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Research article

A novel analytical treatment for the Ambartsumian delay differential equation with a variable coefficient

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Abstract: The Ambartsumian delay differential equation with a variable coefficient is considered in this paper. An effective transformation is produced to convert the extended Ambartsumian equation to the pantograph model. Two kinds of analytical solutions are determined. The first solution is expressed as an exponential function multiplied by an infinite power series. The second solution is obtained as an infinite series in terms of exponential functions. Several exact solutions are established for different forms of the extended Ambartsumian equation under specific relations. In addition, the convergence analysis is addressed theoretically. Moreover, numeric calculations are conducted to estimate the accuracy. The results reveal that the present analysis is efficient and accurate and can be further applied to similar delay models in a straightforward manner.

Keywords: Ambartsumian equation; delay; exact solution; series solution
Mathematics Subject Classification: 34K06, 65L03

1. Introduction

The Ambartsumian equation is of practical interest in astrophysics [1]. It describes the surface brightness in the Milky Way. This paper focuses on an extended version of this equation. The extended Ambartsumian delay differential equation (EADDE) is considered in the following form:

$$y'(t) = -y(t) + \frac{\alpha}{\xi} e^{\sigma t} y\left(\frac{t}{\xi}\right), \quad y(0) = \lambda, \quad \xi > 1, \quad t \geq 0, \tag{1.1}$$

where α, ξ, σ, and λ are constants. If $\sigma = 0$ and $\alpha = 1$, the EADDE (1.1) becomes the standard Ambartsumian delay differential equation (SADDE):

$$y'(t) = -y(t) + \frac{1}{\xi}y\left(\frac{t}{\xi}\right), \quad y(0) = \lambda, \quad \xi > 1, \quad t \geq 0. \tag{1.2}$$

Moreover, the case $\alpha = 0$ transforms Eq. (1.1) to the initial value problem (IVP) $y'(t) = -y(t)$, $y(0) = \lambda$. Such IVP consists of a simple linear ordinary differential equation (ODE) in which the exact solution is well-known as $y(t) = \lambda e^{-t}$. When $\alpha \neq 0$, the exact solution of the EADDE (1.1) is still unavailable. So, the main objective of this work is to report some new results in this regard. In the last decade, several techniques have been discussed and proposed for analyzing the SADDE (1.2) in classical form [2–4] and also in a generalized form; see, for example [5,6]. However, the EADDE (1.1) may be considered for the first time in this paper.

In order to solve the present model, there are many numerical and analytical methods that can be used. For the numerical methods, there are the Taylor method [7], Chebyshev polynomials [8], the Bernoulli operational matrix [9], Bernstein polynomials [10], spectral methods [11, 12] and other numerical approaches [13, 14]. The analytic methods include the Laplace transform (LT) [15–17], the combined Laplace transform-Adomian decomposition method [18], the double integral transform [19], Adomian's method [20–22], the Homotopy perturbation method (HPM) [23–26], the differential transform method [27, 28], and the homotopy analysis method [29–32] and its modifications/extensions [33, 34].

However, a simpler approach is to be developed in this paper to treat the model (1.1) in an analytical sense. Our procedure depends mainly on two basic steps. The first step is to produce an efficient transformation to put Eq (1.1) in a new form that contains no exponential-function coefficient. The second step is to solve the transformed equation in which the coefficients will be constants. It will be demonstrated that the transformed equation possesses the same structure as the Pantograph delay differential equation (PDDE) [35–39]:

$$z'(t) = a\,z(t) + b\,z\,(ct)\,, \quad z(0) = \lambda, \quad t \geq 0, \tag{1.3}$$

where a, b, and c are constants. To the best of our knowledge, there are several available analytical solutions for the PDDE (1.3) in different forms. Such ready solutions of the PDDE are to be invested in constructing the analytical solution of the present model in different forms. In addition, it will be revealed that the exact solution of the model (1.1) is still available when specific constraints on the involved parameters are satisfied. The next section highlights the basic transformation that is capable of converting the EADDE to the PDDE.

2. Formulation and mathematical analysis

Theorem 1. *The transformation:*

$$y(t) = e^{\mu t}z(t), \tag{2.1}$$

converts the EADDE (1.1) to

$$z'(t) = -(1 + \mu)z(t) + \frac{\alpha}{\xi}z\left(\frac{t}{\xi}\right), \quad z(0) = \lambda, \tag{2.2}$$

where

$$\mu = \frac{\xi\sigma}{\xi - 1}. \tag{2.3}$$

Proof. Suppose that

$$y(t) = e^{\mu t} z(t), \tag{2.4}$$

where μ is an unknown parameter to be determined later. Substituting Eq (2.4) into Eq (1.1), then

$$e^{\mu t} z'(t) + \mu e^{\mu t} z(t) = -e^{\mu t} z(t) + \frac{\alpha}{\xi} e^{\left(\sigma + \frac{\mu}{\xi}\right)t} z\left(\frac{t}{\xi}\right), \quad z(0) = \lambda, \tag{2.5}$$

i.e.,

$$z'(t) + \mu z(t) = -z(t) + \frac{\alpha}{\xi} e^{\left[\sigma + \left(\frac{1}{\xi} - 1\right)\mu\right]t} z\left(\frac{t}{\xi}\right), \quad z(0) = \lambda, \tag{2.6}$$

or

$$z'(t) = -(1 + \mu) z(t) + \frac{\alpha}{\xi} e^{\left[\sigma + \left(\frac{1}{\xi} - 1\right)\mu\right]t} z\left(\frac{t}{\xi}\right), \quad z(0) = \lambda. \tag{2.7}$$

