the proposed approaches is significantly better at smaller γ values.

Table 2. Different measures of iterative convergence for test problem $p_1(z) = 0$.

γ	**Methods**	**M/P**	**NC (%)**	M_C/C
10^{-2}	FM_1	4.3224	0.0000	4.3224
10^{-2}	FM_2	6.1212	9.0425	4.2444
10^{-2}	SM_1	6.3036	33.3406	6.7266
10^{-2}	SM_2	7.9442	19.6862	3.7635
10^{-2}	MM_1	12.0488	30.6586	6.3226
10^{-2}	LM_1	7.8721	20.5879	3.4316
10^{-2}	LM_2	7.7204	20.2854	3.3232
10^{-2}	LM_3	6.4615	33.3045	3.2973
10^{-4}	FM_1	4.3224	0.0000	4.3224
10^{-4}	FM_2	6.1212	9.0425	4.2444
10^{-4}	SM_1	7.1947	2.9802	7.1566
10^{-4}	SM_2	3.3625	0.0092	3.3605
10^{-4}	MM_1	5.0118	0.6593	4.8791
10^{-4}	LM_1	3.2445	2.6249	2.6581
10^{-4}	LM_2	3.2631	2.6379	2.6742
10^{-4}	LM_3	2.9392	2.7215	2.6344
10^{-6}	FM_1	4.3224	$0,0000$	4.3224
10^{-6}	FM_2	6.1212	9.0425	4.2444
10^{-6}	SM_1	7.3083	0.7630	7.2817
10^{-6}	SM_2	3.3590	0.0000	3.3593
10^{-6}	MM_1	4.8903	0.0403	4.8822
10^{-6}	LM_1	3.1610	1.8289	2.7541
10^{-6}	LM_2	3.1601	1.8262	2.7539
10^{-6}	LM_3	2.9596	1.8946	2.7501

Table 3. Different measures of iterative convergence for test problem $p_2(z) = 0$.

γ	**Methods**	**M/P**	**NC (%)**	M_C/C
10^{-2}	FM_1	4.3196	0.0027	4.3191
10^{-2}	FM_2	13.1313	43.2737	4.0773
10^{-2}	SM_1	21.9086	84.4655	5.1974
10^{-2}	SM_2	18.9200	71.0817	4.1045
10^{-2}	MM_1	21.7949	84.0213	5.0422
10^{-2}	LM_1	18.6034	70.0892	3.8040
10^{-2}	LM_2	19.3298	73.4156	3.7913
10^{-2}	LM_3	11.2078	77.1364	3.5877
10^{-4}	FM_1	4.3196	0.0027	4.3191
10^{-4}	FM_2	13.1313	43.2737	4.0773
10^{-4}	SM_1	12.0181	30.0681	7.3592
10^{-4}	SM_2	4.8651	12.0227	3.4618
10^{-4}	MM_1	14.9290	49.3531	6.0382
10^{-4}	LM_1	5.1886	14.1236	3.1477
10^{-4}	LM_2	5.1637	14.1799	3.1548
10^{-4}	LM_3	4.4466	13.0958	3.1389
10^{-6}	FM_1	4.3196	0.0027	4.3191
10^{-6}	FM_2	13.1313	43.2737	4.0773
10^{-6}	SM_1	12.6317	34.8122	8.3879
10^{-6}	SM_2	5.4710	19.8234	3.4935
10^{-6}	MM_1	11.4979	33.2152	7.1989
10^{-6}	LM_1	5.42989	18.5421	3.1306
10^{-6}	LM_2	5.4593	18.7222	3.1285
10^{-6}	LM_3	5.0120	18.6869	3.1274

Table 4. Different measures of iterative convergence for test problem $p_3(z) = 0$.

γ	**Methods**	**M/P**	**NC (%)**	M_C/C
10^{-2}	FM_1	5.7004	0.0289	5.6948
10^{-2}	FM_2	6.5795	10.3819	4.4460
10^{-2}	SM_1	24.0285	94.6878	6.7120
10^{-2}	SM_2	22.0260	85.8817	3.9354
10^{-2}	MM_1	23.1753	90.6551	5.4738
10^{-2}	LM_1	21.9881	86.2613	3.0770
10^{-2}	LM_2	22.4859	88.5366	3.0684
10^{-2}	LM_3	12.7140	89.3171	2.9839
10^{-4}	FM_1	5.7004	0.0289	5.6948
10^{-4}	FM_2	6.5795	10.3819	4.4456
10^{-4}	SM_1	23.6691	92.2901	7.7383
10^{-4}	SM_2	18.2325	65.5750	5.3413
10^{-4}	MM_1	20.8471	78.5020	5.6822
10^{-4}	LM_1	17.7304	66.4600	3.3255
10^{-4}	LM_2	19.0186	72.4094	3.3210
10^{-4}	LM_3	11.5767	72.1510	3.3508
10^{-6}	FM_1	5.7004	0.0289	5.6948
10^{-6}	FM_2	6.5795	10.3819	4.4455
10^{-6}	SM_1	23.6912	92.8491	6.7024
10^{-6}	SM_2	13.4588	35.4586	7.1190
10^{-6}	MM_1	16.9522	58.6414	5.5425
10^{-6}	LM_1	8.3798	22.6486	3.5143
10^{-6}	LM_2	10.7761	33.7749	3.5235
10^{-6}	LM_3	7.7696	34.1925	3.4390

