AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(12): 35645–35654.
DOI: 10.3934/math.20241691
Received: 17 October 2024
Revised: 04 December 2024
Accepted: 10 December 2024
Published: 23 December 2024

Research article

Multi-solitons in the model of an inhomogeneous optical fiber

Jinfang Li[1], Chunjiang Wang[2,*], Li Zhang[1] and Jian Zhang[3]

[1] School of Fundamental Education, Beijing Polytechnic College, Beijing 100042, China
[2] V.C. & V.R. Key Lab of Sichuan Province, Sichuan Normal University, Chengdu 610068, China
[3] School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China

* **Correspondence:** Email: wangchunjiangmath@163.com.

Abstract: This paper was concerned with the inhomogeneous optical fiber model, which was governed by a nonlinear Schrödinger equation with variable coefficients. By spectral analysis for Lax pair of the equation, a corresponding Riemann-Hilbert problem was formulated. By solving the Riemann-Hilbert problem with simple poles, the formula of multi-soliton solutions was derived. Finally, we considered a soliton control system and obtained the one-soliton and two-soliton.

Keywords: nonlinear Schrödinger equation; soliton solutions; Riemann-Hilbert problem; soliton control; optical fibers
Mathematics Subject Classification: 35C08, 35Q51, 37K15

1. Introduction

Multi-solitons are a type of wave packet with soliton properties in nonlinear systems, which have self-similarity and stable transmission properties. Different from simple solitons, multi-solitons exhibit more complex interactions and nonlinear dynamic properties [1, 2]. In 1972, Zakharov and Shabat [3] obtained multi-soliton solutions of the nonlinear Schrödinger equation by the inverse scattering method. Subsequently, different types of multi-soliton solutions for nonlinear integrable systems were extensively studied [4–7].

The soliton control system is a technology that effectively manages and controls the state of nonlinear physical systems by adjusting and controlling solitons [8]. The optical fiber communication system controls and regulates the transmission of solitons [9, 10]. Since the first soliton dispersion management experiment [11], the various soliton management mechanisms have been theoretically predicted through different methods [12–15].

The realistic optical fiber is inhomogeneous, which can influence various effects such as the amplification or absorption, group velocity dispersion, and self-phase modulation [16]. The problem of soliton control in nonlinear systems is described by the following nonlinear Schrödinger equation with variable coefficients:

$$iq_z + \frac{1}{2}D(z)q_{tt} + R(z)|q|^2 q = i\Gamma(z)q, \quad (t, z) \in \mathbb{R}^2, \tag{1.1}$$

where $q(t, z)$ is the complex envelope of the electrical field in a co-moving frame, $D(z)$ is the group velocity dispersion, $R(z)$ is the nonlinearity coefficient, $\Gamma(z)$ is the amplification or absorption coefficient, z is the propagation distance, and t is the retarded time. Equation (1.1) describes the amplification or absorption of pulse propagation in a single-mode optical fiber with distributed dispersion and nonlinearity [17].

Joshi [18] obtained the integrability constraint of (1.1) by the Painlevé test. Serkin and Hasegawa [19] studied the one-soliton solution of (1.1) from the integrable point of view. Then, Hao et al. [20] studied soliton solutions of (1.1) by Darboux transformation. Tian and Gao [21] obtained soliton solutions of (1.1) by symbolic computation. Lü et al. [22, 23] studied soliton solutions of (1.1) by the Hirota method. Sun et al. [24] obtained rogue-wave solutions of (1.1) by hierarchy reduction.

In this paper, we obtained the new multi-soliton solutions of (1.1) by using the Riemann-Hilbert method [25–30]. These solutions are useful not only in designing transmission lines for soliton management, but also in some femtosecond laser experiments [14, 15]. Additionally, the inverse scattering transform presented in this paper will pave a way for investigating the long-time asymptotic behavior of the solution to (1.1) by the nonlinear steepest descent method.

This paper is organized as follows: In Section 2, we provide some elementary preliminaries for constructing a Riemann-Hilbert problem. In Section 3, we solve the Riemann-Hilbert problem with simple pole and obtain soliton solutions for (1.1). Moreover, we discuss the properties of optical solitons by numerical simulations. Section 4 gives our conclusions.

