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Research article

Quantitative analysis and stability results in β-normed space for sequential differential equations with variable coefficients involving two fractional derivatives

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Abstract: This article conducted an analysis on quantitative properties and stability in a β-normed space for a category of boundary value problems of nonlinear two-term fractional-order sequential differential equations with variable coefficients. The original problem was converted into an equivalent integral equation. Banach's fixed-point principle and Shaefer's fixed-point theorem were exploited to ensure that two existence conditions of the solutions for the problems were established. In addition, the stability known as β-Ulam-Hyers for such problems has also been analyzed. Illustrative examples demonstrated practical applications of the work.

Keywords: quantitative analysis; sequential fractional differential equations; variable coefficients; β-Ulam-Hyers stability
Mathematics Subject Classification: 34A08, 34B10, 34B15, 34D20

1. Introduction

Over the past few decades, the theories of fractional calculus (FC) and the related problems of fractional differential equations (FDEs) have attracted increasing research interest. The non-local intrinsic property of fractional integrators and differential operators enhances their effectiveness in reflecting the memory and genetic properties of various materials and change processes, and FDEs have seen wide use in various scientific and engineering fields to simulate problems and phenomena [1–4]. Frackiewicz and Palus [5] introduced the application of FC in image quality assessment indices. Etemad et al. [6] used these kinds of differential equations to simulate the changes of elastic beams. Meral et al. [7] discussed the applications of FCs in viscoelastic problems. Zhang and Huang [8] discussed the various applications of fractional calculus theory and FDEs in real-world science and engineering. The literature on the application of FC can also be referred to [9–12].

The study of FDEs has aspects such as fractional calculus theory, quantitative properties, and the stability of solutions [13,14]. A number of publications have investigated the conclusions of existence, existence and uniqueness, and/or stability of solutions to FDEs with various initial conditions and boundary conditions [15–18]. Wang et al. [19] investigated the existence and Ulam stability of an implicit multi-term FDE subject to boundary conditions. Yaseen et al. [20] analyzed the criteria for the existence and uniqueness of solutions to a class of sequential differential equations with Caputo–Hadamard fractional-order derivatives by exploiting Darbo's fixed-point theorem. Shah et al. [21] conducted a quantitative analysis of a nonlinear system of pantograph impulsive FDEs, derived the sufficient conditions for the existence and uniqueness of the solution, and investigated the Ulam–Hyers stability (UHS) of the solution.

Researchers have also seen progress regarding FDEs with constant and/or variable coefficients. In some literatures, people use numerical methods to deal with such FDEs [22,23]. Some sufficient or necessary conditions were established for the oscillation of FDEs with constant coefficients [24,25]. Yi and Huang [26] developed an effective and accurate method to solve FDEs with variable coefficients, obtaining approximate solutions using the Haar wavelet operational matrix.

Some literature derives explicit formulas for the solution of linear FDEs with constant and/or variable coefficients. The authors in [27] derived an explicit formula of solutions to Hilfer linear fractional integro–differential equations with a variable coefficient in a weighted space. Restrepo and Suragan [28] studied a kind of Hilfer–type FDEs with continuous variable coefficients and attained the solution represented by convergent infinite series. As is well known, it is difficult to obtain analytical solutions to nonlinear differential equations [29–32], due to their complexity. Therefore, researchers have turned to analytical techniques to explore the quantitative and stable properties of solutions to nonlinear FDEs with varying coefficients [33–36]. For example, the authors in [33] dealt with the quantitative property of positive solutions for FDEs with polynomial coefficients. Bai in [35] explored the following category of questions:

$$D^{\varsigma}v(t) = \mu r(t)g(v(t)), 0 < t < 1,$$

where $0 < \varsigma < 1, D^{\varsigma}$ represented the standard Riemann-Liouville fractional derivative, the parameter $\mu > 0$, and the mappings $g(t)$ and $r(t)$ satisfied the following conditions:

$$g : [0,\infty) \to [0,\infty), g(0) > 0; r : [0,1] \to (-\infty,+\infty).$$

By using the nonlinear alternative of Leray-Schauder type, the author obtained the existence condition of positive solutions to the above problem. Some results on the linear FDEs with variable coefficients can also be found in [37–39].

Recently, several papers in the literature have investigated the related problems of fractional differential equations in a kind of space known as β-normed space. To be specific, Du [40], in such a weighted function space, proved β-UHS and existence of solutions for a type of non-instantaneous impulsive FDEs. Yu [41] discussed the following FDEs with non-instantaneous impulse terms:

$$\begin{gathered} {}^{C}D^{\alpha}_{s_i,\tau}x(\tau) = -\lambda x(\tau) + f(\tau, x(\tau)), \tau \in [s_i, t_{i+1}], i = 0,1,2,\cdots,m, \alpha \in (0,1), \lambda \geq 0, \\ x(\tau) = q + I^{\gamma}_{\tau_i,\tau}g_i(\tau, x(\tau)) - I^{\alpha}_{0,s}f(s_i, x(s_i)), i = 1,2,\cdots,m, \gamma \neq \alpha, \end{gathered}$$

where $q \in R$, $f : [0,T] \times R \longrightarrow R$ is continuous, $g_i : [\tau_i, s_i] \times R \longrightarrow R$ is continuous, and $I^{\gamma}_{\tau_i,\tau}$ and $I^{\alpha}_{0,s}$ are fractional integral operators. The author obtained the expression of piecewise continuous solutions and results of existence and UHS.

Inspired by the above studies, we investigate the conditions in the β-normed Banach space for the existence, existence and uniqueness, and β-UHS of solutions to the boundary value (BV) problem of a category of nonlinear sequential FDEs with variable coefficients,

$$ {}^{C}D_{0^+}^{\omega}\lambda(t) + {}^{C}D_{0^+}^{\omega-1}[p'(t)\lambda(t)] = h(t, \lambda(t)), 0 < t < 1, \tag{1.1} $$

subject to the conditions of boundary as follows:

$$ \lambda(0) = \lambda'(0) = \lambda'(1) = 0, \tag{1.2} $$

where $2 < \omega \leq 3$, the mapping h is continuous in $[0, 1]$, and $p = p(t) \in C^3([0, 1], R)$ satisfying $1 - p'(1)\int_0^1 se^{p(s)-p(1)}ds \neq 0$. The above fractional system can be used to simulate the transmission of signals and images, and can also be used to explain the laws of the underground hydrological cycle.

The remainder of this work is structured as follows. Section 2 discusses notations and preliminary concepts. Section 3 discusses the existence of solutions, and Section 4 addresses β-Ulam-Hyers stability, for the BV problem of (1.1)–(1.2). Section 5 provides examples, and Section 6 is mainly devoted to summarizing our main results.

2. Preliminary concepts

We introduce some symbols and conclusions of fractional calculus theory which will be used in this paper.

Definition 2.1. [13] Given a set $[a_1, a_2](-\infty < a_1 < a_2 < +\infty) \subset R$, a common fractional integral of order $\gamma > 0$, called the Riemann–Liouville type, can be defined as follows:

$$ (I_{a_1^+}^{\gamma}g)(x) := \frac{1}{\Gamma(\gamma)}\int_{a_1}^{x}\frac{g(\tau)}{(x-\tau)^{1-\gamma}}d\tau(x > a_1; \gamma > 0) $$

and

$$ (I_{a_2^-}^{\gamma}g)(x) := \frac{1}{\Gamma(\gamma)}\int_{x}^{a_2}\frac{g(\tau)}{(\tau-x)^{1-\gamma}}d\tau(x < a_2; \gamma > 0), $$

where $\Gamma(\cdot)$ stands for the second type of Euler integral (also known as the gamma function), and both of the above integrals are considered to exist.