6. Numerical experimentation

In this section, we examine the performance of the derivative-free proposed methods LM_1, LM_2 and LM_3 on test problems by fixing $\beta = -\frac{25}{26}$, and $\gamma = \frac{1}{2}$. We have considered several practical and standard academic problems to assess the execution of iterative schemes. For this, first we have determined the comparison on three well-known real-life problems stated in Examples 1, 2, and 3, in which nonlinear equations are established consisting of multiple roots. Next, we choose a problem related to eigenvalues as in Example 4. At last, we check three academic Examples 5, 6, and 7 having multiple roots of multiplicity $m = 5$, $m = 6$ and $m = 8$, respectively. Along with the methods considered for the sake of comparison in the previous section, we have included one more iterative method to show that the present method is faster than the classic second-order method (TM_1 for $\gamma = \frac{1}{2}$).

We utilize computer specifications and the programming application Mathematica 11 to perform multiple precision arithmetic: CPU speed: 2.80GHz Intel (R) i7-7600U (64-bit operating system) 8 GB of RAM and Microsoft Windows 10 Pro.

The computational performance of the iterative techniques is measured in terms of the iteration number (t), consecutive error approximations $|e'_{t+1}| = |x_{t+1} - x_t|$, absolute residual functional error $|\Theta(x_t)|$, and the order of convergence (ρ), which is calculated computationally [8, 14] using formula:

$$\rho \approx \frac{\ln |\Theta(x_{t+1})/\Theta(x_t)|}{\ln |\Theta(x_t)/\Theta(x_{t-1})|}, \ t = 1, 2, \ldots.$$

Further, in order to reduce round-off errors, the results are evaluated with a minimum of 3000 significant digits in Tables 5–11. Furthermore, the data displayed corresponding to the column of $|x_{t+1} - x_t|$ and $|\Theta(x_t)|$ in the comparison tables are up to the first two significant digits along with its exponent power. The scientific notation $a \times 10^{\pm b}$ is represented as $a(\pm b)$. In addition, the ρ is displayed up to five significant digits. For each example, we have plotted the basins of attraction in Figures 12–18 to know convergence domains of all fourth-order iterative methods for solving the nonlinear equation $\Theta_i(z) = 0, \ i = 1, 2, \ldots, 7$ in Examples 1–7.

Table 5. Computational outcomes of iterative methods for Example 1.

Methods	t	$\|e'_2\|$	$\|e'_3\|$	$\|e'_4\|$	$\|\Theta_1(x_t)\|$	ρ
TM_1	3	2.7(−3)	1.5(−7)	4.4(−16)	8.5(−17)	1.9998
FM_1	3	3.2(−6)	1.7(−26)	1.5(−107)	2.8(−108)	4.0000
FM_2	3	5.3(−3)	3.6(−10)	8.1(−39)	1.6(−39)	4.0010
SM_1	3	3.1(−6)	1.6(−26)	9.3(−108)	1.8(−108)	4.0000
SM_2	3	3.1(−6)	1.3(−26)	4.7(−108)	9.1(−109)	4.0000
MM_1	3	1.2(−4)	1.4(−11)	1.6(−25)	3.1(−26)	2.0000
LM_1	3	3.3(−6)	2.0(−26)	2.5(−107)	4.8(−108)	4.0000
LM_2	3	3.3(−6)	2.0(−26)	2.7(−107)	5.2(−108)	4.0000
LM_3	3	3.4(−6)	2.1(−26)	3.1(−107)	5.9(−108)	4.0000

Example 1. *We study a problem pertaining to Planck's radiation law [3]. The following nonlinear equation that appears during the mathematical modeling:*

$$G(y) = \frac{8\pi c h y^{-5}}{e^{\frac{ch}{ykT}} - 1},$$

Here, c, T, y, k,, and h stand for the speed of light, the black body's absolute temperature, the radiation wavelength, the Boltzmann value, and Planck's constant, respectively. This law gives the spectrum distribution of radiations from a black body at a given temperature in thermal equilibrium. In order to determine the wavelength y, which corresponds to the maximum energy density $G(y)$, we solve the equation $G'(y) = 0$.
For doing so, we perform some algebraic calculations and arrive at the following equation:

$$\frac{\left(\frac{ch}{ykT}\right) e^{\frac{ch}{ykT}}}{e^{\frac{ch}{ykT}} - 1} = 5.$$