Let

$$\sigma + \left(\frac{1}{\xi} - 1\right)\mu = 0, \tag{2.8}$$

then

$$\mu = \frac{\xi\sigma}{\xi - 1}. \tag{2.9}$$

Hence, Eq (2.7) becomes

$$z'(t) = -(1 + \mu) z(t) + \frac{\alpha}{\xi} z\left(\frac{t}{\xi}\right), \quad z(0) = \lambda, \tag{2.10}$$

and this completes the proof. □

Remark 1. *Based on Theorem 1, we have*

$$y(t) = e^{\frac{\xi\sigma t}{\xi - 1}} z(t), \tag{2.11}$$

as a solution for the EADDE (1.1) such that $z(t)$ *is a solution of the PDDE:*

$$z'(t) = a z(t) + b z(ct), \quad z(0) = \lambda, \tag{2.12}$$

where a, b, and c are defined as

$$a = -(1 + \mu), \quad b = \frac{\alpha}{\xi}, \quad c = \frac{1}{\xi}, \tag{2.13}$$

and μ *is already given by* $\mu = \frac{\xi\sigma}{\xi - 1}$.

3. Solutions of the EADDE

In the literature, the solutions for the PDDE (1.3) have been obtained in different forms. Two kinds of solutions were found and addressed below. The first kind expresses the solution in the form of a power series, i.e., a power series solution (PSS). The second kind uses the exponential function solution (EFS) in closed form. Both the PSS and the EFS will be implemented in this section to formulate the solution of the current model.

3.1. The PSS

In [36], the author solved the PDDE (1.3) and obtained the PSS:

$$z(t) = \lambda\left[1 + \sum_{i=1}^{\infty}\left(\prod_{k=1}^{i}\left(a + bc^{k-1}\right)\right)\frac{t^i}{i!}\right]. \tag{3.1}$$

Implementing the values of a, b, and c given by Eq (2.13), then Eq (3.1) yields

$$z(t) = \lambda\left[1 + \sum_{i=1}^{\infty}\left(\prod_{k=1}^{i}\left(-1 - \mu + \alpha\xi^{-k}\right)\right)\frac{t^i}{i!}\right], \tag{3.2}$$

i.e.,

$$z(t) = \lambda\sum_{i=0}^{\infty}\left(\prod_{k=1}^{i}\left(-1 - \mu + \alpha\xi^{-k}\right)\right)\frac{t^i}{i!}. \tag{3.3}$$

Note that $\prod_{k=1}^{i}\left(-1 - \mu + \alpha\xi^{-k}\right) = 1$ when $i = 0$. Substituting (3.3) into (2.11) leads to the following solution for the model (1.1):

$$y(t) = \lambda\, e^{\frac{\xi\sigma t}{\xi-1}}\sum_{i=0}^{\infty}\left(\prod_{k=1}^{i}\left(-1 - \mu + \alpha\xi^{-k}\right)\right)\frac{t^i}{i!}. \tag{3.4}$$

It should be noted that the solution (3.4) reduces to the corresponding solution of the SADDE (1.2) when $\sigma = 0$ and $\alpha = 1$. For declaration, utilizing these values in (3.4) gives

$$y(t) = \lambda\sum_{i=0}^{\infty}\left(\prod_{k=1}^{i}\left(\xi^{-k} - 1\right)\right)\frac{t^i}{i!}, \tag{3.5}$$

which agrees with the obtained PSS in [2] for the SADDE (1.2). It may be important to mention that the series (3.3) converges in the whole domain for all real values of σ, α, and $\xi > 1$. Consequently, the solution (3.4) is convergent; this issue is discussed by Theorem 2 below.

3.2. Convergence analysis

Theorem 2. *For $\sigma, \alpha \in \mathbb{R}$, the series $z(t) = \lambda\sum_{i=0}^{\infty}\left(\prod_{k=1}^{i}\left(-1 - \mu + \alpha\xi^{-k}\right)\right)\frac{t^i}{i!}$ has an infinite radius of convergence $\forall\ \xi > 1$ and hence the series is uniformly convergent on any compact interval on $\mathbb{R}$.*

Proof. Let us rewrite Eq (3.3) as

$$z(t) = \sum_{i=0}^{\infty} h_i(t), \tag{3.6}$$

where h_i is

$$h_i(t) = \lambda \frac{t^i}{i!} \prod_{k=1}^{i} \left(-1 - \mu + \alpha\xi^{-k}\right), \quad i \geq 1. \tag{3.7}$$

