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Research article

Intra-regular semihypergroups characterized by Fermatean fuzzy bi-hyperideals

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Abstract: The concept of Fermatean fuzzy sets was introduced by Senapati and Yager in 2019 as a generalization of fuzzy sets, intuitionistic fuzzy sets, and Pythagorean fuzzy sets. In this article, we apply the notions of Fermatean fuzzy left (resp., right) hyperideals and Fermatean fuzzy (resp., generalized) bi-hyperideals in semihypergroups to characterize intra-regular semihypergroups, such as S is an intra-regular semihypergroup if and only if $\mathcal{L} \cap \mathcal{R} \subseteq \mathcal{L} \circ \mathcal{R}$, for every Fermatean fuzzy left hyperideal $\mathcal{L}$ and Fermatean fuzzy right hyperideal $\mathcal{R}$ of a semihypergroup S. Moreover, we introduce the concept of Fermatean fuzzy interior hyperideals of semihypergroups and use these properties to describe the class of intra-regular semihypergroups. Next, we demonstrate that Fermatean fuzzy interior hyperideals coincide with Fermatean fuzzy hyperideals in intra-regular semihypergroups. However, in general, Fermatean fuzzy interior hyperideals do not necessarily have to be Fermatean fuzzy hyperideals in semihypergroups. Finally, we discuss some characterizations of semihypergroups when they are both regular and intra-regular by means of different types of Fermatean fuzzy hyperideals in semihypergroups.

Keywords: Fermatean fuzzy hyperideal; Fermatean fuzzy bi-hyperideal; Fermatean fuzzy interior hyperideal; regular semihypergroup; intra-regular semihypergroup
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1. Introduction

In the opinion of Zadeh [45], fuzzy set theory, which was introduced in 1965, says that decision-makers should take membership degree into account while settling ambiguous situations. It is a method of conveying and presenting vague or ill-defined information. The concept of fuzzy sets has been explored by several researchers (see, e.g., [9, 11, 38, 42, 46]). In mathematics, the concept of a fuzzy set is a generalization of classical sets. There are various extensions of fuzzy sets, such as intuitionistic fuzzy sets [4], Pythagorean fuzzy sets [44], Fermatean fuzzy sets [34], spherical fuzzy sets [3], picture fuzzy sets [7], and linear Diophantine fuzzy sets [33], among others. In this research, we will review the extensions of fuzzy sets relevant to this study, namely intuitionistic fuzzy sets, Pythagorean fuzzy sets, and Fermatean fuzzy sets. In 1986, Atanassov [4] introduced the notion of intuitionistic fuzzy sets as a generalization of fuzzy sets. These sets consist of an element's degree of membership and non-membership in a universe set, with the rule that sum of these degrees not be greater than one. Currently, the concept of intuitionistic fuzzy sets is still being studied continuously [10, 22, 23, 41]. Subsequently, Yager [44] introduced the notion of Pythagorean fuzzy sets, where the sum of the squares of membership and non-membership is constrained to the unit interval $[0, 1]$. This concept generalizes both fuzzy sets and intuitionistic fuzzy sets. In addition, Senapati and Yager [34] first introduced the concept of Fermatean fuzzy sets in 2019, defining them as the cube sum of their membership and non-membership degrees in $[0, 1]$. The fuzzy sets, intuitionistic fuzzy sets, and Pythagorean fuzzy sets are all generalized by Fermatean fuzzy sets. For example, consider two real numbers, 0.7 and 0.8, in the interval $[0, 1]$. We can observe that $0.7 + 0.8 > 1$ and $(0.7)^2 + (0.8)^2 > 1$, but $(0.7)^3 + (0.8)^3 < 1$. This means that the Fermatean fuzzy sets have a better information space than the intuitionistic fuzzy sets and the Pythagorean fuzzy sets.

The concepts of various types of fuzzy set mentioned above are applied to the classes of algebras, helping develop the basic properties of these algebras. The semigroup is an essential structure in abstract algebra and has applications in automata theory, numerical theory, functional analysis, and optimization, among other mathematical and theoretical fields. The study of the regularity of semigroups is an important and trending area of research. This article will briefly review the classification of semigroups using various types of fuzzy sets. Kehayopulu and Tsingelis [18] used fuzzy quasi-ideals and fuzzy left (resp., right) ideals to characterize regular ordered semigroups. Xie and Tang [43] later developed fuzzy left (resp., right) ideals, fuzzy (resp., generalized) bi-ideals, and fuzzy quasi-ideals that characterized the classes of regular and intra-regular ordered semigroups. Further characterizations of regular, intra-regular, and left weakly regular ordered semigroups were then provided by Khan and Shabir [19], using their intuitionistic fuzzy left (resp., right) ideals. Subsequently, Hussain et al. [13] introduced the concept of rough Pythagorean fuzzy ideals in semigroups, which extends to the lower and upper approximations of bi-ideals, interior ideals, $(1, 2)$-ideals, and Pythagorean fuzzy left (resp., right) ideals of semigroups. Afterwards, the concepts of Pythagorean fuzzy prime ideals and semi-prime ideals of ordered semigroups, together with some of the essential features of Pythagorean fuzzy regular and intra-regular ordered semigroup ideals, were examined by Adak et al. [2]. A review of relations is provided for the family of Fermatean fuzzy regular ideals of ordered semigroups, and Adak et al. [2] determined the concept of Fermatean fuzzy semi-prime (resp., prime) ideals. For using different types of fuzzy sets to classify the regularity of semigroups, see [5, 17, 20, 21, 36].

As a generalization to ordinary algebraic structures, Marty [24] gave algebraic hyperstructures in 1934. In an algebraic hyperstructure, the composition of two elements is a nonempty set, but in an ordinary algebraic structure, the composition of two elements is an element. The notion of a semigroup is generalized to form a semihypergroup. Several authors have investigated various facets of semihypergroups; for instance, see [8, 12, 31, 32]. Fuzzy set theory gives a novel field of study called fuzzy hyperstructures. In 2014, Hila and Abdullah [16] characterized various classes of Γ-semihypergroups using intuitionistic fuzzy left (resp., right, two-sided) Γ-hyperideals and intuitionistic bi-Γ-hyperideals. Afterwards, the characteristics of fuzzy quasi-Γ-hyperideals were used by Tang et al. [39] in 2017 to study characterizations of regular and intra-regular ordered Γ-semihypergroups. Additional characterizations of regular semihypergroups and intra-regular semihypergroups were given by Shabir et al. [35], based on the properties of their $(\in, \in \vee q)$-bipolar fuzzy hyperideals and $(\in, \in \vee q)$-bipolar fuzzy bi-hyperideals. Furthermore, Masmali [25] used Pythagorean picture fuzzy sets hyperideals to characterize the class of regular semihypergroups. More recently, Nakkhasen [28] introduced Fermatean fuzzy subsemihypergroups, Fermatean fuzzy (resp., left, right) hyperideals, and Fermatean fuzzy (resp., generalized) bi-hyperideals of semihypergroups in 2023. Additionally, some characterizations of regular semihypergroups were made using their corresponding types of Fermatean fuzzy hyperideals. Further, Nakkhasen has also studied the characterizations of different types of regularities in algebraic structures involving semigroups using the concept of generalized fuzzy sets, such as picture fuzzy sets, spherical fuzzy sets, and Pythagorean fuzzy sets, which can be found in the following references [26, 27, 29, 30].

As previously mentioned, there are various types of regularities in algebra that are related to semigroups, such as regular, intra-regular, completely regular, left regular, right regular, and generalized regular. However, the most popular are the regular and intra-regular types. It is known that the algebraic structure of semihypergroups is an extension of semigroups and ordered semigroups. The objective of this research is to classify the regularity of semihypergroups using the properties of Fermatean fuzzy set theory. For usage in the following section, we review the fundamental ideas and features of Fermatean fuzzy sets in semihypergroups in Section 2. In Section 3, which is the main section of our paper, we characterize intra-regular semihypergroups by Fermatean fuzzy left (resp., right) hyperideals, and Fermatean (resp., generalized) bi-hyperideals. Additionally, the notion of Fermatean fuzzy interior hyperideals of semihypergroups is defined, and the class of intra-regular semihypergroups is characterized by Fermatean fuzzy interior hyperideals. Finally, Section 4 delves into the features of Fermatean fuzzy left (resp., right) hyperideals and Fermatean (resp., generalized) bi-hyperideals of semihypergroups, which are used to characterize both regular and intra-regular semihypergroups.

2. Preliminaries

A map $\circ : X \times X \to P^*(X)$ is called a *hyperoperation* (see [24]) on a nonempty set X where $P^*(X)$ is the set of all nonempty subsets of X. The pair $(X, \circ)$ is called a *hypergroupoid*. Let X be a nonempty set and let $A, B \in P^*(X)$ and $x \in X$. Then, we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

A hypergroupoid $(S, \circ)$ is said to be a *semihypergroup* (see [6]) if for every $x, y, z \in S$, $(x \circ y) \circ z =$

$x \circ (y \circ z)$, which means that $\bigcup\limits_{u \in x \circ y} u \circ z = \bigcup\limits_{v \in y \circ z} x \circ v$. For simplicity, we represent a semihypergroup as S instead of a semihypergroup $(S, \circ)$, AB represents $A \circ B$, for all nonempty subsets A and B of S, and xy represents $x \circ y$, for all $x, y \in S$.

Now, we will review the notions of various types of hyperideals in semihypergroups, taken from [14] and [37]. A nonempty subset A of a semihypergroup S is called:

(i) a *subsemihypergroup* of S if $AA \subseteq A$;
(ii) a *left hyperideal* of S if $SA \subseteq A$;
(iii) a *right hyperideal* of S if $AS \subseteq A$;
(iv) a *hyperideal* of S if it is both a left and a right hyperideal of S;
(v) a *bi-hyperideal* of S if $AA \subseteq A$ and $ASA \subseteq A$;
(vi) a *generalized bi-hyperideal* of S if $ASA \subseteq A$;
(vii) an *interior hyperideal* of S if $AA \subseteq A$ and $SAS \subseteq A$.

A map $f : X \to [0, 1]$ from a nonempty set X into the unit interval is called a *fuzzy set* [45]. Let f and g be any two fuzzy sets of a nonempty set X. The notions $f \cap g$ and $f \cup g$ are defined by $(f \cap g)(x) = \min\{f(x), g(x)\}$ and $(f \cup g)(x) = \max\{f(x), g(x)\}$, for all $x \in X$, respectively.