Additionally, it can be expressed as the following nonlinear function:

$$\Theta_1(x) = \left(e^{-x} - 1 + \frac{x}{5}\right), \tag{6.1}$$

where $x = \frac{ch}{ykT}$. *Now, the approximated root of an Eq* (6.1) *of multiplicity* 1 *is computed using iterative techniques with initial guess* $x_0 = 5.4$, *which turns out to be* 4.96511423174428. *Further, the wavelength of the nonlinear model can be obtained via the expression* $x = \frac{ch}{ykT}$. *The outcomes of our testing of this problem are shown in Table 5.*

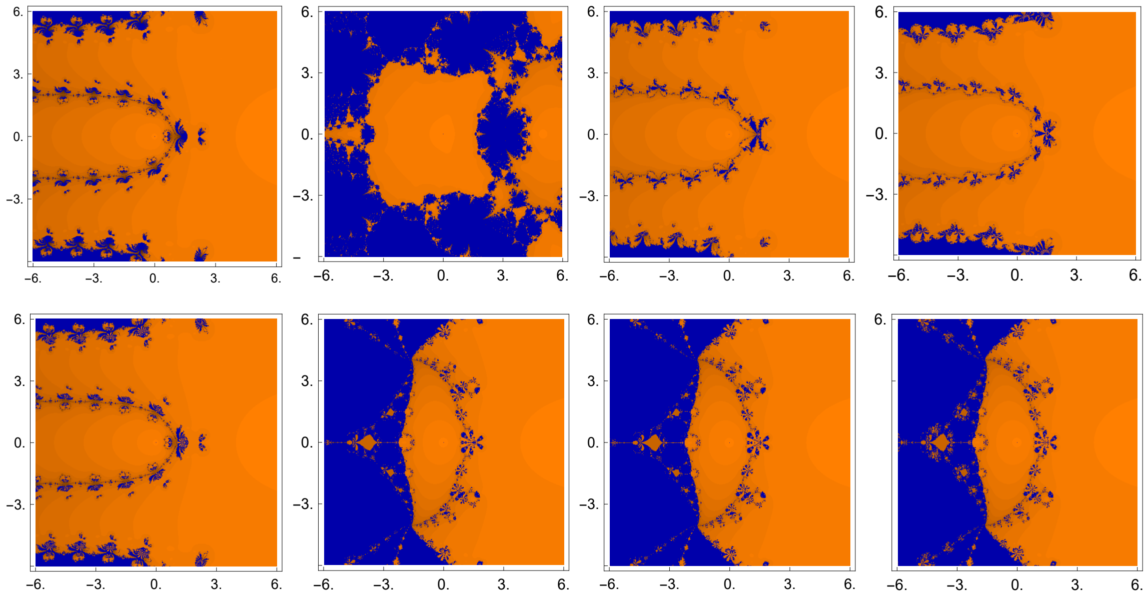

Figure 12. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for $\gamma = 10^{-6}$ in Example 1.

Example 2. *The Van der Waals equation [4] for the ideal gas is studied in this instance, whose mathematical representation is given by*

$$\left(P_1 + \frac{a_1 n_1^2}{V_1}\right)(V_1 - n_1 b_1) = n_1 R_1 T_1,$$

indicating how a real gas with two constants, a_1 *and* b_1, *behaves. To simplify this equation, the problem at hand is to determine the optimal value of volume in the above expression. After giving values to pressure* (P_1) *and the number of moles* (n_1), *the following nonlinear equation is obtained in form of volume* (x)

$$\Theta_2(x) = x^3 - (522/100)x^2 + (90825/10000)x - (52675/10000) = 0.$$

The multiple root for this model is 1.75 *of multiplicity two which is approximated by using initial guess* $x_0 = 2$, *and the corresponding outcomes are provided in Table 6.*

Table 6. Computational outcomes of iterative methods for Example 2.

Methods	t	$\lvert e_2' \rvert$	$\lvert e_3' \rvert$	$\lvert e_4' \rvert$	$\lvert \Theta_2(x_t) \rvert$	ρ
TM_1	3	6.0(−2)	1.9(−2)	3.8(−3)	5.4(−7)	1.1565
FM_1	3	3.4(−2)	2.1(−3)	6.8(−7)	1.4(−14)	2.5369
FM_2	3	2.7(−2)	5.5(−4)	1.8(−11)	9.3(−24)	4.0681
SM_1	3	2.7(−2)	6.7(−4)	2.1(−9)	1.3(−19)	3.1312
SM_2	3	2.4(−2)	3.7(−4)	1.2(−10)	4.7(−22)	3.3247
MM_1	3	3.7(−2)	2.6(−3)	1.3(−6)	5.2(−14)	2.4417
LM_1	3	1.6(−2)	5.1(−5)	1.8(−14)	9.3(−30)	3.6445
LM_2	3	1.8(−2)	7.3(−5)	7.7(−14)	1.8(−28)	3.5977
LM_3	3	1.4(−2)	3.1(−5)	2.5(−15)	1.8(−31)	3.6983

