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Research article

An efficient numerical method for 2D elliptic singularly perturbed systems with different magnitude parameters in the diffusion and the convection terms, part II

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Abstract: This work is the continuation of [11], where a two-dimensional elliptic singularly perturbed weakly system, for which small parameters affected both the diffusion and the convection terms, was solved; moreover, all perturbation parameters could have different orders of magnitude, which is the most interesting and difficult case for this type of problem. It is well known that then, in general, overlapping regular or parabolic boundary layers appear in the solution of the continuous problem. To solve numerically the problem, the classical upwind finite difference scheme, defined on special piecewise uniform Shsihkin meshes, was used, proving its uniform convergence, with respect to all parameters, for four different ratios between them. In this paper, we complete the previous analysis, considering the two cases for these possible ratios, that were not considered in [11]. To see in practice the efficiency of the numerical method, we show the numerical results obtained with our algorithm for a test problem, when the cases analyzed in this work are fixed; from those results, the uniform convergence of the numerical algorithm follows, in agreement with the theoretical results.

Keywords: 2D elliptic coupled systems; singularly perturbed problems; upwind scheme; Shishkin meshes; uniform convergence
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1. Introduction

In this work, we are interested in solve numerically a two-dimensional, singularly perturbed, elliptic, weakly–coupled system, which is given by

$$\begin{cases} \vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}\vec{\mathbf{z}}(x,y) \equiv \vec{\varepsilon}\Delta\vec{\mathbf{z}}(x,y) + \vec{\mu}\vec{\mathbf{A}}(x,y)\cdot\nabla\vec{\mathbf{z}}(x,y) - \vec{\mathbf{B}}(x,y)\vec{\mathbf{z}}(x,y) = \vec{\mathbf{f}}(x,y), \quad \forall (x,y)\in\Omega, \\ \vec{\mathbf{z}}(x,y) = \vec{\mathbf{g}}_1(x,y), \quad \forall (x,y)\in\Gamma_1, \quad \vec{\mathbf{z}}(x,y) = \vec{\mathbf{g}}_2(x,y), \quad \forall (x,y)\in\Gamma_2, \\ \vec{\mathbf{z}}(x,y) = \vec{\mathbf{g}}_3(x,y), \quad \forall (x,y)\in\Gamma_3, \quad \vec{\mathbf{z}}(x,y) = \vec{\mathbf{g}}_4(x,y), \quad \forall (x,y)\in\Gamma_4, \end{cases} \tag{1.1}$$

where the domain is $\Omega = (0,1)^2$ and its boundaries are denoted by

$$\partial\Omega = \begin{cases} \Gamma_1 = \{(0,y)\,|\,(0\le y\le 1)\}, \quad \Gamma_2 = \{(x,0)\,|\,(0\le x\le 1)\}, \\ \Gamma_3 = \{(1,y)\,|\,(0\le y\le 1)\}, \quad \Gamma_4 = \{(x,1)\,|\,(0\le x\le 1)\}. \end{cases}$$

Below we denote the whole boundary by $\Gamma = \Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$.

The convection matrix is $\vec{\mathbf{A}}(x,y) = (\vec{\mathbf{A}}_1(x,y), \vec{\mathbf{A}}_2(x,y))$ and the reaction matrix is $\vec{\mathbf{B}}(x,y)$. We suppose that the convection matrices are diagonal and the reaction matrix is a full matrix, i.e., we have

$$\vec{\mathbf{A}}_1(x,y) = \begin{pmatrix} a_1^1(x,y) & 0 \\ 0 & a_1^2(x,y) \end{pmatrix}, \vec{\mathbf{A}}_2(x,y) = \begin{pmatrix} a_2^1(x,y) & 0 \\ 0 & a_2^2(x,y) \end{pmatrix}, \vec{\mathbf{B}}(x,y) = \begin{pmatrix} b_{11}(x,y) & b_{12}(x,y) \\ b_{21}(x,y) & b_{22}(x,y) \end{pmatrix}.$$

Moreover, the differential operator, the diffusion parameters, the convection parameters, the source term, and the boundary conditions are given by

$$\begin{aligned} &\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}} = (\mathcal{L}^1_{\varepsilon_1,\mu_1}, \mathcal{L}^2_{\varepsilon_2,\mu_2})^T, \vec{\varepsilon} = (\varepsilon_1,\varepsilon_2)^T, \vec{\mu} = (\mu_1,\mu_2)^T, \vec{\mathbf{z}}(x,y) = (z_1(x,y), z_2(x,y))^T, \\ &\vec{\mathbf{f}}(x,y) = (f_1(x,y), f_2(x,y))^T, \vec{\mathbf{g}}_i(x,y) = (g_{i1}(x,y), g_{i2}(x,y))^T, \ i = 1,2,3,4. \end{aligned}$$

We suppose that $0 < \varepsilon_1, \varepsilon_2, \mu_1, \mu_2 \ll 1$ and, without loss of generality, we assume that $\varepsilon_1 \le \varepsilon_2$ and $\mu_1 \le \mu_2$. Moreover, the coefficients of the reaction and the convection matrices satisfy

$$\begin{cases} a_1^i(x,y) \ge \vartheta_1 > 0, \ a_2^i(x,y) \ge \vartheta_2 > 0, \ i = 1,2, \\ b_{ii}(x,y) \ge \beta > 0, \ i = 1,2, \\ b_{ii}(x,y) > |b_{ij}(x,y)|, \quad b_{ij}(x,y) \le 0, \ i,j = 1,2, \ i \ne j, \end{cases} \tag{1.2}$$

for some positive constants ϑ_1, ϑ_2, and β. From the previous values, we define the constants

$$\vartheta = \min(\vartheta_1,\vartheta_2), \ \Lambda = \min_{i,j}\left\{\frac{b_{ii}-b_{ij}}{2a_1^i}, \frac{b_{ii}-b_{ij}}{2a_2^i}\right\}, \ \text{for } i,j = 1,2, \ i\ne j, \tag{1.3}$$

which will play an important role posteriorly. Finally, we assume that the components of $\vec{\mathbf{A}}_1$, $\vec{\mathbf{A}}_2$, $\vec{\mathbf{B}}$ and $\vec{\mathbf{f}}$ are sufficiently smooth functions on Ω, $\vec{\mathbf{g}}_i \in C^{3,\gamma}(\Gamma_i)$, $i = 1,2,3,4$, for some $\gamma \in (0,1]$, and they satisfy sufficient compatibility conditions in order that the continuous problem has a solution $\vec{\mathbf{z}}$; moreover, this solution satisfies $\vec{\mathbf{z}} \in C^{3,\gamma}(\bar{\Omega})$ (in [17] and [15], Theorem 3.2, appear the compatibility conditions that guarantee this regularity).

Problems of type (1.1) are interesting because they appear as models for many physical problems in different areas, as transport phenomena in chemistry and biology, turbulent interactions of waves and currents, bio-fluids mechanics, saturated flow in fractured porous media, quantum mechanics, or elasticity (see [2, 3, 5, 16, 27, 30]). For instance, from [2], consider the following model for saturated flow in fractured porous media:

$$\begin{cases} (\gamma_{c_1} + n_1\gamma)\frac{\partial\rho_1}{\partial t} - \frac{k_1}{\nu}\Delta\rho_1 + \frac{\gamma}{\nu}(\rho_1 - \rho_2) = f_1(x,t), \\ (\gamma_{c_2} + n_2\gamma)\frac{\partial\rho_2}{\partial t} - \frac{k_2}{\nu}\Delta\rho_2 + \frac{\gamma}{\nu}(\rho_2 - \rho_1) = f_2(x,t), \end{cases}$$

where ρ_1 and ρ_2 are the pressures of the liquid in the pores of the first and second-order, respectively, γ is the coefficient of compressibility of the liquid, and ν is the viscosity of the liquid. Here γ_{c_1} and γ_{c_2} are positive constants, whereas k_1 and k_2 are the porosity of the system of pores of first and second-order respectively, and n_1 and n_2 are the values of the first and second-order porosity at standard pressure.

It is well known that the exact solution of singularly perturbed problems has, in general, boundary layers and/or internal layers of different types when the positive parameters are sufficiently small. Then, the use of standard numerical methods defined on uniform meshes does not give good approximations unless the step size of the mesh is very small (depending on the value of the parameters), that is not useful from a numerical point of view. So, uniformly convergent methods are needed, i.e., numerical methods which calculate a good approximation of the exact solution of the continuous problem independently of the value of the parameters.

In the last years there has been an increasing interest in problems having small parameters that affect both the convection and the diffusion terms of the differential equation (see, for instance, [13, 14, 24, 25]). The case of singularly perturbed systems with parameters at the diffusion and the convection terms is a particular case where difficulties are added due to the complex structure of the boundary layers that appear in the exact solution.

Elliptic and parabolic singularly perturbed coupled systems, for which the small parameters appear only in the diffusion term, are well studied; see, for instance, [4, 7, 8, 20, 21, 26, 28, 29], where 1D and 2D convection–diffusion or reaction-diffusion systems were considered and uniformly convergent methods were constructed to solve them. Nevertheless, the case where small parameters appear at both the diffusion and the convection terms is a special case, which is less analyzed in the literature. In [1], a parabolic 1D weakly–coupled system of convection–diffusion type was studied. In [9], a 2D elliptic singularly perturbed weakly–coupled system of convection–reaction–diffusion type, was analyzed. In [22], a 1D elliptic singularly perturbed weakly–coupled system, for which the diffusion parameters at each equation are different and the convection parameters are the same at both equations, was considered.

In this work we use similar ideas and techniques to those in [10,11] for the same type of problems as (1.1). Nevertheless, in [10], for the 2D elliptic system considered, it was assumed that the diffusion parameters can be distinct and with a different order of magnitude, but the convection parameter was the same in both equations of the coupled system. On the other hand, in [11], the same problem that (1.11) was studied; nevertheless, in that work, the authors analyzed the cases $\vartheta{\mu_1}^2 \le \vartheta{\mu_2}^2 \le \Lambda\varepsilon_1 \le \Lambda\varepsilon_2$, $\Lambda\varepsilon_1 \le \Lambda\varepsilon_2 < \vartheta{\mu_1}^2 \le \vartheta{\mu_2}^2$, $\Lambda\varepsilon_1 < \vartheta{\mu_1}^2 \le \vartheta{\mu_2}^2 < \Lambda\varepsilon_2$ and $\vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \le \Lambda\varepsilon_2 < \vartheta{\mu_2}^2$ were analyzed; note that in this case, the asymptotic behavior of the exact solution is more complicated. In the two previous works, a uniformly convergent method was constructed based on the use of the classical upwind scheme defined on appropriate Shishkin meshes, which depend on the value and the ratio between the four discretization parameters; in both works, it was proved that the numerical algorithm has almost first order of convergence. Note that analysis of the uniform convergence is considerably more difficult in the second work. Here, our main motivation is to complete the study made in [11], assuming now that $\vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \le \vartheta{\mu_2}^2 < \Lambda\varepsilon_2$ or $\Lambda\varepsilon_1 \le \vartheta{\mu_1}^2 < \Lambda\varepsilon_2 \le \vartheta{\mu_2}^2$ hold. Note that due to $\varepsilon_1 \neq \varepsilon_2$ and $\mu_1 \neq \mu_2$, they can have different orders of magnitude, and the structure of overlapping boundary layers, on the inflow and the outflow boundary of the domain, is a difficult task.

The paper is structured as follows. In Section 2, we study the asymptotic behavior of the exact solution, and we prove adequate estimates for its partial derivatives; note that the analysis shows the behavior of the exact solution with respect to the four singular perturbation parameters $\varepsilon_1, \varepsilon_2, \mu_1$ and μ_2, and the ratio between them in the two cases analyzed in this work. In Section 3, we construct the numerical method; in its construction, it is crucial to define a special nonuniform mesh of Shsihkin type, adapted to the type of boundary layers that the exact solution has. In Section 4, we prove the main result of the work, i.e., we obtain the uniform convergence of the numerical method, with respect to the four singular perturbation parameters, proving that it is an almost first-order uniformly convergent method. In Section 5, we show the numerical results obtained for a test problem, for which the two cases associated with the ratios of the parameters are taken; from these results, clearly the uniform convergence of the numerical algorithm follows, in agreement with the theoretical results. Finally, in Section 6, some conclusions are given.

Henceforth, we denote by $\|\cdot\|$ the continuous maximum norm; moreover, for a function $\vec{\boldsymbol{\Psi}} = (\Psi_1, \Psi_2)^T$, $|\vec{\boldsymbol{\Psi}}| = (|\Psi_1|, |\Psi_2|)^T$, and C denotes a generic positive constant which is independent of the diffusion parameters, ε_1 and ε_2, the convection parameters, μ_1 and μ_2; and also of the discretization parameter N.

2. Asymptotic behavior of the exact solution of the continuous problem

In this section, we prove which is the asymptotic behavior of the exact solution of the problem (1.1) and also adequate estimates for its partial derivatives with respect to the diffusion and the convection parameters. These estimates are crucial to obtain the uniform convergence of the numerical method defined posteriorly.

As we have indicated previously, in [11] the authors studied four different cases related to the ratio between the four parameters $\varepsilon_1, \varepsilon_2, \mu_1$, and μ_2. Now, we consider the other cases, which are given by

$$\textbf{Case 1: } \text{it holds } \vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \le \vartheta{\mu_2}^2 < \Lambda\varepsilon_2, \quad \textbf{Case 2: } \text{it holds } \Lambda\varepsilon_1 \le \vartheta{\mu_1}^2 < \Lambda\varepsilon_2 \le \vartheta{\mu_2}^2. \tag{2.1}$$

Before to obtaining the estimates, we remember some results, which hold for the exact solution of the continuous problem, which were proved in [11].

Lemma 2.1 (Minimum principle)**.** *Let $\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}$ be the differential operator given in (1.1), and we assume that (1.2) holds. If $\vec{\boldsymbol{\Phi}}(x,y) \ge \vec{\mathbf{0}}$ on $\partial\Omega$ and $\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}\vec{\boldsymbol{\Phi}}(x,y) \le \vec{\mathbf{0}}$ for all $(x,y) \in \Omega$, then it holds $\vec{\boldsymbol{\Phi}}(x,y) \ge \vec{\mathbf{0}}$ for all $(x,y) \in \overline{\Omega}$.*

Proof. The lemma can be proven using a similar methodology to this one used in [11] (Lemma 2.1). □

Lemma 2.2 (Stability result)**.** *Let $\vec{\boldsymbol{\Phi}} \in C^{3,\gamma}(\bar{\Omega})$; then, it holds*

$$|\vec{\boldsymbol{\Phi}}(x,y)| \le \frac{1}{\vartheta}\|\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}\vec{\boldsymbol{\Phi}}\| + \max\{\|\vec{\boldsymbol{\Phi}}\|_{\Gamma_1}, \|\vec{\boldsymbol{\Phi}}\|_{\Gamma_2}, \|\vec{\boldsymbol{\Phi}}\|_{\Gamma_3}, \|\vec{\boldsymbol{\Phi}}\|_{\Gamma_4}\},$$

where ϑ is the constant defined in (1.3).