A *Fermatean fuzzy set* [34] (briefly, FFS) on a nonempty set X is defined as:

$$\mathcal{A} := \{\langle x, \mu_{\mathcal{A}}(x), \lambda_{\mathcal{A}}(x)\rangle \mid x \in X\},$$

where $\mu_{\mathcal{A}} : X \to [0, 1]$ and $\lambda_{\mathcal{A}} : X \to [0, 1]$ represent the degree of membership and non-membership of each $x \in X$ to the set $\mathcal{A}$, respectively, with satisfy $0 \leq (\mu_{\mathcal{A}}(x))^3 + (\lambda_{\mathcal{A}}(x))^3 \leq 1$, for all $x \in X$. Throughout this paper, we will use the symbol $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ instead of the FFS $\mathcal{A} = \{\langle x, \mu_{\mathcal{A}}(x), \lambda_{\mathcal{A}}(x)\rangle \mid x \in X\}$.

In 2023, Nakkhasen [28] defined the concepts of many types of Fermatean fuzzy hyperideals in semihypergroups as follows. Let S be a semihypergroup, and $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ be an FFS on S. Then:

(i) $\mathcal{A}$ is called a *Fermatean fuzzy subsemihypergroup* (briefly, FFSub) of S if for every $x, y \in S$,

$$\inf_{z \in xy} \mu_{\mathcal{A}}(z) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y)\} \text{ and } \sup_{z \in xy} \lambda_{\mathcal{A}}(z) \leq \max\{\lambda_{\mathcal{A}}(x), \lambda_{\mathcal{A}}(y)\};$$

(ii) $\mathcal{A}$ is called a *Fermatean fuzzy left hyperideal* (briefly, FFL) of S if for every $x, y \in S$,

$$\inf_{z \in xy} \mu_{\mathcal{A}}(z) \geq \mu_{\mathcal{A}}(y) \text{ and } \sup_{z \in xy} \lambda_{\mathcal{A}}(z) \leq \lambda_{\mathcal{A}}(y);$$

(iii) $\mathcal{A}$ is called a *Fermatean fuzzy right hyperideal* (briefly, FFR) of S if for every $x, y \in S$,

$$\inf_{z \in xy} \mu_{\mathcal{A}}(z) \geq \mu_{\mathcal{A}}(x) \text{ and } \sup_{z \in xy} \lambda_{\mathcal{A}}(z) \leq \lambda_{\mathcal{A}}(x);$$

(iv) $\mathcal{A}$ is called a *Fermatean fuzzy hyperideal* (briefly, FFH) of S if it is both an FFL and an FFR of S;

(v) an FFSub $\mathcal{A}$ of S is called a *Fermatean fuzzy bi-hyperideal* (briefly, FFB) of S if for every $w, x, y \in S$,

$$\inf_{z \in xwy} \mu_{\mathcal{A}}(z) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y)\} \text{ and } \sup_{z \in xwy} \lambda_{\mathcal{A}}(z) \leq \max\{\lambda_{\mathcal{A}}(x), \lambda_{\mathcal{A}}(y)\};$$

(vi) a FFS $\mathcal{A}$ of S is called a *Fermatean fuzzy generalized bi-hyperideal* (briefly, FFGB) of S if for every $w, x, y \in S$,

$$\inf_{z\in xwy} \mu_{\mathcal{A}}(z) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y)\} \text{ and } \sup_{z\in xwy} \lambda_{\mathcal{A}}(z) \leq \max\{\lambda_{\mathcal{A}}(x), \lambda_{\mathcal{A}}(y)\}.$$

For any FFSs $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ on a nonempty set X, we denote:

(i) $\mathcal{A} \subseteq \mathcal{B}$ if and only if $\mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(x)$ and $\lambda_{\mathcal{A}}(x) \geq \lambda_{\mathcal{B}}(x)$, for all $x \in X$;
(ii) $\mathcal{A} = \mathcal{B}$ if and only if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$;
(iii) $\mathcal{A} \cap \mathcal{B} := \{\langle x, (\mu_{\mathcal{A}} \cap \mu_{\mathcal{B}})(x), (\lambda_{\mathcal{A}} \cup \lambda_{\mathcal{B}})(x)\rangle \mid x \in X\}$;
(iv) $\mathcal{A} \cup \mathcal{B} := \{\langle x, (\mu_{\mathcal{A}} \cup \mu_{\mathcal{B}})(x), (\lambda_{\mathcal{A}} \cap \lambda_{\mathcal{B}})(x)\rangle \mid x \in X\}$.

We observe that $\mathcal{A} \cap \mathcal{B}$ and $\mathcal{A} \cup \mathcal{B}$ are FFSs of X if $\mathcal{A}$ and $\mathcal{B}$ are FFSs on X.

Let $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ be any FFSs of a semihypergroup S. Then, the *Fermatean fuzzy product* of $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$\mathcal{A} \circ \mathcal{B} := \{\langle x, (\mu_{\mathcal{A}} \circ \mu_{\mathcal{B}})(x), (\lambda_{\mathcal{A}} \circ \lambda_{\mathcal{B}})(x)\rangle \mid x \in S\},$$

where

$$(\mu_{\mathcal{A}} \circ \mu_{\mathcal{B}})(x) = \begin{cases} \sup\limits_{x\in ab}[\min\{\mu_{\mathcal{A}}(a), \mu_{\mathcal{B}}(b)\}] & \text{if } x \in S^2, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\lambda_{\mathcal{A}} \circ \lambda_{\mathcal{B}})(x) = \begin{cases} \inf\limits_{x\in ab}[\max\{\lambda_{\mathcal{A}}(a), \lambda_{\mathcal{B}}(b)\}] & \text{if } x \in S^2, \\ 1 & \text{otherwise.} \end{cases}$$

For any semihypergroup S, we determine the FFSs $\mathcal{S} := \{\langle x, 1, 0\rangle \mid x \in S\}$ and $\mathcal{O} := \{\langle x, 0, 1\rangle \mid x \in S\}$ on S. This obtains that $\mathcal{A} \subseteq \mathcal{S}$ and $\mathcal{O} \subseteq \mathcal{A}$, for all FFS $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ on S. The *Fermatean characteristic function* of a subset A of a semihypergroup S, as an FFS on S, defined by $C_A = \{\langle x, \mu_{C_A}(x), \lambda_{C_A}(x)\rangle \mid x \in S\}$, where

$$\mu_{C_A}(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases} \text{ and } \lambda_{C_A}(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{otherwise.} \end{cases}$$

We note that if for each subset A of S such that $A = S$ (resp., $A = \emptyset$), then $C_A = \mathcal{S}$ (resp., $C_A = \mathcal{O}$). All the above-mentioned notions are presented in [28].

Lemma 2.1. [28] *Let $C_A = (\mu_{C_A}, \lambda_{C_A})$ and $C_B = (\mu_{C_B}, \lambda_{C_B})$ be FFSs of a semihypergroup S with respect to nonempty subsets A and B of S, respectively. Then the following axioms hold:*

(i) $C_{A\cap B} = C_A \cap C_B$;
(ii) $C_{AB} = C_A \circ C_B$.

Lemma 2.2. [28] *Let $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$, $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$, $\mathcal{C} = (\mu_{\mathcal{C}}, \lambda_{\mathcal{C}})$ and $\mathcal{D} = (\mu_{\mathcal{D}}, \lambda_{\mathcal{D}})$ be any FFSs of a semihypergroup S. If $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{C} \subseteq \mathcal{D}$, then $\mathcal{A} \circ \mathcal{C} \subseteq \mathcal{B} \circ \mathcal{D}$.*

Lemma 2.3. [28] *Let $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ be an FFS on a semihypergroup S. The following conditions hold:*

(i) $\mathcal{A}$ is an FFSub of S if and only if $\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A}$;
(ii) $\mathcal{A}$ is an FFL of S if and only if $\mathcal{S} \circ \mathcal{A} \subseteq \mathcal{A}$;
(iii) $\mathcal{A}$ is an FFR of S if and only if $\mathcal{A} \circ \mathcal{S} \subseteq \mathcal{A}$;
(iv) $\mathcal{A}$ is an FFGB of S if and only if $\mathcal{A} \circ \mathcal{S} \circ \mathcal{A} \subseteq \mathcal{A}$;
(v) $\mathcal{A}$ is an FFB of S if and only if $\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A}$ and $\mathcal{A} \circ \mathcal{S} \circ \mathcal{A} \subseteq \mathcal{A}$.

Lemma 2.4. [28] *For any nonempty subset A of a semihypergroup S, the following statements hold:*

(i) A is a subsemihypergroup of S if and only if $C_A = (\mu_{C_A}, \lambda_{C_A})$ is an FFSub of S;
(ii) A is a left hyperideal of S if and only if $C_A = (\mu_{C_A}, \lambda_{C_A})$ is an FFL of S;
(iii) A is a right hyperideal of S if and only if $C_A = (\mu_{C_A}, \lambda_{C_A})$ is an FFR of S;
(iv) A is a hyperideal of S if and only if $C_A = (\mu_{C_A}, \lambda_{C_A})$ is an FFH of S;
(v) A is a generalized bi-hyperideal of S if and only if $C_A = (\mu_{C_A}, \lambda_{C_A})$ is an FFGB of S;
(vi) A is a bi-hyperideal of S if and only if $C_A = (\mu_{C_A}, \lambda_{C_A})$ is an FFB of S.

A semihypergroup S is called *regular* (see [15]) if for every element a in S, there exists $x \in S$ such that $a \in axa$. Equivalently, $a \in aSa$, for all $a \in S$ or $A \subseteq ASA$, for any $A \subseteq S$. A semihypergroup S is called *intra-regular* (see [35]) if, for any element a in S, there exist $x, y \in S$ such that $a \in xa^2y$. Equivalently, $a \in Sa^2S$, for all $a \in S$ or $A \subseteq SA^2S$, for each $A \subseteq S$.

Example 2.5. Let $\mathbb{N}$ denote the set of all natural numbers. Define a hyperoperation $\circ$ on $\mathbb{N}$ by $a \circ b := \{x \in \mathbb{N} \mid x \leq ab\}$, for all $a, b \in \mathbb{N}$. Next, we claim that the hyperoperation $\circ$ on $\mathbb{N}$ is consistent with the associative property. Let $a, b \in \mathbb{N}$ and $x \in (a \circ b) \circ c$. Then, $x \in u \circ c$, for some $u \in a \circ b$. So, $x \leq uc$ and $u \leq ab$. It follows that $x \leq uc \leq (ab)c = a(bc)$. Also, $x \in a \circ (bc) \subseteq a \circ (b \circ c)$, since $bc \in b \circ c$. Thus, $(a \circ b) \circ c \subseteq a \circ (b \circ c)$. Similarly, we can prove that $a \circ (b \circ c) \subseteq (a \circ b) \circ c$. Hence, $(a \circ b) \circ c = a \circ (b \circ c)$. Therefore, $(\mathbb{N}, \circ)$ is a semihypergroup. Now, for every $a \in \mathbb{N}$, we have $a \leq axa$ and $a \leq ya^2z$, for some $x, y, z \in \mathbb{N}$. This implies that $a \in a \circ x \circ a$ and $a \in y \circ a \circ a \circ z$. It turns out that $(\mathbb{N}, \circ)$ is a regular and intra-reular semihypergroup.