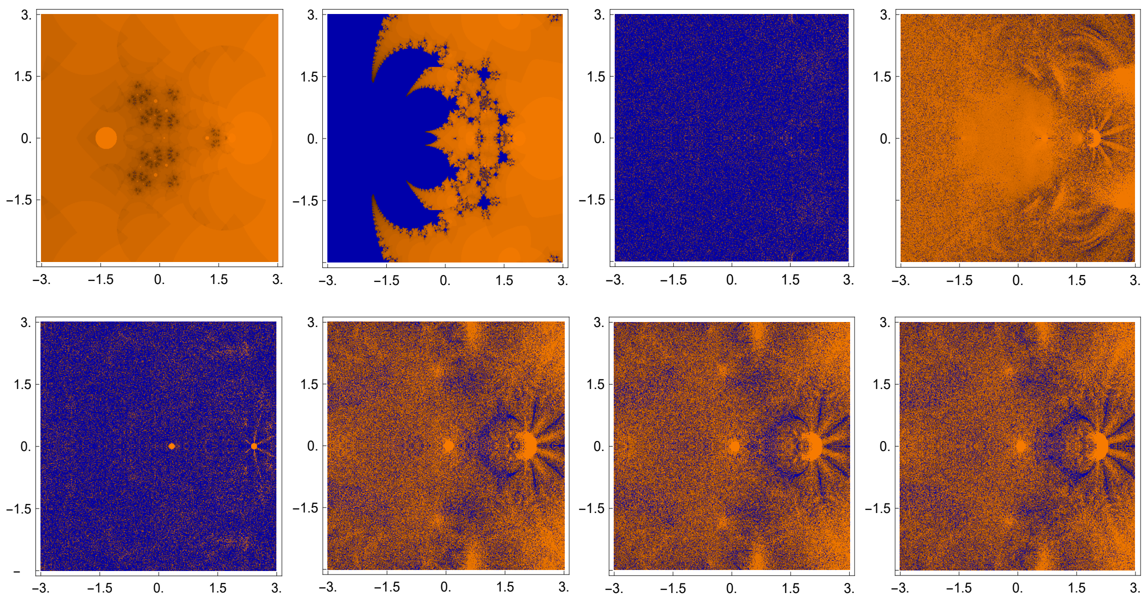

Figure 13. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for γ =10^{-6} in Example 2.

Example 3. *An unwanted radio frequency breakdown can occur in high-power microwave media running with the condition of vacuum, termed as a multifactor effect. In general, it can be found inside a parallel plate that guides the waves. Between the plates, an electric field is generated with voltage, which allows the electron to move within the plates. So, here we study a mathematical model of the trajectory of an electron in the air gap between two parallel plates, given by*

$$y(t) = y_0 + e\frac{E_0}{m_a\omega^2}(\cos(\omega t + \Delta) + \left(v_0 + e\frac{E_0}{m_a\omega}\sin(\omega t_0 + \delta)\right)(t - t_0) - \cos(\omega t_0 + \Delta)),$$

where e and m_a denote the charge and mass of an electron at rest, whereas $E_0\sin(\omega t + \Delta)$ corresponds to a radio frequency electric field between the plates. Here, v_0 and y_0 denote the velocity and position *of an electron at time t_0. By considering the particular values of the parameters in the above equation, one can obtain its normalized form as given below:*

$$\Theta_3(x) = \cos(x) + x - \frac{\pi}{2}.$$

We are looking for multiple root $\frac{\pi}{2}$ of multiplicity three which is approximated by using an initial guess of $x_0 = 1$, Table 7 lists the numerical performance of the considered approaches.

Table 7. Computational outcomes of iterative methods for Example 3.

Methods	t	$\lvert e'_2\rvert$	$\lvert e'_3\rvert$	$\lvert e'_4\rvert$	$\lvert\Theta_3(x_t)\rvert$	ρ
TM_1	3	8.5(−3)	3.0(−8)	1.4(−24)	4.4(−73)	3.0000
FM_1	3	Diverges	–	–	–	–
FM_2	3	Diverges	–	–	–	–
SM_1	3	3.2(−5)	1.9(−26)	1.5(−132)	5.9(−397)	5.0000
SM_2	3	3.4(−5)	2.5(−26)	5.3(−132)	2.5(−395)	5.0000
MM_1	3	5.0(−5)	3.8(−25)	9.6(−126)	1.5(−376)	5.0000
LM_1	3	2.5(−6)	6.1(−32)	5.1(−160)	2.2(−479)	5.0000
LM_2	3	1.6(−6)	5.8(−33)	4.1(−165)	1.2(−494)	5.0000
LM_3	3	2.8(−7)	1.1(−36)	1.0(−183)	1.9(−550)	5.0000