Proof. The proof follows straightforwardly from Lemma 2.1; the technique used for the proof is well known in the literature in the context of singularly perturbed problems. □

Theorem 2.3. *Let $\vec{z}$ be the exact solution of the continuous problem (1.1). Then, its derivatives satisfy the following bounds on $\overline{\Omega}$:*

$$\left|\frac{\partial^{(l_1+l_2)} z_i(x,y)}{\partial x^{l_1} \partial y^{l_2}}\right| \le C(\varepsilon_i)^{(-l_1-l_2)/2}\left\{1+\left(\frac{\mu_i}{\sqrt{\varepsilon_i}}\right)^{(l_1+l_2)}\right\}\max\{z_i,\ f_i\},\ 1\le l_1+l_2\le 2,\ i=1,2, \tag{2.2a}$$

$$\left|\frac{\partial^{(l_1+l_2)} z_1(x,y)}{\partial x^{l_1} \partial y^{l_2}}\right| \le C(\varepsilon_1)^{(-l_1-l_2)/2}\left\{1+\left(\frac{\mu_1}{\sqrt{\varepsilon_1}}\right)^{(l_1+l_2)}\right\}\max\{z_1,\ f_1\}+$$
$$C\varepsilon_1^{2-l_1-l_2}\max\left\{z_1,\ f_1,\ \frac{\partial f_1}{\partial x},\ \frac{\partial f_1}{\partial y}\right\},\ 3\le l_1+l_2\le 4, \tag{2.2b}$$

$$\left|\frac{\partial^{(l_1+l_2)} z_2(x,y)}{\partial x^{l_1} \partial y^{l_2}}\right| \le C(\varepsilon_2)^{(-l_1-l_2)/2}\left\{1+\left(\frac{\mu_2}{\sqrt{\varepsilon_2}}\right)^{(l_1+l_2)}\right\}\max\{z_2,\ f_2\}+$$
$$C\varepsilon_1^{1-(l_1+l_2)/2}\varepsilon_2^{-1}\max\left\{z_2,\ f_2,\ \frac{\partial f_2}{\partial x},\ \frac{\partial f_2}{\partial y}\right\},\ 3\le l_1+l_2\le 4. \tag{2.2c}$$

Proof. We can use the similar methodology given in [11] (Theorem 2.3). □

Previous estimates are not adequate because they do not reflect the presence of boundary layers in the exact solution of the continuous problem. To obtain better estimates, we propose a decomposition of the exact solution $\vec{\mathbf{z}}$ of the problem (1.1) into its regular component $\vec{\mathbf{r}}$, boundary layer components $\vec{\mathbf{w}}$ and corner layer components $\vec{\mathbf{s}}$. Moreover, those functions can be decomposed into $\vec{\mathbf{w}}_l, \vec{\mathbf{w}}_r, \vec{\mathbf{w}}_b, \vec{\mathbf{w}}_t$ and $\vec{\mathbf{s}}_{lb}, \vec{\mathbf{s}}_{br}, \vec{\mathbf{s}}_{rt}, \vec{\mathbf{s}}_{lt}$, respectively. These components are obtained as the solution of the following problems:

$$\begin{cases} \vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}\vec{\mathbf{v}}(x,y) = \vec{\mathbf{f}}, \quad \forall (x,y)\in\Omega, \\ \vec{\mathbf{v}}(x,y) = \vec{\xi}_1(y), \quad \forall (x,y)\in\Gamma_1, \quad \vec{\mathbf{v}}(x,y) = \vec{\xi}_2(x), \quad \forall (x,y)\in\Gamma_2, \\ \vec{\mathbf{v}}(x,y) = \vec{\xi}_3(y), \quad \forall (x,y)\in\Gamma_3, \quad \vec{\mathbf{v}}(x,y) = \vec{\xi}_4(x), \quad \forall (x,y)\in\Gamma_4, \end{cases} \tag{2.3}$$

where $\vec{\xi}_i,\ i = 1,2,3,4$, are specially chosen functions (see the analysis below), for the regular component,

$$\begin{cases} \vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}\vec{\mathbf{w}}_k(x,y) = 0,\ k = l,r,b,t, \quad \forall (x,y)\in\Omega, \\ \vec{\mathbf{w}}_l(x,y) = (\vec{\mathbf{z}}-\vec{\mathbf{r}})(x,y),\ \forall (x,y)\in\Gamma_1,\ \vec{\mathbf{w}}_l(x,y) = 0,\ \forall (x,y)\in\Gamma_2\cup\Gamma_3\cup\Gamma_4, \\ \vec{\mathbf{w}}_r(x,y) = (\vec{\mathbf{z}}-\vec{\mathbf{r}})(x,y),\ \forall (x,y)\in\Gamma_3,\ \vec{\mathbf{w}}_r(x,y) = 0,\ \forall (x,y)\in\Gamma_1\cup\Gamma_2\cup\Gamma_4, \\ \vec{\mathbf{w}}_t(x,y) = (\vec{\mathbf{z}}-\vec{\mathbf{r}})(x,y),\ \forall (x,y)\in\Gamma_4,\ \vec{\mathbf{w}}_t(x,y) = 0,\ \forall (x,y)\in\Gamma_1\cup\Gamma_2\cup\Gamma_3, \end{cases} \tag{2.4}$$

for the boundary layer components and

$$\begin{cases} \vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}\vec{\mathbf{s}}_k(x,y)=0,\ k=lb,br,rt,lt, \quad \forall(x,y)\in\Omega, \\ \vec{\mathbf{s}}_{lb}(x,y)=-\vec{\mathbf{w}}_l(x,y),\ \forall(x,y)\in\Gamma_1,\ \vec{\mathbf{s}}_{lb}(x,y)=-\vec{\mathbf{w}}_b(x,y),\ \forall(x,y)\in\Gamma_2, \\ \vec{\mathbf{s}}_{br}(x,y)=-\vec{\mathbf{w}}_b(x,y),\ \forall(x,y)\in\Gamma_2,\ \vec{\mathbf{s}}_{br}(x,y)=-\vec{\mathbf{w}}_r(x,y),\ \forall(x,y)\in\Gamma_3, \\ \vec{\mathbf{s}}_{rt}(x,y)=-\vec{\mathbf{w}}_r(x,y),\ \forall(x,y)\in\Gamma_3,\ \vec{\mathbf{s}}_{rt}(x,y)=-\vec{\mathbf{w}}_t(x,y),\ \forall(x,y)\in\Gamma_4, \\ \vec{\mathbf{s}}_{lt}(x,y)=-\vec{\mathbf{w}}_l(x,y),\ \forall(x,y)\in\Gamma_1,\ \vec{\mathbf{s}}_{lt}(x,y)=-\vec{\mathbf{w}}_t(x,y),\ \forall(x,y)\in\Gamma_4, \\ \vec{\mathbf{s}}_{lb}(x,y)=0,\ \forall(x,y)\in\Gamma_3\cup\Gamma_4,\ \vec{\mathbf{s}}_{br}(x,y)=0,\ \forall(x,y)\in\Gamma_1\cup\Gamma_4, \\ \vec{\mathbf{s}}_{rt}(x,y)=0,\ \forall(x,y)\in\Gamma_1\cup\Gamma_2,\ \vec{\mathbf{s}}_{lt}(x,y)=0,\ \forall(x,y)\in\Gamma_2\cup\Gamma_3, \end{cases} \tag{2.5}$$

for the corner layer components, respectively.

First, we study the behavior of the smooth component $\vec{\mathbf{r}}=(r_1,r_2)^T$. To accomplish this, we distinguish two cases.

Case 1: If $\vartheta{\mu_1}^2<\Lambda\varepsilon_1\leq\vartheta{\mu_2}^2<\Lambda\varepsilon_2$, we decompose the components of $\vec{\mathbf{r}}$ as

$$r_1=v_{01}+\sqrt{\varepsilon_1}v_{11}+(\sqrt{\varepsilon_1})^2v_{21}+(\sqrt{\varepsilon_1})^3v_{31}, \tag{2.6}$$

$$r_2=v_{02}+\sqrt{\varepsilon_2}v_{12}+(\sqrt{\varepsilon_2})^2v_{22}+(\sqrt{\varepsilon_2})^3v_{32}, \tag{2.7}$$

where $\vec{\mathbf{v}}_i=(v_{i1},v_{i2})^T$, $i=0,1,2,3$, and their corresponding defining equations on $\overline{\Omega}$ are given by

$$-b_{11}v_{01}-b_{12}v_{02}=f_1,\quad b_{11}v_{11}+b_{12}v_{12}=\Delta v_{01}+\frac{\mu_1}{\sqrt{\varepsilon_1}}(a_1^1,a_2^1)\nabla v_{01}, \tag{2.8a}$$

$$b_{11}v_{21}+b_{12}v_{22}=\Delta v_{11}+\frac{\mu_1}{\sqrt{\varepsilon_1}}(a_1^1,a_2^1)\nabla v_{11}, \tag{2.8b}$$

$$\mathcal{L}^1_{\varepsilon_1,\mu_1}v_{31}=-\Delta v_{21}-\frac{\mu_1}{\sqrt{\varepsilon_1}}(a_1^1,a_2^1)\nabla v_{21},\quad v_{31}(x,y)=0,\ \forall(x,y)\in\Gamma, \tag{2.8c}$$

and

$$-b_{21}v_{01}-b_{22}v_{02}=f_2,\quad b_{21}v_{11}+b_{22}v_{12}=\Delta v_{02}+\frac{\mu_2}{\sqrt{\varepsilon_2}}(a_1^2,a_2^2)\nabla v_{02}, \tag{2.9a}$$

$$b_{21}v_{21}+b_{22}v_{22}=\Delta v_{12}+\frac{\mu_2}{\sqrt{\varepsilon_2}}(a_1^2,a_2^2)\nabla v_{12}, \tag{2.9b}$$

$$\mathcal{L}^2_{\varepsilon_2,\mu_2}v_{32}=-\Delta v_{22}-\frac{\mu_2}{\sqrt{\varepsilon_2}}(a_1^2,a_2^2)\nabla v_{22},\quad v_{32}(x,y)=0,\ \forall(x,y)\in\Gamma, \tag{2.9c}$$

respectively.

Then, following the same technique that we have used for the analysis in [11], and using the results of Lemmas 2.2 and 2.3, applied now to both problems (2.8c) and (2.9c), we can obtain

$$\left\|\frac{\partial^{l_1+l_2}\vec{\mathbf{r}}}{\partial x^{l_1}\partial y^{l_2}}\right\|\leq C,\ 0\leq l_1+l_2\leq 2,\ \left\|\frac{\partial^{l_1+l_2}r_1}{\partial x^{l_1}\partial y^{l_2}}\right\|\leq C\varepsilon_1^{-1/2},\ l_1+l_2=3, \tag{2.10a}$$

$$\left\|\frac{\partial^{l_1+l_2}r_2}{\partial x^{l_1}\partial y^{l_2}}\right\|\leq C\varepsilon_2^{-1/2},\ l_1+l_2=3. \tag{2.10b}$$

Then, from previous bounds, the required result for the regular component follows.

Case 2: If $\Lambda\varepsilon_1 \leq \vartheta{\mu_1}^2 < \Lambda\varepsilon_2 \leq \vartheta{\mu_2}^2$, we now decompose the components of $\vec{\mathbf{r}}$ in a different way; then, we have

$$r_1 = v_{01} + \varepsilon_1 v_{11} + \varepsilon_1^2 v_{21} + \varepsilon_1^3 v_{31}, \tag{2.11}$$

$$r_2 = v_{02} + \varepsilon_2 v_{12} + \varepsilon_2^2 v_{22} + \varepsilon_2^3 v_{32}, \tag{2.12}$$

where $\vec{\mathbf{v}}_i = (v_{i1}, v_{i2})^T$, $i = 0, 1, 2, 3$, and their corresponding defining equations on $\overline{\Omega}$ are given by

$$\mathcal{L}^1_{\mu_1} v_{01} = f_1,\ v_{01}(x,y) = z_1(x,y),\ \forall(x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.13a}$$

$$\mathcal{L}^1_{\mu_1} v_{11} = -\Delta v_{01},\ v_{11}(x,y) = z_1(x,y),\ \forall(x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.13b}$$

$$\mathcal{L}^1_{\mu_1} v_{21} = -\Delta v_{11},\ v_{21}(x,y) = 0,\ \forall(x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.13c}$$

$$\mathcal{L}^1_{\varepsilon_1,\mu_1} v_{31} = -\Delta v_{21}, \quad v_{31}(x,y) = 0,\ \forall(x,y) \in \Gamma, \tag{2.13d}$$

and

$$\mathcal{L}^2_{\mu_2} v_{02} = f_2,\ v_{02}(x,y) = z_2(x,y),\ \forall(x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.14a}$$

$$\mathcal{L}^2_{\mu_2} v_{12} = -\Delta v_{02},\ v_{12}(x,y) = z_2(x,y),\ \forall(x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.14b}$$

$$\mathcal{L}^2_{\mu_2} v_{22} = -\Delta v_{12},\ v_{22}(x,y) = 0,\ \forall(x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.14c}$$

$$\mathcal{L}^2_{\varepsilon_2,\mu_2} v_{32} = -\Delta v_{22}, \quad v_{32}(x,y) = 0,\ \forall(x,y) \in \Gamma, \tag{2.14d}$$

respectively.

Then, using a similar methodology to that in [11], and applying Lemmas 2.2 and 2.3 to both problems (2.13d) and (2.14b), it holds

$$\left\|\frac{\partial^{l_1+l_2}\vec{\mathbf{r}}}{\partial x^{l_1}\partial y^{l_2}}\right\| \leq C,\ 0 \leq l_1 + l_2 \leq 2,\ \left\|\frac{\partial^{l_1+l_2} r_1}{\partial x^{l_1}\partial y^{l_2}}\right\| \leq C\varepsilon_1^{-1},\ l_1 + l_2 = 3, \tag{2.15a}$$

$$\left\|\frac{\partial^{l_1+l_2} r_2}{\partial x^{l_1}\partial y^{l_2}}\right\| \leq C\varepsilon_2^{-1},\ l_1 + l_2 = 3. \tag{2.15b}$$

In second place, to study the asymptotic behavior of the layer functions, we define $\mathcal{B}^l_i(x)$, $\mathcal{B}^r_i(x)$, and $\mathcal{B}^l_i(y)$, $\mathcal{B}^r_i(y)$, $i = 1, 2$, which, on the domain Ω, are given by

$$\mathcal{B}^l_1(x) = \begin{cases} e^{-\theta_1 x},\ \vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \leq \vartheta{\mu_2}^2 < \Lambda\varepsilon_2, \\ e^{-\lambda_1 x},\ \Lambda\varepsilon_1 \leq \vartheta{\mu_1}^2 < \Lambda\varepsilon_2 \leq \vartheta{\mu_2}^2, \end{cases} \qquad \mathcal{B}^r_1(x) = \begin{cases} e^{-\theta_1(1-x)},\ \vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \leq \vartheta{\mu_2}^2 < \Lambda\varepsilon_2, \\ e^{-\kappa_1(1-x)},\ \vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \leq \vartheta{\mu_2}^2 < \Lambda\varepsilon_2, \end{cases} \tag{2.16a}$$

$$\mathcal{B}^l_2(x) = \begin{cases} e^{-\theta_2 x},\ \vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \leq \vartheta{\mu_2}^2 < \Lambda\varepsilon_2, \\ e^{-\lambda_2 x},\ \vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \leq \vartheta{\mu_2}^2 < \Lambda\varepsilon_2, \end{cases} \qquad \mathcal{B}^r_2(x) = \begin{cases} e^{-\theta_2(1-x)},\ \vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \leq \vartheta{\mu_2}^2 < \Lambda\varepsilon_2, \\ e^{-\kappa_2(1-x)},\ \vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \leq \vartheta{\mu_2}^2 < \Lambda\varepsilon_2, \end{cases} \tag{2.16b}$$

where $\theta_i = \sqrt{\dfrac{\vartheta\Lambda}{\varepsilon_i}}$, $\lambda_i = \dfrac{\vartheta\mu_i}{\varepsilon_i}$, $\kappa_i = \dfrac{\Lambda}{2\mu_i}$, for $i = 1, 2$. Similarly, we can describe the functions corresponding to the y-direction $\mathcal{B}^b_i(y)$, $\mathcal{B}^t_i(y)$, $i = 1, 2$.