Lemma 2.6. [28] *Let S be a semihypergroup. Then, S is regular if and only if $\mathcal{R} \cap \mathcal{L} = \mathcal{R} \circ \mathcal{L}$, for any FFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and any FFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ of S.*

Lemma 2.7. [35] *Let S be a semihypergroup. Then, S is intra-regular if and only if $L \cap R \subseteq LR$, for every left hyperideal L and every right hyperideal R of S.*

3. Intra-regular semihypergroups

In this section, we present results about the characterizations of intra-regular semihypergroups by properties of FFLs, FFRs, FFBs, and FFGBs of semihypergroups.

Theorem 3.1. *Let S be a semihypergroup. Then, S is intra-regular if and only if $\mathcal{L} \cap \mathcal{R} \subseteq \mathcal{L} \circ \mathcal{R}$, for every FFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and every FFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ of S.*

Proof. Assume that S is intra-regular. Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ be an FFL and an FFR of S, respectively. For any $a \in S$, there exist $x, y \in S$ such that $a \in xa^2y$. Then, we have

$$(\mu_{\mathcal{L}} \circ \mu_{\mathcal{R}})(a) = \sup_{a \in pq} [\min\{\mu_{\mathcal{L}}(p), \mu_{\mathcal{R}}(q)\}]$$

$$\geq \min\left\{\inf_{p\in xa}\mu_{\mathcal{L}}(p), \inf_{q\in ay}\mu_{\mathcal{R}}(q)\right\}$$
$$\geq \min\{\mu_{\mathcal{L}}(a), \mu_{\mathcal{R}}(a)\}$$
$$= (\mu_{\mathcal{L}} \cap \mu_{\mathcal{R}})(a),$$

and

$$\begin{aligned}(\lambda_{\mathcal{L}} \circ \lambda_{\mathcal{R}})(a) &= \inf_{a\in pq}[\max\{\lambda_{\mathcal{L}}(p), \lambda_{\mathcal{R}}(q)\}]\\ &\leq \max\left\{\sup_{p\in xa}\lambda_{\mathcal{L}}(p), \sup_{q\in ay}\lambda_{\mathcal{R}}(q)\right\}\\ &\leq \max\{\lambda_{\mathcal{L}}(a), \lambda_{\mathcal{R}}(a)\}\\ &= (\lambda_{\mathcal{L}} \cup \lambda_{\mathcal{R}})(a).\end{aligned}$$

Hence, $\mathcal{L} \cap \mathcal{R} \subseteq \mathcal{L} \circ \mathcal{R}$.

Conversely, let L and R be any left hyperideal and any right hyperideal of S, respectively. By Lemma 2.4, we have $C_L = (\mu_{C_L}, \lambda_{C_L})$ and $C_R = (\mu_{C_R}, \lambda_{C_R})$ are an FFL and an FFR of S, respectively. By the given assumption and Lemma 2.1, we get

$$C_{L\cap R} = C_L \cap C_R \subseteq C_L \circ C_R = C_{LR}.$$

Now, let $a \in L \cap R$. Thus, we have $\mu_{C_{LR}}(a) \geq \mu_{C_{L\cap R}}(a) = 1$. Also, $\mu_{C_{LR}}(a) = 1$; that is, $a \in LR$. This implies that $L \cap R \subseteq LR$. By Lemma 2.7, we conclude that S is intra-regular. □

Theorem 3.2. *Let S be a semihypergroup. Then the following statements are equivalent:*

(i) S is intra-regular;
(ii) $\mathcal{L} \cap \mathcal{G} \subseteq \mathcal{L} \circ \mathcal{G} \circ \mathcal{S}$, for each FFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and each FFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;
(iii) $\mathcal{L} \cap \mathcal{B} \subseteq \mathcal{L} \circ \mathcal{B} \circ \mathcal{S}$, for each FFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and each FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.

Proof. (i)⇒(ii) Assume that S is intra-regular. Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ be an FFL and an FFGB of S, respectively. Let $a \in S$. Then, there exist $x, y \in S$ such that $a \in xa^2y$. It follows that $a \in (x^2a)(ayay)$. Thus, we have

$$\begin{aligned}(\mu_{\mathcal{L}} \circ \mu_{\mathcal{G}} \circ \mu_{\mathcal{S}})(a) &= \sup_{a\in pq}[\min\{\mu_{\mathcal{L}}(p), (\mu_{\mathcal{G}} \circ \mu_{\mathcal{S}})(q)\}]\\ &= \sup_{a\in pq}\left[\min\left\{\mu_{\mathcal{L}}(p), \sup_{q\in mn}[\min\{\mu_{\mathcal{G}}(m), \mu_{\mathcal{S}}(n)\}]\right\}\right]\\ &\geq \min\left\{\inf_{p\in x^2a}\mu_{\mathcal{L}}(p), \min\left\{\inf_{m\in aya}\mu_{\mathcal{G}}(m), \mu_{\mathcal{S}}(y)\right\}\right\}\\ &\geq \min\{\mu_{\mathcal{L}}(a), \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}\}\\ &= \min\{\mu_{\mathcal{L}}(a), \mu_{\mathcal{G}}(a)\}\\ &= (\mu_{\mathcal{L}} \cap \mu_{\mathcal{G}})(a),\end{aligned}$$

and

$$(\lambda_{\mathcal{L}} \circ \lambda_{\mathcal{G}} \circ \lambda_{\mathcal{S}})(a) = \inf_{a\in pq}[\max\{\lambda_{\mathcal{L}}(p), (\lambda_{\mathcal{G}} \circ \lambda_{\mathcal{S}})(q)\}]$$

$$
\begin{aligned}
&= \inf_{a\in pq}\left[\max\left\{\lambda_{\mathcal{L}}(p), \inf_{q\in mn}\left[\max\{\lambda_{\mathcal{G}}(m), \lambda_{\mathcal{S}}(n)\}\right]\right\}\right] \\
&\leq \max\left\{\sup_{p\in x^2a}\lambda_{\mathcal{L}}(p), \max\left\{\sup_{m\in aya}\lambda_{\mathcal{G}}(m), \lambda_{\mathcal{S}}(y)\right\}\right\} \\
&\leq \max\{\lambda_{\mathcal{L}}(a), \max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{G}}(a)\}\} \\
&= \max\{\lambda_{\mathcal{L}}(a), \lambda_{\mathcal{G}}(a)\} \\
&= (\lambda_{\mathcal{L}} \cup \lambda_{\mathcal{G}})(a).
\end{aligned}
$$

This means that $\mathcal{L} \cap \mathcal{G} \subseteq \mathcal{L} \circ \mathcal{G} \circ \mathcal{S}$.

(ii)⇒(iii) Since every FFB is also an FFGB of S, it follows that (iii) holds.

(iii)⇒(i) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ be an FFL and an FFR of S, respectively. We obtain that R is also an FFB of S. By assumption, we have $\mathcal{L} \cap \mathcal{R} \subseteq \mathcal{L} \circ (\mathcal{R} \circ \mathcal{S}) \subseteq \mathcal{L} \circ \mathcal{R}$. By Theorem 3.1, it turns out that S is intra-regular. □

Theorem 3.3. *Let S be a semihypergroup. Then the following statements are equivalent:*

(i) S is intra-regular;
(ii) $\mathcal{G} \cap \mathcal{R} \subseteq \mathcal{S} \circ \mathcal{G} \circ \mathcal{R}$, for each FFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and each FFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;
(iii) $\mathcal{B} \cap \mathcal{R} \subseteq \mathcal{S} \circ \mathcal{B} \circ \mathcal{R}$, for each FFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and each FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.

Proof. (i)⇒(ii) Assume that S is intra-regular. Let $a \in S$. Then, there exist $x, y \in S$ such that $a \in (xaxa)(ay^2)$. Hence, we have

$$
\begin{aligned}
(\mu_{\mathcal{S}} \circ \mu_{\mathcal{G}} \circ \mu_{\mathcal{R}})(a) &= \sup_{a\in pq}\left[\min\{(\mu_{\mathcal{S}} \circ \mu_{\mathcal{G}})(p), \mu_{\mathcal{R}}(q)\}\right] \\
&= \sup_{a\in pq}\left[\min\left\{\sup_{p\in mn}\left[\min\{\mu_{\mathcal{S}}(m), \mu_{\mathcal{G}}(n)\}\right], \mu_{\mathcal{R}}(q)\right\}\right] \\
&\geq \min\left\{\min\left\{\mu_{\mathcal{S}}(x), \inf_{n\in axa}\mu_{\mathcal{G}}(n)\right\}, \inf_{q\in ay^2}\mu_{\mathcal{R}}(q)\right\} \\
&\geq \min\{\min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}, \mu_{\mathcal{R}}(a)\} \\
&= \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{R}}(a)\} \\
&= (\mu_{\mathcal{R}} \cap \mu_{\mathcal{G}})(a),
\end{aligned}
$$

and

$$
\begin{aligned}
(\lambda_{\mathcal{S}} \circ \lambda_{\mathcal{G}} \circ \lambda_{\mathcal{R}})(a) &= \inf_{a\in pq}\left[\max\{(\lambda_{\mathcal{S}} \circ \lambda_{\mathcal{G}})(p), \lambda_{\mathcal{R}}(q)\}\right] \\
&= \inf_{a\in pq}\left[\max\left\{\inf_{p\in mn}\left[\max\{\lambda_{\mathcal{S}}(m), \lambda_{\mathcal{G}}(n)\}\right], \lambda_{\mathcal{R}}(q)\right\}\right] \\
&\leq \max\left\{\max\left\{\lambda_{\mathcal{S}}(x), \sup_{n\in axa}\lambda_{\mathcal{G}}(n)\right\}, \sup_{q\in ay^2}\lambda_{\mathcal{R}}(q)\right\} \\
&\leq \max\{\max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{G}}(a)\}, \lambda_{\mathcal{R}}(a)\} \\
&= \max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{R}}(a)\}
\end{aligned}
$$

$$= (\lambda_{\mathcal{R}} \cup \lambda_{\mathcal{G}})(a).$$

This shows that $\mathcal{R} \cap \mathcal{G} \subseteq \mathcal{S} \circ \mathcal{G} \circ \mathcal{R}$.

(ii)⇒(iii) Since every FFB is also an FFGB of S, it is well done.