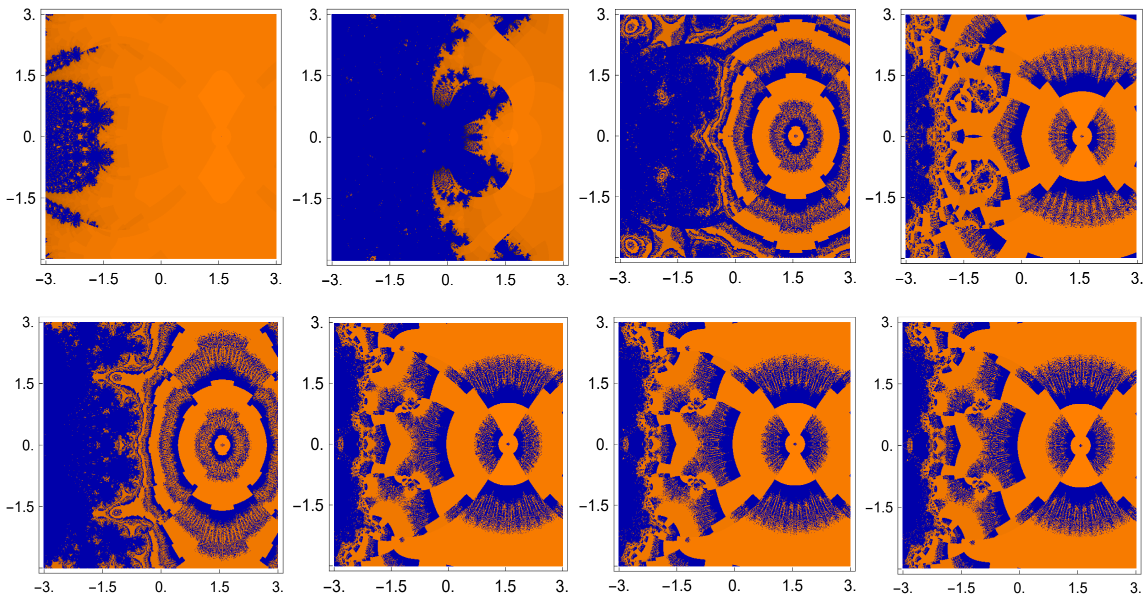

Figure 14. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for $\gamma = 10^{-6}$ in Example 3.

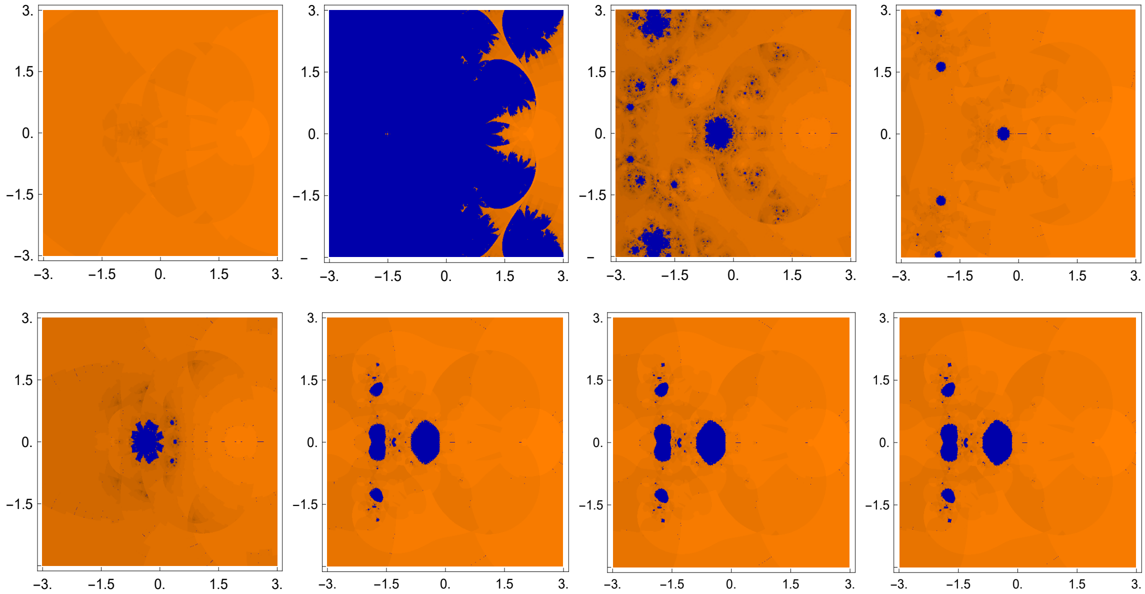

Figure 15. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for $\gamma = 10^{-6}$ in Example 4.

Example 4. *Let us consider a* 5×5 *matrix:*

$$\begin{bmatrix} 29 & 14 & 2 & 6 & -9 \\ -47 & -22 & -1 & -11 & 13 \\ 19 & 10 & 5 & 4 & -8 \\ -19 & -10 & -3 & -2 & 8 \\ 7 & 4 & 3 & 1 & -3 \end{bmatrix}.$$

The matrix shown above has a characteristic polynomial in terms of x *as*

$$\Theta_4(x) = (1 + x)(2 - x)^4.$$

This polynomial contains a multiple root 2 of multiplicity 4. Using $x_0 = 1.9$*, the computational outcomes are summarized in Table 8.*

Table 8. Computational outcomes of iterative methods for Example 4.