2. Preliminaries

We first construct a Riemann-Hilbert problem by spectral analysis of Lax pair. Generally, Eq (1.1) is not integrable. We consider the following relationship [12]:

$$\Gamma(z) = \frac{1}{2}\frac{R(z)D_z(z) - D(z)R_z(z)}{R(z)D(z)}, \tag{2.1}$$

where the subscript denotes taking the derivative of z. Then, Eq (1.1) admits the Lax pair

$$\phi_t = X\phi, \quad \phi_z = T\phi, \tag{2.2}$$

where $\phi = \phi(t, z, k)$ is a 2×2 matrix function, $k \in \mathbb{C}$ is a spectral parameter, and

$$X = -ik\sigma_3 + Q, \quad Q = \sqrt{\frac{R}{D}}\begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$T = -ik^2 D(z)\sigma_3 + kD(z)Q + \frac{i}{2}(R(z)|q|^2\sigma_3 + D(z)\sigma_3 Q_t).$$

Here and in the following content, the asterisk indicates complex conjugate.

Considering the initial condition $\lim_{z\to\pm\infty} q(0,z)=0$, the Lax pair (2.2) becomes

$$\phi_t = X_0\phi, \quad \phi_z = T_0\phi, \tag{2.3}$$

where $X_0 = -ik\sigma_3$ and $T_0 = kD(z)X_0$. We obtain the Jost solutions $\phi_\pm(t,z,k)$ of Eq (2.3)

$$\phi_\pm(t,z,k) = e^{i\theta(t,z,k)\sigma_3} + o(1), \quad z\to\pm\infty,$$

where $\theta(t,z,k) = -k(t+kzD(z))$. By making transformation

$$\phi_\pm(t,z,k) = \mu_\pm(t,z,k)e^{i\theta(t,z,k)\sigma_3}, \tag{2.4}$$

it has

$$\mu_\pm(t,z,k)\to I, \quad as\ \ z\to\pm\infty.$$

Moreover, $\mu(t,z,k)$ satisfies the following Lax pair:

$$\begin{aligned}\mu_t &= -\,ik[\sigma_3,\mu]+Q\mu,\\ \mu_z &= -\,ik^2D(z)[\sigma_3,\mu]+\Delta T\mu,\end{aligned} \tag{2.5}$$

where $[\sigma_3,\mu]=\sigma_3\mu-\mu\sigma_3$, $\Delta T = T-T_0$. The Jost solution $\mu(t,z,k)$ can be solved by the following Volterral integrable equations:

$$\mu_\pm(t,z,k) = I + \int_{\pm\infty}^{t} e^{-ik(t-y)\hat{\sigma}_3}(Q(y,z)\mu_\pm(y,z,k))dy, \tag{2.6}$$

where $e^{\hat{\sigma}_3}A = e^{\sigma_3}Ae^{-\sigma_3}$. For convenience, we define D^+, D^-, and Σ on the $\mathbb{C}$-plane as

$$D^\pm = \{k\in\mathbb{C}\,|\ \pm\mathrm{Im}k>0\}, \quad \Sigma = i\mathbb{R}\cup\mathbb{R}.$$

Proposition 2.1. *Suppose that $u(t,z)\in L^1(\mathbb{R})$ and $\mu_{\pm,j}(t,z,k)$ represent the j-th column of $\mu_\pm(t,z,k)$, Then, the Jost solutions $\mu_\pm(t,z,k)$ have the following properties:*

- *$\mu_{-,1}$ and $\mu_{+,2}$ can be analytically extended to D^+ and continuously extended to $D^+\cup\Sigma$.*
- *$\mu_{+,1}$ and $\mu_{-,2}$ can be analytically extended to D^- and continuously extended to $D^-\cup\Sigma$.*

Proof. We define $\mu_\pm = \begin{pmatrix}\mu_{\pm,11} & \mu_{\pm,12}\\ \mu_{\pm,21} & \mu_{\pm,22}\end{pmatrix}$. Then, taking μ_- as an example, Eq (2.6) can be rewritten as

$$\begin{pmatrix}\mu_{-,11} & \mu_{-,12}\\ \mu_{-,21} & \mu_{-,22}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \sqrt{\frac{R}{D}}\int_{-\infty}^{t}\begin{pmatrix} q\mu_{-,21} & e^{-2ik(t-y)}q\mu_{-,22}\\ -e^{2ik(t-y)}q^*\mu_{-,11} & -q^*\mu_{-,12}\end{pmatrix}dy.$$