Assume that ρ is the radius of convergence, and applying the ratio test, then

$$\begin{aligned} \frac{1}{\rho} &= \lim_{i\to\infty} \left|\frac{h_{i+1}(t)}{h_i(t)}\right| = \lim_{i\to\infty} \left|\frac{\frac{t^{i+1}}{(i+1)!} \prod_{k=1}^{i+1} \left(-1 - \mu + \alpha\xi^{-k}\right)}{\frac{t^i}{i!} \prod_{k=1}^{i} \left(-1 - \mu + \alpha\xi^{-k}\right)}\right|, \\ &= |t| \lim_{i\to\infty} \left|\frac{-1 - \mu + \alpha\xi^{-(i+1)}}{i+1}\right|. \end{aligned} \tag{3.8}$$

For $\xi > 1$, we have $\lim_{i\to\infty} \xi^{-(i+1)} = 0$, thus

$$\frac{1}{\rho} = |t| \lim_{i\to\infty} \left|\frac{1 + \mu}{i+1}\right| = 0, \quad \forall\, \mu = \frac{\xi\sigma}{\xi - 1} \in \mathbb{R},\ t \geq 0, \tag{3.9}$$

which completes the proof. □

3.3. The EFS

In [37], the authors determined the following solution for the PDDE (1.3)

$$z(t) = \lambda \sum_{i=0}^{\infty} \left(\frac{b}{a}\right)^i \sum_{j=0}^{i} \frac{(-1)^j c^{\frac{1}{2}(i-j)(i-j-1)} e^{ac^j t}}{(c:c)_{i-j}(c:c)_j}, \tag{3.10}$$

in terms of the exponential functions. Implementing the values of a, b, and c in (2.13), then the solution of the EADDE (1.1) reads

$$y(t) = \lambda\, e^{\frac{\xi\sigma t}{\xi-1}} \sum_{i=0}^{\infty} \left(-\frac{\alpha}{\xi(1+\mu)}\right)^i \sum_{j=0}^{i} \frac{(-1)^j \xi^{-\frac{1}{2}(i-j)(i-j-1)} e^{-(1+\mu)\xi^{-j} t}}{(1/\xi : 1/\xi)_{i-j}(1/\xi : 1/\xi)_j}, \tag{3.11}$$

where $\mu = \frac{\xi\sigma}{\xi-1}$ and $(1/\xi : 1/\xi)_j$ is the Pochhammer symbol:

$$(1/\xi : 1/\xi)_j = \prod_{k=0}^{j-1} \left(1 - \xi^{-(k+1)}\right) = \prod_{k=1}^{j} \left(1 - \xi^{-k}\right). \tag{3.12}$$

In general, $(p : q)_j$ is defined by the product:

$$(p : q)_j = \prod_{k=0}^{j-1} \left(1 - pq^k\right) = \prod_{k=1}^{j} \left(1 - pq^{k-1}\right). \tag{3.13}$$

In addition, El-Zahar and Ebaid [38] introduced the following solution for Eq (1.3)

$$z(t) = \lambda\,(-b/a : c)_\infty \sum_{i=0}^{\infty} \frac{(-b/a)^i\, e^{ac^i t}}{(c : c)_i}. \tag{3.14}$$

Hence, the solution of the EADDE (1.1) is

$$y(t) = \lambda\, e^{\frac{\xi\sigma t}{\xi-1}} \left(\frac{\alpha}{\xi(1+\mu)} : \frac{1}{\xi}\right)_\infty \sum_{i=0}^{\infty} \frac{\left(\frac{\alpha}{\xi(1+\mu)}\right)^i e^{-(1+\mu)\xi^{-i} t}}{(\frac{1}{\xi} : \frac{1}{\xi})_i}. \tag{3.15}$$

Moreover, the authors [38] showed that the convergence of $z(t)$ holds if the conditions $|b/a| < 1$ and $|c| < 1$ are satisfied. For our model (1.1), the condition $|c| < 1$ is already satisfied for $|c| = |1/\xi| < 1$, where $\xi > 1$. The other condition $|b/a| < 1$ becomes $\left|\frac{\alpha}{\xi(1+\mu)}\right| < 1$, i.e., $\left|\frac{\alpha}{1+\mu}\right| < \xi$.

4. Exact solution at special cases

One of the main advantages of the PSS is that it can be used to generate several exact solutions at specific cases of the model's parameters. This section focuses on this issue. Before launching to the target of this section, we put the solution (3.4) in the form:

$$y(t) = \lambda\, e^{\frac{\xi\sigma t}{\xi-1}} \sum_{i=0}^{\infty} v_i \frac{t^i}{i!}, \qquad v_i = \prod_{k=1}^{i} \left(-1 - \mu + \alpha\xi^{-k}\right). \tag{4.1}$$