Methods	t	$\lvert e'_2\rvert$	$\lvert e'_3\rvert$	$\lvert e'_4\rvert$	$\lvert\Theta_4(x_t)\rvert$	ρ
TM_1	3	1.1(−3)	9.8(−8)	8.0(−16)	1.3(−60)	1.9999
FM_1	3	1.8(−3)	6.5(−14)	1.2(−55)	1.3(−220)	3.9999
FM_2	3	1.4(−2)	2.4(−8)	6.7(−16)	6.2(−61)	1.3076
SM_1	3	2.2(−3)	1.2(−14)	2.4(−29)	1.0(−114)	1.3046
SM_2	3	2.2(−3)	2.6(−14)	1.1(−28)	5.0(−112)	1.3140
MM_1	3	2.2(−3)	1.5(−13)	3.7(−54)	5.8(−214)	3.9980
LM_1	3	2.2(−3)	5.0(−14)	1.6(−56)	1.9(−223)	3.9931
LM_2	3	2.2(−3)	4.9(−14)	1.6(−56)	1.9(−223)	3.9930
LM_3	3	2.2(−3)	5.0(−14)	1.6(−56)	2.0(−223)	3.9931

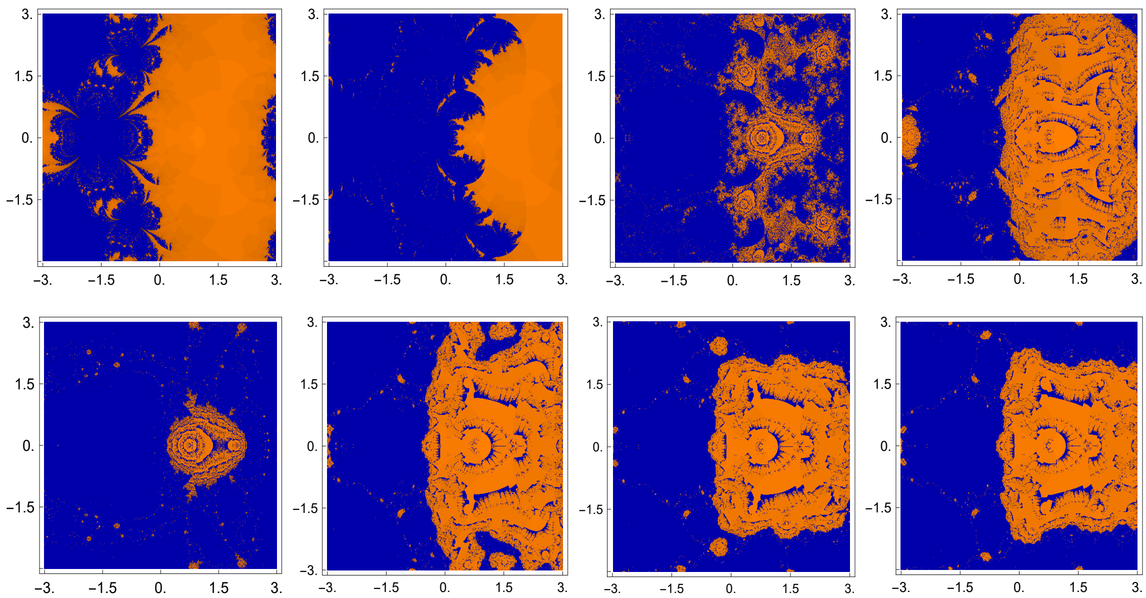

Figure 16. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for $\gamma = 10^{-6}$ in Example 5.

In the next three examples, we consider some academic problems to check the efficacy of proposed algorithms compared to the existing schemes.

Example 5. *Consider the following nonlinear function carrying multiple zero at* $x = 0.739085133$ *of multiplicity 5:*

$$\Theta_5(x) = (cos(x) - x)^5.$$

Here, the numerical outcomes are shown in Table 9 by taking $x_0 = 0.8$.

Table 9. Computational outcomes of iterative methods for Example 5.

Methods	t	$\lvert e_2' \rvert$	$\lvert e_3' \rvert$	$\lvert e_4' \rvert$	$\lvert \Theta_5(x_t) \rvert$	ρ
TM_1	3	7.6(−4)	1.3(−7)	3.5(−15)	7.1(−72)	1.9999
FM_1	3	1.4(−6)	4.3(−25)	4.1(−99)	1.5(−491)	3.9999
FM_2	3	Diverges	–	–	–	–
SM_1	3	Diverges	–	–	–	–
SM_2	3	Diverges	–	–	–	–
MM_1	3	1.4(−6)	5.1(−25)	8.3(−99)	5.2(−490)	3.9997
LM_1	3	9.7(−7)	5.6(−26)	6.2(−103)	1.2(−510)	4.0000
LM_2	3	1.1(−6)	1.1(−25)	8.4(−102)	5.4(−505)	4.0000
LM_3	3	7.9(−7)	2.5(−26)	2.6(−104)	1.5(−517)	4.0000

Example 6. *Consider the function*

$$\Theta_6(x) = \left(1 + (1 - x)^3\right)^6,$$

with zero at $x = 2$ having multiplicity 6. With the value $x_0 = 1.9$, we presented the numerical outcomes in Table 10.