Note that $e^{2ik(t-y)} = e^{2i(t-y)\mathrm{Re}(k)}e^{2(y-t)\mathrm{Im}(k)}$. Since $y-t<0$ and $\mathrm{Im}(k)>0$, we obtain that $\mu_{-,1}$ is analytically extended to D^+. Moreover, since $\mathrm{Im}(k)=0$, when $k\in\Sigma$, it is shown that $\mu_{-,1}$ is continuously extended to $D^+\cup\Sigma$. In the same way, we also obtain the analyticity and continuity of $\mu_{-,2}$, $\mu_{+,1}$, and $\mu_{+,2}$. This completes the proof. □

According to the method in [30], the Jost solution $\mu_\pm(t,z,k)$ admits the symmetry

$$\mu_\pm(t,z,k) = \sigma_2\mu_\pm^*(t,z,k^*)\sigma_2, \tag{2.7}$$

where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and its asymptotic behavior is

$$\mu_\pm(t,z,k) \to I, \ as\ k \to \infty. \tag{2.8}$$

Since $\phi_\pm(t,z,k)$ are two fundamental matrix solutions of Lax pair (2.2), one can define a constant scattering matrix $S(k) = (s_{ij}(k))_{2\times 2}$ such that

$$\phi_+(t,z,k) = \phi_-(t,z,k)S(k), \tag{2.9}$$

where $s_{ij}(k)$ is called scattering coefficients and $\det S(k) = 1$. From Eq (2.7), one has

$$S(k) = \sigma_2 S^*(k^*)\sigma_2. \tag{2.10}$$

According to the analyticities of $\mu_\pm(t,z,k)$, we obtain that $s_{11}(k)$ analytic in D^- and $s_{22}(k)$ analytic in D^+. From Eq (2.8), the scattering matrix $S(k)$ satisfies $S(k) \to I$, as $k \to \pm\infty$.

Now, we construct the Riemann-Hilbert problem for Eq (1.1). Define the following sectionally meromorphic matrices:

$$M(t,z,k) = \begin{cases} M^- = \left(\dfrac{\mu_{+,1}}{s_{11}}, \mu_{-,2}\right), & k \in D^-, \\ M^+ = \left(\mu_{-,1}, \dfrac{\mu_{+,2}}{s_{22}}\right), & k \in D^+. \end{cases} \tag{2.11}$$

Then, a multiplicative matrix Riemann-Hilbert problem is proposed:

$$\begin{cases} M^\pm(t,z,k) \ are\ respectively\ analytic\ in\ D^\pm; \\ M^-(t,z,k) = M^+(t,z,k)(I - G(t,z,k)); \\ M(t,z,k) \sim I, \quad k \to \infty, \end{cases} \tag{2.12}$$

where

$$G(t,z,k) = \begin{pmatrix} \rho(k)\tilde{\rho}(k) & e^{2i\theta(k)}\tilde{\rho}(k) \\ -e^{-2i\theta(k)}\rho(k) & 0 \end{pmatrix},$$

$$\rho(k) = \frac{s_{21}(k)}{s_{11}(k)}\ and\ \tilde{\rho}(k) = -\rho^*(k^*).$$

3. Multi-soliton solutions

In what follows, we will solve the Riemann-Hilbert problem with simple poles and present the multi-soliton solutions for Eq (1.1).

We suppose that $s_{22}(k)$ has N simple zeros k_n $(n = 1, 2, \cdots, N)$ in D^+, which means $s_{22}(k_n) = 0$ and $s_{22}'(k_n) \neq 0$. Here and in the following $'$ represents taking the derivative of a function variable.

According to Eq (2.10), one has $s_{22}(k_n) = s_{11}(k_n^*) = 0$. Then, the corresponding discrete spectrum can be collected as

$$K = \{k_n,\ \ k_n^*\}_{n=1}^N. \tag{3.1}$$

Solving the above Riemann-Hilbert problem requires us to regularize it by subtracting out the asymptotic behaviors and the pole contributions. Then, one has

$$M^- - I - \sum_{n=1}^N \left\{ \frac{\operatorname*{Res}_{k=k_n} M^+}{k-k_n} + \frac{\operatorname*{Res}_{k=k_n^*} M^-}{k-k_n^*} \right\} = M^+ - I - \sum_{n=1}^N \left\{ \frac{\operatorname*{Res}_{k=k_n} M^+}{k-k_n} + \frac{\operatorname*{Res}_{k=k_n^*} M^-}{k-k_n^*} \right\} - M^+ G,$$

where

$$\operatorname*{Res}_{k=k_n} M^+ = \left(0\ \ \tilde{C}_n e^{2i\theta(k_n)} \mu_{-,1}(t,z,k_n)\right), \quad n = 1, 2, \cdots, N,$$
$$\operatorname*{Res}_{k=k_n^*} M^- = \left(C_n e^{-2i\theta(k_n^*)} \mu_{-,2}(t,z,k_n^*)\ \ 0\right), \quad n = 1, 2, \cdots, N,$$