The second equation in (4.1) reveals that

$$\begin{aligned}
v_0 &= 1,\\
v_1 &= -1 - \mu + \alpha\xi^{-1},\\
v_2 &= \left(-1 - \mu + \alpha\xi^{-1}\right)\left(-1 - \mu + \alpha\xi^{-2}\right),\\
v_3 &= \left(-1 - \mu + \alpha\xi^{-1}\right)\left(-1 - \mu + \alpha\xi^{-2}\right)\left(-1 - \mu + \alpha\xi^{-3}\right),\\
&.,\\
&.,\\
v_i &= \left(-1 - \mu + \alpha\xi^{-1}\right)\left(-1 - \mu + \alpha\xi^{-2}\right)\left(-1 - \mu + \alpha\xi^{-3}\right)\ldots\left(-1 - \mu + \alpha\xi^{-i}\right),\ i \geq 1.
\end{aligned} \tag{4.2}$$

This section implements Eqs (4.1) and (4.2) to determine several exact solutions of the model (1.1) under different constraints such as $-1 - \mu + \alpha\xi^{-1} = 0$, $-1 - \mu + \alpha\xi^{-2} = 0$, $-1 - \mu + \alpha\xi^{-3} = 0, \ldots$, and $-1 - \mu + \alpha\xi^{-n} = 0$ $(n \in \mathbb{N}^+)$.

4.1. $-1 - \mu + \alpha\xi^{-1} = 0$

From Eqs (4.1) and (4.2), it will be shown in the next theorem that only the first term v_0 in series (4.1) has a non-zero value when $-1 - \mu + \alpha\xi^{-1} = 0$, while the other higher-order terms $v_i, i \geq 1$ are zeros. So, the series (4.1) transforms to the exact solution for the EADDE (1.1).

Lemma 1. *If* $-1-\mu+\alpha\xi^{-1}=0$*, then the EADDE (1.1) becomes*

$$y'(t) = -y(t) + \frac{\alpha}{\xi}e^{-\left(1-\frac{1}{\xi}\right)\left(1-\frac{\alpha}{\xi}\right)t}y\left(\frac{t}{\xi}\right), \quad y(0)=\lambda, \quad t \geq 0, \tag{4.3}$$

with exact solution:

$$y(t) = \lambda e^{\left(\frac{\alpha}{\xi}-1\right)t}. \tag{4.4}$$

Proof. Consider $-1-\mu+\alpha\xi^{-1}=0$; in this case we have $\mu=-1+\frac{\alpha}{\xi}$ which implies $\sigma=-\left(1-\frac{1}{\xi}\right)\left(1-\frac{\alpha}{\xi}\right)$ and the model (1.1) takes the form:

$$y'(t) = -y(t) + \frac{\alpha}{\xi}e^{-\left(1-\frac{1}{\xi}\right)\left(1-\frac{\alpha}{\xi}\right)t}y\left(\frac{t}{\xi}\right), \quad y(0)=\lambda. \tag{4.5}$$

On using the relation $\mu=-1+\frac{\alpha}{\xi}$ in Eq (4.2) gives $v_i=0\ \forall\ i\geq 1$. Hence, the series (4.1) contains only the first term v_0 (which equals one), consequently

$$y(t) = \lambda e^{\left(\frac{\alpha}{\xi}-1\right)t}, \tag{4.6}$$

and this completes the proof. □

Remark 2. *As a direct result of this lemma, we have at* $\alpha=\xi$ *the constant function* $y(t)=\lambda$ *as a solution of the 1st-order delay equation* $y'(t)+y(t)=y\left(\frac{t}{\xi}\right)$ *whatever the value of* ξ*. For a further validation of the solution (4.6), we consider the additional special case* $\alpha=0$*. Then Eq (4.5) becomes* $y'(t)+y(t)=0$ *and the solution is derived directly by setting* $\alpha=0$ *into Eq (4.4); this gives* $y(t)=\lambda e^{-t}$ *which is the well-known solution.*

4.2. $-1-\mu+\alpha\xi^{-2}=0$

This case gives the solution as a product of the exponential function and a polynomial of first degree in t.

Lemma 2. *If* $-1-\mu+\alpha\xi^{-2}=0$*, then the EADDE (1.1) becomes*

$$y'(t) = -y(t) + \frac{\alpha}{\xi}e^{-\left(1-\frac{1}{\xi}\right)\left(1-\frac{\alpha}{\xi^2}\right)t}y\left(\frac{t}{\xi}\right), \quad y(0)=\lambda, \quad t \geq 0, \tag{4.7}$$

and the exact solution is

$$y(t) = \lambda e^{\left(\frac{\alpha}{\xi^2}-1\right)t}\left[1+\frac{\alpha}{\xi}\left(1-\frac{1}{\xi}\right)t\right]. \tag{4.8}$$

Proof. Let $-1-\mu+\alpha\xi^{-2}=0$, then $\sigma=-\left(1-\frac{1}{\xi}\right)\left(1-\frac{\alpha}{\xi^2}\right)$. Hence, Eq (1.1) yields

$$y'(t) = -y(t) + \frac{\alpha}{\xi}e^{-\left(1-\frac{1}{\xi}\right)\left(1-\frac{\alpha}{\xi^2}\right)t}y\left(\frac{t}{\xi}\right), \quad y(0)=\lambda, \quad t \geq 0. \tag{4.9}$$