Definition 2.2. [13] If $g = g(x) \in AC^n[a_1, a_2]$, a common fractional derivative of order $\gamma > 0$, called Caputo's derivative, can be computed almost on the interval $[a_1, a_2]$.
(a) When $\gamma \notin N_0$, N_0 stands for the set of positive integers, then

$$ ({}^{c}D_{a_1^+}^{\gamma}g)(x) = \frac{1}{\Gamma(n-\gamma)}\int_{a_1}^{x}\frac{g^{(n)}(\tau)}{(x-\tau)^{\gamma-n+1}}d\tau $$

and

$$ ({}^{c}D_{a_2^-}^{\gamma}g)(x) = \frac{(-1)^n}{\Gamma(n-\gamma)}\int_{x}^{a_2}\frac{g^{(n)}(\tau)}{(\tau-x)^{\gamma-n+1}}d\tau, $$

where $n = [\gamma] + 1$.
(b) If $\gamma \in N_0$, then

$$ ({}^{c}D_{a_1^+}^{n}g)(x) = g^{(n)}(x) $$

and

$$({}^cD^n_{a_2^-}g)(x) = (-1)^{(n)}g^{(n)}(x).$$

Lemma 2.1. [14] Suppose $d = d(t) \in C([a_1, a_2], R)$, and $e = e(t) \in C^k([a_1, a_2], R)$, $l_i \in R$, are some constants, $i = 0, 1, \cdots, k-1, k \in N_0$. Then,

$${}^cD^\gamma_{a_1^+}(I^\gamma_{a_1^+}d(t)) = d(t),$$

and

$$I^\gamma_{a_1^+}({}^cD^\gamma_{a_1^+}e(t)) = e(t) + \sum_{i=0}^{k-1} l_i(t-a_1)^i.$$

Lemma 2.2. [14] The general solution of the equation $({}^cD^\xi_{\varsigma^+}e)(t) = 0, n-1 < \xi < n$, can be given by

$$e(t) = \sum_{j=0}^{n-1} \frac{e^{(j)}(\varsigma)}{j!}(t-\varsigma)^j.$$

In particular, if $\varsigma = 0$, the above conclusion can be rephrased as

$$e(t) = d_0 + d_1t + d_2t^2 + \cdots + d_{n-1}t^{n-1},$$

where $e_j = \frac{e^{(j)}(0)}{j!}(j = 0, 1, \cdots n-1)$ are constants.

Lemma 2.3. Suppose that $u(t) \in C[0, 1]$ represents one mapping. Then equation

$${}^CD^\omega_{0^+}\lambda(t) +^C D^{\omega-1}_{0^+}[p'(t)\lambda(t)] = u(t), 0 < t < 1, \tag{2.1}$$

with conditions (1.2) has the unique solution

$$\begin{aligned}\lambda(t) =& \frac{1}{\Gamma(\omega-1)}\int_0^t e^{p(s)-p(t)}\int_0^s (s-\tau)^{\omega-2}u(\tau)d\tau ds \\ &+ \frac{q(t)}{\theta\Gamma(\omega-1)}[p'(1)\int_0^1 e^{p(s)-p(1)}\int_0^s (s-\tau)^{\omega-2}u(\tau)d\tau ds \\ &- \int_0^1 (1-\tau)^{\omega-2}u(\tau)d\tau],\end{aligned} \tag{2.2}$$

where $2 < \omega \leq 3, t \in [0, 1]$, and $q(t) = \int_0^t se^{p(s)-p(t)}ds, \theta = 1 - p'(1)\int_0^1 se^{p(s)-p(1)}ds \neq 0$.

Proof. Taking the integral of order $\omega - 1$ on both sides of Eq (2.1), and we can use the conclusions of Lemmas 2.1 and 2.2 to get the following formula:

$$\lambda'(t) + p'(t)\lambda(t) = \frac{1}{\Gamma(\omega-1)}\int_0^t (t-\tau)^{\omega-2}u(\tau)d\tau + c_0 + c_1t. \tag{2.3}$$

By multiplying $e^{p(t)}$ on both sides of Eq (2.3), one gets

$$[e^{p(t)}\lambda(t)]' = \frac{1}{\Gamma(\omega-1)}e^{p(t)}\int_0^t (t-\tau)^{\omega-2}u(\tau)d\tau + c_0e^{p(t)} + c_1te^{p(t)}. \tag{2.4}$$

Integrating Eq (2.4) over the interval [0, 1] gives

$$e^{p(t)}\lambda(t) - e^{p(0)}\lambda(0) = \frac{1}{\Gamma(\omega-1)}\int_0^t e^{p(s)}\int_0^s (s-\tau)^{\omega-2}u(\tau)d\tau ds + c_0\int_0^t e^{p(s)}ds + c_1\int_0^t se^{p(s)}ds. \quad (2.5)$$

Noticing $\lambda(0) = 0$ in (1.2), one can obtain from (2.5) that

$$\lambda(t) = \frac{1}{\Gamma(\omega-1)}\int_0^t e^{p(s)-p(t)}\int_0^s (s-\tau)^{\omega-2}u(\tau)d\tau ds + c_0\int_0^t e^{p(s)-p(t)}ds + c_1\int_0^t se^{p(s)-p(t)}ds. \quad (2.6)$$

Differentiating Eq (2.6) gives

$$\begin{aligned}\lambda'(t) = &-\frac{p'(t)}{\Gamma(\omega-1)}\int_0^t e^{p(s)-p(t)}\int_0^s (s-\tau)^{\omega-2}u(\tau)d\tau ds + \frac{1}{\Gamma(\omega-1)}\int_0^t (t-\tau)^{\omega-2}u(\tau)d\tau\\ &- c_0p'(t)\int_0^t e^{p(s)-p(t)}ds + c_0 - c_1p'(t)\int_0^t se^{p(s)-p(t)}ds + c_1t.\end{aligned}$$

Utilizing $\lambda'(0) = \lambda'(1) = 0$ in (1.2), we have from the above equation that $c_0 = 0$, and

$$c_1 = \frac{p'(1)}{\theta\Gamma(\omega-1)}\int_0^1 e^{p(s)-p(1)}\int_0^s (s-\tau)^{\omega-2}u(\tau)d\tau ds - \frac{1}{\theta\Gamma(\omega-1)}\int_0^1 (1-\tau)^{\omega-2}u(\tau)d\tau.$$

Substituting the expressions of c_0 and c_1 in Eq (2.6) gives

$$\begin{aligned}\lambda(t) = &\frac{1}{\Gamma(\omega-1)}\int_0^t e^{p(s)-p(t)}\int_0^s (s-\tau)^{\omega-2}u(\tau)d\tau ds\\ &+ \frac{\int_0^t se^{p(s)-p(t)}ds}{\theta\Gamma(\omega-1)}[p'(1)\int_0^1 e^{p(s)-p(1)}\int_0^s (s-\tau)^{\omega-2}u(\tau)d\tau ds\\ &- \int_0^1 (1-\tau)^{\omega-2}u(\tau)d\tau].\end{aligned} \quad (2.7)$$

Letting $q(t) = \int_0^t se^{p(s)-p(t)}ds$, Lemma 2.3 is proven. □

The conclusions in this work are mainly based on Banach's fixed-point principle and Schaefer's fixed-point principle (FPP).

Lemma 2.4. [42] (Schaefer's FPP) Suppose X is a Banach space. Let $\Upsilon : X \to X$ denote a completely continuous operator, while the set $\Omega(\Upsilon) = \{u \in X \mid u = \mu\Upsilon u, 0 < \mu < 1\}$ is finite. Then Υ attains at least one fixed point in X.