$C_n = -\tilde{C}_n^* = \dfrac{b_n}{s'_{11}(k_n^*)}$, and b_n is a constant.

With the help of Plemelj's formula, the solution of Eq (2.12) can be written as

$$M = I + \sum_{n=1}^N \left\{ \frac{\operatorname*{Res}_{k=k_n} M^+}{k-k_n} + \frac{\operatorname*{Res}_{k=k_n^*} M^-}{k-k_n^*} \right\} + \frac{1}{2\pi i} \int_\Sigma \frac{M^+(\xi)G(\xi)}{\xi - k} d\xi. \tag{3.2}$$

Taking $M = M^-$ and comparing the (1,2) position element of matrices (3.2), we get

$$q(t,z) = 2i\sqrt{\frac{D(z)}{R(z)}} \sum_{n=1}^N \tilde{C}_n e^{2i\theta(k_n)} \mu_{-,11}(t,z,k_n) - \frac{1}{\pi}\sqrt{\frac{D(z)}{R(z)}} \int_\Sigma (M^+G)_{12}(\xi) d\xi,$$

where $\tilde{C}_n = \dfrac{b_n}{s'_{11}(k_n^*)}$.

Now, we focus on the potentials $q(t,z)$ with the reflection coefficient $\rho(k) = 0$. By some algebraic calculations, we obtain the multi-soliton solutions formula

$$q(t,z) = -2i\sqrt{\frac{D(z)}{R(z)}} \frac{\det \hat{H}}{\det H}, \tag{3.3}$$

where

$$\hat{H} = \begin{bmatrix} 0 & \mathbf{P} \\ B & H \end{bmatrix}_{(N+1)\times(N+1)}, \quad H = \left(I + \sum_{j=1}^N c_j(k_n) c_l^*(k_j) \right)_{N\times N},$$
$$c_j(t,z,k) = \frac{C_j}{k - k_j^*} e^{-2i\theta(t,z,k_j^*)}, \quad B = (1, 1, \cdots, 1)^T_{1\times n},$$
$$\mathbf{P} = (P_1, \cdots P_N)\ \mathit{and}\ P_n = \tilde{C}_n e^{2i\theta(t,z,k_n)},\ n = 1, \cdots, N.$$

Now, we consider a periodic distributed amplification system with the varying group velocity dispersion parameter

$$D(z) = \frac{1}{d_0} e^{\gamma z} R(z), \tag{3.4}$$

and the nonlinearity parameter

$$R(z) = r_0 + r_1 \sin(cz), \tag{3.5}$$

where r_0, r_1, and c are the parameters described by the Kerr nonlinearity, and d_0 is the parameter related to initial peak power in the system.

Now, as an application of formula (3.3), we first present the one-soliton solution. Let $N = 1$, $k_1 = \alpha + i\beta$, and one has

$$q(t,z) = -2i\sqrt{\frac{D(z)}{R(z)}} \frac{4\beta^2 C_1^* e^{2i\theta(k_1)}}{4\beta^2 + |C_1|^2 e^{-2i\theta(k_1^*)} e^{2i\theta(k_1)}}, \tag{3.6}$$

where $C_1 = \dfrac{b_1}{s'_{11}(k_1^*)}$, $\theta(k_1) = -k_1(t + k_1 z D(z))$.

Figure 1 exhibits the dynamical structures of the one-soliton solution (3.6). Due to the value of parameter γ, the soliton group velocity is changed in propagating along the fiber, but the shape of the soliton remains unchanged. This is an important property of solitons.