From Eqs (4.2), we find

$$v_0 = 1, \quad v_1 = \frac{\alpha}{\xi}\left(1-\frac{1}{\xi}\right), \quad v_i = 0\ \forall\ i \geq 2, \tag{4.10}$$

and accordingly,

$$y(t) = \lambda e^{\left(\frac{\alpha}{\xi^2}-1\right)t}\left(v_0 + v_1 t\right). \tag{4.11}$$

Inserting the values (4.10) into (4.11) completes the proof. □

Remark 3. *An interesting case arises from this lemma when $\alpha = \xi^2$. This case implies the 1st-order delay equation $y'(t) + y(t) = \xi y\left(\frac{t}{\xi}\right)$. The corresponding solution does not contain the term of the exponential function; the solution is a pure linear polynomial given by $y(t) = \lambda\left[1 + (\xi - 1)\, t\right]$.*

4.3. $-1 - \mu + \alpha\xi^{-3} = 0$

Lemma 3. *If $-1 - \mu + \alpha\xi^{-3} = 0$, the corresponding equation is*

$$y'(t) = -y(t) + \frac{\alpha}{\xi} e^{-\left(1-\frac{1}{\xi}\right)\left(1-\frac{\alpha}{\xi^3}\right)t} y\left(\frac{t}{\xi}\right), \quad y(0) = \lambda, \quad t \geq 0, \tag{4.12}$$

with the exact solution:

$$y(t) = \lambda e^{\left(\frac{\alpha}{\xi^3}-1\right)t}\left[1 + \frac{\alpha}{\xi}\left(1 - \frac{1}{\xi^2}\right)t + \frac{\alpha^2}{\xi^3}\left(1 - \frac{1}{\xi}\right)\left(1 - \frac{1}{\xi^2}\right)\frac{t^2}{2}\right]. \tag{4.13}$$

Proof. The proof follows immediately by repeating the above analysis. □

Remark 4. *Choosing $\alpha = \xi^3$ yields the 1st-order delay equation $y'(t) + y(t) = \xi^2 y\left(\frac{t}{\xi}\right)$ and the corresponding solution is the polynomial given by $y(t) = \lambda\left[1 + \left(\xi^2 - 1\right)t + (\xi - 1)\left(\xi^2 - 1\right)\frac{t^2}{2}\right]$.*

4.4. $-1 - \mu + \alpha\xi^{-n} = 0$*,* $n \in \mathbb{N}^+$

This case generalizes the previous cases. The present case expresses the solution as a product of an exponential function and a polynomial of degree n.

Theorem 3. *If $-1 - \mu + \alpha\xi^{-n} = 0$, the corresponding equation is*

$$y'(t) = -y(t) + \frac{\alpha}{\xi} e^{-\left(1-\frac{1}{\xi}\right)\left(1-\frac{\alpha}{\xi^n}\right)t} y\left(\frac{t}{\xi}\right), \quad y(0) = \lambda, \quad t \geq 0, \tag{4.14}$$

with the exact solution:

$$y(t) = \lambda e^{\left(\frac{\alpha}{\xi^n}-1\right)t}\sum_{i=0}^{n-1} v_i \frac{t^i}{i!}, \tag{4.15}$$

where v_i is

$$v_i = \alpha^i \xi^{-\frac{1}{2}i(i+1)}\prod_{k=1}^{i}\left(1 - \xi^{-n+k}\right) = \alpha^i \xi^{-\frac{1}{2}i(i+1)}\left(\xi^{-n} : \xi\right)_i. \tag{4.16}$$

Proof. Since $-1 - \mu + \alpha\xi^{-n} = 0$, then we have from Eq (4.1) that

$$v_i = \prod_{k=1}^{i}\left(-1 - \mu + \alpha\xi^{-k}\right) = \prod_{k=1}^{i}\left(-\alpha\xi^{-n} + \alpha\xi^{-k}\right), \tag{4.17}$$

which implies $v_i = 0\ \forall\ i \geq n$. Hence, v_i exists $\forall\ 0 \leq i \leq n-1$. Thus, the infinite series in (4.1) is truncated to

$$y(t) = \lambda e^{\left(\frac{\alpha}{\xi^n}-1\right)t} \sum_{i=0}^{n-1} v_i \frac{t^i}{i!}. \tag{4.18}$$

On the other hand, v_i can be rewritten as

$$v_i = \prod_{k=1}^{i} \left(\alpha\xi^{-k}\right) \prod_{k=1}^{i} \left(1 - \xi^{-n+k}\right) = \alpha^i \xi^{-\frac{1}{2}i(i+1)} \prod_{k=1}^{i} \left(1 - \xi^{-n+k}\right) = \alpha^i \xi^{-\frac{1}{2}i(i+1)} \left(\xi^{-n} : \xi\right)_i, \tag{4.19}$$