(iii)⇒(i) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ be an FFL and an FFR of S, respectively. Then, $\mathcal{L}$ is also an FFB of S. By the hypothesis, we have $\mathcal{L} \cap \mathcal{R} \subseteq (\mathcal{L} \circ \mathcal{S}) \circ \mathcal{R} \subseteq \mathcal{L} \circ \mathcal{R}$. By Theorem 3.1, we obtain that S is intra-regular. □

Theorem 3.4. *The following statements are equivalent in a semihypergroup S:*

(i) S is intra-regular;
(ii) $\mathcal{G}_1 \cap \mathcal{G}_2 \subseteq \mathcal{S} \circ \mathcal{G}_1 \circ \mathcal{G}_2 \circ \mathcal{S}$, for any FFGBs $\mathcal{G}_1 = (\mu_{\mathcal{G}_1}, \lambda_{\mathcal{G}_1})$ and $\mathcal{G}_2 = (\mu_{\mathcal{G}_2}, \lambda_{\mathcal{G}_2})$ of S;
(iii) $\mathcal{B}_1 \cap \mathcal{B}_2 \subseteq \mathcal{S} \circ \mathcal{B}_1 \circ \mathcal{B}_2 \circ \mathcal{S}$, for any FFBs $\mathcal{B}_1 = (\mu_{\mathcal{B}_1}, \lambda_{\mathcal{B}_1})$ and $\mathcal{B}_2 = (\mu_{\mathcal{B}_2}, \lambda_{\mathcal{B}_2})$ of S.

Proof. (i)⇒(ii) Let $a \in S$. Then, there exist $x, y \in S$ such that $a \in xa^2y$. Thus, we have

$$\begin{aligned}
(\mu_{\mathcal{S}} \circ \mu_{\mathcal{G}_1} \circ \mu_{\mathcal{G}_2} \circ \mu_{\mathcal{S}})(a) &= \sup_{a \in pq} [\min\{(\mu_{\mathcal{S}} \circ \mu_{\mathcal{G}_1})(p), (\mu_{\mathcal{G}_2} \circ \mu_{\mathcal{S}})(q)\}] \\
&= \sup_{a \in pq} \left[\min \left\{ \sup_{p \in mn} [\min\{\mu_{\mathcal{S}}(m), \mu_{\mathcal{G}_1}(n)\}], \sup_{q \in kl} [\min\{\mu_{\mathcal{G}_2}(k), \mu_{\mathcal{S}}(l)\}] \right\} \right] \\
&\geq \min\{\min\{\mu_{\mathcal{S}}(x), \mu_{\mathcal{G}_1}(a)\}, \min\{\mu_{\mathcal{G}_2}(a), \mu_{\mathcal{S}}(y)\}\} \\
&= \min\{\mu_{\mathcal{G}_1}(a), \mu_{\mathcal{G}_2}(a)\} \\
&= (\mu_{\mathcal{G}_1} \cap \mu_{\mathcal{G}_2})(a),
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_{\mathcal{S}} \circ \lambda_{\mathcal{G}_1} \circ \lambda_{\mathcal{G}_2} \circ \lambda_{\mathcal{S}})(a) &= \inf_{a \in pq} [\max\{(\lambda_{\mathcal{S}} \circ \lambda_{\mathcal{G}_1})(p), (\lambda_{\mathcal{G}_2} \circ \lambda_{\mathcal{S}})(q)\}] \\
&= \inf_{a \in pq} \left[\max \left\{ \inf_{p \in mn} [\max\{\lambda_{\mathcal{S}}(m), \lambda_{\mathcal{G}_1}(n)\}], \inf_{q \in kl} [\max\{\lambda_{\mathcal{G}_2}(k), \lambda_{\mathcal{S}}(l)\}] \right\} \right] \\
&\leq \max\{\max\{\lambda_{\mathcal{S}}(x), \lambda_{\mathcal{G}_1}(a)\}, \max\{\lambda_{\mathcal{G}_2}(a), \lambda_{\mathcal{S}}(y)\}\} \\
&= \max\{\lambda_{\mathcal{G}_1}(a), \lambda_{\mathcal{G}_2}(a)\} \\
&= (\lambda_{\mathcal{G}_1} \cup \lambda_{\mathcal{G}_2})(a).
\end{aligned}$$

This implies that $\mathcal{G}_1 \cap \mathcal{G}_2 \subseteq \mathcal{S} \circ \mathcal{G}_1 \circ \mathcal{G}_2 \circ \mathcal{S}$.

(ii)⇒(iii) It is obvious.

(iii)⇒(i) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ be any FFL of S, and $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ be any FFR of S. Then, $\mathcal{L}$ and $\mathcal{R}$ are also FFBs of S. By the hypothesis, we have $\mathcal{L} \cap \mathcal{R} \subseteq (\mathcal{S} \circ \mathcal{L}) \circ (\mathcal{R} \circ \mathcal{S}) \subseteq \mathcal{L} \circ \mathcal{R}$. By Theorem 3.1, it follows that S is intra-regular. □

Corollary 3.5. Let S be a semihypergroup. Then, the following conditions are equivalent:

(i) S is intra-regular;
(ii) $\mathcal{G} \subseteq \mathcal{S} \circ \mathcal{G} \circ \mathcal{G} \circ \mathcal{S}$, for any FFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;
(iii) $\mathcal{B} \subseteq \mathcal{S} \circ \mathcal{B} \circ \mathcal{B} \circ \mathcal{S}$, for any FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.

Proof. (i)⇒(ii) It follows by Theorem 3.4.

(ii)⇒(iii) It is clear.

(iii)⇒(i) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ be an FFL and an FFR of S, respectively. It is not difficult to see that $\mathcal{L} \cap \mathcal{R}$ is also an FFB of S. By the given assumption, we have $\mathcal{L} \cap \mathcal{R} \subseteq \mathcal{S} \circ (\mathcal{L} \cap \mathcal{R}) \circ (\mathcal{L} \cap \mathcal{R}) \circ \mathcal{S} \subseteq (\mathcal{S} \circ \mathcal{L}) \circ (\mathcal{R} \circ \mathcal{S}) \subseteq \mathcal{L} \circ \mathcal{R}$. By Theorem 3.1, we conclude S is intra-regular. □

The following corollary is obtained by Corollary 3.5.

Corollary 3.6. Let S be a semihypergroup. Then, S is intra-regular if and only if $\mathcal{B} \cap \mathcal{G} \subseteq (\mathcal{S} \circ \mathcal{B} \circ \mathcal{B} \circ \mathcal{S}) \cap (\mathcal{S} \circ \mathcal{G} \circ \mathcal{G} \circ \mathcal{S})$, for every FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ and every FFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S.

Theorem 3.7. *If S is an intra-regular semihypergroup, then $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \circ \mathcal{B}$, for each FFHs $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.*

Proof. Assume that S is an intra-regular semihypergroup. Let $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ be FFHs of S. Then, $\mathcal{A} \circ \mathcal{B} \subseteq \mathcal{A} \circ \mathcal{S} \subseteq \mathcal{A}$ and $\mathcal{A} \circ \mathcal{B} \subseteq \mathcal{S} \circ \mathcal{B} \subseteq \mathcal{B}$, it follows that $\mathcal{A} \circ \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B}$. Next, let $a \in S$. By assumption, there exist $x, y \in S$ such that $a \in xa^2y = (xa)(ay)$; that is, $a \in pq$, for some $p \in xa$ and $q \in ay$. Thus, we have

$$\begin{aligned}(\mu_{\mathcal{A}} \circ \mu_{\mathcal{B}})(a) &= \sup_{a \in pq} [\min\{\mu_{\mathcal{A}}(p), \mu_{\mathcal{B}}(q)\}] \\ &\geq \min\left\{\inf_{p \in xa} \mu_{\mathcal{A}}(p), \inf_{q \in ay} \mu_{\mathcal{B}}(q)\right\} \\ &\geq \min\{\mu_{\mathcal{A}}(a), \mu_{\mathcal{B}}(a)\} \\ &= (\mu_{\mathcal{A}} \cap \mu_{\mathcal{B}})(a),\end{aligned}$$

and

$$\begin{aligned}(\lambda_{\mathcal{A}} \circ \lambda_{\mathcal{B}})(a) &= \inf_{a \in pq} [\max\{\lambda_{\mathcal{A}}(p), \lambda_{\mathcal{B}}(q)\}] \\ &\leq \max\left\{\sup_{p \in xa} \lambda_{\mathcal{A}}(p), \sup_{q \in ay} \lambda_{\mathcal{B}}(q)\right\} \\ &\leq \max\{\lambda_{\mathcal{A}}(a), \lambda_{\mathcal{B}}(a)\} \\ &= (\lambda_{\mathcal{A}} \cup \lambda_{\mathcal{B}})(a).\end{aligned}$$

Hence, $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \circ \mathcal{B}$. Therefore, $\mathcal{A} \cap \mathcal{B} = \mathcal{A} \circ \mathcal{B}$. □

Theorem 3.8. *Let S be a semihypergroup. Then the following properties are equivalent:*

(i) S is intra-regular;
(ii) $\mathcal{L} \cap \mathcal{G} \cap \mathcal{R} \subseteq \mathcal{L} \circ \mathcal{G} \circ \mathcal{R}$, for every FFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$, every FFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and every FFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;
(iii) $\mathcal{L} \cap \mathcal{B} \cap \mathcal{R} \subseteq \mathcal{L} \circ \mathcal{B} \circ \mathcal{R}$, for every FFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$, every FFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and every FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.