Theorem 2.4. *Let $\vec{w}_k$, $k = l, r, b, t$, where $\vec{w}_k = (w_{k_1}, w_{k_2})^T$ satisfy the problem (2.4) for the two cases defined in (2.1). Then, the following bounds hold for the singular components.*
For **Case 1***, it holds*

$$
\begin{aligned}
&|w_{l_1}(x,y)| \le C\mathcal{B}_1^l(x),\ |w_{l_2}(x,y)| \le C\mathcal{B}_2^l(x),\ \ |w_{b_1}(x,y)| \le C\mathcal{B}_1^b(y),\ |w_{b_2}(x,y)| \le C\mathcal{B}_2^b(y),\\
&\left|\frac{\partial^{i+j} w_{l_1}}{\partial x^i \partial y^j}\right| \le C(\varepsilon_1^{-i/2}\mathcal{B}_1^l(x) + \varepsilon_2^{-i/2}\mathcal{B}_2^l(x)),\ \left|\frac{\partial^{i+j} w_{b_1}}{\partial x^i \partial y^j}\right| \le C(\varepsilon_1^{-j/2}\mathcal{B}_1^b(y) + \varepsilon_2^{-j/2}\mathcal{B}_2^b(y)),\ i,j = 1,2,3,\\
&\left|\frac{\partial^{i+j} w_{l_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-i/2}\mathcal{B}_2^l(x),\ i,j=1,2,\ \left|\frac{\partial^{i+j} w_{l_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-1}(\mu_1\varepsilon_1^{-1}\mathcal{B}_1^l(x) + \varepsilon_2^{-1/2}\mathcal{B}_2^l(x)),\ i,j=3,\\
&\left|\frac{\partial^{i+j} w_{b_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-j/2}\mathcal{B}_2^b(y),\ i,j=1,2,\ \left|\frac{\partial^{i+j} w_{b_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-1}(\mu_1\varepsilon_1^{-1}\mathcal{B}_1^b(y) + \varepsilon_2^{-1/2}\mathcal{B}_2^b(y)),\ i,j=3,\\
&|w_{r_1}(x,y)| \le C\mathcal{B}_1^r(x),\ |w_{r_2}(x,y)| \le C\mathcal{B}_2^r(x),\ \ |w_{t_1}(x,y)| \le C\mathcal{B}_1^t(y),\ |w_{t_2}(x,y)| \le C\mathcal{B}_2^l(y),\\
&\left|\frac{\partial^{i+j} w_{r_1}}{\partial x^i \partial y^j}\right| \le C({\varepsilon_1}^{-i/2}\mathcal{B}_1^r(x) + \varepsilon_2^{-i/2}\mathcal{B}_2^r(x)),\ \left|\frac{\partial^{i+j} w_{t_1}}{\partial x^i \partial y^j}\right| \le C(\varepsilon_1^{-j/2}\mathcal{B}_1^t(y) + \varepsilon_2^{-j/2}\mathcal{B}_2^t(y)),\ i,j = 1,2,3,\\
&\left|\frac{\partial^{i+j} w_{r_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-i/2}\mathcal{B}_2^r(x),\ i,j=1,2,\ \left|\frac{\partial^{i+j} w_{r_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-1}({\mu_1}^{-3}\mathcal{B}_1^r(x) + \varepsilon_2^{-1/2}\mathcal{B}_2^r(x)),\ i,j=3,\\
&\left|\frac{\partial^{i+j} w_{t_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-j/2}\mathcal{B}_2^t(y),\ i,j=1,2,\ \left|\frac{\partial^{i+j} w_{t_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-1}({\mu_1}^{-3}\mathcal{B}_1^t(y) + \varepsilon_2^{-1/2}\mathcal{B}_2^t(y)),\ i,j=3.
\end{aligned}
$$

For **Case 2***, we have*

$$
\begin{aligned}
&|w_{l_1}(x,y)| \le C\mathcal{B}_1^l(x),\ |w_{l_2}(x,y)| \le C\mathcal{B}_2^l(x),\ \ |w_{b_1}(x,y)| \le C\mathcal{B}_1^b(y),\ |w_{b_2}(x,y)| \le C\mathcal{B}_2^b(y),\\
&\left|\frac{\partial^{i+j} w_{l_1}}{\partial x^i \partial y^j}\right| \le C({\mu_1}^i\varepsilon_1^{-i}\mathcal{B}_1^l(x) + \varepsilon_2^{-i/2}\mathcal{B}_2^l(x)),\ \left|\frac{\partial^{i+j} w_{b_1}}{\partial x^i \partial y^j}\right| \le C({\mu_1}^j\varepsilon_1^{-j}\mathcal{B}_1^b(y) + \varepsilon_2^{-j/2}\mathcal{B}_2^b(y)),\ i,j = 1,2,\\
&\left|\frac{\partial^{i+j} w_{l_2}}{\partial x^i \partial y^j}\right| \le C{\mu_2}^i\varepsilon_2^{-i}\mathcal{B}_2^l(x),\ i,j=1,2,3,\ \left|\frac{\partial^{i+j} w_{l_1}}{\partial x^i \partial y^j}\right| \le C\varepsilon_1^{-1}(\mu_1\varepsilon_2^{-1}\mathcal{B}_1^l(x) + \varepsilon_2^{-1/2}\mathcal{B}_2^l(x)),\ i,j=3,
\end{aligned}
$$

$$
\begin{aligned}
&\left|\frac{\partial^{i+j} w_{b_2}}{\partial x^i \partial y^j}\right| \le C{\mu_2}^j\varepsilon_2^{-j}\mathcal{B}_2^b(y),\ i,j=1,2,3,\ \left|\frac{\partial^{i+j} w_{b_1}}{\partial x^i \partial y^j}\right| \le C\varepsilon_1^{-1}(\mu_1\varepsilon_1^{-1}\mathcal{B}_1^b(y) + \varepsilon_2^{-1/2}\mathcal{B}_2^b(y)),\ i,j=3,\\
&|w_{r_1}(x,y)| \le C\mathcal{B}_1^r(x),\ |w_{r_2}(x,y)| \le C\mathcal{B}_2^r(x),\ \ |w_{t_1}(x,y)| \le C\mathcal{B}_1^t(y),\ |w_{t_2}(x,y)| \le C\mathcal{B}_2^l(y),\\
&\left|\frac{\partial^{i+j} w_{r_1}}{\partial x^i \partial y^j}\right| \le C({\mu_1}^{-i}\mathcal{B}_1^r(x) + \varepsilon_2^{-i/2}\mathcal{B}_2^r(x)),\ \left|\frac{\partial^{i+j} w_{t_1}}{\partial x^i \partial y^j}\right| \le C({\mu_1}^{-j}\mathcal{B}_1^t(y) + \varepsilon_2^{-j/2}\mathcal{B}_2^t(y)),\ i,j = 1,2,\\
&\left|\frac{\partial^{i+j} w_{r_2}}{\partial x^i \partial y^j}\right| \le C{\mu_2}^{-i}\mathcal{B}_2^r(x),\ i,j=1,2,\ \left|\frac{\partial^{i+j} w_{r_1}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-1}({\mu_1}^{-3}\mathcal{B}_1^r(x) + \varepsilon_2^{-1/2}\mathcal{B}_2^r(x)),\ i,j=3,\\
&\left|\frac{\partial^{i+j} w_{t_2}}{\partial x^i \partial y^j}\right| \le C{\mu_2}^{-j}\mathcal{B}_2^t(y),\ i,j=1,2,\ \left|\frac{\partial^{i+j} w_{t_1}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-1}({\mu_1}^{-3}\mathcal{B}_1^t(y) + \varepsilon_2^{-1/2}\mathcal{B}_2^t(y)),\ i,j=3.
\end{aligned}
$$

Proof. To prove the estimates for the derivatives of the layer components, we use the concept of extended domains (see, for instance, [18, 23, 24] for more details). In this context, we focus on the bounds for the left layer component for the two cases defined in (2.1). To do that, we consider the change of variable for both components, $\Psi_1 = x/\nu_1$, $\Psi_2 = x/\nu_2$, where $\nu_1 = \varepsilon_1(\mu_1 + \sqrt{\varepsilon_1})^{-1}$, $\nu_2 = \varepsilon_2(\mu_2 + \sqrt{\varepsilon_2})^{-1}$. Then, the resulting problems after this transformation of variables are given by

$$\begin{cases} \left(\varepsilon_1\nu_1^{-2}\dfrac{\partial^2 w_{l_1}^*}{\partial\Psi_1^2} + \varepsilon_1\dfrac{\partial^2 w_{l_1}^*}{\partial y^2}\right) + \mu_1\nu_1^{-1}a_1^{1^*}(\Psi_1,y)\dfrac{\partial w_{l_1}^*}{\partial\Psi_1} + \mu_1 a_2^{1^*}(\Psi_1,y)\dfrac{\partial w_{l_1}^*}{\partial y} - \quad b_{11}^*(\Psi_1,y)w_{l_1}^* - b_{12}^*(\Psi_1,y)w_{l_2}^* \\ \qquad = f_1^*(\Psi_1,y), \quad \forall(\Psi_1,y)\in\Omega_{\nu_1}^{*,1}, \\ w_{l_1}^*(\Psi_1,y) = (z_1^* - v_1^*)(\Psi_1,y), \quad \forall(\Psi_1,y)\in\Gamma_{1,\nu_1}^{*,1}, \\ w_{l_1}^*(\Psi_1,y) = 0, \quad \forall(\Psi_1,y)\in\Gamma_{2,\nu_1}^{*,1}\cup\Gamma_{3,\nu_1}^{*,1}\cup\Gamma_{4,\nu_1}^{*,1}, \end{cases} \tag{2.17a}$$

$$\begin{cases} \left(\varepsilon_2\nu_2^{-2}\dfrac{\partial^2 w_{l_2}^*}{\partial\Psi_2^2} + \varepsilon_2\dfrac{\partial^2 w_{l_2}^*}{\partial y^2}\right) + \mu_2\nu_2^{-1}a_1^{2^*}(\Psi_2,y)\dfrac{\partial w_{l_2}^*}{\partial\Psi_2} + \mu_2 a_2^{2^*}(\Psi_2,y)\dfrac{\partial w_{l_2}^*}{\partial y} - \quad b_{21}^*(\Psi_2,y)w_{l_1}^* - b_{22}^*(\Psi_2,y)w_{l_2}^* \\ \qquad = f_2^*(\Psi_2,y), \quad \forall(\Psi_2,y)\in\Omega_{\nu_2}^{*,1}, \\ w_{l_2}^*(\Psi_2,y) = (z_2^* - v_2^*)(\Psi_2,y), \quad \forall(\Psi_2,y)\in\Gamma_{1,\nu_2}^{*,1}, \\ w_{l_2}^*(\Psi_2,y) = 0, \quad \forall(\Psi_2,y)\in\Gamma_{2,\nu_2}^{*,1}\cup\Gamma_{3,\nu_2}^{*,1}\cup\Gamma_{4,\nu_2}^{*,1}, \end{cases} \tag{2.17b}$$

respectively, where $\Omega_{\nu_1}^{*,1} = (0,1/\nu_1)\times(0,1)$, $\Omega_{\nu_2}^{*,1} = (0,1/\nu_2)\times(0,1)$ and $\Gamma_{i,\nu_1}^{*,1}$, $\Gamma_{i,\nu_2}^{*,1}$, $i = 1,2,3,4$, are the corresponding boundaries of the extended domain $\Omega_{\nu_1}^{*,1}$ and $\Omega_{\nu_2}^{*,1}$, respectively.

Then, using a similar methodology to this one in [6, 19], we can get the bounds for the left layer component $\vec{\mathbf{w}}_l^{\,*}$.

Analogously, we can consider the relative boundary layer components $\vec{\mathbf{w}}_k$, $k = r,b,t$, for the two cases defined in (2.1). □

Theorem 2.5. *Let* $\vec{s}_{k_1k_2}$, $k_1,k_2 = l,b,r,t$, *where* $\vec{s}_{k_1k_2} = (s_{(k_1k_2)_1}, s_{(k_1k_2)_2})^T$ *satisfy problem (2.5) for the two cases defined in (2.1). Then, the following bounds are satisfied by the corner components.*
For **Case 1**, *it holds*

$$\begin{aligned} &|\vec{s}_{(lb)_1}(x,y)| \le C\mathcal{B}_1^l(x)\mathcal{B}_1^b(y),\ |\vec{s}_{(lb)_2}(x,y)| \le C\mathcal{B}_2^l(x)\mathcal{B}_2^b(y),\ |\vec{s}_{(br)_1}(x,y)| \le C\mathcal{B}_1^r(x)\mathcal{B}_1^b(y),\\ &|\vec{s}_{(br)_2}(x,y)| \le C\mathcal{B}_2^r(x)\mathcal{B}_2^b(y),\ |\vec{s}_{(rt)_1}(x,y)| \le C\mathcal{B}_1^r(x)\mathcal{B}_1^t(y),\ |\vec{s}_{(rt)_2}(x,y)| \le C\mathcal{B}_2^r(x)\mathcal{B}_2^t(y),\\ &|\vec{s}_{(lt)_1}(x,y)| \le C\mathcal{B}_1^l(x)\mathcal{B}_1^t(y),\ |\vec{s}_{(lt)_2}(x,y)| \le C\mathcal{B}_2^l(x)\mathcal{B}_2^t(y),\\ &\left|\frac{\partial^{i+j}s_{lb_1}}{\partial x^i\partial y^j}\right| \le C(\varepsilon_1^{(-i-j)/2}\mathcal{B}_1^l(x)\mathcal{B}_1^b(y) + \varepsilon_2^{(-i-j)/2}\mathcal{B}_2^l(x)\mathcal{B}_2^b(y)),\ 1\le i+j\le 3,\\ &\left|\frac{\partial^{i+j}s_{(lb)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^l(x)\mathcal{B}_2^b(y),\ 1\le i+j\le 2, \end{aligned}$$

$$\begin{aligned} &\left|\frac{\partial^{i+j}s_{(lb)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{-2}({\mu_1}^2\varepsilon_1^{-2}\mathcal{B}_1^l(x)\mathcal{B}_1^b(y) + \varepsilon_2^{-1}\mathcal{B}_2^l(x)\mathcal{B}_2^b(y),\ i+j = 3,\\ &\left|\frac{\partial^{i+j}s_{(br)_1}}{\partial x^i\partial y^j}\right| \le C(\varepsilon_1^{(-i-j)/2}\mathcal{B}_1^r(x)\mathcal{B}_1^b(y) + \varepsilon_2^{(-1-j)/2}\mathcal{B}_2^r(x)\mathcal{B}_2^b(y),\ 1\le i+j\le 3, \end{aligned}$$

$$\left|\frac{\partial^{i+j}s_{(br)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^r(x)\mathcal{B}_2^b(y),\ 0\le i+j\le 2,$$
$$\left|\frac{\partial^{i+j}s_{(br)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{-2}(\mu_2{}^{-2}\varepsilon_1^{-1}\mathcal{B}_1^b(y)+\varepsilon_2^{-1}\mathcal{B}_2^r(x)\mathcal{B}_2^b(y),\ i+j=3,$$
$$\left|\frac{\partial^{i+j}s_{(rt)_1}}{\partial x^i\partial y^j}\right| \le C(\varepsilon_1{}^{(-i-j)/2}\mathcal{B}_1^r(x)\mathcal{B}_1^t(y)+\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^r(x)\mathcal{B}_2^t(y)),\ 1\le i+j\le 3,$$
$$\left|\frac{\partial^{i+j}s_{(rt)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^r(x)\mathcal{B}_2^t(y),\ 1\le i+j\le 2,$$
$$\left|\frac{\partial^{i+j}s_{(rt)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{-2}(\mu_2{}^{-6}\mathcal{B}_1^r(x)\mathcal{B}_1^t(y)+\varepsilon_2^{-1}\mathcal{B}_2^r(x)\mathcal{B}_2^t(y)),\ i+j=3,$$
$$\left|\frac{\partial^{i+j}s_{(lt)_1}}{\partial x^i\partial y^j}\right| \le C(\varepsilon_1^{(-i-j)/2}\mathcal{B}_1^l(x)\mathcal{B}_1^t(y)+\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^l(x)\mathcal{B}_2^t(y)),\ 1\le i+j\le 3,$$
$$\left|\frac{\partial^{i+j}s_{(lt)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^l(x)\mathcal{B}_2^t(y),\ 1\le i+j\le 2,$$
$$\left|\frac{\partial^{i+j}s_{(lt)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{-2}(\mu_2{}^{-2}\varepsilon_1^{-1}\mathcal{B}_1^l(x)\mathcal{B}_1^t(y)+\varepsilon_2^{-1}\mathcal{B}_2^l(x)\mathcal{B}_2^t(y)),\ i+j=3.$$