By setting $N = 2$, from (3.3), we obtain the two-soliton solution

$$q(t,z) = -2i\sqrt{\frac{D(z)}{R(z)}} \frac{\det\begin{pmatrix} 0 & P_1 & P_2 \\ 1 & 1 + A_{11} & A_{12} \\ 1 & A_{21} & 1 + A_{22} \end{pmatrix}}{\det\begin{pmatrix} 1 + A_{11} & A_{12} \\ A_{21} & 1 + A_{22} \end{pmatrix}}, \tag{3.7}$$

where

$$P_n = \tilde{C}_n e^{2i\theta(t,z,k_n)},\ A_{nl} = \sum_{j=1}^{2} c_j(k_n) c_l^*(k_j),\ n, l = 1, 2,$$
$$\theta(t,z,k_j) = -k_j(t + k_j z D(z)),\ c_j(t,z,k) = \frac{C_j}{k - k_j^*} e^{-2i\theta(k_j^*)},\ j = 1, 2.$$

Figure 2 exhibits the dynamical structures of the two-soliton solution (3.7). From it we observe that two solitons propagate at the same speed in the fiber and exhibit periodic oscillations.

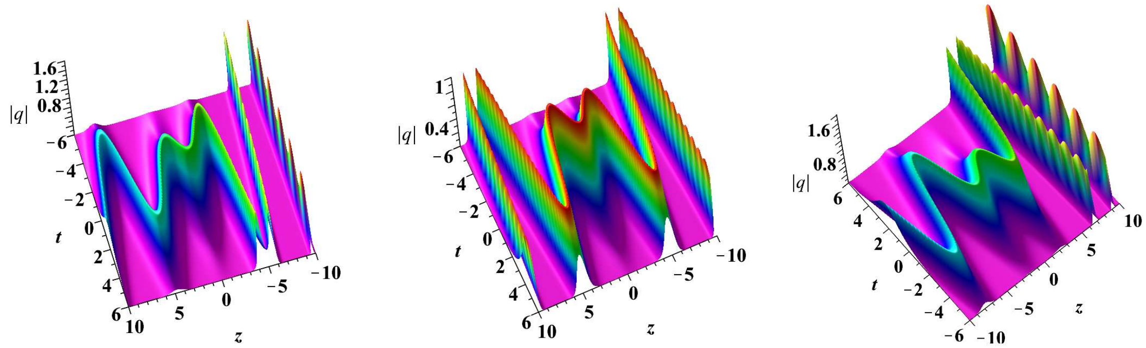

Figure 1. The one-soliton solution given by (3.6) for system parameters $d_0 = r_1 = c = 1$, $r_0 = 0$. The other parameters adopted are $C_1 = 1$, $k_1 = \frac{1}{2}(1 + i)$. Left: $\gamma = -0.5$; Middle: $\gamma = 0$; Right: $\gamma = 0.5$.

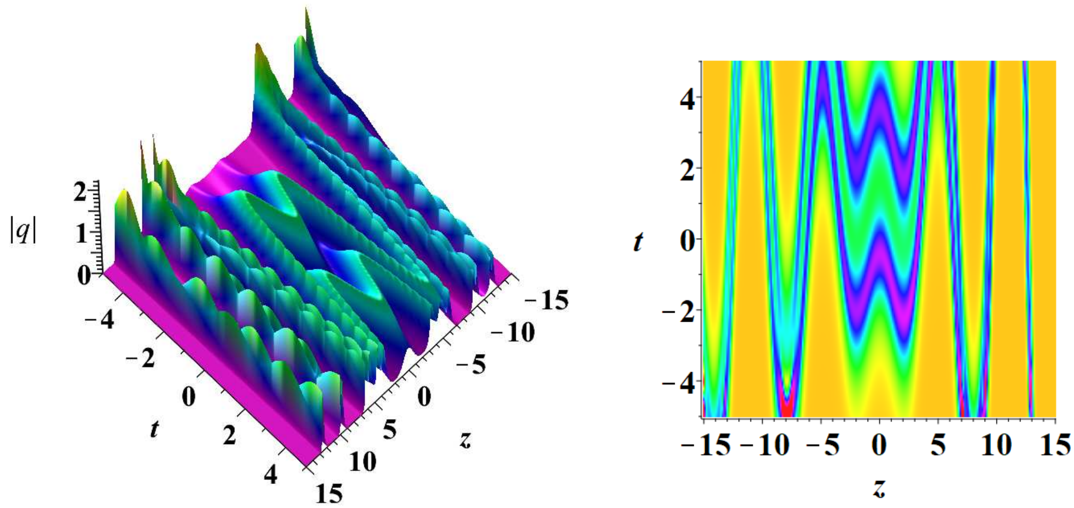

Figure 2. The two-soliton solution given by (3.7) for system parameters $d_0 = r_1 = c = 1$, $r_0 = \gamma = 0.03$. The other parameters adopted are $C_1 = C_2 = 1$, $k_1 = \frac{1}{2}(1 + i)$, $k_2 = \frac{1}{2}(1 + \sqrt{2}i)$. Left: Three-dimensional plot; Right: Density plot.