which finalizes the proof. □

5. Results and discussion

This section explores some numerical results for the obtained PSS and the EFS in the previous sections. The current discussion focuses on several issues, such as the behavior of the PSS and the EFS, their accuracy, the domain of the involved parameters to ensure the convergence, and also the advantages of each solution over the other. It was shown in a previous section that the PSS can be used to generate exact solutions for the EADDE (1.1) under the restriction $-1 - \mu + \alpha\xi^{-n} = 0$ ($n \in \mathbb{N}^+$) or, equivalently, $\sigma = -\left(1 - \frac{1}{\xi}\right)\left(1 - \frac{\alpha}{\xi^n}\right)$. Regarding the obtained exact solution in Theorem 3, it depends mainly on n. The behavior of such exact solution is plotted in Figures 1 and 2 at different values of n. In the general case, in which the restriction $\sigma = -\left(1 - \frac{1}{\xi}\right)\left(1 - \frac{\alpha}{\xi^n}\right)$ is not satisfied, the series form (3.4) is used to obtain the m-term approximate solution of the PSS as

$$\Phi_m(t) = \lambda\, e^{\frac{\xi\sigma t}{\xi-1}} \sum_{i=0}^{m-1} \left(\prod_{k=1}^{i} \left(-1 - \mu + \alpha\xi^{-k}\right) \right) \frac{t^i}{i!}. \tag{5.1}$$

Figures 3 and 4 show the convergence of the approximations $\Phi_m(t)$ at selected values of the parameters α, ξ, and σ, where $\lambda = 1$ is fixed in all computations. These figures also indicate the difference in the behavior of the PSS when the exponent σ is changed from negative to positive. In order to estimate the accuracy of these approximations, we construct the residuals:

$$RE_m(t) = \left| \Phi'_m(t) + \Phi_m(t) - \frac{\alpha}{\beta} e^{\sigma t} \Phi_m\left(\frac{t}{\xi}\right) \right|, \quad m \geq 1. \tag{5.2}$$

The numerical results displayed in Figures 5 and 6 declare that the residuals $RE_m(t)$ are acceptable; especially, they approach zero as t tends to infinity. It may be important here to mention that the PSS (3.4) is convergent for all real values of α, σ, and ξ (> 1) as proved by Theorem 2.

However, the situation for the EFS (3.15) is different because of the condition of convergence given by $\left|\frac{\alpha}{1+\mu}\right| < \xi$, where $\mu = \frac{\xi\sigma}{\xi-1}$. So, the m-term approximate solution of the EFS:

$$\Psi_m(t) = \lambda\, e^{\frac{\xi\sigma t}{\xi-1}} \left(\frac{\alpha}{\xi(1+\mu)} : \frac{1}{\xi} \right)_\infty \sum_{i=0}^{m-1} \frac{\left(\frac{\alpha}{\xi(1+\mu)}\right)^i e^{-(1+\mu)\xi^{-i} t}}{\left(\frac{1}{\xi} : \frac{1}{\xi}\right)_i}, \tag{5.3}$$

converges in certain domains of the parameters α, σ, and ξ. Figures 7 and 8 determine the domain of the parameters ξ and α for which the EFS converges at selected values of σ, where $\sigma = 1$ in Figure 7 and $\sigma = -1$ in Figure 8. Similarly, Figures 9 and 10 show the domains of σ and α for which the EFS converges at selected values of ξ, where $\xi = 1.7$ in Figure 9 and $\xi = 5.7$ in Figure 10.

Figures 11 and 12 show the convergence of the approximations $\Psi_m(t)$ at selected values of the parameters α, ξ, and σ. In addition, the residuals corresponding to the approximations $\Psi_m(t)$ are introduced in Figures 13–16, which confirm the accuracy of the EFS.

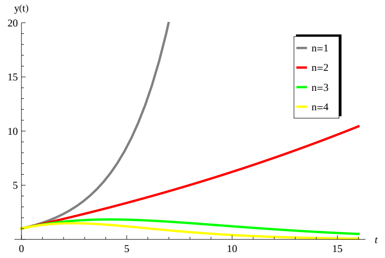

Figure 1. Plots of the exact solution (4.15-4.16) for the EADDE $y'(t) = -y(t) + \frac{\alpha}{\xi} e^{-\left(1-\frac{1}{\xi}\right)\left(1-\frac{\alpha}{\xi^n}\right)t} y\left(\frac{t}{\xi}\right)$, $y(0) = \lambda$ when $\lambda = 1$, $\alpha = 2$, and $\xi = 1.4$ at different values of n, $n = 1, 2, 3, 4$.

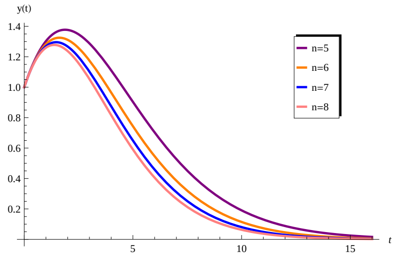

Figure 2. Plots of the exact solution (4.15-4.16) for the EADDE $y'(t) = -y(t) + \frac{\alpha}{\xi} e^{-\left(1-\frac{1}{\xi}\right)\left(1-\frac{\alpha}{\xi^n}\right)t} y\left(\frac{t}{\xi}\right)$, $y(0) = \lambda$ when $\lambda = 1$, $\alpha = 2$, and $\xi = 1.4$ at different values of n, $n = 5, 6, 7, 8$.