For the **Case 2***, we have*

$$|\vec{s}_{(lb)_1}(x,y)|\le C\mathcal{B}_1^l(x)\mathcal{B}_1^b(y),\ |\vec{s}_{(lb)_2}(x,y)|\le C\mathcal{B}_2^l(x)\mathcal{B}_2^b(y),\ |\vec{s}_{(br)_1}(x,y)|\le C\mathcal{B}_1^r(x)\mathcal{B}_1^b(y),$$
$$|\vec{s}_{(br)_2}(x,y)|\le C\mathcal{B}_2^r(x)\mathcal{B}_2^b(y),\ |\vec{s}_{(rt)_1}(x,y)|\le C\mathcal{B}_1^r(x)\mathcal{B}_1^t(y),\ |\vec{s}_{(rt)_2}(x,y)|\le C\mathcal{B}_2^r(x)\mathcal{B}_2^t(y),$$
$$|\vec{s}_{(lt)_1}(x,y)|\le C\mathcal{B}_1^l(x)\mathcal{B}_1^t(y),\ |\vec{s}_{lt_2}(x,y)|\le C\mathcal{B}_2^l(x)\mathcal{B}_2^t(y),$$
$$\left|\frac{\partial^{i+j}s_{(lb)_1}}{\partial x^i\partial y^j}\right| \le C(\mu_1^{i+j}\varepsilon_1^{-i-j}\mathcal{B}_1^l(x)\mathcal{B}_1^b(y)+\varepsilon_2^{(-i-j)/2}\mathcal{B}_1^l(x)\mathcal{B}_1^b(y)),\ 1\le i+j\le 2,$$
$$\left|\frac{\partial^{i+j}s_{(lb)_2}}{\partial x^i\partial y^j}\right| \le C\mu_2{}^{i+j}\varepsilon_2^{-i-j}\mathcal{B}_2^l(x)\mathcal{B}_1^b(y),\ 1\le i+j\le 3,$$
$$\left|\frac{\partial^{i+j}s_{(lb)_1}}{\partial x^i\partial y^j}\right| \le C\varepsilon_1^{-2}(\mu_1^2\varepsilon_1^{-2}\mathcal{B}_1^l(x)\mathcal{B}_1^b(y)+\varepsilon_2^{-1}\mathcal{B}_2^l(x)\mathcal{B}_2^b(y),\ i+j=3,$$
$$\left|\frac{\partial^{i+j}s_{(br)_1}}{\partial x^i\partial y^j}\right| \le C(\mu_1^{-i+j}\varepsilon_1^{-j}\mathcal{B}_1^r(x)\mathcal{B}_1^b(y)+\varepsilon_2^{(-i-j)/2}\mathcal{B}_1^r(x)\mathcal{B}_1^b(y)),\ 1\le i+j\le 2,$$
$$\left|\frac{\partial^{i+j}s_{(br)_2}}{\partial x^i\partial y^j}\right| \le C\mu_1{}^{-i+j}\varepsilon_1^{-j}\mathcal{B}_1^b(y),\ 0\le i+j\le 3,$$
$$\left|\frac{\partial^{i+j}s_{(br)_1}}{\partial x^i\partial y^j}\right| \le C\varepsilon_1^{-2}(\mu_1^{-2}\varepsilon_1^{-1}\mathcal{B}_1^r(x)\mathcal{B}_1^b(y)+\varepsilon_2^{-1}\mathcal{B}_2^r(x)\mathcal{B}_2^b(y)),\ i+j=3,$$
$$\left|\frac{\partial^{i+j}s_{(rt)_1}}{\partial x^i\partial y^j}\right| \le C(\mu_1^{-i-j}\mathcal{B}_1^r(x)\mathcal{B}_1^t(y)+\varepsilon_2^{(-i-j)/2}\mathcal{B}_1^r(x)\mathcal{B}_1^t(y)),\ 1\le i+j\le 2,$$
$$\left|\frac{\partial^{i+j}s_{(rt)_2}}{\partial x^i\partial y^j}\right| \le C\mu_2{}^{-i-j}\mathcal{B}_2^r(x)\mathcal{B}_2^t(y),\ 1\le i+j\le 2,$$
$$\left|\frac{\partial^{i+j}s_{(rt)_1}}{\partial x^i\partial y^j}\right| \le C\varepsilon_1^{-2}(\mu_2{}^{-6}\mathcal{B}_2^r(x)\mathcal{B}_2^t(y)+\varepsilon_2^{-1}\mathcal{B}_2^r(x)\mathcal{B}_2^t(y)),\ i+j=3,$$

$$\left|\frac{\partial^{i+j}s_{(lt)_1}}{\partial x^i\partial y^j}\right| \le C(\mu_1^{i-j}\varepsilon_1^{-i}\mathcal{B}_1^l(x)\mathcal{B}_1^t(y)+\varepsilon_2^{(-i-j)/2}\mathcal{B}_1^l(x)\mathcal{B}_1^t(y)),\ 1\le i+j\le 2,$$

$$\left|\frac{\partial^{i+j}s_{(lt)_2}}{\partial x^i\partial y^j}\right| \le C({\mu_1}^{i-j}\varepsilon_1^{-i}\mathcal{B}_1^l(x)+\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^l(x)\mathcal{B}_2^t(y)),\ 1\le i+j\le 3,$$

$$\left|\frac{\partial^{i+j}s_{(lt)_1}}{\partial x^i\partial y^j}\right| \le C\varepsilon_1^{-2}({\mu_1}^{-2}\varepsilon_1^{-1}\mathcal{B}_1^l(x)+\varepsilon_2^{-1}\mathcal{B}_2^l(x)\mathcal{B}_2^t(y)),\ i+j=3.$$

Proof. Here, we only discuss the bounds on the derivatives of the corner layer component $\vec{\mathbf{s}}_{lb}$. For this component, we consider the new variables $\Psi_1 = x/\nu_1$, $\varphi_1 = y/\nu_1$, and $\Psi_2 = x/\nu_2$, $\varphi_2 = y/\nu_2$, where $\nu_1 = \varepsilon_1(\mu_1+\sqrt{\varepsilon_1})^{-1}$, $\nu_2 = \varepsilon_2(\mu_2+\sqrt{\varepsilon_2})^{-1}$. After changing the variables, the resulting continuous problems are given by

$$\begin{cases}\varepsilon_1\nu_1^{-2}\Big(\dfrac{\partial^2 s^*_{lb_1}}{\partial\Psi_1^2}+\dfrac{\partial^2 s^*_{lb_1}}{\partial\varphi_1^2}\Big)+\mu_1\nu_1^{-1}\Big(a_1^{1^*}(\Psi_1,\varphi_1)\dfrac{\partial s^*_{lb_1}}{\partial\Psi_1}+a_2^{1^*}(\Psi_1,\varphi_1)\dfrac{\partial s^*_{lb_1}}{\partial\varphi_1}\Big)-b^*_{11}(\Psi_1,\varphi_1)s^*_{lb_1}-b^*_{12}(\Psi_1,\varphi_1)s^*_{lb_2}\\ \qquad\qquad\qquad\qquad = f_1^*(\Psi_1,\varphi_1),\ \forall(\Psi_1,\varphi_1)\in\Omega^{*,2}_{\nu_1},\\ s^*_{lb_1}(\Psi_1,\varphi_1)=-w^*_{l_1}(\Psi_1,\varphi_1),\quad \forall(\Psi_1,\varphi_1)\in\Gamma^{*,2}_{1,\nu_1},\\ s^*_{lb_1}(\Psi_1,\varphi_1)=-w^*_{b_1}(\Psi_1,\varphi_1),\quad \forall(\Psi_1,\varphi_1)\in\Gamma^{*,2}_{2,\nu_1},\quad s^*_{lb_1}(\Psi_1,\varphi_1)=0,\quad \forall(\Psi_1,\varphi_1)\in\Gamma^{*,2}_{3,\nu_1}\cup\Gamma^{*,2}_{4,\nu_1},\end{cases}$$

and

$$\begin{cases}\varepsilon_2\nu_2^{-2}\Big(\dfrac{\partial^2 s^*_{lb_2}}{\partial\Psi_2^2}+\dfrac{\partial^2 s^*_{lb_2}}{\partial\varphi_2^2}\Big)+\mu_2\nu_2^{-1}\Big(a_1^{2^*}(\Psi_2,\varphi_2)\dfrac{\partial s^*_{lb_2}}{\partial\Psi_2}+a_2^{2^*}(\Psi_2,\varphi_2)\dfrac{\partial s^*_{lb_2}}{\partial\varphi_2}\Big)-b^*_{21}(\Psi_2,\varphi_2)s^*_{lb_1}-b^*_{22}(\Psi_2,\varphi_2)s^*_{lb_2}\\ \qquad\qquad\qquad\qquad = f_2^*(\Psi_2,\varphi_2),\ \forall(\Psi_2,\varphi_2)\in\Omega^{*,2}_{\nu_2},\\ s^*_{lb_2}(\Psi_2,\varphi_2)=-w^*_{l_2}(\Psi_2,\varphi_2),\quad \forall(\Psi_2,\varphi_2)\in\Gamma^{*,2}_{1,\nu_2},\\ s^*_{lb_2}(\Psi_2,\varphi_2)=-w^*_{b_2}(\Psi_2,\varphi_2),\quad \forall(\Psi_2,\varphi_2)\in\Gamma^{*,2}_{2,\nu_2},\quad s^*_{lb_2}(\Psi_2,\varphi_2)=0,\quad \forall(\Psi_2,\varphi_2)\in\Gamma^{*,2}_{3,\nu_2}\cup\Gamma^{*,2}_{4,\nu_2},\end{cases}$$

respectively, where $\Omega^{*,2}_{\nu_1} = (0,1/\nu_1)^2$, $\Omega^{*,2}_{\nu_2} = (0,1/\nu_2)^2$ and $\Gamma^{*,2}_{i,\nu_1}$, $\Gamma^{*,2}_{i,\nu_2}$ $i = 1,...,4$ are the relative boundaries of the extended domains $\Omega^{*,2}_{\nu_1}$, and $\Omega^{*,2}_{\nu_2}$, respectively.

Then, using a similar methodology to this one in [6, 19, 23, 24], we can obtain the bounds for the corner layer component $\vec{\mathbf{s}}^*{}_{lb}$.

Analogously, we can consider the relative corner layer components $\vec{\mathbf{s}}_k$, $k = br, rt, lt$, for which we can get analogous bounds for the same two cases defined in (2.1). □

3. Discretization of the problem

In this section, we construct the numerical method to approximate the solution of (1.1). The first step is to define a special nonuniform mesh of Shishkin type, adapted to the behavior of the exact solution for each one of the cases given in (2.1). Then, when we have the mesh, we will define the finite difference method (FDM) used to solve the problem (1.1) on the domain $\bar{\Omega}^{N,N} = \{(x_i, y_j) : 0 \le i, j \le N\}$. For simplicity in the presentation, we take the same number of mesh points for the x and y variables.

3.1. The Shishkin mesh

The mesh is given as a tensorial product of appropriate one-dimensional Shishkin meshes. We define the one-dimensional mesh associated to the x-variable, x_i, $i = 0, 1, \ldots, N$, and similarly, we can define the mesh points associated to the y-variable, $y_j,\ j = 0, 1, \ldots, N$.