Proof. (i)⇒(ii) Assume that S is intra-regular. Let $a \in S$. Then, there exist $x, y \in S$ such that $a \in xa^2y$, which implies that $a \in (x^2a)(ayxaxa)(ay^3)$. Thus, $a \in uvq$, for some $u \in x^2a$, $v \in ayxaxa$ and $q \in ay^3$. Also, there exists $p \in S$ such that $p \in uv$, and so $a \in pq$. So, we have

$$\begin{aligned}(\mu_{\mathcal{L}} \circ \mu_{\mathcal{G}} \circ \mu_{\mathcal{R}})(a) &= \sup_{a \in pq} [\min\{(\mu_{\mathcal{L}} \circ \mu_{\mathcal{G}})(p), \mu_{\mathcal{R}}(q)\}] \\ &= \sup_{a \in pq} \left[\min\left\{\sup_{p \in uv} [\min\{\mu_{\mathcal{L}}(u), \mu_{\mathcal{G}}(v)\}], \mu_{\mathcal{R}}(q)\right\}\right] \\ &\geq \min\left\{\min\left\{\inf_{u \in x^2a} \mu_{\mathcal{L}}(u), \inf_{v \in ayxaxa} \mu_{\mathcal{G}}(v)\right\}, \inf_{q \in ay^3} \mu_{\mathcal{R}}(q)\right\} \\ &\geq \min\{\min\{\mu_{\mathcal{L}}(a), \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}\}, \mu_{\mathcal{R}}(a)\} \\ &= \min\{\mu_{\mathcal{L}}(a), \mu_{\mathcal{G}}(a), \mu_{\mathcal{R}}(a)\} \\ &= (\mu_{\mathcal{L}} \cap \mu_{\mathcal{G}} \cap \mu_{\mathcal{R}})(a),\end{aligned}$$

and

$$\begin{aligned}(\lambda_{\mathcal{L}} \circ \lambda_{\mathcal{G}} \circ \lambda_{\mathcal{R}})(a) &= \inf_{a \in pq} [\max\{(\lambda_{\mathcal{L}} \circ \lambda_{\mathcal{G}})(p), \lambda_{\mathcal{R}}(q)\}] \\ &= \inf_{a \in pq} \left[\max\left\{\inf_{p \in uv} [\max\{\lambda_{\mathcal{L}}(u), \lambda_{\mathcal{G}}(v)\}], \lambda_{\mathcal{R}}(q)\right\}\right] \\ &\leq \max\left\{\max\left\{\sup_{u \in x^2a} \lambda_{\mathcal{L}}(u), \sup_{v \in ayxaxa} \lambda_{\mathcal{G}}(v)\right\}, \sup_{q \in ay^3} \lambda_{\mathcal{R}}(q)\right\} \\ &\leq \max\{\max\{\lambda_{\mathcal{L}}(a), \max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{G}}(a)\}\}, \lambda_{\mathcal{R}}(a)\} \\ &= \max\{\lambda_{\mathcal{L}}(a), \lambda_{\mathcal{G}}(a), \lambda_{\mathcal{R}}(a)\} \\ &= (\lambda_{\mathcal{L}} \cup \lambda_{\mathcal{G}} \cup \lambda_{\mathcal{R}})(a).\end{aligned}$$

This shows that $\mathcal{L} \cap \mathcal{G} \cap \mathcal{R} \subseteq \mathcal{L} \circ \mathcal{G} \circ \mathcal{R}$.

(ii)⇒(iii) It is clear.

(iii)⇒(i) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ be any FFL and any FFR of S, respectively. Then, $\mathcal{R}$ is also an FFB of S. By the given assumption, we have $\mathcal{L} \cap \mathcal{R} = \mathcal{L} \cap \mathcal{R} \cap \mathcal{R} \subseteq \mathcal{L} \circ \mathcal{R} \circ \mathcal{R} \subseteq \mathcal{L} \circ \mathcal{R}$. By Theorem 3.1, we get that S is intra-regular. □

Now, we introduce the notion of Fermatean fuzzy interior hyperideals in semihypergroups and investigate some properties of this notion. Moreover, we use the properties of Fermatean fuzzy interior hyperideals to study the characterizations of intra-regular semihypergroups.

Definition 3.9. An FFsub $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ is said to be a *Fermatean fuzzy interior hyperideal* (briefly, FFInt) of a semihypergroup S if for every $w, x, y \in S$, $\inf_{z \in wxy} \mu_{\mathcal{A}}(z) \geq \mu_{\mathcal{A}}(x)$ and $\sup_{z \in wxy} \lambda_{\mathcal{A}}(z) \leq \lambda_{\mathcal{A}}(x)$.

Theorem 3.10. *Let S be a semihypergroup, and $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ be an FFS of S. Then, $\mathcal{A}$ is an FFInt of S if and only if $\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A}$ and $\mathcal{S} \circ \mathcal{A} \circ \mathcal{S} \subseteq \mathcal{A}$.*

Proof. Assume that $\mathcal{A}$ is an FFInt of S. Then, $\mathcal{A}$ is an FFSub of S. By Lemma 2.3, we have $\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A}$. Now, let $a \in S$. If $a \notin bcd$, for all $b, c, d \in S$, then $\mathcal{S} \circ \mathcal{A} \circ \mathcal{S} \subseteq \mathcal{A}$. Suppose that there exist $p, q, x, y \in S$

such that $a \in xy$ and $x \in pq$. It follows that $a \in pqy$. Thus, we have

$$\begin{aligned}(\mu_{\mathcal{S}} \circ \mu_{\mathcal{A}} \circ \mu_{\mathcal{S}})(a) &= \sup_{a\in xy} [\min\{(\mu_{\mathcal{S}} \circ \mu_{\mathcal{A}})(x), \mu_{\mathcal{S}}(y)\}] \\ &= \sup_{a\in xy} [(\mu_{\mathcal{S}} \circ \mu_{\mathcal{A}})(x)] \\ &= \sup_{a\in xy} \left[\sup_{x\in pq} [\min\{\mu_{\mathcal{S}}(p), \mu_{\mathcal{A}}(q)\}]\right] \\ &= \sup_{a\in xy} \left[\sup_{x\in pq} [\mu_{\mathcal{A}}(q)]\right] \\ &\leq \mu_{\mathcal{A}}(a),\end{aligned}$$

and

$$\begin{aligned}(\lambda_{\mathcal{S}} \circ \lambda_{\mathcal{A}} \circ \lambda_{\mathcal{S}})(a) &= \inf_{a\in xy} [\max\{(\lambda_{\mathcal{S}} \circ \lambda_{\mathcal{A}})(x), \lambda_{\mathcal{S}}(y)\}] \\ &= \inf_{a\in xy} [(\lambda_{\mathcal{S}} \circ \lambda_{\mathcal{A}})(x)] \\ &= \inf_{a\in xy} \left[\inf_{x\in pq} [\max\{\lambda_{\mathcal{S}}(p), \lambda_{\mathcal{A}}(q)\}]\right] \\ &= \inf_{a\in xy} \left[\inf_{x\in pq} [\lambda_{\mathcal{A}}(q)]\right] \\ &\geq \lambda_{\mathcal{A}}(a).\end{aligned}$$

Hence, $\mathcal{S} \circ \mathcal{A} \circ \mathcal{S} \subseteq \mathcal{A}$. Conversely, let $x, y, z \in S$, and let $w \in xyz$. Then, there exists $u \in xy$ such that $w \in uz$. By assumption, we have

$$\begin{aligned}\mu_{\mathcal{A}}(w) \geq (\mu_{\mathcal{S}} \circ \mu_{\mathcal{A}} \circ \mu_{\mathcal{S}}) &= \sup_{w\in pq} [\min\{(\mu_{\mathcal{S}} \circ \mu_{\mathcal{A}})(p), \mu_{\mathcal{S}}(q)\}] \\ &\geq \{(\mu_{\mathcal{S}} \circ \mu_{\mathcal{A}})(u), \mu_{\mathcal{S}}(z)\} \\ &= \sup_{u\in st} [\min\{\mu_{\mathcal{S}}(s), \mu_{\mathcal{A}}(t)\}] \\ &\geq \min\{\mu_{\mathcal{S}}(x), \mu_{\mathcal{A}}(y)\} = \mu_{\mathcal{A}}(y),\end{aligned}$$

and

$$\begin{aligned}\lambda_{\mathcal{A}}(w) \leq (\lambda_{\mathcal{S}} \circ \lambda_{\mathcal{A}} \circ \lambda_{\mathcal{S}}) &= \inf_{w\in pq} [\max\{(\lambda_{\mathcal{S}} \circ \lambda_{\mathcal{A}})(p), \lambda_{\mathcal{S}}(q)\}] \\ &\leq \{(\lambda_{\mathcal{S}} \circ \lambda_{\mathcal{A}})(u), \lambda_{\mathcal{S}}(z)\} \\ &= \inf_{u\in st} [\max\{\lambda_{\mathcal{S}}(s), \lambda_{\mathcal{A}}(t)\}] \\ &\leq \max\{\lambda_{\mathcal{S}}(x), \lambda_{\mathcal{A}}(y)\} = \lambda_{\mathcal{A}}(y).\end{aligned}$$

This shows that $\mu_{\mathcal{A}}(w) \geq \mu_{\mathcal{A}}(y)$ and $\lambda_{\mathcal{A}}(w) \leq \lambda_{\mathcal{A}}(y)$, for all $w \in xyz$. It follows that $\inf_{w\in xyz} \mu_{\mathcal{A}}(z) \geq \mu_{\mathcal{A}}(y)$ and $\sup_{w\in xyz} \lambda_{\mathcal{A}}(z) \leq \lambda_{\mathcal{A}}(y)$. Therefore, $\mathcal{A}$ is an FFInt of S. □

Theorem 3.11. *Let S be a semihypergroup, and A be a nonempty subset of S. Then, A is an interior hyperideal of S if and only if $C_A = (\mu_{C_A}, \lambda_{C_A})$ is an FFInt of S.*

Proof. Assume that A is an interior hyperideal of S. Then A is a subsemihypergroup of S. By Lemma 2.4, we have C_A is an FFSub of S. Now, let $x, y, z \in S$. If $y \notin A$, then $\inf_{w \in xyz} \mu_{C_A}(w) \geq 0 = \mu_{C_A}(y)$ and $\sup_{w \in xyz} \lambda_{C_A}(w) \leq 1 = \lambda_{C_A}(y)$. On the other hand, suppose that $y \in A$. Thus, $xyz \subseteq A$, which implies that for every $w \in xyz$, we have $\mu_{C_A}(w) = 1$ and $\lambda_{C_A}(w) = 0$. This means that $\mu_{C_A}(w) \geq \mu_{C_A}(y)$ and $\lambda_{C_A}(w) \leq \lambda_{C_A}(y)$, for all $w \in xyz$. That is, $\inf_{w \in xyz} \mu_{C_A}(w) \geq \mu_{C_A}(y)$ and $\sup_{w \in xyz} \lambda_{C_A}(w) \leq \lambda_{C_A}(y)$. Hence, C_A is an FFInt of S.

Conversely, assume that $C_A = (\mu_{C_A}, \lambda_{C_A})$ is an FFInt of S. Then, C_A is an FFSub of S. By Lemma 2.4, we have that A is a subsemihypergroup of S. Let $x, z \in S$ and $y \in A$. By assumption, we get $\inf_{w \in xyz} \mu_{C_A}(w) \geq \mu_{C_A}(y) = 1$ and $\sup_{w \in xyz} \lambda_{C_A}(w) \leq \lambda_{C_A}(y) = 0$. This implies that $\mu_{C_A}(w) \geq 1$ and $\lambda_{C_A}(w) \leq 0$, for all $w \in xyz$. Otherwise, $\mu_{C_A}(w) \leq 1$ and $\lambda_{C_A}(w) \geq 0$. So, $\mu_{C_A}(w) = 1$ and $\lambda_{C_A}(w) = 0$, for all $w \in xyz$. It turns out that $w \in A$. This shows that $SAS \subseteq A$. Therefore, A is an interior hyperideal of S. □

Example 3.12. Let $S = \{a, b, c, d\}$ be a set with the hyperoperation $\circ$ on S defined by the following table:

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a, b\}$
d	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$

It follows that $(S, \circ)$ is a semihypergroup, [40]. We see that $A = \{a, c\}$ is an interior hyperideal of S. After that, the FFS $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ of S defined by

$$\mu_{\mathcal{A}}(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases} \text{ and } \lambda_{\mathcal{A}}(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{otherwise,} \end{cases}$$

for all $x \in S$. Applying Theorem 3.11, we have $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ is a FFInt of S.