At the end of this section, we introduce some conclusions about Hölder's inequality [43] and a β-normed space [40,41].

Suppose $\Omega \subset R^n$ is an open (or measurable) set, and $\zeta(t)$ is a mapping with measurable property on Ω. $|\zeta(t)|^s$ is measurable on Ω for $1 \le s < \infty$, and $\int_\Omega |\zeta(t)|^s dt$ exists. A function space $L^s(\Omega)$ is introduced as follows:

$$L^s(\Omega) = \{\zeta(t)|\zeta(t) \text{ is measurable on } \Omega, \int_\Omega |\zeta(t)|^s dt < \infty\}.$$

For $\zeta \in L^s(\Omega)$, the subsequent norm is

$$\|\zeta\|_s = (\int_\Omega |\zeta(t)|^s dt)^{1/s}.$$

$1 < s_1$ and $s_2 < \infty$ are called conjugate exponentials of each other if $\frac{1}{s_1} + \frac{1}{s_2} = 1$.

Lemma 2.5. [43] (Hölder's inequality) Let $\Omega \subset R^n$ denote an open set, s_1 and s_2 represent conjugate exponentials, and $\zeta_1(t) \in L^{s_1}(\Omega), \zeta_2(t) \in L^{s_2}(\Omega)$, where $\zeta_1(t)\zeta_2(t)$ is integrable on Ω. Then it holds that

$$\int_\Omega | \zeta_1(t)\zeta_2(t) | dx \le \|\zeta_1\|_{s_1} \|\zeta_2\|_{s_2}.$$

Subsequently, we let $\|\cdot\|_s$ denote $(\int_0^1 |\cdot|^s dt)^{1/s}$ for any $s > 1$.

Definition 2.3. [44] Assume M is a vector space in some field N, and a number $\beta \in (0, 1]$. If a mapping $\|\cdot\|_\beta$: $M \longrightarrow [0, +\infty)$ satisfies the following conditions for any $u, v \in M$:
(a) $\|v\|_\beta = 0 \Longleftrightarrow v = 0$;
(b) $\|lv\|_\beta = | l |^\beta \| v \|_\beta, l \in N$;
(c) $\|u + v\|_\beta \le \| u \|_\beta + \| v \|_\beta$,
then $\|\cdot\|_\beta$ is named a β-norm. The space M on which the norm $\|\cdot\|_\beta$ endowed is called a β-normed space, and is represented by $(M, \|\cdot\|_\beta)$. Accordingly, a completed β-normed space is named a β-Banach space.

Subsequently, we let $\|\cdot\|_s$ denote $(\int_0^1 |\cdot|^s dt)^{1/s}$, and $\|\cdot\|_s^\beta$ denote $(\int_0^1 |\cdot|^s dt)^{\beta/s}$, for any $s > 1$ and $0 < \beta \le 1$.

3. Existence result for problem (1.1)–(1.2)

In the sequel of this paper, we denote the interval $[0, 1]$ by I.

We assume that $C(I, R)$ is the space of all continuous mappings from $I \to R$, and its norm can be described as $\|\lambda\|_\beta = max_{t\in I}|\lambda(t)|^\beta$ for $\lambda = \lambda(t) \in C(I, R)$. Therefore, $\Theta = (C(I, R), \|\cdot\|_\beta)$ is a β-normed Banach space.

Remark 3.1. For $\lambda = \lambda(t) \in C(I, R)$, let $\|\lambda\| = max_{t\in I} | \lambda(t) |$, then we have the following relation holds:

$$\|\lambda\|_\beta = max_{t\in I} | \lambda(t) |^\beta = \|\lambda\|^\beta.$$

Given Lemma 2.3, we can equivalently convert problem (1.1)–(1.2) into the following integral equation,

$$\lambda = \Upsilon\lambda,$$

where $\Upsilon : \Theta \to \Theta$ is defined as

$$\begin{aligned}(\Upsilon\lambda)(t) = &\frac{1}{\Gamma(\omega - 1)} \int_0^t e^{p(s)-p(t)} \int_0^s (s - \tau)^{\omega-2} h(\tau, \lambda(\tau)) d\tau ds \\ &+ \frac{q(t)}{\theta\Gamma(\omega - 1)} [p'(1) \int_0^1 e^{p(s)-p(1)} \int_0^s (s - \tau)^{\omega-2} h(\tau, \lambda(\tau)) d\tau ds - \int_0^1 (1 - \tau)^{\omega-2} h(\tau, \lambda(\tau)) d\tau].\end{aligned}$$

Therefore, problem (1.1)–(1.2) has solutions if and only if the operator Υ has fixed points.

In order to prove the results of existence, we first prove several related lemmas.

Lemma 3.1. For the function $p = p(t) \in C(I, R)$, there is a positive constant, $m_p = max_{s,t\in I} e^{p(s)-p(t)}$. So,

$$\int_0^t e^{p(s)-p(t)}dt \le m_p; q(t) = \int_0^t se^{p(s)-p(t)}dt \le \frac{1}{2}m_p, s, t \in I.$$

Lemma 3.2. For the function $p = p(t) \in C^1(I, R)$, there are two positive constants m_p and m'_p such that $e^{p(s)-p(t)} \le m_p$, $m'_p = max \mid p'(t) \mid$, $s, t \in I$. Thus,

$$|q'(t)| \le N_1, s, t \in I,$$

where $N_1 = 1 + \frac{1}{2}m_p m'_p$.

Proof. For $q(t) = \int_0^t se^{p(s)-p(t)}ds$, we can differentiate it to get $q'(t) = t - p'(t)\int_0^t se^{p(s)-p(t)}ds = t - p'(t)q(t)$. Then

$$\mid q'(t) \mid \le 1 + \mid p'(t) \mid q(t) \le 1 + m'_p \cdot \frac{1}{2}m_p =: N_1.$$

This finishes the proof. □

Lemma 3.3. For the functions $p = p(t) \in C(I, R)$ and $r = r(t) \in L^s(I, R^+)(s > 1)$, the following inequalities hold for $s, t \in I$:

(i) $\int_0^t e^{p(s)-p(t)} \int_0^s (s-\tau)^{\omega-2} r(\tau)d\tau ds \le \frac{m_p \|r\|_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}}$;

(ii) $\int_0^t e^{p(s)-p(t)} \int_0^s (s-\tau)^{\omega-2} d\tau ds \le \frac{m_p}{\omega-1}$;

(iii) $\int_0^t (t-\tau)^{\omega-2} r(\tau)d\tau \le \frac{\|r\|_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}}$,

where $1 < s_1, s_2 < +\infty$ satisfy $\frac{1}{s_1} + \frac{1}{s_2} = 1$, and $2 < \omega \le 3$.