Case 1: If $\vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \leq \vartheta{\mu_2}^2 < \Lambda\varepsilon_2$, then, the piecewise uniform Shishkin mesh is developed and split the unit interval $[0, 1]$ into seven subintervals in the form

$$[0,1] = [0,\tau_1] \cup [\tau_1,\tau_2] \cup [\tau_2,\tau_3] \cup [\tau_3, 1-\tau_3] \cup [1-\tau_3, 1-\tau_2] \cup [1-\tau_2, 1-\sigma_1] \cup [1-\sigma_1, 1],$$

where the transition points $\tau_i,\ i = 1, 2, 3$ and σ_1 are defined as

$$\tau_1 = \min\left\{\frac{\tau_2}{2}, \frac{2\varepsilon_1}{\mu_2\vartheta}\ln N\right\},\ \tau_2 = \min\left\{\frac{2\tau_3}{3}, 2\sqrt{\frac{\varepsilon_1}{\Lambda\vartheta}}\ln N\right\},\ \tau_3 = \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon_2}{\Lambda\vartheta}}\ln N\right\}, \tag{3.1a}$$

$$\sigma_1 = \min\left\{\frac{\tau_2}{2}, \frac{2\mu_2}{\Lambda}\ln N\right\}. \tag{3.1b}$$

On each of the subintervals $[0,\tau_1], [\tau_1,\tau_2], [\tau_2,\tau_3], [1-\tau_3, 1-\tau_2], [1-\tau_2, 1-\sigma_1], [1-\sigma_1, 1]$, there are $N/12 + 1$ uniformly spaced grid points, while in the remaining subinterval $[\tau_3, 1-\tau_3]$, there are $N/2 + 1$ grid points also uniformly spaced.
Then, the grid points along the x-axis are given by

$$x_i = \begin{cases} \frac{12}{N}\tau_1 i, & \text{if } \ 0 \leq i \leq N/12, \\ \tau_1 + \frac{12}{N}(\tau_2 - \tau_1)(i - \frac{N}{12}), & \text{if } \ N/12 + 1 \leq i \leq N/6, \\ \tau_2 + \frac{12}{N}(\tau_3 - \tau_2)(i - \frac{N}{6}), & \text{if } \ N/6 + 1 \leq i \leq N/4, \\ \tau_3 + \frac{2}{N}(1 - 2\tau_3)(i - \frac{N}{4}), & \text{if } \ N/4 + 1 \leq i \leq 3N/4, \\ 1 - \tau_3 + \frac{12}{N}(\tau_3 - \tau_2)(i - \frac{3N}{4}), & \text{if } \ 3N/4 + 1 \leq i \leq 5N/6, \\ 1 - \tau_2 + \frac{12}{N}(\tau_2 - \sigma_1)(i - \frac{5N}{6}), & \text{if } \ 5N/6 + 1 \leq i \leq 11N/12, \\ 1 - \sigma_1 + \frac{12}{N}\sigma_1(i - \frac{11N}{12}), & \text{if } \ 11N/12 + 1 \leq i \leq N. \end{cases}$$

Case 2: If $\Lambda\varepsilon_1 \leq \vartheta{\mu_1}^2 < \Lambda\varepsilon_2 \leq \vartheta{\mu_2}^2$, then the piecewise uniform Shishkin mesh is developed and splits the unit interval $[0, 1]$ into seven subintervals in the form

$$[0,1] = [0,\tau_1] \cup [\tau_1,\tau_2] \cup [\tau_2,\tau_3] \cup [\tau_3, 1-\tau_3] \cup [1-\tau_3, 1-\sigma_2] \cup [1-\sigma_2, 1-\sigma_1] \cup [1-\sigma_1, 1],$$

where the transition points $\tau_i,\ i = 1, 2, 3$, and σ_1 now are defined as

$$\tau_1 = \min\left\{\frac{\tau_2}{2}, \frac{2\varepsilon_1}{\mu_1\vartheta}\ln N\right\},\ \tau_2 = \min\left\{\frac{2\tau_3}{3}, \frac{2\varepsilon_2}{\mu_2\vartheta}\ln N\right\},\ \tau_3 = \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon_2}{\Lambda\vartheta}}\ln N\right\}, \tag{3.2a}$$

$$\sigma_1 = \min\left\{\frac{\tau_2}{2}, \frac{2\mu_2}{\Lambda}\ln N\right\}. \tag{3.2b}$$

On each of the subintervals $[0,\tau_1], [\tau_1,\tau_2], [\tau_2,\tau_3], [1-\tau_3, 1-\sigma_2], [1-\sigma_2, 1-\sigma_1], [1-\sigma_1, 1]$, there are $N/12 + 1$ uniformly spaced grid points, while the rest subinterval $[\tau_3, 1-\tau_3]$ there are $N/2+1$ grid points also uniformly spaced.

Then, the grid points along the x-axis are given by

$$x_i = \begin{cases} \frac{12}{N}\tau_1 i, & \text{if} \quad 0 \le i \le N/12, \\ \tau_1 + \frac{12}{N}(\tau_2 - \tau_1)(i - \frac{N}{12}), & \text{if} \quad N/12 + 1 \le i \le N/6, \\ \tau_2 + \frac{12}{N}(\tau_3 - \tau_2)(i - \frac{N}{6}), & \text{if} \quad N/6 + 1 \le i \le N/4, \\ \tau_3 + \frac{2}{N}(1 - 2\tau_3)(i - \frac{N}{4}), & \text{if} \quad N/4 + 1 \le i \le 3N/4, \\ 1 - \tau_3 + \frac{12}{N}(\tau_3 - \sigma_2)(i - \frac{3N}{4}), & \text{if} \quad 3N/4 + 1 \le i \le 5N/6, \\ 1 - \sigma_2 + \frac{12}{N}(\sigma_2 - \sigma_1)(i - \frac{5N}{6}), & \text{if} \quad 5N/6 + 1 \le i \le 11N/12, \\ 1 - \sigma_1 + \frac{12}{N}\sigma_1(i - \frac{11N}{12}), & \text{if} \quad 11N/12 + 1 \le i \le N. \end{cases}$$

For each one of the cases, the step sizes are defined as $h_i = x_i - x_{i-1},\ i = 1, 2, ..., N,\ \bar{h}_i = h_i + h_{i+1},\ i = 1, 2, ..., N - 1$.

The boundaries of the domain $\bar{\Omega}^{N,N}$ are denoted by

$$\Gamma_1^{N,N} = \Big\{(0, y_j)\,\Big|\,0 \le j \le N\Big\}, \Gamma_2^{N,N} = \Big\{(x_i, 0)\,\Big|\,0 \le i \le N\Big\},$$
$$\Gamma_3^{N,N} = \Big\{(1, y_j)\,\Big|\,0 \le j \le N\Big\}, \Gamma_4^{N,N} = \Big\{(x_i, 1)\,\Big|\,0 \le i \le N\Big\},$$

and $\Gamma^{N,N} = \Gamma_1^{N,N} \cup \Gamma_2^{N,N} \cup \Gamma_3^{N,N} \cup \Gamma_4^{N,N}$.

3.2. Finite difference method (FDM)

On an arbitrary mesh, $\bar{\Omega}^{N,N}$, to discretize (1.1), we consider the classical upwind finite difference scheme, which is given by

$$\begin{cases} \vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N}\vec{\mathbf{Z}}(x_i, y_j) = \vec{\mathbf{f}}(x_i, y_j), \ \forall (x_i, y_j) \in \Omega^{N,N}, \\ \vec{\mathbf{Z}}(x_i, y_j) = \vec{\mathbf{g}}_1(y_j), \ (x_i, y_j) \in \Gamma_1^{N,N}, \quad \vec{\mathbf{Z}}(x_i, y_j) = \vec{\mathbf{g}}_2(x_i), \ (x_i, y_j) \in \Gamma_2^{N,N}, \\ \vec{\mathbf{Z}}(x_i, y_j) = \vec{\mathbf{g}}_3(y_j), \ (x_i, y_j) \in \Gamma_3^{N,N}, \quad \vec{\mathbf{Z}}(x_i, y_j) = \vec{\mathbf{g}}_4(x_i), \ (x_i, y_j) \in \Gamma_4^{N,N}, \end{cases} \tag{3.3}$$

where

$$\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N}\vec{\mathbf{Z}}(x_i, y_j) = \vec{\varepsilon}(\delta_{xx}^2 + \delta_{yy}^2)\vec{\mathbf{Z}}(x_i, y_j) + \vec{\mu}(\vec{\mathbf{A}}_1(x_i, y_j)D_x^+ + \vec{\mathbf{A}}_2(x_i, y_j)D_y^+)\vec{\mathbf{Z}}(x_i, y_j) - \vec{\mathbf{B}}(x_i, y_j)\vec{\mathbf{Z}}(x_i, y_j).$$

As it is usual, the discrete differential operators D_x^+, D_y^+, δ_{xx}^2, and δ_{yy}^2 are given by

$$D_x^+\vec{\mathbf{Z}}(x_i, y_j) = \frac{\vec{\mathbf{Z}}(x_{i+1}, y_j) - \vec{\mathbf{Z}}(x_i, y_j)}{h_{i+1}}, \ D_x^-\vec{\mathbf{Z}}(x_i, y_j) = \frac{\vec{\mathbf{Z}}(x_i, y_j) - \vec{\mathbf{Z}}(x_{i-1}, y_j)}{h_i},$$
$$D_y^+\vec{\mathbf{Z}}(x_i, y_j) = \frac{\vec{\mathbf{Z}}(x_i, y_{j+1}) - \vec{\mathbf{Z}}(x_i, y_j)}{k_{j+1}}, \ D_y^-\vec{\mathbf{Z}}(x_i, y_j) = \frac{\vec{\mathbf{Z}}(x_i, y_j) - \vec{\mathbf{Z}}(x_i, y_{j-1})}{k_j},$$
$$\delta_{xx}^2\vec{\mathbf{Z}}(x_i, y_j) = \frac{2}{\bar{h}_i}(D_x^+\vec{\mathbf{Z}}(x_i, y_j) - D_x^-\vec{\mathbf{Z}}(x_i, y_j)), \ \delta_{yy}^2\vec{\mathbf{Z}}(x_i, y_j) = \frac{2}{\bar{k}_j}(D_y^+\vec{\mathbf{Z}}(x_i, y_j) - D_y^-\vec{\mathbf{Z}}(x_i, y_j)),$$

for $i, j = 1, 2, \dots, N-1$.

On the piecewise-uniform mesh $\bar{\Omega}^{N,N}$, the elements of the system matrix $\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}$ are given by

$$\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}\vec{Z}(x_i, y_j) \equiv \vec{q}_1(x_i, y_j)\vec{Z}(x_{i+1}, y_j) + \vec{q}_2(x_i, y_j)\vec{Z}(x_{i-1}, y_j) + \vec{q}_3(x_i, y_j)\vec{Z}(x_i, y_{j+1}) + \vec{q}_4(x_i, y_j)\vec{Z}(x_i, y_{j-1}) + \vec{q}_5(x_i, y_j)\vec{Z}(x_i, y_j), \tag{3.4}$$

where

$$\vec{q}_1(x_i, y_j) = \frac{\vec{\varepsilon}}{\bar{h}_i h_{i+1}} + \frac{\vec{\mu}\vec{A}_1(x_i, y_j)}{h_{i+1}},\ \vec{q}_2(x_i, y_j) = \frac{\vec{\varepsilon}}{\bar{h}_i h_i},\ \vec{q}_3(x_i, y_j) = \frac{\vec{\varepsilon}}{\bar{k}_j k_{j+1}} + \frac{\vec{\mu}\vec{A}_2(x_i, y_j)}{k_{j+1}},\ \vec{q}_4(x_i, y_j) = \frac{\vec{\varepsilon}}{\bar{k}_j k_j},$$

$$\vec{q}_5(x_i, y_j) = -\frac{\vec{\varepsilon}}{\bar{h}_i h_{i+1}} - \frac{\vec{\varepsilon}}{\bar{h}_i h_i} - \frac{\vec{\varepsilon}}{\bar{k}_j k_{j+1}} - \frac{\vec{\varepsilon}}{\bar{k}_j k_j} - \frac{\vec{\mu}\vec{A}_1(x_i, y_j)}{h_{i+1}} - \frac{\vec{\mu}\vec{A}_2(x_i, y_j)}{k_{j+1}} - \vec{B}(x_i, y_j).$$

Similarly to the continuous problem, we can prove a discrete minimum principle for the discrete operator $\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}$.

Lemma 3.1 (Discrete minimum principle)**.** *Let $\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}$ be the discrete operator given in (3.3), If $\vec{\Phi}(x_i, y_j) \geq \vec{0}$ on $\Gamma^{N,N}$ and $\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}\vec{\Phi}(x_i, y_j) \leq \vec{0}$, $\forall (x_i, y_j) \in \Omega^{N,N}$, then $\vec{\Phi}(x_i, y_j) \geq \vec{0}$, $\forall (x_i, y_j) \in \overline{\Omega}^{N,N}$.*

Proof. The proof is standard in the context of singularly perturbed problems, and it follows the ideas of [7, 26, 28, 29], where parabolic singularly perturbed systems of convection–diffusion type were considered. □

Lemma 3.2 (Discrete stability result)**.** *Let $\vec{Z}(x_i, y_j)$ be the solution of (3.3). Then it holds*

$$||\vec{Z}(x_i, y_j)||_{\overline{\Omega}^{N,N}} \leq \frac{1}{\vartheta}||\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}\vec{Z}||^{N,N}_{\Omega} + \max\left\{||\vec{Z}||_{\Gamma_1^{N,N}}, ||\vec{Z}||_{\Gamma_2^{N,N}},\ ||\vec{Z}||_{\Gamma_3^{N,N}}, ||\vec{Z}||_{\Gamma_4^{N,N}}\right\},$$

where $||.||^{N,N}_{\overline{\Omega}}$ denotes the discrete pointwise maximum norm on $\overline{\Omega}^{N,N}$ and ϑ is defined in (1.3).

Proof. It is straightforward from Lemma 3.1. □

4. Uniform convergence of the numerical method

In this section, we analyze the error of our numerical method, and we prove that it is a uniformly convergent method. To approximate the nodal errors, we decompose the discrete solution into regular (smooth), layer, and corner components, similarly to the approach used for the continuous solutions. Then, we have $\vec{Z}(x_i, y_j) = \vec{R}(x_i, y_j) + \vec{W}(x_i, y_j) + \vec{S}(x_i, y_j)$. Moreover, as for the continuous problem, the layer and the corner layer components are decomposed in the form

$$\vec{W}(x_i, y_j) = \vec{W}_l(x_i, y_j) + \vec{W}_r(x_i, y_j) + \vec{W}_b(x_i, y_j) + \vec{W}_t(x_i, y_j),$$
$$\vec{S}(x_i, y_j) = \vec{S}_{lb}(x_i, y_j) + \vec{S}_{br}(x_i, y_j) + \vec{S}_{rt}(x_i, y_j) + \vec{S}_{lt}(x_i, y_j).$$

The regular component $\vec{\mathbf{R}}(x_i, y_j)$ is the solution of the discrete problem

$$\begin{cases} \vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}\vec{\mathbf{R}}(x_i, y_j) = \vec{\mathbf{f}}(x_i, y_j), \quad \forall (x_i, y_j) \in \Omega^{N,N}, \\ \vec{\mathbf{R}}(x_i, y_j) = \vec{\mathbf{r}}(x_i, y_j), \quad \forall (x_i, y_j) \in \Gamma^{N,N}, \end{cases} \tag{4.1}$$

and the layer and corner components $\vec{\mathbf{W}}(x_i, y_j)$ and $\vec{\mathbf{S}}(x_i, y_j)$ are the solutions of the following discrete problems

$$\begin{cases} \vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}\vec{\mathbf{W}}(x_i, y_j) = \vec{\mathbf{0}}, \quad \forall (x_i, y_j) \in \Omega^{N,N}, \\ \vec{\mathbf{W}}(x_i, y_j) = \vec{\mathbf{w}}(x_i, y_j), \quad \forall (x_i, y_j) \in \Gamma^{N,N}, \end{cases} \tag{4.2}$$

for the layer component and

$$\begin{cases} \vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}\vec{\mathbf{S}}(x_i, y_j) = 0, \quad \forall (x_i, y_j) \in \Omega^{N,N}, \\ \vec{\mathbf{S}}(x_i, y_j) = \vec{\mathbf{s}}(x_i, y_j), \quad \forall (x_i, y_j) \in \Gamma^{N,N}, \end{cases} \tag{4.3}$$

for the corner component, respectively.

Then, we study the contribution to the error of each one of these components. First, we consider the error associated with the regular component.