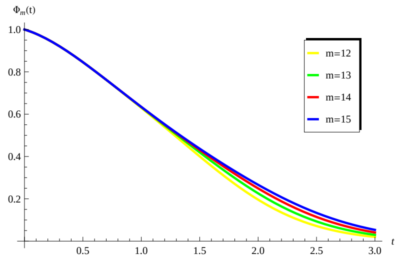

Figure 3. Convergence of the PSS approximations $\Phi_m(t)$, $m = 12, 13, 14, 15$ at $\lambda = 1$, $\xi = 1.2$, $\alpha = 1$, and $\sigma = -1$.

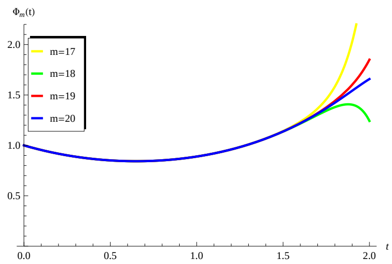

Figure 4. Convergence of the PSS approximations $\Phi_m(t)$, $m = 17, 18, 19, 20$ at $\lambda = 1$, $\xi = 1.2$, $\alpha = 1$, and $\sigma = 1$.

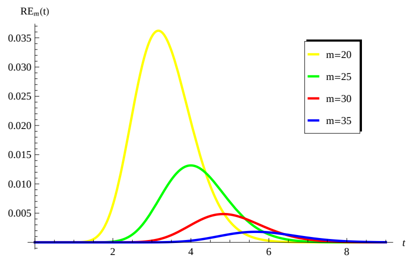

Figure 5. Plots of the PSS-residuals $RE_m(t)$, $m = 20, 25, 30, 35$ at $\xi = 1.2$, $\alpha = 1$, and $\sigma = -1$.

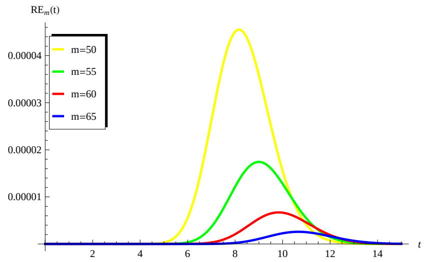

Figure 6. Plots of the PSS-residuals $RE_m(t)$, $m = 50, 55, 60, 65$ at $\xi = 2$, $\alpha = 1$, and $\sigma = -3$.

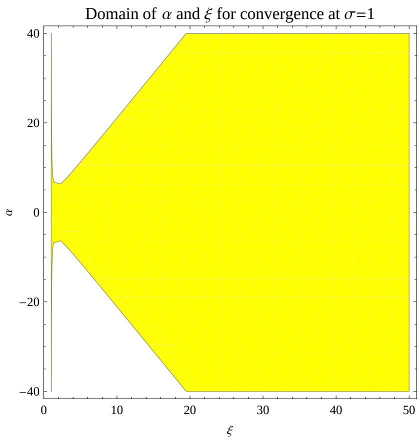

Figure 7. Domain of ξ and α for the convergence of the EFS (3.15) at $\sigma = 1$.

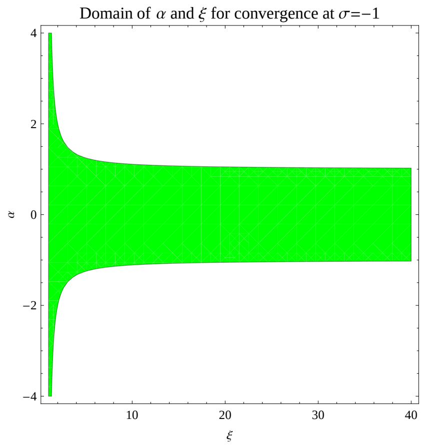

Figure 8. Domain of ξ and α for the convergence of the EFS (3.15) at $\sigma = -1$.

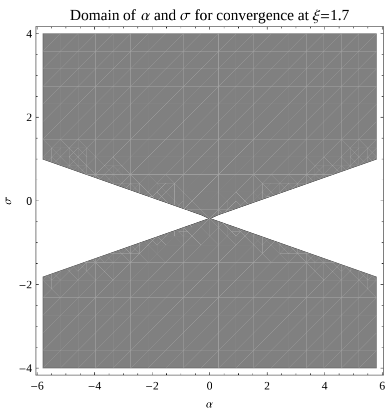

Figure 9. Domain of α and σ for the convergence of the EFS (3.15) at $\xi = 1.7$.

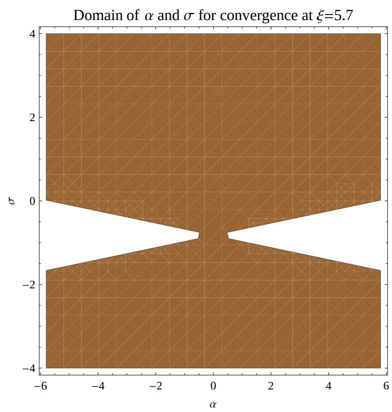

Figure 10. Domain of α and σ for the convergence of the EFS (3.15) at $\xi = 5.7$.