Proof. (i) The following result is obtained by using the conclusions of the Hölder's inequality and Lemma 3.1:

$$\begin{aligned}
&\int_0^t e^{p(s)-p(t)} \int_0^s (s-\tau)^{\omega-2} r(\tau)d\tau ds \\
\le &\int_0^t e^{p(s)-p(t)} [\int_0^s (s-\tau)^{s_1(\omega-2)} d\tau]^{\frac{1}{s_1}} [\int_0^s r^{s_2}(\tau)d\tau]^{\frac{1}{s_2}} ds \\
\le &\frac{\| r \|_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}} \int_0^t e^{p(s)-p(t)} s^{\frac{1+s_1(\omega-2)}{s_1}} ds \le \frac{\| r \|_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}} \int_0^t e^{p(s)-p(t)} ds \le \frac{m_p \|r\|_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}}.
\end{aligned}$$

(ii) By the results of Lemma 3.1, we have

$$\int_0^t e^{p(s)-p(t)} \int_0^s (s-\tau)^{\omega-2} d\tau ds = \frac{1}{\omega-1} \int_0^t e^{p(s)-p(t)} s^{\omega-1} ds \le \frac{m_p}{\omega-1}.$$

The result of (iii) can be proved similarly to (i) and (ii); the proof is omitted here. □

For the function $p(t)$ in (1.1), let $M_p = max_{t\in I} \mid p(t) \mid$. For computational convenience, we use the following notations in the subsequent sections:

$$\Lambda_1 = \frac{2 \mid \theta \mid + m_p \mid p'(1) \mid +1}{2 \mid \theta \mid \Gamma(\omega-1) \sqrt[s_1]{1+s_1(\omega-2)}} m_p, \Lambda_2 = \frac{(M_p m_p + 1) \mid \theta \mid + N_1 m_p \mid p'(1) \mid + N_1}{\mid \theta \mid \Gamma(\omega-1) \sqrt[s_1]{1+s_1(\omega-2)}},$$

$$\Lambda_3 = \frac{2 \mid \theta \mid + m_p \mid p'(1) \mid + 1}{2 \mid \theta \mid \Gamma(\omega)} m_p,$$

where $s_1 > 1$ is some positive number.

We make the following assumptions.

(S1) $h(t, \lambda(t)) : I \times R \to R$ is a jointly continuous mapping, and a mapping $r(t) \in L^s(I, R^+)(s > 1)$ exists such that

$$|h(t, \lambda_1(t)) - h(t, \lambda_2(t))| \leq r(t) \mid \lambda_1 - \lambda_2 \mid,$$

for any $\lambda_1, \lambda_2 \in \Theta$ and $t \in I$;

(S2) For $r(t)$ in (S1), and conjugate exponentials $s_1 > 1, s_2 > 1$, the following condition is met:

$$1 - \Lambda_1^{\beta} \|r\|_{s_2}^{\beta} > 0;$$

(S3) $h(t, \lambda(t)) : I \times R \to R$ is a jointly continuous mapping, and there exist mappings $\mu_1 = \mu_1(t), \mu_2 = \mu_2(t) \in L^s(I, R^+)(s > 1)$ such that

$$|h(t, \lambda(t))| \leq \mu_1(t) + \mu_2(t) \mid \lambda \mid;$$

(S4) For $\mu_2(t)$ in (S3), and conjugate exponentials $s_1 > 1, s_2 > 1$, the following condition is met:

$$1 - \Lambda_1^{\beta} \parallel \mu_2 \parallel_{s_2}^{\beta} > 0.$$

In the first step, we show the uniqueness of the solution for problem (1.1)–(1.2).

Theorem 3.1. For the assumptions (S1) and (S2), the problem (1.1)–(1.2) has a unique solution on Θ.

Proof. Suppose that $C_0 = max_{t \in I} |h(t, 0)|$, $B_\rho = \{\lambda \in \Theta : \parallel \lambda \parallel_\beta \leq \rho\}$, and ρ satisfies

$$\rho \geq \frac{(\Lambda_3 C_0)^{\beta}}{1 - \Lambda_1^{\beta} \parallel r \parallel_{s_2}^{\beta}}. \tag{3.1}$$

We will show that $\Upsilon B_\rho \subseteq B_\rho$. For $x \in B_\rho$, we have

$$\begin{aligned}
|(\Upsilon\lambda)(t)| \leq & \frac{1}{\Gamma(\omega - 1)} \int_0^t e^{p(s)-p(t)} \int_0^s (s - \tau)^{\omega-2} \mid h(\tau, \lambda(\tau)) \mid d\tau ds \\
& + \frac{q(t)}{\mid \theta \mid \Gamma(\omega - 1)} [\mid p'(1) \mid \int_0^1 e^{p(s)-p(1)} \int_0^s (s - \tau)^{\omega-2} \mid h(\tau, \lambda(\tau)) \mid d\tau ds \\
& + \int_0^1 (1 - \tau)^{\omega-2} \mid h(\tau, \lambda(\tau)) \mid d\tau] \\
\leq & \frac{1}{\Gamma(\omega - 1)} \int_0^t e^{p(s)-p(t)} \int_0^s (s - \tau)^{\omega-2} [\mid h(\tau, \lambda(\tau)) - h(\tau, 0) \mid + \mid h(\tau, 0) \mid] d\tau ds \\
& + \frac{q(t)}{\mid \theta \mid \Gamma(\omega - 1)} [\mid p'(1) \mid \int_0^1 e^{p(s)-p(1)} \int_0^s (s - \tau)^{\omega-2} [\mid h(\tau, \lambda(\tau)) - h(\tau, 0) \mid + \mid h(\tau, 0) \mid] d\tau ds \\
& + \int_0^1 (1 - \tau)^{\omega-2} [\mid h(\tau, \lambda(\tau)) - h(\tau, 0) \mid + \mid h(\tau, 0) \mid] d\tau]
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\omega-1)}\int_0^t e^{p(s)-p(t)}\int_0^s (s-\tau)^{\omega-2}[r(\tau)\mid\lambda(\tau)\mid+C_0]d\tau ds$$
$$+\frac{q(t)}{\mid\theta\mid\Gamma(\omega-1)}\{\mid p'(1)\mid\int_0^1 e^{p(s)-p(1)}\int_0^s (s-\tau)^{\omega-2}[r(\tau)\mid\lambda(\tau)\mid+C_0]d\tau ds$$
$$+\int_0^1 (1-\tau)^{\omega-2}[r(\tau)\mid\lambda(\tau)\mid+C_0]d\tau\}.$$

Noticing the definition of B_ρ, and by the results of Lemmas 3.1 and 3.3, we have

$$\mid(\Upsilon\lambda)(t)\mid\leq \frac{\sqrt[\beta]{\rho}}{\Gamma(\omega-1)}\frac{m_p\parallel r\parallel_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}}+\frac{C_0m_p}{\Gamma(\omega)}$$
$$+\frac{m_p}{2\mid\theta\mid\Gamma(\omega-1)}\{\mid p'(1)\mid\frac{m_p\parallel r\parallel_{s_2}\sqrt[\beta]{\rho}}{\sqrt[s_1]{1+s_1(\omega-2)}}+\mid p'(1)\mid\frac{C_0m_p}{\omega-1}+\frac{\parallel r\parallel_{s_2}\sqrt[\beta]{\rho}}{\sqrt[s_1]{1+s_1(\omega-2)}}+\frac{C_0}{\omega-1}\}$$
$$\leq\Lambda_1\parallel r\parallel_{s_2}\sqrt[\beta]{\rho}+\Lambda_3C_0,$$

which implies that

$$\mid(\Upsilon\lambda)(t)\mid^\beta\leq\Lambda_1^\beta\parallel r\parallel_{s_2}^\beta\rho+(\Lambda_3C_0)^\beta.$$

So, the condition (3.1) ensures that $\Upsilon B_\rho\subseteq B_\rho$.