Lemma 4.1. *Let $\vec{r}(x, y)$ be the solution of (2.3) and $\vec{R}(x_i, y_j)$ the numerical solution of (4.1) at the grid point (x_i, y_j). Then, for the four cases defined in (2.1), it holds*

$$|\vec{R}(x_i, y_j) - \vec{r}(x_i, y_j)| \leq CN^{-1}.$$

Proof. Using standard Taylor expansions, it is easy to see that the truncation error associated with the regular component (4.1) satisfies

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{R}}(x_i, y_j) - \vec{\mathbf{r}}(x_i, y_j))| \leq C\Big[(h_i + h_{i+1})\Big(\vec{\varepsilon}\Big\|\frac{\partial^3\vec{\mathbf{r}}}{\partial x^3}\Big\| + \vec{\mu}\Big\|\frac{\partial^2\vec{\mathbf{r}}}{\partial x^2}\Big\|\Big) + (k_j + k_{j+1})\Big(\vec{\varepsilon}\Big\|\frac{\partial^3\vec{\mathbf{r}}}{\partial y^3}\Big\| + \vec{\mu}\Big\|\frac{\partial^2\vec{\mathbf{r}}}{\partial y^2}\Big\|\Big)\Big].$$

Therefore, from (2.10) and (2.15), it is straightforward to prove that it holds

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{R}}(x_i, y_j) - \vec{\mathbf{r}}(x_i, y_j))| \leq \begin{cases} \begin{pmatrix} CN^{-1}(\sqrt{\varepsilon_1} + \mu_1) \\ CN^{-1}(\sqrt{\varepsilon_2} + \mu_2) \end{pmatrix}, & \text{if } \vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \leq \vartheta{\mu_2}^2 < \Lambda\varepsilon_2, \\ \begin{pmatrix} CN^{-1}(1 + \mu_1) \\ CN^{-1}(1 + \mu_2) \end{pmatrix}, & \text{if } \Lambda\varepsilon_1 \leq \vartheta{\mu_1}^2 < \Lambda\varepsilon_2 \leq \vartheta{\mu_2}^2. \end{cases}$$

Further, using Lemma 3.1, for all cases, we can obtain

$$|\vec{\mathbf{R}}(x_i, y_j) - \vec{\mathbf{r}}(x_i, y_j)| \leq CN^{-1}, \tag{4.4}$$

which is the required result. □

Next, we study the error associated with the layer components.

Lemma 4.2. *Let $\vec{\mathbf{w}}_k(x_i, y_j)$ be the true solution of* (2.4) *and $\vec{\mathbf{W}}_k(x_i, y_j)$ be the numerical solution of* (4.2) *at the grid point (x_i, y_j). Then, for* **Case 1** *defined in (2.1), we have*

$$|\vec{\mathbf{W}}_k(x_i, y_j) - \vec{\mathbf{w}}_k(x_i, y_j)| \leq CN^{-1}\ln N, \ \ k = l, b, r, t.$$

Proof. If $\tau_1 = 1/12$, $\tau_2 = 1/6$, and $\tau_3 = 1/4$, the proof follows by using standard methods for uniform meshes, because it holds $\mu_2\varepsilon_1^{-1} \leq C\ln N$, $\varepsilon_2^{-1/2} \leq C\ln N$ and $\mu_2^{-1} \leq C\ln N$. Hence, by using (4.2) and Theorem 2.4, we have

$$\begin{aligned}\|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)\| &\leq C\Big[(h_i + h_{i+1})\Big(\vec{\varepsilon}\Big\|\frac{\partial^3\vec{\mathbf{w}}_l}{\partial x^3}\Big\| + \vec{\mu}\Big\|\frac{\partial^2\vec{\mathbf{w}}_l}{\partial x^2}\Big\|\Big) + (k_j + k_{j+1})\Big(\vec{\varepsilon}\Big\|\frac{\partial^3\vec{\mathbf{w}}_l}{\partial y^3}\Big\| + \vec{\mu}\Big\|\frac{\partial^2\vec{\mathbf{w}}_l}{\partial y^2}\Big\|\Big)\Big]\\ &\leq CN^{-1}\ln N.\end{aligned}$$

Further, if $\tau_1 = \dfrac{2\varepsilon_1}{\mu_2\vartheta}\ln N$, $\tau_2 = 1/6$, $\tau_3 = 1/4$, $\sigma_1 = 1/12$, and we consider the intervals $(x_i, y_j) \in (\tau_1, \tau_2)\times(0,1), (\tau_2, \tau_3)\times(0,1), (\tau_3, 1-\tau_3)\times(0,1)$, $(1-\tau_3, 1-\tau_2)\times(0,1)$ or $(1-\tau_2, 1-\sigma_1)\times(0,1)$, then the truncation error satisfies

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \leq CN^{-1}\begin{pmatrix}\varepsilon_2^{-1/2}\mathcal{B}_2^l(x_{i-1})\\ \varepsilon_2^{-1/2}\mathcal{B}_2^l(x_{i-1})\end{pmatrix} \leq CN^{-1}\ln N.$$

On the other hand, when $(x_i, y_j) \in (0, \tau_1]\times(0,1)$ or $[1-\sigma_1, 1]\times(0,1)$, it follows

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \leq CN^{-1}\ln N\varepsilon_1\mu_1^{-1}\begin{pmatrix}\varepsilon_1^{-1/2} + \varepsilon_2^{-1/2}\\ \varepsilon_1^{-1/2} + \varepsilon_2^{-1/2}\end{pmatrix} \leq CN^{-1}\ln N.$$

In the case that $\tau_2 = \dfrac{2\varepsilon_1}{\mu_2\vartheta}$, $\tau_1 = 1/12$, $\tau_3 = 1/4$, $\sigma_1 = 1/12$, and we consider the intervals $(x_i, y_j) \in (\tau_2, \tau_3)\times(0,1)$, $(\tau_3, 1-\tau_3)\times(0,1), (1-\tau_2, 1-\sigma_1)\times(0,1)$, or $(1-\tau_3, 1-\tau_2)\times(0,1)$, we have

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \leq CN^{-1}\begin{pmatrix}\varepsilon_2^{-1/2}\mathcal{B}_2^l(x_{i-1})\\ \varepsilon_2^{-1/2}\mathcal{B}_2^l(x_{i-1})\end{pmatrix} \leq CN^{-1}\ln N.$$

When $(x_i, y_j) \in (0, \tau_1)\times(0,1)$, $(\tau_1, \tau_2]\times(0,1)$, $(1-\tau_2, 1-\sigma_1)\times(0,1)$ or $(1-\sigma_1, 1)\times(0,1)$, it follows

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \leq CN^{-1}\ln N\varepsilon_1\mu_2^{-1}\begin{pmatrix}\varepsilon_1^{-1/2} + \varepsilon_2^{-1/2}\\ \varepsilon_1^{-1/2} + \varepsilon_2^{-1/2}\end{pmatrix} \leq CN^{-1}\ln N.$$

In the case that $\tau_3 = \sqrt{\dfrac{\varepsilon_2}{\Lambda\vartheta}}\ln N$, $\tau_2 = \dfrac{2\tau_3}{3}$ and $\tau_1 = \dfrac{\tau_2}{2}$, $\dfrac{\sqrt{\varepsilon_2}}{2} \leq \sqrt{\varepsilon_1} < \sqrt{\varepsilon_2}$, it follows that $\tau_3 \leq C\sqrt{\varepsilon_1}\ln N$.

To prove the error estimate on the region $(x_i, y_j) \in [\tau_3, 1-\tau_3]\times(0,1)$, we consider the following barrier functions:

$$\mathcal{B}_1^{l,N}(x_i) = \prod_{\iota=1}^{i}\left(1 + \left(\frac{\vartheta\mu}{2\varepsilon_1}\right)h_\iota\right)^{-1}, \quad \mathcal{B}_2^{l,N}(x_i) = \prod_{\iota=1}^{i}\left(1 + \sqrt{\left(\frac{\Lambda\vartheta}{\varepsilon_2}\right)}h_\iota\right)^{-1}, \tag{4.5}$$

with $\mathcal{B}_1^{l,N}(x_0) = \mathcal{B}_2^{l,N}(x_0) = 1$. After applying Theorem 2.4, we can conclude that it holds

$$|(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \leq |\vec{\mathbf{W}}_l(x_i, y_j)| + |\vec{\mathbf{w}}_l(x_i, y_j)| \leq C\mathcal{B}_2^{l,N}(\tau_3) + C\mathcal{B}_2^{l,N}(\tau_3) \leq CN^{-1}.$$

To get appropriate bounds for the error in the region $(x_i, y_j) \in (\tau_1, \tau_2) \times (0, 1)$, $(\tau_2, \tau_3) \times (0, 1)$, $(1 - \tau_3, 1 - \tau_2) \times (0, 1)$ or $(1 - \tau_2, 1 - \sigma_1) \times (0, 1)$ and $(h_i + h_{i+1}) \leq N^{-1} \ln N \sqrt{\varepsilon_2}$, we take into account that now the truncation error satisfies

$$|\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \leq CN^{-1} \ln N \sqrt{\varepsilon_2} \begin{pmatrix} \varepsilon_1^{-1/2} + \varepsilon_2^{-1/2} \\ \varepsilon_1^{-1/2} + \varepsilon_2^{-1/2} \end{pmatrix},$$

and therefore, using the barrier function $\vec{\Psi}^{\pm}(x_i, y_j) = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} N^{-1} \ln N(x_i - \tau_3) \pm (\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)$ and Lemma 3.1, we can obtain that it holds

$$|\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \leq CN^{-1} \ln N.$$

In the cases where (x_i, y_j) belongs to either $(0, \tau_1) \times (0, 1)$ or $(1 - \sigma_1, 1) \times (0, 1)$, we have $h_i + h_{i+1} \leq CN^{-1} \ln N \sqrt{\varepsilon_2}$, following the same approach as mentioned earlier, we obtain the required bounds.

Finally, assuming that $\tau_1 = \dfrac{\varepsilon_1}{\mu_1 \alpha} \ln N$, $\tau_2 = \sqrt{\dfrac{\varepsilon_1}{\Lambda \vartheta}} \ln N$ and $\tau_3 = \sqrt{\dfrac{\varepsilon_2}{\Lambda \vartheta}} \ln N$, in cases where $(x_i, y_j) \in [\tau_3, 1 - \tau_3] \times (0, 1)$ or $(0, \tau_1] \times (0, 1)$, or $(1 - \sigma_1, 1) \times (0, 1)$, we can obtain the required bounds by using a method similar to that used in the corresponding intervals of the previous cases. Also, when (x_i, y_j) is within either $(\tau_1, \tau_2) \times (0, 1)$, $(\tau_2, \tau_3) \times (0, 1)$, $(1 - \tau_3, 1 - \tau_2) \times (0, 1)$, or $(1 - \tau_2, 1 - \sigma_1) \times (0, 1)$, we have $h_i + h_{i+1} \leq CN^{-1} \ln N \mu_1^{-1} \varepsilon_1$. Therefore, from all previous estimates, we obtain

$$|\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \leq CN^{-1} \ln N.$$

So, by using Lemma 3.1, we obtain

$$|\vec{\mathbf{W}}_l(x_i, y_j) - \vec{\mathbf{w}}_l(x_i, y_j)| \leq CN^{-1} \ln N, \tag{4.6}$$

which is the required result. □

Now, we analyze the error associated with the layer components in **Case 2** defined in (2.1).

Lemma 4.3. *Let $\vec{w}_k(x_i, y_j)$ be the true solution of* (2.4) *and $\vec{W}_k(x_i, y_j)$ be the numerical solution of* (4.2) *at the grid point (x_i, y_j). Then, for* **Case 2** *defined in (2.1), the error associated with the boundary layer functions satisfies*

$$|\vec{W}_k(x_i, y_j) - \vec{w}_k(x_i, y_j)| \leq CN^{-1} \ln N, \quad k = l, b, r, t.$$

Proof. Using the similar argument shown in Lemma 4.2, we can prove the error estimates for the case $\Lambda \varepsilon_1 \leq \vartheta {\mu_1}^2 < \Lambda \varepsilon_2 \leq \vartheta {\mu_2}^2$. □

To finish our study, we estimate the error associated with the corner layer components.

Lemma 4.4. *Let $\vec{s}_k(x_i, y_j)$ be the true solution of* (2.5) *and $\vec{S}_k(x_i, y_j)$ the numerical solution of (4.3) at the grid point (x_i, y_j), for $k = lb, br, rt, lt$. Then, for* **Case 1** *defined in (2.1), we have*

$$|\vec{S}_k(x_i, y_j) - \vec{s}_k(x_i, y_j)| \leq CN^{-1} \ln N, \quad k = lb, br, rt, lt.$$

Proof. Here, we present the specific details related to the corner layer component $\vec{\mathbf{s}}_{lb}$ exclusively. The same procedure can be followed for the remaining corner layer components. According to the findings mentioned in Theorem 2.5, the truncation error for the singular component $\vec{\mathbf{s}}_{lb}$ can be estimated as follows:

$$\|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})\| \leq C\Big[(h_i + h_{i+1})\Big(\vec{\varepsilon}\Big\|\frac{\partial^3 \vec{\mathbf{s}}_{lb}}{\partial x^3}\Big\| + \vec{\mu}\Big\|\frac{\partial^2 \vec{\mathbf{s}}_{lb}}{\partial x^2}\Big\|\Big) + (k_j + k_{j+1})\Big(\vec{\varepsilon}\Big\|\frac{\partial^3 \vec{\mathbf{s}}_{lb}}{\partial y^3}\Big\| + \vec{\mu}\Big\|\frac{\partial^2 \vec{\mathbf{s}}_{lb}}{\partial y^2}\Big\|\Big)\Big]. \tag{4.7}$$

If $\tau_1 = 1/12$, $\tau_2 = 1/6$, $\tau_3 = 1/4$, and $\sigma_1 = 1/12$, the proof can be obtained easily by applying standard methods (on uniform meshes) by taking into consideration that $\mu_1\varepsilon_1^{-1} \leq C \ln N$, $\varepsilon_1^{-1/2} \leq C \ln N$, and ${\varepsilon_2}^{-1/2} \leq C \ln N$. Then, using Theorem 2.5 in (4.7), we have

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{S}_{lb} - \vec{s}_{lb})| \leq CN^{-1}\begin{pmatrix} \varepsilon_1^{-1/2}(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1})) + \varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \\ \varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \end{pmatrix}. \tag{4.8}$$

Now, let us consider the appropriate barrier functions, as indicated in Lemma 4.2 and defined in equations (4.5), within the domain $\bar{\Omega}^{N,N}$. Therefore, by using Lemma 3.1, we can deduce that it holds

$$|(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1}\begin{pmatrix} \varepsilon_1^{-1/2} + \varepsilon_2^{-1/2} \\ \varepsilon_2^{-1/2} \end{pmatrix} \leq CN^{-1} \ln N.$$

Next, from (4.3), when $\tau_3 = \sqrt{\frac{\varepsilon_2}{\Lambda\vartheta}} \ln N$, for the grid points $\{(x_i, y_j), | (0 < i, j < N)/(0 < i, j < N/4)\}$, we have

$$\begin{aligned} |(S_{lb_1} - s_{lb_1})(x_i, y_j)| &\leq |S_{lb_1}(x_i, y_j)| + |s_{lb_1}(x_i, y_j)| \leq C \min(\mathcal{G}_1^{l,N}(x_i), \mathcal{G}_2^{b,N}(y_j)) \\ &\leq C \min(\mathcal{G}_1^{l,N}(\tau_3), \mathcal{G}_2^{b,N}(\tau_3)) \leq CN^{-1}. \end{aligned} \tag{4.9}$$

Analogously, we can deduce

$$|(S_{lb_2} - s_{lb_2})(x_i, y_j)| \leq CN^{-1}. \tag{4.10}$$

In the case $\tau_3 = \sqrt{\frac{\varepsilon_2}{\Lambda\vartheta}} \ln N$, for $N/6 \leq i \leq N/4$, $0 \leq j \leq N/4$, $\tau_3 \leq \sqrt{\frac{\varepsilon_2}{\Lambda\vartheta}} \ln N$. Hence, it holds