4. Conclusions

This paper is concerned with the multi-solitons in an inhomogeneous optical fiber model. The formula of multi-soliton solutions and inverse scattering transform are obtained by the Riemann-Hilbert method. Furthermore, we consider a soliton control system and obtain the one-soliton and two-soliton. Comparing our results with the solutions in [20, 22], we confirm that the obtained soliton solutions are new. Finally, the inverse scattering transformation presented in this paper will pave a way for the study of the long-time asymptotic behavior of the solution to Eq (1.1).

Author contributions

Jinfang Li: Conceptualization, Investigation, Writing-original draft, Writing-review & editing. Chunjiang Wang: Methodology, Supervision. Li Zhang: Visualization, Data creation, Software, Validation. Jian Zhang: Methodology, Supervision.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research is supported by the National Natural Science Foundation of China 12271080.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. Y. Kivshar, G. Agrawal, *Optical solitons: From fibers to photonic crystals*, Academic Press, 2003.
2. J. Yang, *Nonlinear waves in integrable and nonintegrable systems*, SIAM, 2010.
3. V. E. Zakharov, A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Sov. Phys. JETP*, **34** (1972), 62–69.
4. M. Pichler, G. Biondini, On the focusing non-linear Schrödinger equation with non-zero boundary conditions and double poles, *IMA J. Appl. Math.*, **82** (2017), 131–151. https://doi.org/10.1093/imamat/hxw009
5. Y. Zhang, X. Tao, S. Xu, The bound-state soliton solutions of the complex modified KdV equation, *Inverse Probl.*, **36** (2020), 065003. https://doi.org/10.1088/1361-6420/ab6d59
6. G. Biondini, G. Kovačič, Inverse scattering transform for the focusing nonlinear Schrödinger equation with nonzero boundary conditions, *J. Math. Phys.*, **55** (2014), 031506. https://doi.org/10.1063/1.4868483
7. C. Wang, J. Zhang, Riemann-Hilbert approach and *N*-soliton solutions of the two-component Kundu-Eckhaus equation, *Theor. Math. Phys.*, **212** (2022), 1222–1236. https://doi.org/10.1134/S0040577922090057
8. B. A. Malomed, *Soliton management in periodic systems*, New York: Springer, 2006. https://doi.org/10.1007/0-387-29334-5
9. A. Hasegawa, Quasi-soliton for ultra-high speed communications, *Physica D*, **123** (1998), 267–270. https://doi.org/10.1016/S0167-2789(98)00126-2
10. T. I. Lakoba, D. J. Kaup, Hermite-Gaussian expansion for pulse propagation in strongly dispersion managed fibers, *Phys. Rev. E*, **58** (1998), 6728–6741.

11. V. A. Bogatyrev, M. M. Bubnov, E. M. Dianov, A. S. Kurkov, P. V. Mamyshev, A. M. Prokhorov, A single-mode fiber with chromatic dispersion varying along the length, *J. Lightwave Technol.*, **9** (1991), 561–566. https://doi.org/10.1109/50.79530

12. V. N. Serkin, A. Hasegawa, Exactly integrable nonlinear Schrödinger equation models with varying dispersion, nonlinearity and gain: Application for soliton dispersion, *IEEE J. Sel. Top. Quant.*, **8** (2002), 418–431. https://doi.org/10.1109/JSTQE.2002.1016344

13. V. N. Serkin, T. L. Belyaeva, High-energy optical Schrödinger solitons, *JETP Lett.*, **74** (2001), 573-577. https://doi.org/10.1134/1.1455063

14. R. Hao, L. Li, Z. Li, R. Yang, G. Zhou, A new way to exact quasi-soliton solutions and soliton interaction for the cubic-quintic nonlinear Schrödinger equation with variable coefficients, *Opt. Commun.*, **245** (2005), 383–390. https://doi.org/10.1016/j.optcom.2004.10.001