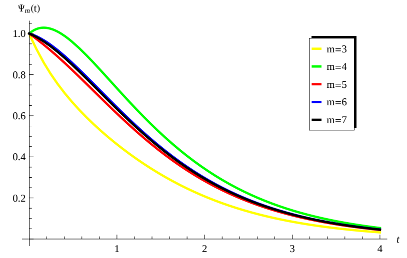

Figure 11. Convergence of the EFS approximations $\Psi_m(t)$, $m = 3, 4, 5, 6, 7$ at $\lambda = 1$, $\xi = 1.2$, $\alpha = 1$, and $\sigma = -1$.

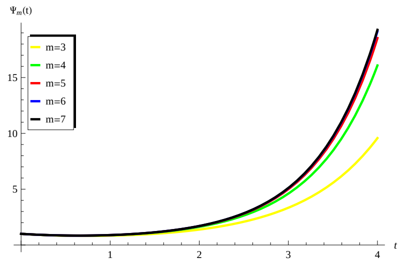

Figure 12. Convergence of the EFS approximations $\Psi_m(t)$, $m = 3, 4, 5, 6, 7$ at $\lambda = 1$, $\xi = 1.2$, $\alpha = 1$, and $\sigma = 1$.

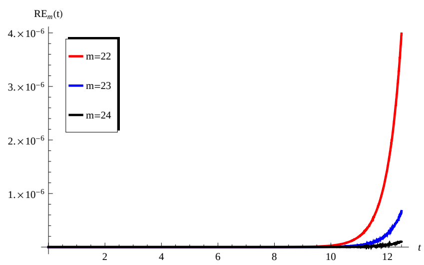

Figure 13. Plots of the EFS-residuals $RE_m(t)$, $m = 22, 23, 24$ at $\xi = 2$, $\alpha = 1$, and $\sigma = 1$.

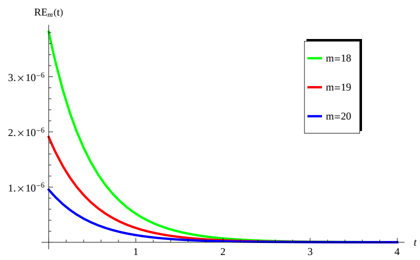

Figure 14. Plots of the EFS-residuals $RE_m(t)$, $m = 18, 19, 20$ at $\xi = 2$, $\alpha = -1$, and $\sigma = -1$.

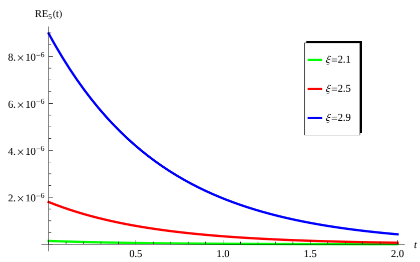

Figure 15. Plots of the EFS-residual $RE_{24}(t)$ when $\alpha = -1$ and $\sigma = -1$ at different values of ξ.

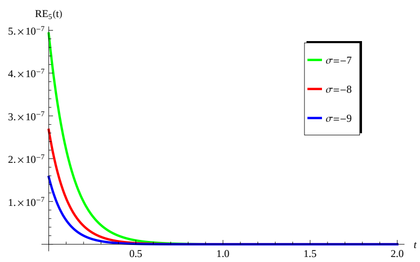

Figure 16. Plots of the EFS-residual $RE_5(t)$ when $\alpha = 2$ and $\xi = 8$ at different values of σ.

6. Conclusions

The extended Ambartsumian delay differential equation with a variable coefficient was analyzed in this paper. By the aid of a suitable transformation, the extended model was converted to the standard pantograph model. The available/known solutions in the literature for the pantograph model were employed to construct two kinds of analytical solutions, mainly, the power series solution PSS and the exponential function solution EFS. The PSS was successfully reformulated to generate several exact solutions for different forms of the present extended Ambartsumian model utilizing certain relations between the variable coefficient and the involved parameters. The obtained exact solutions reflect the advantage of the PSS over the EFS. Additionally, the PSS was found valid and convergent for any real values of the model's parameters. In contrast to the PSS, the EFS requires specific domains for the involved parameters to achieve the convergence criteria. However, the EFS enjoys better accuracy over the PSS. This is simply because the EFS needs a lower number of terms if compared with the EFS. However, the residuals both of the PSS and the EFS tend to zero, which reflect the effectiveness and efficiency of the developed analysis. Perhaps the suggested approach needs a further validation for applied problems such as delay-differential equations in pharmacokinetic compartment modeling [40]. Although the present extended model was analytically analyzed via a transformation approach, it may also be treated numerically via applying the randomized Euler scheme [41] as future work.

Author contributions

Rana M. S. Alyoubi: Conceptualization, methodology, validation, formal analysis, investigation, writing-review and editing, visualization; Abdelhalim Ebaid: Methodology, validation, formal analysis, investigation, writing-original draft preparation; Essam R. El-Zahar: Conceptualization, methodology, validation, formal analysis, investigation, writing-review and editing; Mona D. Aljoufi: Conceptualization, methodology, software, validation, formal analysis, investigation, data curation, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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