Next, we proceed to prove the contraction of the operator Υ on B_ρ. For any $\lambda_1,\lambda_2\in B_\rho$ and $t\in I$, we have the following inequality based on condition (S1):

$$\mid(\Upsilon\lambda_1)(t)-(\Upsilon\lambda_2)(t)\mid\leq\frac{1}{\Gamma(\omega-1)}\int_0^t e^{p(s)-p(t)}\int_0^s (s-\tau)^{\omega-2}\mid h(\tau,\lambda_1(\tau))-h(\tau,\lambda_2(\tau))\mid d\tau ds$$
$$+\frac{q(t)}{\mid\theta\mid\Gamma(\omega-1)}[\mid p'(1)\mid\int_0^1 e^{p(s)-p(1)}\int_0^s (s-\tau)^{\omega-2}\mid h(\tau,\lambda_1(\tau))-h(\tau,\lambda_2(\tau))\mid d\tau ds$$
$$+\int_0^1 (1-\tau)^{\omega-2}\mid h(\tau,\lambda_1(\tau))-h(\tau,\lambda_2(\tau))\mid d\tau]$$
$$\leq\frac{1}{\Gamma(\omega-1)}\int_0^t e^{p(s)-p(t)}\int_0^s (s-\tau)^{\omega-2}r(\tau)\mid\lambda_1(\tau)-\lambda_2(\tau)\mid d\tau ds$$
$$+\frac{q(t)}{\mid\theta\mid\Gamma(\omega-1)}[\mid p'(1)\mid\int_0^1 e^{p(s)-p(1)}\int_0^s (s-\tau)^{\omega-2}r(\tau)\mid\lambda_1(\tau)-\lambda_2(\tau)\mid d\tau ds$$
$$+\int_0^1 (1-\tau)^{\omega-2}r(\tau)\mid\lambda_1(\tau)-\lambda_2(\tau)\mid d\tau].$$

By the results of Lemmas 3.1 and 3.3 again, we have

$$\mid(\Upsilon\lambda_1)(t)-(\Upsilon\lambda_2)(t)\mid\leq\frac{1}{\Gamma(\omega-1)}\int_0^t e^{p(s)-p(t)}\int_0^s (s-\tau)^{\omega-2}r(\tau)d\tau ds\cdot\parallel\lambda_1(\tau)-\lambda_2(\tau)\parallel$$
$$+\frac{q(t)}{\mid\theta\mid\Gamma(\omega-1)}[\mid p'(1)\mid\int_0^1 e^{p(s)-p(1)}\int_0^s (s-\tau)^{\omega-2}r(\tau)d\tau ds$$
$$+\int_0^1 (1-\tau)^{\omega-2}r(\tau)d\tau]\cdot\parallel\lambda_1(\tau)-\lambda_2(\tau)\parallel$$
$$\leq\Lambda_1\parallel r\parallel_{s_2}\cdot\parallel\lambda_1-\lambda_2\parallel,$$

which implies that

$$\mid(\Upsilon\lambda_1)(t)-(\Upsilon\lambda_2)(t)\mid^\beta\leq\Lambda_1^\beta\parallel r\parallel_{s_2}^\beta\cdot\parallel\lambda_1-\lambda_2\parallel^\beta.$$

From Remark 3.1, the above equation can be reduced to

$$\|(\Upsilon\lambda_1)(t) - (\Upsilon\lambda_2)(t)\|_\beta \le \Lambda_1^\beta \parallel r \parallel_{s_2}^\beta \cdot \parallel \lambda_1 - \lambda_2 \parallel_\beta .$$

So, the condition (S2) implies that Υ is a contraction mapping. Hence, Banach's FPP ensures that Υ attains a unique fixed point on $B_\rho \subset \Theta$. This shows equivalently the problem of (1.1)–(1.2) has a unique solution on Θ. This proof is complete. □

We continue to investigate the existence result for the problem (1.1)–(1.2).

Theorem 3.2. For the assumptions (S3) and (S4), the problem (1.1)–(1.2) has at least one solution on I, provided that

$$\frac{\Lambda_1^\beta \parallel \mu_1 \parallel_{s_2}^\beta}{1 - \Lambda_1^\beta \parallel \mu_2 \parallel_{s_2}^\beta} < \infty. \tag{3.2}$$

Proof. We first show the complete continuity of the operator Υ. The continuity of the function h implies that Υ is continuous on Θ. Take $\rho > 0$ and a ball B_ρ as used in the proof of the Theorem 3.1. Then, for $\lambda \in B_\rho, t \in I$, noticing condition (S3), the following inequality holds:

$$\begin{aligned}
|(\Upsilon\lambda)(t)| \le & \frac{1}{\Gamma(\omega-1)} \int_0^t e^{p(s)-p(t)} \int_0^s (s-\tau)^{\omega-2}[\mu_1(\tau) + \mu_2(\tau) \mid \lambda(\tau) \mid] d\tau ds \\
& + \frac{m_p}{2 \mid \theta \mid \Gamma(\omega-1)} \{\mid p'(1) \mid \int_0^1 e^{p(s)-p(1)} \int_0^s (s-\tau)^{\omega-2}[\mu_1(\tau) + \mu_2(\tau) \mid \lambda(\tau) \mid] d\tau ds \\
& + \int_0^1 (1-\tau)^{\omega-2}[\mu_1(\tau) + \mu_2(\tau) \mid \lambda(\tau) \mid] d\tau\}.
\end{aligned}$$

By the results in Lemmas 3.1 and 3.3, we have

$$\begin{aligned}
|(\Upsilon\lambda)(t)| \le & \frac{1}{\Gamma(\omega-1)} \frac{m_p \parallel \mu_1 \parallel_{s_2}}{\sqrt[s_1]{1 + s_1(\omega-2)}} + \frac{1}{\Gamma(\omega-1)} \frac{m_p \parallel \mu_2 \parallel_{s_2}}{\sqrt[s_1]{1 + s_1(\omega-2)}} \sqrt[\beta]{\rho} \\
& + \frac{m_p}{2 \mid \theta \mid \Gamma(\omega-1)} \{\mid p'(1) \mid [\frac{m_p \parallel \mu_1 \parallel_{s_2}}{\sqrt[s_1]{1 + s_1(\omega-2)}} + \frac{m_p \parallel \mu_2 \parallel_{s_2}}{\sqrt[s_1]{1 + s_1(\omega-2)}} \sqrt[\beta]{\rho}] \\
& + \frac{\parallel \mu_1 \parallel_{s_2}}{\sqrt[s_1]{1 + s_1(\omega-2)}} + \frac{\parallel \mu_2 \parallel_{s_2}}{\sqrt[s_1]{1 + s_1(\omega-2)}} \sqrt[\beta]{\rho}\} \\
\le & \frac{m_p}{\Gamma(\omega-1) \sqrt[s_1]{1 + s_1(\omega-2)}} (\parallel \mu_1 \parallel_{s_2} + \parallel \mu_2 \parallel_{s_2} \sqrt[\beta]{\rho}) \\
& + \frac{m_p^2 \mid p'(1) \mid}{2 \mid \theta \mid \Gamma(\omega-1) \sqrt[s_1]{1 + s_1(\omega-2)}} (\parallel \mu_1 \parallel_{s_2} + \parallel \mu_2 \parallel_{s_2} \sqrt[\beta]{\rho}) \\
& + \frac{m_p}{2 \mid \theta \mid \Gamma(\omega-1) \sqrt[s_1]{1 + s_1(\omega-2)}} (\parallel \mu_1 \parallel_{s_2} + \parallel \mu_2 \parallel_{s_2} \sqrt[\beta]{\rho}) \\
= & \Lambda_1(\parallel \mu_1 \parallel_{s_2} + \parallel \mu_2 \parallel_{s_2} \sqrt[\beta]{\rho}),
\end{aligned}$$

which implies

$$\|(\Upsilon\lambda)(t)\|_\beta \le \Lambda_1^\beta(\parallel \mu_1 \parallel_{s_2}^\beta + \parallel \mu_2 \parallel_{s_2}^\beta \rho) =: N_2 < \infty,$$

and it follows that the operator Υ is bounded uniformly on Θ.