15. C. Dai, Y. Wang, J. Chen, Analytic investigation on the similariton transmission control in the dispersion decreasing fiber, *Opt. Commun.*, **248** (2011), 3440–3444. https://doi.org/10.1016/j.optcom.2011.03.033

16. Y. Kubota, T. Odagaki, Numerical study of soliton scattering in inhomogeneous optical fibers, *Phys. Rev. E*, **68** (2003), 026603. https://doi.org/10.1103/PhysRevE.68.026603

17. Y. Kodama, Optical solitons in a monomode fiber, *J. Stat. Phys.*, **39** (1985), 597–614. https://doi.org/10.1007/BF01008354

18. N. Joshi, Painlevé property of general variable-coefficient versions of the Korteweg-de Vries and nonlinear Schrödinger equations, *Phys. Lett. A*, **125** (1987), 456–460. https://doi.org/10.1016/0375-9601(87)90184-8

19. V. N. Serkin, A. Hasegawa, Novel soliton solutions of the nonlinear Schrödinger equation model, *Phys. Rev. Lett.*, **85** (2000), 4502. https://doi.org/10.1103/PhysRevLett.85.4502

20. R. Hao, L. Li, Z. Li, W. Xue, G. Zhou, A new approach to exact soliton solutions and soliton interaction for the nonlinear Schrödinger equation with variable coefficients, *Opt. Commun.*, **236** (2004), 79–86. https://doi.org/10.1016/j.optcom.2004.03.005

21. B. Tian, Y. T. Gao, Symbolic-computation study of the perturbed nonlinear Schrödinger model in inhomogeneous optical fibers, *Phys. Lett. A*, **342** (2005), 228–236. https://doi.org/10.1016/j.physleta.2005.05.041

22. X. Lü, H. W. Zhu, X. H. Meng, Z. C. Yang, B. Tian, Soliton solutions and a Bäcklund transformation for a generalized nonlinear Schrödinger equation with variable coefficients from optical fiber communications, *J. Math. Anal. Appl.*, **336** (2007), 1305–1315. https://doi.org/10.1016/j.jmaa.2007.03.017

23. X. Lü, H. W. Zhu, Z. Z. Yao, X. H. Meng, C. Zhang, C. Y. Zhang, et al., Multisoliton solutions in terms of double Wronskian determinant for a generalized variable-coefficient nonlinear Schrödinger equation from plasma physics, arterial mechanics, fluid dynamics and optical communications, *Ann. Phys.*, **323** (2008), 1947–1955. https://doi.org/10.1016/j.aop.2007.10.007

24. Y. Sun, B. Tian, L. Liu, X. Y. Wu, Rogue waves for a generalized nonlinear Schrödinger equation with distributed coefficients in a monomode optical fiber, *Chao Soliton. Fract.*, **107** (2018), 266–274. https://doi.org/10.1016/j.chaos.2017.12.012

25. P. Deift, *Orthogonal polynomials and random matrices: A Riemann-Hilbert approach*, 2000.
26. L. L. Wen, E. G. Fan, Y. Chen, Multiple-high-order pole solutions for the NLS equation with quartic terms, *Appl. Math. Lett.*, **130** (2022), 108008. https://doi.org/10.1016/j.aml.2022.108008
27. Y. Yang, E. Fan, Riemann-Hilbert approach to the modified nonlinear Schrödinger equation with non-vanishing asymptotic boundary conditions, *Physica D*, **417** (2021), 132811. https://doi.org/10.1016/j.physd.2020.132811
28. W. Weng, Z. Yan, Inverse scattering and N-triple-pole soliton and breather solutions of the focusing nonlinear Schröinger hierarchy with nonzero boundary conditions, *Phys. Lett. A*, **407** (2021), 127472. https://doi.org/10.1016/j.physleta.2021.127472
29. X. Geng, J. Wu, Riemann-Hilbert approach and N-soliton solutions for a generalized Sasa-Satsuma equation, *Wave Motion*, **60** (2016), 62–72. https://doi.org/10.1016/j.wavemoti.2015.09.003
30. C. Wang, J. Zhang, Double-pole solutions in the modified nonlinear Schrödinger equation, *Wave Motion*, **118** (2023), 103102. https://doi.org/10.1016/j.wavemoti.2022.103102