Proposition 3.13. *Every FFH of a semihypergroup S is also an FFInt of S.*

Proof. Let $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ be an FFH of a semihypergroup S. By Lemma 2.3, we have $\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A} \circ \mathcal{S} \subseteq \mathcal{A}$ and $\mathcal{S} \circ \mathcal{A} \circ \mathcal{S} = (\mathcal{S} \circ \mathcal{A}) \circ \mathcal{S} \subseteq \mathcal{A} \circ \mathcal{S} \subseteq \mathcal{A}$. By Theorem 3.10, it follows that $\mathcal{A}$ is an FFInt of S. □

Example 3.14. Let $S = \{a, b, c, d\}$ such that $(S, \circ)$ is a semihypergroup, as defined in Example 3.12. In the next step, we define an FFS $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ on S as follows:

$\mathcal{A}$ \ S	a	b	c	d
$\mu_{\mathcal{A}}$	0.9	0.6	0.8	0.5
$\lambda_{\mathcal{A}}$	0.5	0.8	0.7	0.9

Upon careful inspection, we obtain that the FFS $\mathcal{A}$ is an FFInt of S. However, the FFInt $\mathcal{A}$ of S is not a FFL of S, because

$$\inf_{z\in d\circ c}\mu_{\mathcal{A}}(z) = \mu_{\mathcal{A}}(b) < \mu_{\mathcal{A}}(c) \text{ and } \sup_{z\in d\circ c}\lambda_{\mathcal{A}}(z) = \lambda_{\mathcal{A}}(b) > \lambda_{\mathcal{A}}(c).$$

Furthermore, the FFInt $\mathcal{A}$ of S is not an FFR of S either, since

$$\inf_{z\in c\circ d}\mu_{\mathcal{A}}(z) = \mu_{\mathcal{A}}(b) < \mu_{\mathcal{A}}(c) \text{ and } \sup_{z\in c\circ d}\lambda_{\mathcal{A}}(z) = \lambda_{\mathcal{A}}(b) > \lambda_{\mathcal{A}}(c).$$

It can be concluded that the FFInt of S does not have to be an FFH of S.

Theorem 3.15. *In an intra-regular semihypergroup S, every FFInt of S is also an FFH of S.*

Proof. Let $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ be an FFInt of S, and let $a, b \in S$. Then, there exist $x, y \in S$ such that $a \in xa^2y$. So, $ab \subseteq (xa^2y)b = (xa)a(yb)$. Thus, for every $z \in ab$, there exist $u \in xa$ and $v \in yb$ such that $z \in uav$, which implies that $\mu_{\mathcal{A}}(z) \geq \inf_{z\in uav}\mu_{\mathcal{A}}(z) \geq \mu_{\mathcal{A}}(a)$ and $\lambda_{\mathcal{A}}(z) \leq \sup_{z\in uav}\lambda_{\mathcal{A}}(z) \leq \lambda_{\mathcal{A}}(a)$. We obtain that $\inf_{z\in ab}\mu_{\mathcal{A}}(z) \geq \mu_{\mathcal{A}}(a)$ and $\sup_{z\in ab}\lambda_{\mathcal{A}}(a)$. Hence, $\mathcal{A}$ is an FFR of S. Similarly, we can show that $\mathcal{A}$ is an FFL of S. Therefore, $\mathcal{A}$ is an FFH of S. □

Theorem 3.16. *Let S be a semihypergroup. Then the following results are equivalent:*

(i) S is intra-regular;
(ii) $\mathcal{I} \cap \mathcal{G} \cap \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{G} \circ \mathcal{I}$, for each FFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$, each FFInt $\mathcal{I} = (\mu_{\mathcal{I}}, \lambda_{\mathcal{I}})$ and each FFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;
(iii) $\mathcal{I} \cap \mathcal{B} \cap \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{B} \circ \mathcal{I}$, for each FFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$, each FFInt $\mathcal{I} = (\mu_{\mathcal{I}}, \lambda_{\mathcal{I}})$ and each FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.

Proof. (i)⇒(ii) Assume that S is intra-regular. Let $a \in S$. Then, there exist $x, y \in S$ such that $a \in xa^2y$, and so $a \in (x^2a)a(yay)$. Thus, $a \in waq$, for some $w \in x^2a$ and $q \in yay$, and then $a \in pq$, for some $p \in wa$. So, we have

$$\begin{aligned}
(\mu_{\mathcal{L}} \circ \mu_{\mathcal{G}} \circ \mu_{\mathcal{I}})(a) &= \sup_{a\in pq}\left[\min\{(\mu_{\mathcal{L}} \circ \mu_{\mathcal{G}})(p), \mu_{\mathcal{I}}(q)\}\right] \\
&= \sup_{a\in pq}\left[\min\left\{\sup_{p\in wa}\left[\min\{\mu_{\mathcal{L}}(w), \mu_{\mathcal{G}}(a)\}\right], \mu_{\mathcal{I}}(q)\right\}\right] \\
&\geq \min\left\{\min\left\{\inf_{w\in x^2a}\mu_{\mathcal{L}}(w), \mu_{\mathcal{G}}(a)\right\}, \inf_{q\in yay}\mu_{\mathcal{I}}(q)\right\} \\
&\geq \min\left\{\min\left\{\mu_{\mathcal{L}}(a), \mu_{\mathcal{G}}(a)\right\}, \mu_{\mathcal{I}}(a)\right\} \\
&= \min\{\mu_{\mathcal{L}}(a), \mu_{\mathcal{G}}(a), \mu_{\mathcal{I}}(a)\} \\
&= (\mu_{\mathcal{L}} \cap \mu_{\mathcal{G}} \cap \mu_{\mathcal{I}})(a),
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_{\mathcal{L}} \circ \lambda_{\mathcal{G}} \circ \lambda_{\mathcal{I}})(a) &= \inf_{a\in pq}\left[\max\{(\lambda_{\mathcal{L}} \circ \lambda_{\mathcal{G}})(p), \lambda_{\mathcal{I}}(q)\}\right] \\
&= \inf_{a\in pq}\left[\max\left\{\inf_{p\in wa}\left[\max\{\lambda_{\mathcal{L}}(w), \lambda_{\mathcal{G}}(a)\}\right], \lambda_{\mathcal{I}}(q)\right\}\right] \\
&\leq \max\left\{\max\left\{\sup_{w\in x^2a}\lambda_{\mathcal{L}}(w), \lambda_{\mathcal{G}}(a)\right\}, \sup_{q\in yay}\lambda_{\mathcal{I}}(q)\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \max\{\max\{\lambda_{\mathcal{L}}(a), \lambda_{\mathcal{G}}(a)\}, \lambda_{\mathcal{I}}(a)\} \\
&= \max\{\lambda_{\mathcal{L}}(a), \lambda_{\mathcal{G}}(a), \lambda_{\mathcal{I}}(a)\} \\
&= (\lambda_{\mathcal{L}} \cup \lambda_{\mathcal{G}} \cup \lambda_{\mathcal{I}})(a).
\end{aligned}
$$

Therefore, $\mathcal{I} \cap \mathcal{G} \cap \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{G} \circ \mathcal{I}$.

(ii)⇒(iii) Since every FFB of S is an FFGB of S, it follows that (iii) is obtained.

(iii)⇒(i) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ be an FFL and an FFR of S, respectively. Then, $\mathcal{R}$ is also an FFB of S. By assumption, we have $\mathcal{L} \cap \mathcal{R} = \mathcal{S} \cap \mathcal{L} \cap \mathcal{R} \subseteq \mathcal{L} \circ (\mathcal{R} \circ \mathcal{S}) \subseteq \mathcal{L} \circ \mathcal{R}$. Consequently, S is intra-regular by Theorem 3.1. □

Theorem 3.17. *Let S be a semihypergroup. Then the following results are equivalent:*

(i) S is intra-regular;

(ii) $\mathcal{I} \cap \mathcal{G} \cap \mathcal{R} \subseteq \mathcal{I} \circ \mathcal{G} \circ \mathcal{R}$, for each FFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$, each FFInt $\mathcal{I} = (\mu_{\mathcal{I}}, \lambda_{\mathcal{I}})$ and each FFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;

(iii) $\mathcal{I} \cap \mathcal{B} \cap \mathcal{R} \subseteq \mathcal{I} \circ \mathcal{B} \circ \mathcal{R}$, for each FFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$, each FFInt $\mathcal{I} = (\mu_{\mathcal{I}}, \lambda_{\mathcal{I}})$ and each FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.

Proof. (i)⇒(ii) Assume that S is intra-regular. Let $a \in S$. Then, there exist $x, y \in S$ such that $a \in xa^2y$. This implies that $a \in (xax)a(ay^2)$. Thus, $a \in paw$, for some $p \in xax$ and $w \in ay^2$, and so $a \in pq$, for some $q \in aw$. So, we have

$$
\begin{aligned}
(\mu_{\mathcal{I}} \circ \mu_{\mathcal{G}} \circ \mu_{\mathcal{R}})(a) &= \sup_{a \in pq} [\min\{\mu_{\mathcal{I}}(p), (\mu_{\mathcal{G}} \circ \mu_{\mathcal{R}})(q)\}] \\
&= \sup_{a \in pq} \left[\min\left\{\mu_{\mathcal{I}}(p), \sup_{q \in aw} [\min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{R}}(w)\}]\right\}\right] \\
&\geq \min\left\{\inf_{p \in xax} \mu_{\mathcal{I}}(p), \min\left\{\mu_{\mathcal{G}}(a), \inf_{w \in ay^2} \mu_{\mathcal{R}}(w)\right\}\right\} \\
&\geq \min\{\mu_{\mathcal{I}}(a), \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{R}}(a)\}\} \\
&= \min\{\mu_{\mathcal{I}}(a), \mu_{\mathcal{G}}(a), \mu_{\mathcal{R}}(a)\} \\
&= (\mu_{\mathcal{I}} \cap \mu_{\mathcal{G}} \cap \mu_{\mathcal{R}})(a),
\end{aligned}
$$

and

$$
\begin{aligned}
(\lambda_{\mathcal{I}} \circ \lambda_{\mathcal{G}} \circ \lambda_{\mathcal{R}})(a) &= \inf_{a \in pq} [\max\{\lambda_{\mathcal{I}}(p), (\lambda_{\mathcal{G}} \circ \lambda_{\mathcal{R}})(q)\}] \\
&= \inf_{a \in pq} \left[\max\left\{\lambda_{\mathcal{I}}(p), \inf_{q \in aw} [\max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{R}}(w)\}]\right\}\right] \\
&\leq \max\left\{\sup_{p \in xax} \lambda_{\mathcal{I}}(p), \max\left\{\lambda_{\mathcal{G}}(a), \sup_{w \in ay^2} \lambda_{\mathcal{R}}(w)\right\}\right\} \\
&\leq \max\{\lambda_{\mathcal{I}}(a), \max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{R}}(a)\}\} \\
&= \max\{\lambda_{\mathcal{I}}(a), \lambda_{\mathcal{G}}(a), \lambda_{\mathcal{R}}(a)\} \\
&= (\lambda_{\mathcal{I}} \cup \lambda_{\mathcal{G}} \cup \lambda_{\mathcal{R}})(a).
\end{aligned}
$$

It turns out that $\mathcal{I} \cap \mathcal{G} \cap \mathcal{R} \subseteq \mathcal{I} \circ \mathcal{G} \circ \mathcal{R}$.