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1} \ln N\begin{pmatrix} \varepsilon_1^{-1/2}(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1})) + \varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \\ \varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \end{pmatrix}.$$

For $N/12 \leq i \leq N/6$, $0 \leq j \leq N/4$, $\tau_2 \leq \sqrt{\frac{\varepsilon_1}{\Lambda\vartheta}} \ln N$, it follows that the truncation error satisfies

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1} \ln N\begin{pmatrix} C\varepsilon_1^{-1/2}(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1})) + \varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \\ \varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \end{pmatrix}.$$

Similarly, we can define the local error for the region $0 \le i \le N/12$, $0 \le j \le N/4$. For $0 \le i \le N/4$ and $0 \le j \le N/4$, we consider the suitable barrier functions. Hence, it follows

$$|(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \le CN^{-1} \ln N, \ 0 \le i \le N/4, \ 0 \le j \le N/4. \tag{4.11}$$

From Eqs (4.9), (4.10), and (4.11), for the case $\tau_3 = \sqrt{\frac{\varepsilon_2}{\Lambda\vartheta}} \ln N$, it has been proven that

$$|(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \le CN^{-1} \ln N.$$

Next, the case $\tau_2 = 1/6$, $\tau_3 = 1/4$,, and $\sigma_1 = 1/12$, $\tau_1 = \frac{\varepsilon_1}{\mu_1\vartheta} \ln N$ is considered; then, $\mu_2\varepsilon_1^{-1} \le C \ln N$, $\varepsilon_2^{-1/2} \le C \ln N$ holds. For $(x_i, y_j) \in (0, \tau_1] \times (0, \tau_3]$, $h_i, k_j \le C\varepsilon_1\mu_1^{-1}N^{-1} \ln N$. Therefore, by using the truncation error estimate (4.7), we obtain

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}} (\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \le CN^{-1} \ln N \begin{pmatrix} \varepsilon_1^{-1/2}\mu_1^{-1}(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1})) + \varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \\ \varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \end{pmatrix}.$$

For $(x_i, y_j) \in [\tau_1, \tau_2] \times (0, \tau_3]$, from (4.7) and Theorem 2.5, we have

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}} (\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \le \begin{pmatrix} C\varepsilon_1^{-1/2}\mu_1^{-1}(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1})) + CN^{-1} \ln N\varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \\ C\varepsilon_2^{-1/2}\mu_2^{-1}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) + CN^{-1} \ln N\varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \end{pmatrix}.$$

For $(x_i, y_j) \in [\tau_2, \tau_3]) \times (0, \tau_3]$, from (4.7) and Theorem 2.5, we can deduce

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}} (\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \le \begin{pmatrix} C\varepsilon_1^{-1/2}\mu_1^{-1}(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1})) + CN^{-1} \ln N\varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \\ C\varepsilon_2^{-1/2}\mu_2^{-1}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) + CN^{-1} \ln N\varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \end{pmatrix}.$$

Similarly, we can define the error for the region $(\tau_3, 1) \times (0, 1)$. Taking the suitable barrier function for the corner layer component for $0 \le i \le N$, $0 \le j \le N$, it has been deduced for each of the cases that

$$|(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \le CN^{-1} \ln N.$$

□

Now, we analyze the error associated with the corner layer components in **Case 2** defined in (2.1).

Lemma 4.5. *Let $\vec{s}_k(x_i, y_j)$ be the true solution of* (2.5) *and $\vec{S}_k(x_i, y_j)$ be the numerical solution of* (4.3) *at the grid point (x_i, y_j). Then, for* **Case 2** *defined in (2.1), the error associated with the corner layer functions satisfies*

$$|\vec{S}_k(x_i, y_j) - \vec{s}_k(x_i, y_j)| \le CN^{-1} \ln N, \ k = lb, br, rt, lt.$$

Proof. Using the similar argument shown in Lemma 4.4, we can prove the error estimates for the case $\Lambda\varepsilon_1 \le \vartheta{\mu_1}^2 < \Lambda\varepsilon_2 \le \vartheta{\mu_2}^2$. □

Combining all previous results, we deduce the main result of this work.

Theorem 4.6. *Let $\vec{z}(x_i, y_j)$ be the true solution of the continuous problem (1.1) and $\vec{\mathbf{Z}}(x_i, y_j)$ the numerical solution of (3.3) at the grid point (x_i, y_j), defined on the corresponding Shishkin mesh. Then, the error satisfies*

$$|\vec{\mathbf{Z}}(x_i, y_j) - \vec{z}(x_i, y_j)| \le CN^{-1} \ln N, \tag{4.12}$$

and therefore, the numerical scheme is an almost uniformly convergent method.

5. Numerical experiments

In previous sections we have constructed our numerical algorithm, which is defined on Shishkin meshes. To observe the performance of the algorithm, we focus on one test example of the type (1.1). The problems are solved using MATLAB R2023a on a system with 32 GB of RAM and an i5 processor operating at 1.8 GHz. We have seen that the computational time to obtain results is relatively high due to the complexity of the problem. However, to mitigate this, a sparse matrix representation is employed within the algorithm to reduce both the computational time and the memory usage.

To solve the test problem, we have ordered the numerical solution in the following form:

$$Z = \begin{cases} Z_1(x_0, y_0), Z_1(x_1, y_0), \ldots, Z_1(x_N, y_0), \\ Z_1(x_0, y_1), Z_1(x_1, y_1), \ldots, Z_1(x_N, y_1), \\ Z_1(x_0, y_2), Z_1(x_1, y_2), \ldots, Z_1(x_N, y_2), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Z_1(x_0, y_N), Z_1(x_1, y_N), \ldots, Z_1(x_N, y_N), \\ Z_2(x_0, y_0), Z_2(x_1, y_0), \ldots, Z_2(x_N, y_0), \\ Z_2(x_0, y_1), Z_2(x_1, y_1), \ldots, Z_2(x_N, y_1), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Z_2(x_0, y_N), Z_2(x_1, y_N), \ldots, Z_2(x_N, y_N), \end{cases}$$

where $Z_1(x_0, y_i), Z_1(x_i, y_0), Z_1(x_N, y_i)$ and $Z_1(x_i, y_N)$, $i = 0, ..., N$, are calculated by using the boundary conditions. Then, the resulting linear system can be written as

$$[A]_{(2(N+1)^2, 2(N+1)^2)} [Z]_{(2(N+1)^2, 1)} = [F]_{(2(N+1)^2, 1)},$$

and we solve this system by using MATLAB, taking into account that the matrix A is sparse.

The test problem is defined as follows.

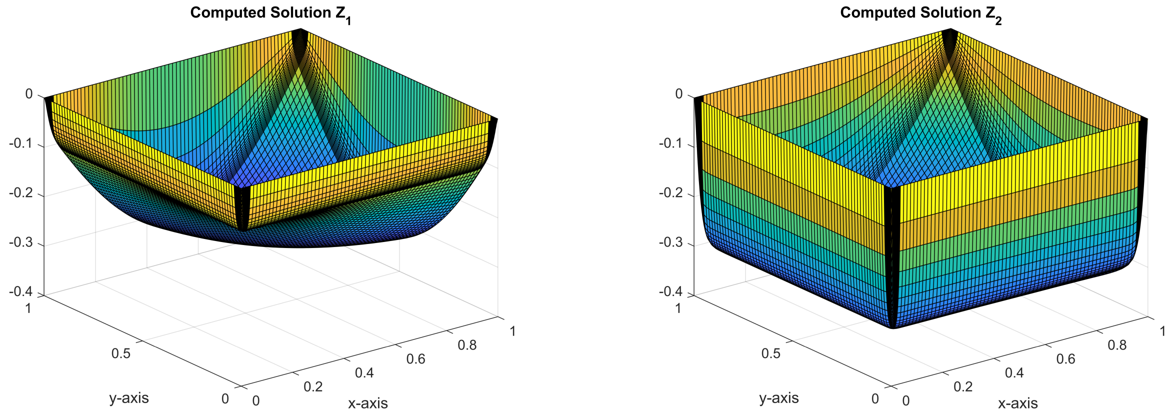
Example 5.1.

$$\vec{\boldsymbol{\varepsilon}}\left(\frac{\partial^2 \vec{z}}{\partial x^2} + \frac{\partial^2 \vec{z}}{\partial y^2}\right) + \vec{\boldsymbol{\mu}}\left(\vec{\boldsymbol{A}}_1(x, y)\frac{\partial \vec{z}}{\partial x} + \vec{\boldsymbol{A}}_2(x, y)\frac{\partial \vec{z}}{\partial y}\right) - \vec{\boldsymbol{B}}(x, y)\vec{z} = \vec{\boldsymbol{f}}(x, y), \quad \forall (x, y) \in \Omega,$$

where the boundary conditions, as well as convection, reaction coefficients, and source terms, are given by

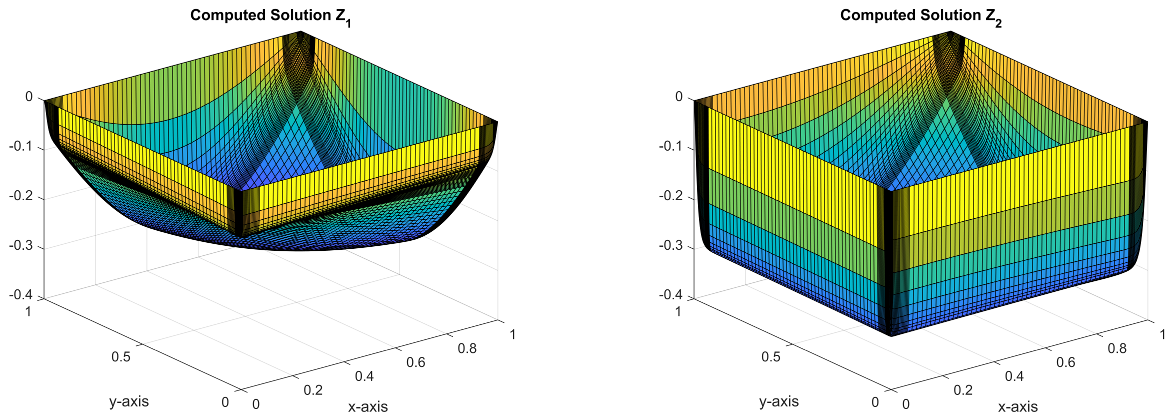
$$\begin{aligned} &\vec{z}(x, 0) = \vec{z}(x, 1) = \vec{z}(0, y) = \vec{z}(1, y) = 0, \\ &\vec{\boldsymbol{A}}_1(x, y) = \begin{pmatrix} 1 + x^2 y^2 & 0 \\ 0 & 2 - xy \end{pmatrix}, \vec{\boldsymbol{A}}_2(x, y) = \begin{pmatrix} 2 + \sin(x + y) & 0 \\ 0 & 2 - \cos(x + y) \end{pmatrix}, \\ &\vec{\boldsymbol{B}}(x, y) = \begin{pmatrix} 3 + x + y & -1 - x^2 y^2 \\ -1 - \exp(xy) & 3 + \exp(xy) \end{pmatrix}, \ \vec{\boldsymbol{f}}(x, y) = \Big(\sin(\pi xy), \cos(\pi xy/2) \Big)^T. \end{aligned}$$

Figures 1 and 2 display the two components of the numerical solution for Example 5.1, taking different values of the diffusion and the convection parameters ε_1, ε_2, μ_1 and μ_2 and a fixed value of the discretization parameter N.



(a) Surface graph of the numerical solution z_1; **(b)** surface graph of the numerical solution z_2;

Figure 1. When $\varepsilon_1 = 2^{-14}, \varepsilon_2 = 2^{-10}$, $\mu_1^2 = 2^{-16}$, $\mu_2^2 = 2^{-12}, N = 96$, for Example 5.1.



(a) Surface graph of the numerical solution z_1; **(b)** surface graph of the numerical solution z_2;

Figure 2. When $\varepsilon_1 = 2^{-16}, \varepsilon_2 = 2^{-12}$, $\mu_1^2 = 2^{-14}$, $\mu_2^2 = 2^{-10}, N = 96$, for Example 5.1.

Figures 3 and 4 display the error for both components, taking the same values of the diffusion, the convection and the discetization parameters; the errors are calculated by using the double mesh technique (see [12]), which is remembered next.

As the exact solution of this problem is unknown, to approximate the maximum point-wise errors we use, in a usual way, the double mesh technique. Then, we calculate

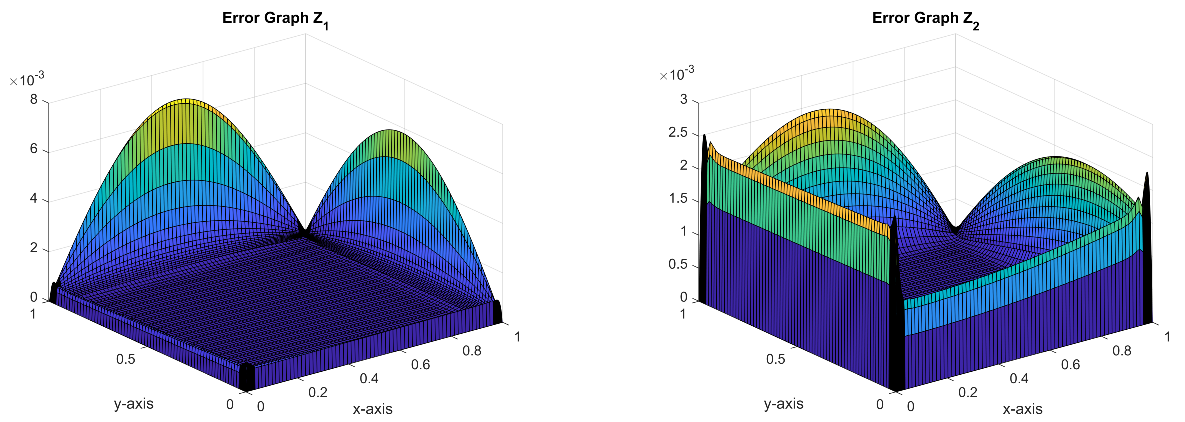
$$E^{N,N}_{\vec{\varepsilon},\vec{\mu}} = \max_{(x_i,y_j)\in\bar{\Omega}^{N,N}} |\widehat{Z}^{2N,2N}(x_{2i}, y_{2j}) - Z^{N,N}(x_i, y_j)|,$$

where $\widehat{Z}^{2N,2N}$ is the numerical solution obtained on a mesh with $2N$ subintervals taking the mesh points of the coarse mesh and also their midpoints on each spatial direction. Then, the parameter uniform maximum point-wise errors are calculated applying the formula

$$E^{N,N} = \max_{\vec{\varepsilon},\vec{\mu}} E^{N,N}_{\vec{\varepsilon},\vec{\mu}}.$$

From the previous values, the uniform numerical orders of convergence are given by

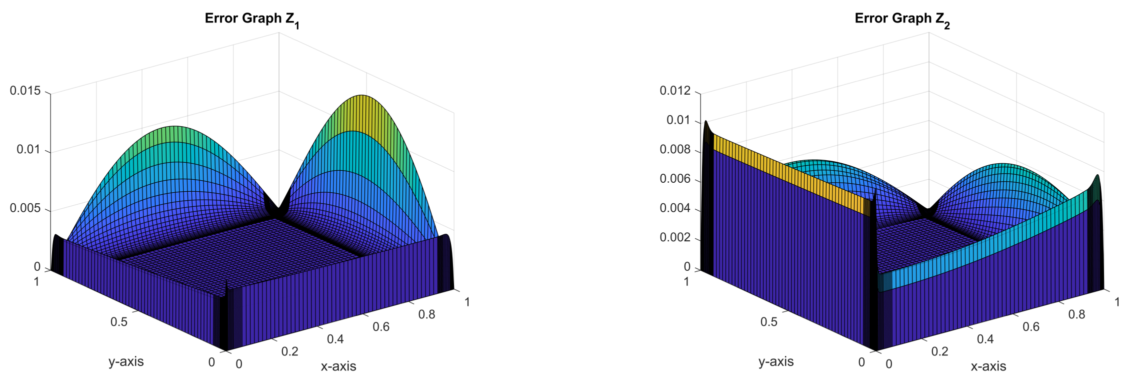
$$Q^{N,N} = \log_2\left(\frac{E^{N,N}}{E^{2N,2N}}\right).$$



(a) Error graph of the numerical solution z_1;

(b) Error graph of the numerical solution z_2;

Figure 3. When $\varepsilon_1 = 2^{-14}, \varepsilon_2 = 2^{-10}$, $\mu_1^2 = 2^{-16}$, $\mu_2^2 = 2^{-12}, N = 96$, for Example 5.1.