Table 10. Computational outcomes of iterative methods for Example 6.

Methods	t	$\lvert e_2' \rvert$	$\lvert e_3' \rvert$	$\lvert e_4' \rvert$	$\lvert \Theta_6(x_t) \rvert$	ρ
TM_1	3	1.2(−2)	1.4(−4)	2.0(−8)	4.7(−44)	1.9912
FM_1	3	2.9(−2)	5.3(−6)	7.5(−21)	1.3(−118)	3.9624
FM_2	3	6.0(−2)	2.4(−6)	1.7(−9)	2.0(−50)	1.2099
SM_1	3	4.5(−2)	2.7(−5)	1.5(−9)	7.8(−51)	1.3175
SM_2	3	4.8(−2)	4.2(−5)	3.5(−9)	1.5(−48)	1.3235
MM_1	3	3.0(−2)	6.3(−6)	1.5(−20)	8.3(−117)	3.9601
LM_1	3	2.1(−2)	5.8(−7)	3.9(−25)	2.5(−144)	3.9830
LM_2	3	1.6(−2)	2.0(−7)	5.2(−27)	1.5(−155)	3.9871
LM_3	3	2.6(−2)	1.4(−6)	1.5(−23)	8.7(−135)	3.9856

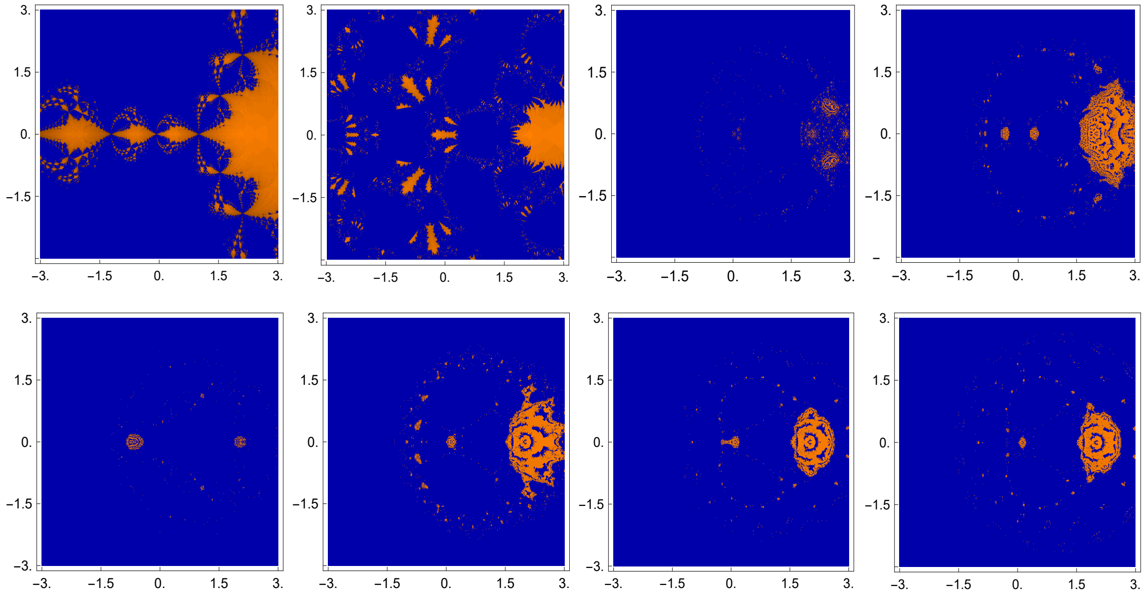

Figure 17. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for $\gamma = 10^{-6}$ in Example 6.

Example 7. *We consider the following nonlinear function:*

$$\Theta_7(x) = (5\arctan(x) - 4x)^8,$$

carrying multiple zero $x = 0.94913461128828951372581521479848875$ *of multiplicity 6. The outcomes for this test problem are presented in Table 11, by taking* $x_0 = 1$.

Table 11. Computational outcomes of iterative methods for Example 7.

Methods	t	$\lvert e'_2\rvert$	$\lvert e'_3\rvert$	$\lvert e'_4\rvert$	$\lvert\Theta_7(x_t)\rvert$	ρ
TM_1	3	2.2(−3)	4.6(−6)	2.0(−11)	3.4(−85)	1.9984
FM_1	3	3.7(−5)	1.7(−17)	7.7(−67)	1.5(−528)	4.0000
FM_2	3	9.0(−6)	2.3(−10)	1.3(−38)	9.6(−303)	6.1661
SM_1	3	5.5(−5)	5.7(−9)	9.2(−33)	6.4(−256)	5.9794
SM_2	3	6.0(−5)	7.0(−9)	2.2(−32)	8.0(−253)	5.9683
MM_1	3	3.7(−5)	1.7(−17)	7.7(−67)	1.5(−528)	4.0000
LM_1	3	3.8(−5)	1.6(−17)	5.5(−67)	1.0(−529)	4.0000
LM_2	3	3.8(−5)	1.6(−17)	5.2(−67)	6.8(−530)	4.0000
LM_3	3	3.8(−5)	1.6(−17)	5.8(−67)	1.5(−529)	4.0000