(ii)⇒(iii) It is obvious.

(iii)⇒(i) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ be an FFL and an FFR of S, respectively. Then, $\mathcal{R}$ is also an FFB of $\mathcal{S}$. By assumption, we have $\mathcal{L} \cap \mathcal{R} = \mathcal{S} \cap \mathcal{L} \cap \mathcal{R} \subseteq (\mathcal{S} \circ \mathcal{L}) \circ \mathcal{R} \subseteq \mathcal{L} \circ \mathcal{R}$. By Theorem 3.1, we obtain that S is intra-regular. □

4. Regular and intra-regular semihypergroups

In this section, we characterize both regular and intra-regular semihypergroups in terms of different types of Fermatean fuzzy hyperideals of semihypergroups.

Lemma 4.1. [35] *Let S be a semihypergroup. Then, S is both regular and intra-regular if and only if $B = BB$, for every bi-hyperideal B of S.*

Theorem 4.2. *Let S be a semihypergroup. Then the following statements are equivalent:*

(i) S is both regular and intra-regular;
(ii) $\mathcal{B} = \mathcal{B} \circ \mathcal{B}$, for any FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S;
(iii) $\mathcal{G} \cap \mathcal{H} \subseteq \mathcal{G} \circ \mathcal{H}$, for all FFGBs $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ and $\mathcal{H} = (\mu_{\mathcal{H}}, \lambda_{\mathcal{H}})$ of S;
(iv) $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \circ \mathcal{B}$, for all FFBs $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.

Proof. (i)⇒(iii) Let $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ and $\mathcal{H} = (\mu_{\mathcal{H}}, \lambda_{\mathcal{H}})$ be FFGBs of S. By assumption, there exist $x, y, z \in S$ such that $a \in axa$ and $a \in ya^2z$. Also, $a \in (axya)(azxa)$, which implies that $a \in pq$, for some $p \in axya$ and $q \in azxa$. Thus, we have

$$\begin{aligned}
(\mu_{\mathcal{G}} \circ \mu_{\mathcal{H}})(a) &= \sup_{a\in pq} [\min\{\mu_{\mathcal{G}}(p), \mu_{\mathcal{H}}(q)\}] \\
&\geq \min\left\{\inf_{p\in axya} \mu_{\mathcal{G}}(p), \inf_{q\in azxa} \mu_{\mathcal{H}}(q)\right\} \\
&\geq \min\{\min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}, \min\{\mu_{\mathcal{H}}(a), \mu_{\mathcal{H}}(a)\}\} \\
&= \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{H}}(a)\} \\
&= (\mu_{\mathcal{G}} \cap \mu_{\mathcal{H}})(a),
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_{\mathcal{G}} \circ \lambda_{\mathcal{H}})(a) &= \inf_{a\in pq} [\max\{\lambda_{\mathcal{G}}(p), \lambda_{\mathcal{H}}(q)\}] \\
&\leq \max\left\{\sup_{p\in axya} \lambda_{\mathcal{G}}(p), \sup_{q\in azxa} \lambda_{\mathcal{H}}(q)\right\} \\
&\leq \max\{\max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{G}}(a)\}, \max\{\lambda_{\mathcal{H}}(a), \lambda_{\mathcal{H}}(a)\}\} \\
&= \max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{H}}(a)\} \\
&= (\lambda_{\mathcal{G}} \cup \lambda_{\mathcal{H}})(a).
\end{aligned}$$

Therefore, $\mathcal{G} \cap \mathcal{H} \subseteq \mathcal{G} \circ \mathcal{H}$.

(iii)⇒(iv) Since every FFB is also an FFGB of S, it follows that (iv) holds.

(iv)⇒(ii) Let $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ be any FFB of S. By the hypothesis, we have $\mathcal{B} = \mathcal{B} \cap \mathcal{B} \subseteq \mathcal{B} \circ \mathcal{B}$. Otherwise, $\mathcal{B} \circ \mathcal{B} \subseteq \mathcal{B}$ always. Hence, $\mathcal{B} = \mathcal{B} \circ \mathcal{B}$.

(ii)⇒(i) Let B be any bi-hyperideal of S. By Lemma 2.4, we have $C_B = (\mu_{C_B}, \lambda_{C_B})$ is an FFB of S. By the given assumption and Lemma 2.1, it follows that $C_B = C_B \circ C_B = C_{BB}$. For every $a \in B$, we have $\mu_{C_{BB}}(a) = \mu_{C_B}(a) = 1$. This means that $a \in BB$. It turns out that $B \subseteq BB$. On the other hand, $BB \subseteq B$. Hence, $B = BB$. By Lemma 4.1, we obtain that S is both regular and intra-regular. □

The next theorem follows by Theorem 4.2.

Theorem 4.3. *The following properties are equivalent in a semihypergroup S:*

(i) S is both regular and intra-regular;
(ii) $\mathcal{B} \cap \mathcal{G} \subseteq \mathcal{B} \circ \mathcal{G}$, for each FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ and each FFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;
(iii) $\mathcal{B} \cap \mathcal{G} \subseteq \mathcal{G} \circ \mathcal{B}$, for each FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ and each FFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S.

Moreover, the following corollary obtained by Theorems 4.2 and 4.3.

Corollary 4.4. For a semihypergroup S, the following conditions are equivalent:

(i) S is both regular and intra-regular;
(ii) $\mathcal{G} \cap \mathcal{H} \subseteq (\mathcal{G} \circ \mathcal{H}) \cap (\mathcal{H} \circ \mathcal{G})$, for all FFGBs $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ and $\mathcal{H} = (\mu_{\mathcal{H}}, \lambda_{\mathcal{H}})$ of S;
(iii) $\mathcal{A} \cap \mathcal{B} \subseteq (\mathcal{A} \circ \mathcal{B}) \cap (\mathcal{B} \circ \mathcal{A})$, for all FFBs $\mathcal{A} = (\mu_{\mathcal{A}}, \lambda_{\mathcal{A}})$ and $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S;
(iv) $\mathcal{B} \cap \mathcal{G} \subseteq (\mathcal{B} \circ \mathcal{G}) \cap (\mathcal{G} \circ \mathcal{B})$, for any FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ and any FFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S.

By Lemma 2.6 and Theorem 3.1, we receive the following theorem.

Theorem 4.5. *Let S be a semihypergroup. Then, S is both regular and intra-regular if and only if $\mathcal{L} \cap \mathcal{R} \subseteq (\mathcal{L} \circ \mathcal{R}) \cap (\mathcal{R} \circ \mathcal{L})$, for every FFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and every FFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ of S.*

The following theorem can be proved by Corollary 4.4 and Theorem 4.5.

Theorem 4.6. *In a semihypergroup S, the following statements are equivalent:*

(i) S is both regular and intra-regular;
(ii) $\mathcal{G} \cap \mathcal{L} \subseteq (\mathcal{G} \circ \mathcal{L}) \cap (\mathcal{L} \circ \mathcal{G})$, for any FFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and any FFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;
(iii) $\mathcal{B} \cap \mathcal{L} \subseteq (\mathcal{B} \circ \mathcal{L}) \cap (\mathcal{L} \circ \mathcal{B})$, for any FFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and any FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S;
(iv) $\mathcal{R} \cap \mathcal{G} \subseteq (\mathcal{G} \circ \mathcal{R}) \cap (\mathcal{R} \circ \mathcal{G})$, for every FFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and any FFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;
(v) $\mathcal{R} \cap \mathcal{B} \subseteq (\mathcal{B} \circ \mathcal{R}) \cap (\mathcal{R} \circ \mathcal{B})$, for every FFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and any FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.

Theorem 4.7. *The following properties are equivalent on a semihypergroup S:*

(i) S is both regular and intra-regular;
(ii) $\mathcal{L} \cap \mathcal{G} \subseteq \mathcal{G} \circ \mathcal{L} \circ \mathcal{G}$, for each FFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and each FFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;
(iii) $\mathcal{L} \cap \mathcal{B} \subseteq \mathcal{B} \circ \mathcal{L} \circ \mathcal{B}$, for each FFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and each FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S;
(iv) $\mathcal{R} \cap \mathcal{G} \subseteq \mathcal{G} \circ \mathcal{R} \circ \mathcal{G}$, for each FFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and each FFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;
(v) $\mathcal{R} \cap \mathcal{B} \subseteq \mathcal{B} \circ \mathcal{R} \circ \mathcal{B}$, for each FFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and each FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.