In the sequel, we continue to show Υ is equicontinuous on Θ. For $t \in I$ and $\lambda \in B_\rho$, we have

$$\begin{aligned}(\Upsilon\lambda)'(t) = &-\frac{p'(t)}{\Gamma(\omega-1)}\int_0^t e^{p(s)-p(t)}\int_0^s (s-\tau)^{\omega-2}h(\tau,\lambda(\tau))d\tau ds + \frac{1}{\Gamma(\omega-1)}\int_0^t (t-\tau)^{\omega-2}h(\tau,\lambda(\tau))d\tau \\ &+ \frac{q'(t)}{\theta\Gamma(\omega-1)}[p'(1)\int_0^1 e^{p(s)-p(1)}\int_0^s (s-\tau)^{\omega-2}h(\tau,\lambda(\tau))d\tau ds - \int_0^1 (1-\tau)^{\omega-2}h(\tau,\lambda(\tau))d\tau].\end{aligned}$$

By (S3) and Lemma 3.2, the following inequality holds:

$$\begin{aligned}|(\Upsilon\lambda)'(t)| \le &\frac{M_p}{\Gamma(\omega-1)}\int_0^t e^{p(s)-p(t)}\int_0^s (s-\tau)^{\omega-2}[\mu_1(\tau)+\mu_2(\tau)\mid\lambda(\tau)\mid]d\tau ds \\ &+ \frac{1}{\Gamma(\omega-1)}\int_0^t (t-\tau)^{\omega-2}[\mu_1(\tau)+\mu_2(\tau)\mid\lambda(\tau)\mid]d\tau \\ &+ \frac{N_1}{\mid\theta\mid\Gamma(\omega-1)}\{\mid p'(1)\mid\int_0^1 e^{p(s)-p(1)}\int_0^s (s-\tau)^{\omega-2}[\mu_1(\tau)+\mu_2(\tau)\mid\lambda(\tau)\mid]d\tau ds \\ &+ \int_0^1 (1-\tau)^{\omega-2}[\mu_1(\tau)+\mu_2(\tau)\mid\lambda(\tau)\mid]d\tau\}.\end{aligned}$$

By Lemma 3.3, we have

$$\begin{aligned}|(\Upsilon\lambda)'(t)| \le &\frac{M_p}{\Gamma(\omega-1)}[\frac{m_p\parallel\mu_1\parallel_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}} + \frac{m_p\parallel\mu_2\parallel_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}}\sqrt[\beta]{\rho}] \\ &+ \frac{1}{\Gamma(\omega-1)}\frac{\parallel\mu_1\parallel_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}} + \frac{1}{\Gamma(\omega-1)}\frac{\parallel\mu_2\parallel_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}}\sqrt[\beta]{\rho} \\ &+ \frac{N_1}{\mid\theta\mid\Gamma(\omega-1)}\{\mid p'(1)\mid[\frac{m_p\parallel\mu_1\parallel_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}} + \frac{m_p\parallel\mu_2\parallel_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}}\sqrt[\beta]{\rho}] \\ &+ \frac{\parallel\mu_1\parallel_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}} + \frac{\parallel\mu_2\parallel_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}}\sqrt[\beta]{\rho}\} \\ \le &\frac{M_p m_p}{\Gamma(\omega-1)\sqrt[s_1]{1+s_1(\omega-2)}}(\parallel\mu_1\parallel_{s_2} + \parallel\mu_2\parallel_{s_2}\sqrt[\beta]{\rho}) \\ &+ \frac{1}{\Gamma(\omega-1)\sqrt[s_1]{1+s_1(\omega-2)}}(\parallel\mu_1\parallel_{s_2} + \parallel\mu_2\parallel_{s_2}\sqrt[\beta]{\rho}) \\ &+ \frac{N_1 m_p\mid p'(1)\mid}{\mid\theta\mid\Gamma(\omega-1)\sqrt[s_1]{1+s_1(\omega-2)}}[\parallel\mu_1\parallel_{s_2} + \parallel\mu_2\parallel_{s_2}\sqrt[\beta]{\rho}] \\ &+ \frac{N_1}{\mid\theta\mid\Gamma(\omega-1)\sqrt[s_1]{1+s_1(\omega-2)}}(\parallel\mu_1\parallel_{s_2} + \parallel\mu_2\parallel_{s_2}\sqrt[\beta]{\rho}) \\ = &\Lambda_2(\parallel\mu_1\parallel_{s_2} + \parallel\mu_2\parallel_{s_2}\sqrt[\beta]{\rho}) =: N_3 < \infty.\end{aligned}$$

So, for any $t_1, t_2 \in I, t_1 < t_2$, and $\lambda \in B_\rho \subset \Theta$, we have

$$|(\Upsilon\lambda)(t_2) - (\Upsilon\lambda)(t_1)| = |\int_{t_1}^{t_2}(\Upsilon\lambda)'(t)dt| \le \int_{t_1}^{t_2}|(\Upsilon\lambda)'(t)|dt \le N_3(t_2-t_1). \tag{3.3}$$

It is easy to obtain $(\Upsilon\lambda)(t_2) - (\Upsilon\lambda)(t_1) \longrightarrow 0$ independently of λ from Eq (3.3), as $t_1 \longrightarrow t_2$. It follows that the operator $\Upsilon : \Theta \longrightarrow \Theta$ is completely continuous by the Arzelá-Ascoli theorem.

We next show that the set $S(\Upsilon) = \{\lambda \in \Theta \mid \lambda = \mu\Upsilon\lambda, 0 < \mu < 1\}$ is bounded. Let $\lambda \in S(\Upsilon)$. Then $\lambda = \mu\Upsilon\lambda$, i.e.,

$$\begin{aligned}\lambda =& \mu\{\frac{1}{\Gamma(\omega-1)}\int_0^t e^{p(s)-p(t)}\int_0^s (s-\tau)^{\omega-2}h(\tau,\lambda(\tau))d\tau ds\\ &+\frac{q(t)}{\theta\Gamma(\omega-1)}[p'(1)\int_0^1 e^{p(s)-p(1)}\int_0^s (s-\tau)^{\omega-2}h(\tau,\lambda(\tau))d\tau ds\\ &-\int_0^1 (1-\tau)^{\omega-2}h(\tau,\lambda(\tau))d\tau]\}.\end{aligned}$$

Then, by (S3), we have

$$\begin{aligned}|\lambda| \leq& \frac{1}{\Gamma(\omega-1)}\int_0^t e^{p(s)-p(t)}\int_0^s (s-\tau)^{\omega-2}[\mu_1(\tau)+\mu_2(\tau)\mid\lambda(\tau)\mid]d\tau ds\\ &+\frac{m_p}{2\mid\theta\mid\Gamma(\omega-1)}\{\mid p'(1)\mid\int_0^1 e^{p(s)-p(1)}\int_0^s (s-\tau)^{\omega-2}[\mu_1(\tau)+\mu_2(\tau)\mid\lambda(\tau)\mid]d\tau ds\\ &+\int_0^1 (1-\tau)^{\omega-2}[\mu_1(\tau)+\mu_2(\tau)\mid\lambda(\tau)\mid]d\tau\},\end{aligned}$$

and by Lemmas 3.1 and 3.3 and Remark 3.1, the following inequality holds:

$$\begin{aligned}|\lambda| \leq& \frac{1}{\Gamma(\omega-1)}\frac{m_p\|\mu_1\|_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}}+\frac{1}{\Gamma(\omega-1)}\frac{m_p\|\mu_2\|_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}}\|\lambda\|\\ &+\frac{m_p}{2\mid\theta\mid\Gamma(\omega-1)}\{\mid p'(1)\mid[\frac{m_p\|\mu_1\|_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}}+\frac{m_p\|\mu_2\|_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}}\|\lambda\|]\\ &+\frac{\|\mu_1\|_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}}+\frac{\|\mu_2\|_{s_2}}{\sqrt[s_1]{1+s_1(\omega-2)}}\|\lambda\|\}\\ \leq& \frac{m_p}{\Gamma(\omega-1)\sqrt[s_1]{1+s_1(\omega-2)}}(\|\mu_1\|_{s_2}+\|\mu_2\|_{s_2}\|\lambda\|)\\ &+\frac{m_p^2\mid p'(1)\mid}{2\mid\theta\mid\Gamma(\omega-1)\sqrt[s_1]{1+s_1(\omega-2)}}(\|\mu_1\|_{s_2}+\|\mu_2\|_{s_2}\|\lambda\|)\\ &+\frac{m_p}{2\mid\theta\mid\Gamma(\omega-1)\sqrt[s_1]{1+s_1(\omega-2)}}(\|\mu_1\|_{s_2}+\|\mu_2\|_{s_2}\|\lambda\|)\\ =& \Lambda_1(\|\mu_1\|_{s_2}+\|\mu_2\|_{s_2}\|\lambda\|),\end{aligned}$$

which implies

$$\|\lambda\|_\beta \leq \Lambda_1^\beta(\|\mu_1\|_{s_2}^\beta+\|\mu_2\|_{s_2}^\beta\|\lambda\|_\beta).$$

So, we have

$$\|\lambda\|_\beta \leq \frac{\Lambda_1^\beta\|\mu_1\|_{s_2}^\beta}{1-\Lambda_1^\beta\|\mu_2\|_{s_2}^\beta}.$$

Therefore, noting assumption (S4), we get the conclusion that the set $S(\Upsilon)$ is bounded. Thus, the conclusion of Lemma 2.4 applies, and hence the operator Υ attains at least one fixed point, which corresponds to a solution of (1.1)–(1.2). □

4. β-Ulam-Hyers stability for problem (1.1)–(1.2)

In this section, we continue to explore the β-UHS of problem (1.1)–(1.2). For the theory of this type of stability, we refer the reader to [44].

We employ a new inequality for some function $\nu = \nu(t) \in C^3(I, R)$,

$$|{}^C D_{0^+}^{\omega} \nu(t) +{}^C D_{0^+}^{\omega-1}[p'(t)\nu(t)] - h(t, \nu(t))| \leq \epsilon,$$

where h is a measure function on Θ, and $\epsilon > 0$ is any real number.

Therefore, we have a BV problem as follows:

$$ {}^C D_{0^+}^{\omega} \nu(t) +{}^C D_{0^+}^{\omega-1}[p'(t)\nu(t)] = h(t, \nu(t)) + \kappa(t), \tag{4.1}$$

with boundary conditions

$$\nu(0) = \nu'(0) = \nu'(1) = 0, \tag{4.2}$$

where the function $\kappa = \kappa(t) \in \Theta$ satisfies $|\kappa(t)| \leq \epsilon$.

Definition 4.1. The BV problem of (1.1)–(1.2) is β-UHS if a constant $C_{\beta,h} > 0$ exists such that for any $\epsilon > 0$ and for every solution $\nu = \nu(t)$ of (4.1)–(4.2), a solution $\lambda = \lambda(t)$ of (1.1)–(1.2) exists, with

$$\|\nu(t) - \lambda(t)\|_{\beta} \leq C_{\beta,h}\epsilon^{\beta}.$$

Theorem 4.1. For the assumptions (S1) and (S2), the problem (1.1)–(1.2) is β-UHS.

Proof. Lemma 2.3 ensures a unique solution $\nu(t)$ of (4.1) − (4.2),

$$\begin{aligned}
\nu(t) = & \frac{1}{\Gamma(\omega-1)} \int_0^t e^{p(s)-p(t)} \int_0^s (s-\tau)^{\omega-2}[h(\tau, \nu(\tau)) + \kappa(\tau)]d\tau ds \\
& + \frac{q(t)}{\theta\Gamma(\omega-1)}\{p'(1) \int_0^1 e^{p(s)-p(1)} \int_0^s (s-\tau)^{\omega-2}[h(\tau, \nu(\tau)) + \kappa(\tau)]d\tau ds \\
& - \int_0^1 (1-\tau)^{\omega-2}[h(\tau, \nu(\tau)) + \kappa(\tau)]d\tau\} \\
=: & L(t) + G(t),
\end{aligned}$$

where

$$\begin{aligned}
L(t) = & \frac{1}{\Gamma(\omega-1)} \int_0^t e^{p(s)-p(t)} \int_0^s (s-\tau)^{\omega-2} h(\tau, \nu(\tau))d\tau ds \\
& + \frac{q(t)}{\theta\Gamma(\omega-1)}[p'(1) \int_0^1 e^{p(s)-p(1)} \int_0^s (s-\tau)^{\omega-2} h(\tau, \nu(\tau))d\tau ds - \int_0^1 (1-\tau)^{\omega-2} h(\tau, \nu(\tau))d\tau]; \\
G(t) = & \frac{1}{\Gamma(\omega-1)} \int_0^t e^{p(s)-p(t)} \int_0^s (s-\tau)^{\omega-2}\kappa(\tau)d\tau ds \\
& + \frac{q(t)}{\theta\Gamma(\omega-1)}[p'(1) \int_0^1 e^{p(s)-p(1)} \int_0^s (s-\tau)^{\omega-2}\kappa(\tau)d\tau ds - \int_0^1 (1-\tau)^{\omega-2}\kappa(\tau)d\tau].
\end{aligned}$$

Then, by Lemma 3.3, we have

$$
\begin{aligned}
|G(t)| = |\nu(t) - L(t)| \leq & \frac{\epsilon}{\Gamma(\omega-1)} \int_0^t e^{p(s)-p(t)} \int_0^s (s-\tau)^{\omega-2} d\tau ds \\
& + \frac{m_p \epsilon}{2 \mid \theta \mid \Gamma(\omega-1)} [\mid p'(1) \mid \int_0^1 e^{p(s)-p(1)} \int_0^s (s-\tau)^{\omega-2} d\tau ds + \int_0^1 (1-\tau)^{\omega-2} d\tau] \\
\leq & \frac{m_p \epsilon}{\Gamma(\omega)} + \frac{m_p \epsilon}{2 \mid \theta \mid \Gamma(\omega)} (\mid p'(1) \mid m_p + 1) \\
= & \Lambda_3 \cdot \epsilon.
\end{aligned}
$$

For $\lambda = \lambda(t), \nu = \nu(t) \in \Theta$, representing the solutions of (1.1)−(1.2) and (4.1)−(4.2), by condition (S1), we have

$$
\begin{aligned}
|L - \lambda| \leq & \frac{1}{\Gamma(\omega-1)} \int_0^t e^{p(s)-p(t)} \int_0^s (s-\tau)^{\omega-2} \mid h(\tau, \nu(\tau)) - h(\tau, \lambda(\tau)) \mid d\tau ds \\
& + \frac{\mid q(t) \mid}{\mid \theta \mid \Gamma(\omega-1)} \{\mid p'(1) \mid \int_0^1 e^{p(s)-p(1)} \int_0^s (s-\tau)^{\omega-2} \mid h(\tau, \nu(\tau)) - h(\tau, \lambda(\tau)) \mid d\tau ds \\
& + \int_0^1 (1-\tau)^{\omega-2} \mid h(\tau, \nu(\tau)) - h(\tau, \lambda(\tau)) \mid d\tau\} \\
\leq & \frac{1}{\Gamma(\omega-1)} \int_0^t e^{p(s)-p(t)} \int_0^s (s-\tau)^{\omega-2} r(\tau) \mid \nu(\tau) - \lambda(\tau) \mid d\tau ds \\
& + \frac{\mid m_p \mid}{2 \mid \theta \mid \Gamma(\omega-1)} \{\mid p'(1) \mid \int_0^1 e^{p(s)-p(1)} \int_0^s (s-\tau)^{\omega-2} r(\tau) \mid \nu(\tau) - \lambda(\tau) \mid d\tau ds \\
& + \int_0^1 (1-\tau)^{\omega-2} r(\tau) \mid \nu(\tau) - \lambda(\tau) \mid d\tau\}.
\end{aligned}
$$