(a) Error graph of the numerical solution z_1;

(b) Error graph of the numerical solution z_2;

Figure 4. When $\varepsilon_1 = 2^{-16}, \varepsilon_2 = 2^{-12}$, $\mu_1^2 = 2^{-14}$, $\mu_2^2 = 2^{-10}, N = 96$, for Example 5.1.

Table 1 shows the maximum errors for the first component, for different values of the parameters of convection and diffusion corresponding to **Case 1** and some values of the discretization parameter

N; also it gives the maximum uniform errors and the uniform orders of convergence. Similarly, Table 2 shows the results for the second component for same values of all parameters. In same way, Tables 3 and 4 show the maximum errors, and numerical orders of convergence for the first and the second component, respectively, when the restriction associated to **Case 2** holds. From these four tables, we clearly can deduce the almost first order of uniform convergence of our numerical algorithm, in agreement with the theoretical result given in Theorem 4.6. Tables 1-4 also illustrate the efficiency of the proposed method, implemented on a Shishkin mesh, in obtaining accurate solutions for Example 5.1 under both cases. Notably, a sparse matrix representation was utilized in all computations to minimize CPU time defined for all diffusion and convection parameter values with fixed N.

Table 1. For Example 5.1, maximum point-wise errors $E^{N,N}$ and orders of convergence $Q^{N,N}$ calculated for z_1 when $\vartheta\mu_1{}^2 < \Lambda\varepsilon_1 \leq \vartheta\mu_2{}^2 < \Lambda\varepsilon_2$.

$\varepsilon_1 = 2^{-6}\eta$ $\varepsilon_2 = 2^{-2}\eta$ $\mu_1{}^2 = 2^{-8}\eta$ $\mu_2{}^2 = 2^{-4}\eta$					
η/N	48	96	192	384	768
2^{0}	1.969e-03	9.520e-04	4.658e-04	2.302e-04	1.144e-04
2^{-1}	2.792e-03	1.336e-03	6.492e-04	3.195e-04	1.583e-04
2^{-2}	3.981e-03	1.897e-03	9.180e-04	4.488e-04	2.218e-04
2^{-3}	5.686e-03	2.745e-03	1.312e-03	6.370e-04	3.133e-04
2^{-4}	8.114e-03	3.953e-03	1.883e-03	9.100e-04	4.451e-04
2^{-5}	1.153e-02	5.662e-03	2.737e-03	1.308e-03	6.349e-04
2^{-6}	1.184e-02	6.749e-03	3.905e-03	1.881e-03	9.087e-04
2^{-7}	1.184e-02	6.749e-03	3.906e-03	2.134e-03	1.150e-03
2^{-8}	1.183e-02	6.749e-03	3.906e-03	2.135e-03	1.150e-03
2^{-9}	1.182e-02	6.749e-03	3.906e-03	2.135e-03	1.150e-03
2^{-10}	1.182e-02	6.749e-03	3.906e-03	2.135e-03	1.150e-03
2^{-11}	1.182e-02	6.748e-03	3.906e-03	2.135e-03	1.150e-03
2^{-12}	1.182e-02	6.748e-03	3.906e-03	2.135e-03	1.150e-03
2^{-13}	1.182e-02	6.748e-03	3.906e-03	2.135e-03	1.150e-03
2^{-14}	1.182e-02	6.748e-03	3.906e-03	2.135e-03	1.150e-03
2^{-15}	1.182e-02	6.748e-03	3.906e-03	2.135e-03	1.150e-03
2^{-16}	1.182e-02	6.748e-03	3.906e-03	2.135e-03	1.150e-03
2^{-17}	1.182e-02	6.748e-03	3.906e-03	2.135e-03	1.150e-03
2^{-18}	1.182e-02	6.748e-03	3.906e-03	2.135e-03	1.150e-03
2^{-19}	1.182e-02	6.748e-03	3.906e-03	2.135e-03	1.150e-03
2^{-20}	1.182e-02	6.748e-03	3.906e-03	2.135e-03	1.150e-03
$E^{N,N}$	1.183e-02	6.749e-03	3.906e-03	2.135e-03	1.150e-03
$Q^{N,N}$	0.8097	0.7890	0.8715	0.8926	-
CPU time (seconds)	2.534042	9.177485	26.732375	46.913353	92.163421

Table 2. For Example 5.1, maximum point-wise errors $E^{N,N}$ and orders of convergence $Q^{N,N}$ calculated for z_2 when $\vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \leq \vartheta{\mu_2}^2 < \Lambda\varepsilon_2$.

$\varepsilon_1 = 2^{-6}\eta$ $\varepsilon_2 = 2^{-2}\eta$ ${\mu_1}^2 = 2^{-8}\eta$ ${\mu_2}^2 = 2^{-4}\eta$					
η/N	48	96	192	384	768
2^{0}	1.091e-03	5.367e-04	2.661e-04	1.325e-04	6.612e-05
2^{-1}	1.410e-03	6.948e-04	3.449e-04	1.718e-04	8.575e-05
2^{-2}	1.791e-03	8.779e-04	4.346e-04	2.163e-04	1.079e-04
2^{-3}	2.415e-03	1.178e-03	5.815e-04	2.890e-04	1.441e-04
2^{-4}	3.334e-03	1.618e-03	7.960e-04	3.950e-04	1.968e-04
2^{-5}	4.651e-03	2.246e-03	1.102e-03	5.458e-04	2.722e-04
2^{-6}	4.687e-03	2.677e-03	1.527e-03	7.711e-04	3.848e-04
2^{-7}	4.572e-03	2.656e-03	1.527e-03	8.612e-04	4.797e-04
2^{-8}	4.505e-03	2.649e-03	1.528e-03	8.619e-04	4.801e-04
2^{-9}	4.506e-03	2.647e-03	1.528e-03	8.625e-04	4.805e-04
2^{-10}	4.508e-03	2.648e-03	1.529e-03	8.630e-04	4.809e-04
2^{-11}	4.509e-03	2.649e-03	1.529e-03	8.635e-04	4.811e-04
2^{-12}	4.510e-03	2.650e-03	1.530e-03	8.639e-04	4.813e-04
2^{-13}	4.511e-03	2.651e-03	1.531e-03	8.641e-04	4.815e-04
2^{-14}	4.512e-03	2.651e-03	1.531e-03	8.643e-04	4.816e-04
2^{-15}	4.512e-03	2.652e-03	1.531e-03	8.645e-04	4.817e-04
2^{-16}	4.513e-03	2.652e-03	1.531e-03	8.646e-04	4.817e-04
2^{-17}	4.513e-03	2.652e-03	1.531e-03	8.646e-04	4.818e-04
2^{-18}	4.513e-03	2.652e-03	1.531e-03	8.647e-04	4.818e-04
2^{-19}	4.513e-03	2.652e-03	1.532e-03	8.647e-04	4.818e-04
2^{-20}	4.513e-03	2.652e-03	1.532e-03	8.647e-04	4.818e-04
$E^{N,N}$	4.513e-03	2.652e-03	1.532e-03	8.647e-04	4.818e-04
$Q^{N,N}$	0.7670	0.7917	0.8251	0.8438	-
CPU time (seconds)	2.806159	10.551836	32.139680	93.906843	112.084236

Table 3. For Example 5.1, maximum point-wise errors $E^{N,N}$ and orders of convergence $Q^{N,N}$ calculated for z_1 when $\Lambda\varepsilon_1 \leq \vartheta{\mu_1}^2 < \Lambda\varepsilon_2 \leq \vartheta{\mu_2}^2$.

$\varepsilon_1 = 2^{-8}\eta$ $\varepsilon_2 = 2^{-4}\eta$ ${\mu_1}^2 = 2^{-6}\eta$ ${\mu_2}^2 = 2^{-2}\eta$					
η/N	48	96	192	384	768
2^{0}	6.713e-03	4.159e-03	2.572e-03	1.443e-03	7.650e-04
2^{-1}	7.036e-03	5.029e-03	3.133e-03	1.780e-03	9.691e-04
2^{-2}	8.050e-03	5.734e-03	3.568e-03	2.205e-03	1.236e-03
2^{-3}	1.119e-02	6.103e-03	4.377e-03	2.730e-03	1.552e-03
2^{-4}	1.470e-02	7.930e-03	4.990e-03	3.129e-03	1.938e-03
2^{-5}	1.755e-02	1.113e-02	5.949e-03	3.242e-03	2.201e-03
2^{-6}	1.773e-02	1.310e-02	7.833e-03	4.225e-03	2.109e-03
2^{-7}	1.774e-02	1.311e-02	7.836e-03	4.650e-03	2.629e-03
2^{-8}	1.774e-02	1.311e-02	7.840e-03	4.653e-03	2.630e-03
2^{-9}	1.772e-02	1.311e-02	7.841e-03	4.654e-03	2.631e-03
2^{-10}	1.771e-02	1.311e-02	7.841e-03	4.654e-03	2.631e-03
2^{-11}	1.771e-02	1.311e-02	7.842e-03	4.654e-03	2.631e-03
2^{-12}	1.771e-02	1.311e-02	7.842e-03	4.654e-03	2.631e-03
2^{-13}	1.770e-02	1.311e-02	7.841e-03	4.654e-03	2.631e-03
2^{-14}	1.770e-02	1.311e-02	7.841e-03	4.654e-03	2.631e-03
2^{-15}	1.770e-02	1.310e-02	7.841e-03	4.654e-03	2.631e-03
2^{-16}	1.770e-02	1.310e-02	7.841e-03	4.654e-03	2.631e-03
2^{-17}	1.770e-02	1.310e-02	7.841e-03	4.654e-03	2.631e-03
2^{-18}	1.770e-02	1.310e-02	7.841e-03	4.654e-03	2.631e-03
2^{-19}	1.770e-02	1.310e-02	7.841e-03	4.654e-03	2.631e-03
2^{-20}	1.770e-02	1.310e-02	7.841e-03	4.654e-03	2.631e-03
$E^{N,N}$	1.774e-02	1.311e-02	7.842e-03	4.654e-03	2.631e-03
$Q^{N,N}$	0.4363	0.7414	0.7528	0.8229	-
CPU time (seconds)	3.113208	9.411691	28.740375	96.230456	138.686516

Table 4. For Example 5.1, maximum point-wise errors $E^{N,N}$ and orders of convergence $Q^{N,N}$ calculated for z_2 when $\Lambda\varepsilon_1 \leq \vartheta{\mu_1}^2 < \Lambda\varepsilon_2 \leq \vartheta{\mu_2}^2$.

$\varepsilon_1 = 2^{-8}\eta$ $\varepsilon_2 = 2^{-4}\eta$ ${\mu_1}^2 = 2^{-6}\eta$ ${\mu_2}^2 = 2^{-2}\eta$					
η/N	48	96	192	384	768
2^{0}	7.932e-03	4.301e-03	2.246e-03	1.148e-03	5.806e-04
2^{-1}	1.028e-02	5.763e-03	3.068e-03	1.585e-03	8.062e-04
2^{-2}	1.296e-02	7.518e-03	4.087e-03	2.137e-03	1.095e-03
2^{-3}	1.573e-02	9.636e-03	5.419e-03	2.888e-03	1.493e-03
2^{-4}	1.590e-02	1.116e-02	7.117e-03	3.921e-03	2.051e-03
2^{-5}	1.544e-02	1.111e-02	7.095e-03	4.298e-03	2.509e-03
2^{-6}	1.558e-02	1.104e-02	7.085e-03	4.293e-03	2.504e-03
2^{-7}	1.535e-02	1.103e-02	7.078e-03	4.291e-03	2.504e-03
2^{-8}	1.533e-02	1.103e-02	7.077e-03	4.291e-03	2.503e-03
2^{-9}	1.534e-02	1.103e-02	7.079e-03	4.292e-03	2.504e-03
2^{-10}	1.534e-02	1.103e-02	7.082e-03	4.294e-03	2.505e-03
2^{-11}	1.535e-02	1.104e-02	7.084e-03	4.295e-03	2.506e-03
2^{-12}	1.535e-02	1.104e-02	7.086e-03	4.296e-03	2.507e-03
2^{-13}	1.536e-02	1.104e-02	7.088e-03	4.297e-03	2.507e-03
2^{-14}	1.536e-02	1.105e-02	7.089e-03	4.298e-03	2.508e-03
2^{-15}	1.536e-02	1.105e-02	7.090e-03	4.299e-03	2.508e-03
2^{-16}	1.536e-02	1.105e-02	7.090e-03	4.299e-03	2.508e-03
2^{-17}	1.536e-02	1.105e-02	7.091e-03	4.299e-03	2.508e-03
2^{-18}	1.537e-02	1.105e-02	7.091e-03	4.300e-03	2.508e-03
2^{-19}	1.537e-02	1.105e-02	7.092e-03	4.300e-03	2.508e-03
2^{-20}	1.537e-02	1.105e-02	7.092e-03	4.300e-03	2.508e-03
$E^{N,N}$	1.537e-02	1.105e-02	7.092e-03	4.300e-03	2.508e-03
$Q^{N,N}$	0.4761	0.6398	0.7219	0.7778	-
CPU time (seconds)	2.826123	13.861913	43.906536	98.107984	135.031256

6. Conclusions

In this work we solve a 2D elliptic coupled singularly perturbed system of convection–diffusion; we analyze the cases remaining in [11] for the ratio between the diffusion and the convection parameters. It is well known that different types of overlapping boundary layers appear on the outflow and the inflow boundary, depending on the value and the ratio between the diffusion and the convection parameters. The numerical algorithm constructed to solve this type of problem is the classical upwind scheme, which is defined on adequate nonuniform meshes of Shishkin type. We prove that the method is an almost first-order uniformly convergent method, in the maximum norm, with respect to all singular perturbation parameters. The numerical results obtained with the numerical algorithm for a test example clearly show the presence of overlapping boundary layers and also the order of uniform convergence theoretically proved.

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Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Professor Carmelo Clavero is a Guest Editor for AIMS Mathematics and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

Author's contribution

Both authors are responsible of all sections and mathematical details in the manuscript; they wrote and reviewed the all manuscript. Ram Shiromani prepared the Figures and Tables included in the manuscript.

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