Proof. (i)⇒(ii) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$ and $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ be an FFL and an FFGB of S, respectively. Let $a \in S$. Then, there exist $x, y, z \in S$ such that $a \in axa$ and $a \in ya^2z$. This implies that

$a \in (axya)(azxya)(azxa)$; that is, $a \in pq$, for some $p \in axya$ and $q \in uv$, where $u \in azxya$ and $v \in azxa$. Thus, we have

$$\begin{aligned}(\mu_{\mathcal{G}} \circ \mu_{\mathcal{L}} \circ \mu_{\mathcal{G}})(a) &= \sup_{a\in pq} [\min\{\mu_{\mathcal{G}}(p), (\mu_{\mathcal{L}} \circ \mu_{\mathcal{G}})(q)\}] \\ &= \sup_{a\in pq} \left[\min\left\{\mu_{\mathcal{G}}(p), \sup_{q\in uv} [\min\{\mu_{\mathcal{L}}(u), \mu_{\mathcal{G}}(v)\}]\right\}\right] \\ &\geq \min\left\{\inf_{p\in axya} \mu_{\mathcal{G}}(p), \min\left\{\inf_{u\in azxya} \mu_{\mathcal{L}}(u), \inf_{v\in azxa} \mu_{\mathcal{G}}(v)\right\}\right\} \\ &\geq \min\{\min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}, \min\{\mu_{\mathcal{L}}(a), \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}\}\} \\ &= \min\{\mu_{\mathcal{L}}(a), \mu_{\mathcal{G}}(a)\} \\ &= (\mu_{\mathcal{L}} \cap \mu_{\mathcal{G}})(a),\end{aligned}$$

and

$$\begin{aligned}(\lambda_{\mathcal{G}} \circ \lambda_{\mathcal{L}} \circ \lambda_{\mathcal{G}})(a) &= \inf_{a\in pq} [\max\{\lambda_{\mathcal{G}}(p), (\lambda_{\mathcal{L}} \circ \lambda_{\mathcal{G}})(q)\}] \\ &= \inf_{a\in pq} \left[\max\left\{\lambda_{\mathcal{G}}(p), \inf_{q\in uv} [\max\{\lambda_{\mathcal{L}}(u), \lambda_{\mathcal{G}}(v)\}]\right\}\right] \\ &\leq \max\left\{\sup_{p\in axya} \lambda_{\mathcal{G}}(p), \max\left\{\sup_{u\in azxya} \lambda_{\mathcal{L}}(u), \sup_{v\in azxa} \lambda_{\mathcal{G}}(v)\right\}\right\} \\ &\leq \max\{\max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{G}}(a)\}, \max\{\lambda_{\mathcal{L}}(a), \max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{G}}(a)\}\}\} \\ &= \max\{\lambda_{\mathcal{L}}(a), \lambda_{\mathcal{G}}(a)\} \\ &= (\lambda_{\mathcal{L}} \cup \lambda_{\mathcal{G}})(a).\end{aligned}$$

We obtain that $\mathcal{L} \cap \mathcal{G} \subseteq \mathcal{G} \circ \mathcal{L} \circ \mathcal{G}$.

(ii)⇒(iii) It follows by the fact that every FFB is also an FFGB of S.

(iii)⇒(i) Let $a \in S$. It is easy to verify that $a \cup Sa$ and $a \cup aa \cup aSa$ are a left hyperideal and a bi-hyperideal of S with containing a, respectively. Then, $C_{a\cup Sa}$ and $C_{a\cup aa\cup aSa}$ are an FFL and an FFB of S, respectively. By the given hypothesis and Lemma 2.1, we obtain:

$$\begin{aligned}C_{(a\cup Sa)\cap(a\cup aa\cup aSa)} &= C_{a\cup Sa} \cap C_{a\cup aa\cup aSa} \\ &\subseteq C_{a\cup aa\cup aSa} \circ C_{a\cup Sa} \circ C_{a\cup aa\cup aSa} \\ &= C_{(a\cup aa\cup aSa)(a\cup Sa)(a\cup aa\cup aSa)}.\end{aligned}$$

This means that $\mu_{C_{(a\cup aa\cup aSa)(a\cup Sa)(a\cup aa\cup aSa)}}(a) \geq \mu_{C_{(a\cup Sa)\cap(a\cup aa\cup aSa)}}(a) = 1$. Also, $a \in (a \cup aa \cup aSa)(a \cup Sa)(a \cup aa \cup aSa)$. It turns out that $a \in (aSa) \cap (Sa^2S)$. Consequently, S is both regular and intra-regular.

Similarly, we can prove that (i)⇒(iv)⇒(v)⇒(i) obtain. □

Theorem 4.8. *Let S be a semihypergroup. Then the following statements are equivalent:*

(i) S is both regular and intra-regular;

(ii) $\mathcal{L} \cap \mathcal{R} \cap \mathcal{G} \subseteq \mathcal{G} \circ \mathcal{R} \circ \mathcal{L}$, for every FFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$, every FFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and every FFGB $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ of S;

(iii) $\mathcal{L} \cap \mathcal{R} \cap \mathcal{B} \subseteq \mathcal{B} \circ \mathcal{R} \circ \mathcal{L}$, for every FFL $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$, every FFR $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and every FFB $\mathcal{B} = (\mu_{\mathcal{B}}, \lambda_{\mathcal{B}})$ of S.

Proof. (i)⇒(ii) Let $\mathcal{L} = (\mu_{\mathcal{L}}, \lambda_{\mathcal{L}})$, $\mathcal{R} = (\mu_{\mathcal{R}}, \lambda_{\mathcal{R}})$ and $\mathcal{G} = (\mu_{\mathcal{G}}, \lambda_{\mathcal{G}})$ be an FFL, FFR, and FFGB of S, respectively. Then, for any $a \in S$, there exist $x, y, z \in S$ such that $a \in axa$ and $a \in ya^2z$. So, $a \in (axya)(az)(xa)$. Also, $a \in pq$, for some $p \in axya$ and $q \in uv$, where $u \in az$ and $v \in xa$. Thus, we have

$$\begin{aligned}
(\mu_{\mathcal{G}} \circ \mu_{\mathcal{R}} \circ \mu_{\mathcal{L}})(a) &= \sup_{a \in pq} [\min\{\mu_{\mathcal{G}}(p), (\mu_{\mathcal{R}} \circ \mu_{\mathcal{L}})(q)\}] \\
&= \sup_{a \in pq} \left[\min \left\{ \mu_{\mathcal{G}}(p), \sup_{q \in uv} [\min\{\mu_{\mathcal{R}}(u), \mu_{\mathcal{L}}(v)\}] \right\} \right] \\
&\geq \min \left\{ \inf_{p \in axya} \mu_{\mathcal{G}}(p), \min \left\{ \inf_{u \in az} \mu_{\mathcal{R}}(u), \inf_{v \in xa} \mu_{\mathcal{L}}(v) \right\} \right\} \\
&\geq \min\{\min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}, \min\{\mu_{\mathcal{R}}(a), \mu_{\mathcal{L}}(a)\}\} \\
&= \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{R}}(a), \mu_{\mathcal{L}}(a)\} \\
&= (\mu_{\mathcal{G}} \cap \mu_{\mathcal{R}} \cap \mu_{\mathcal{L}})(a),
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_{\mathcal{G}} \circ \lambda_{\mathcal{R}} \circ \lambda_{\mathcal{L}})(a) &= \inf_{a \in pq} [\max\{\lambda_{\mathcal{G}}(p), (\lambda_{\mathcal{R}} \circ \lambda_{\mathcal{L}})(q)\}] \\
&= \inf_{a \in pq} \left[\max \left\{ \lambda_{\mathcal{G}}(p), \inf_{q \in uv} [\max\{\lambda_{\mathcal{R}}(u), \lambda_{\mathcal{L}}(v)\}] \right\} \right] \\
&\leq \max \left\{ \sup_{p \in axya} \lambda_{\mathcal{G}}(p), \max \left\{ \sup_{u \in az} \lambda_{\mathcal{R}}(u), \sup_{v \in xa} \lambda_{\mathcal{L}}(v) \right\} \right\} \\
&\leq \max\{\max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{G}}(a)\}, \max\{\lambda_{\mathcal{R}}(a), \lambda_{\mathcal{L}}(a)\}\} \\
&= \max\{\lambda_{\mathcal{G}}(a), \lambda_{\mathcal{R}}(a), \lambda_{\mathcal{L}}(a)\} \\
&= (\lambda_{\mathcal{G}} \cup \lambda_{\mathcal{R}} \cup \lambda_{\mathcal{L}})(a).
\end{aligned}$$

It follows that $\mathcal{L} \cap \mathcal{R} \cap \mathcal{G} \subseteq \mathcal{G} \circ \mathcal{R} \circ \mathcal{L}$.

(ii)⇒(iii) It is obvious.

(iii)⇒(i) Let $s \in S$. It is not difficult to show that the sets $a \cup Sa$, $a \cup aS$, and $a \cup aa \cup aSa$ are a left hyperideal, a right hyperideal, and a bi-hyperideal of S with containing a, respectively. By Lemma 2.4, we have $C_{a \cup Sa}$, $C_{a \cup aS}$, and $C_{a \cup aa \cup aSa}$ are an FFL, an FFR, and an FFB of S, respectively. Using the assumption and Lemma 2.1, we have

$$\begin{aligned}
C_{(a \cup Sa) \cap (a \cup aS) \cap (a \cup aa \cup aSa)} &= C_{a \cup Sa} \cap C_{a \cup aS} \cap C_{a \cup aa \cup aSa} \\
&\subseteq C_{a \cup aa \cup aSa} \circ C_{a \cup aS} \circ C_{a \cup Sa} \\
&= C_{(a \cup aa \cup aSa)(a \cup aS)(a \cup Sa)}.
\end{aligned}$$

It turns out that $\mu_{C_{(a \cup aa \cup aSa)(a \cup aS)(a \cup Sa)}}(a) \geq \mu_{C_{(a \cup Sa) \cap (a \cup aS) \cap (a \cup aa \cup aSa)}}(a) = 1$; that is, $a \in (a \cup aa \cup aSa)(a \cup aS)(a \cup Sa)$. Thus, $a \in (aSa) \cap (Sa^2S)$. Therefore, S is both regular and intra-regular. □

5. Conclusions

In 2023, Nakkhasen [28] applied the concept of Fermatean fuzzy sets to characterize the class of regular semihypergroups. In this research, we discussed the characterizations of intra-regular semihypergroups using the properties of Fermatean fuzzy left hyperideals, Fermatean fuzzy right hyperideals, Fermatean fuzzy generalized bi-hyperideals, and Fermatean fuzzy bi-hyperideals of semihypergroups, which are shown in Section 3. In addition, we introduced the concept of Fermatean fuzzy interior hyperideals of semihypergroups and used this concept to characterize intra-regular semihypergroups and proved that Fermatean fuzzy interior hyperideals and Fermatean fuzzy hyperideals coincide in intra-regular semihypergroups. Furthermore, in Section 4, the characterizations of both regular and intra-regular semihypergroups by many types of their Fermatean fuzzy hyperideals are presented. In our next paper, we will investigate the characterization of weakly regular semihypergroups using different types of Fermatean fuzzy hyperideals of semihypergroups. Additionally, we will use the attributes of Fermatean fuzzy sets to describe various regularities (e.g., left regular, right regular, and completely regular) in semihypergroups.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

Warud Nakkhasen: conceptualization, investigation, original draft preparation, writing-review & editing, supervision; Teerapan Jodnok: writing-review & editing, supervision; Ronnason Chinram: writing-review & editing, supervision. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflict of interest.

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