By the results in Lemma 3.3, we have the following inequalities holds

$$
\begin{aligned}
|L - \lambda| \leq & \frac{\| \nu(\tau) - \lambda(\tau) \|}{\Gamma(\omega-1)} \{\int_0^t e^{p(s)-p(t)} \int_0^s (s-\tau)^{\omega-2} r(\tau) d\tau ds \\
& + \frac{\mid m_p \mid}{2 \mid \theta \mid} [\mid p'(1) \mid \int_0^1 e^{p(s)-p(1)} \int_0^s (s-\tau)^{\omega-2} r(\tau) d\tau ds + \int_0^1 (1-\tau)^{\omega-2} r(\tau) d\tau]\} \\
\leq & \frac{\| r \|_{s_2} \, m_p}{\Gamma(\omega-1) \sqrt[s_1]{1 + s_1(\omega-2)}} [1 + \frac{\mid p'(1) \mid m_p + 1}{2 \mid \theta \mid}] \cdot \| \nu(\tau) - \lambda(\tau) \| \\
= & \Lambda_1 \| r \|_{s_2} \cdot \| \nu(\tau) - \lambda(\tau) \| .
\end{aligned}
$$

Now, we have

$$
|\nu - \lambda| = |\nu - L + L - \lambda| \leq |\nu - L| + |L - \lambda| \leq \Lambda_3 \cdot \epsilon + \Lambda_1 \| r \|_{s_2} \cdot \| \nu(\tau) - \lambda(\tau) \|,
$$

which implies

$$
\mid \nu - \lambda \mid^{\beta} \leq \Lambda_3^{\beta} \cdot \epsilon^{\beta} + \Lambda_1^{\beta} \| r \|_{s_2}^{\beta} \cdot \| \nu(\tau) - \lambda(\tau) \|^{\beta},
$$

so, we have

$$
\|\nu(\tau) - \lambda(\tau)\|_{\beta} \leq \frac{\Lambda_3^{\beta} \epsilon^{\beta}}{1 - \Lambda_1^{\beta} \| r \|_{s_2}^{\beta}}.
$$

Noting the condition (S2), we take the positive constant $C_{\beta,h} = \frac{\Lambda_3^\beta}{1-\Lambda_1^\beta\|r\|_{s_2}^\beta}$. So, we obtain the following inequality:

$$\|\nu(\tau) - \lambda(\tau)\|_\beta \leq C_{\beta,h}\epsilon^\beta.$$

This shows the problem (1.1)–(1.2) is β-UHS. The proof is finished. □

5. Illustrative examples

We provide two examples to illustrate the use of Theorems 3.1, 3.2, and 4.1.

Example 5.1. Consider a fractional BV problem,

$$\begin{cases} {}^C D_{0^+}^{2.51}\lambda(t) + {}^C D_{0^+}^{1.51}[sint \cdot \lambda(t)] = h(t, \lambda(t)), 0 < t < 1, \\ \lambda(0) = \lambda'(0) = \lambda'(1) = 0, \end{cases} \tag{5.1}$$

where $\omega = 2.51$ and $p(t) = -cost$. Therefore, we have $\Gamma(1.51) = 0.8866$, $\Gamma(2.51) = 1.3388$, and $p'(1) = 0.8415$. If we take $e = 2.71828$, then $m_p = \sqrt{e} = 1.6487$, $m_p' = max_{0\leq t\leq 1}|p'(t)| = sin1 = 0.8415$, $q(t) \leq \frac{1}{2}m_p = 0.82435$, $N_1 = 1.6937$, and $\theta = 0.6604$.

For the conjugate exponentials of each other, we assume $s_1 = s_2 = 2$. Then

$$\Lambda_1 = 3.6733, \Lambda_2 = 5.6525, \Lambda_3 = 3.4575.$$

To illustrate Theorems 3.1 and 4.1, let us take

$$h(t, \lambda(t)) = \frac{1}{e(1+t)^{10}} e^{-sin\lambda(t)}. \tag{5.2}$$

For any $\lambda, \nu \in \Theta$, Lagrange's mean value theorem ensures that there exists a function $\xi(t)$ whose value is between $\lambda(t)$ and $\nu(t)$, such that

$$|h(t,\lambda) - h(t,\nu)| = \frac{1}{e(1+t)^{10}}|e^{-sin\lambda(t)} - e^{-sin\nu(t)}]| = \frac{1}{e(1+t)^{10}} e^{-sin\xi(t)}|cos\xi(t)| \ |\lambda - \nu| \leq \frac{1}{(1+t)^{10}}|\lambda - \nu|.$$

Therefore, $r(t) = \frac{1}{(1+t)^{10}}$. Take $\beta = 0.5$, and we have $\|r\|_{s_2}^\beta = 0.2469$, $1 - \Lambda_1^\beta\|r\|_{s_2}^\beta = 0.1502 > 0$, and $C_{\beta,h} = 21.6352$. Then conditions (S1) and (S2) are satisfied, so theorems 3.1 and 4.1 ensure problem (5.1) attains a unique solution on Θ , with h given by (5.2), and it is β-UHS.

Example 5.2. To demonstrate the application of Theorem 3.2, let us take the nonlinear term as follows:

$$h(t, \lambda(t)) = \frac{e^t}{\sqrt{1+t^n}} cos\lambda(t) + e^{-10t} sin\lambda(t), \tag{5.3}$$

where n is some positive number. Clearly, h satisfies condition (S3), with $\mu_1(t) = e^t$ and $\mu_2(t) = e^{-10t}$. We calculate $\| \mu_1 \|_{s_2}^\beta = 1.3369$, $\| \mu_2 \|_{s_2}^\beta = 0.2436 \mid A \mid$, and $1 - \Lambda_1^\beta \| \mu_2 \|_{s_2}^\beta > 0$. Thus, (S4) is true. Condition (3.2) is also satisfied, with $\frac{\Lambda_1^\beta\|\mu_1\|_{s_2}^\beta}{1-\Lambda_1^\beta\|\mu_2\|_{s_2}^\beta} = 4.8154 < \infty$. Thus, Theorem 3.2 guarantees one solution on Θ for problem (5.1), with h given by (5.3).

6. Conclusions

We investigated a category of BV problems of nonlinear sequential FDEs, consisting of two-term fractional derivatives, and with continuous variable coefficients. The considered system was first transformed into an equivalent integral equation, by giving the nonlinear term f to satisfy two different properties, combining Hölder's inequality, Banach's FPP, and Schaefer's FPP, we developed quantitative properties such as criteria guaranteeing the existence and uniqueness of the solutions in β-normed space, and also discussed the stability of the problem in the sense of β-UHS. Two concrete examples were introduced to test the validity of the main conclusions. In particular, it is worth pointing out that problems related to nonlinear sequential FDEs with multi-term derivatives and/or general variable coefficients warrant further study.

Use of Generative-AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares he has no conflict of interest.

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