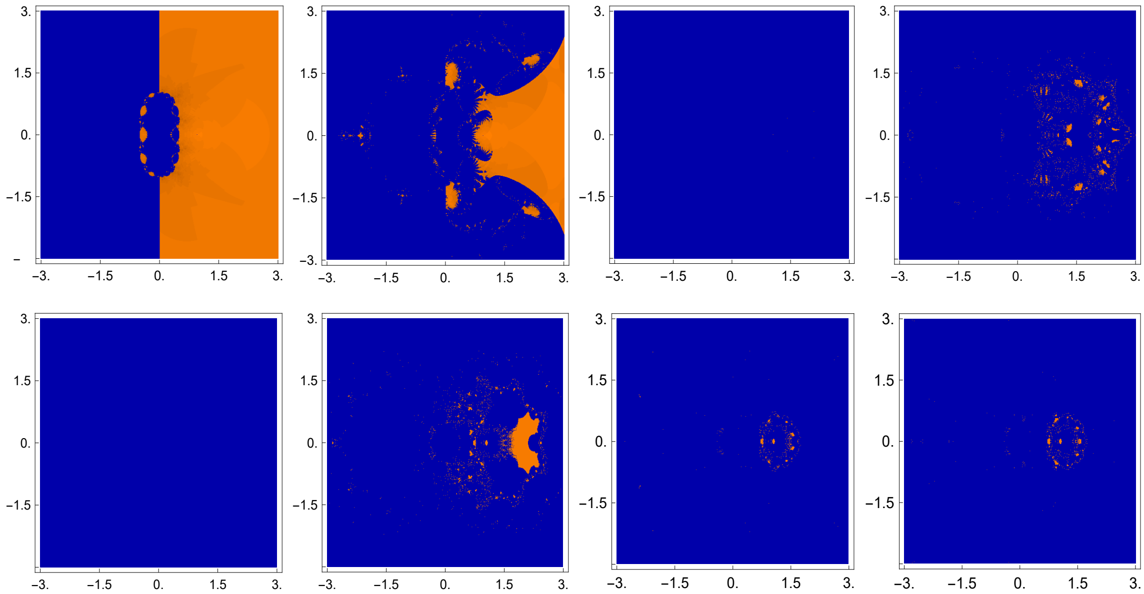

Figure 18. Convergence planes of FM_1, FM_2, SM_1, SM_2, MM_1, LM_1, LM_2 and LM_3 for $\gamma = 10^{-6}$ in Example 7.

The numerical findings of several techniques for calculating multiple roots of nonlinear functions Θ_i, $i = 1, 2, \ldots, 7$ are presented in Tables 5–11, respectively. From these results, one can observe that the proposed schemes work more adequately in terms of accuracy in Examples 2–6. It is necessary to keep in mind that the desired results are obtained by each algorithm. However, in Examples 2, 4, and 6, the convergence order of certain existing methods is lower than their theoretical convergence order. This disparity occurs because the convergence domains can differ for different iterative processes. As a result, a specific method may converge more slowly for a particular initial guess, highlighting its sensitivity to the choice of initial conditions that are close to the desired solution. Henceforth, the numerical outcomes in Tables 5–11 reveal that the schemes LM_1, LM_2, and LM_3 outperform the current robust approaches. Moreover, compared to existing methods, our schemes yield lower absolute errors in the consecutive iterations and functional errors. While drawing basins of attraction, we find that the iterative method MM_1 does not support well in the complex domain in the case of Examples 2 and 5, which shows its unstable nature toward complex initial guesses; however, the proposed solvers show better convergence plane than the existing derivative-free techniques.

7. Conclusions

We have introduced a novel two-point derivative-free iterative technique using weight functions that approximate the multiple roots of the nonlinear equations. We have discussed a detailed theoretical analysis for $m = 1$, 2, and 3. For $m \geq 4$, it is additionally provided in generalized form, confirming that the convergence order is at least four. Furthermore, a few special cases are shown via the use of distinct weight functions. We have also demonstrated the basins of attraction of our methods for various parametric values in the complex plane to verify their stability. The numerical results illustrate the higher performance of the proposed family. Also, it shows that the proposed derivative-free family performs significantly better than the current ones for academic problems as well as for real-life applications. Therefore, we can conclude that the suggested class would be a valuable alternative for numerically calculating multiple zeros of nonlinear functions.

Author contributions

Munish Kansal: Methodology, supervision, writing-review & editing, validation; Vanita Sharma: Conceptualization, methodology, writing-review & editing; Litika Rani: Methodology, Formal analysis, writing-original draft, writing-review & editing, software; Lorentz Jäntschi: Methodology, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interests

The authors declare no potential conflict of interests.

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