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Research article

On (p, q)-fractional linear Diophantine fuzzy sets and their applications via MADM approach

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Abstract: The integration of internationally sustainable practices into supply chain management methodologies is known as "green supply chain management". Reducing the supply chain's overall environmental impact is the main objective in order to improve corporate connections and the social, ecological, and economic ties with other nations. To accomplish appropriate and accurate measures to address the issue of emergency decision-making, the paper is divided into three major sections. First, the (p, q)-fractional linear Diophantine fuzzy set represents a new generalization of several fuzzy set theories, including the Pythagorean fuzzy set, q-rung orthopair fuzzy set, linear Diophantine fuzzy set, and q-rung linear Diophantine fuzzy set, with its key features thoroughly discussed. Additionally, aggregation operators are crucial for handling uncertainty in decision-making scenarios. Consequently, algebraic norms for (p, q)-fractional linear Diophantine fuzzy sets were established based on operational principles. In the second part of the study, we introduced a range of geometric aggregation operators and a series of averaging operators under the (p, q)-fractional linear Diophantine fuzzy set, all grounded in established operational rules. We also explained some flexible aspects for the invented operators. Furthermore, using the newly developed operators for (p, q)-fractional linear Diophantine fuzzy information, we constructed the multi-attribute decision-making $(MADM)$ technique to assess the green supply chain management challenge. Last, we compared the ranking results of the produced approaches with the obtained ranking results of the techniques using several numerical instances to demonstrate the validity and superiority of the developed techniques. Finally, a few comparisons between the findings were made.

Keywords: (p, q)-fractional linear Diophantine fuzzy set; operations and relations; sensitivity and comparison analysis; averaging and geometric aggregation operators; $MADM$ problem
Mathematics Subject Classification: 03B52, 03E72, 28E10, 68T27, 94D05

1. Introduction

Classical mathematics often falls short in addressing the complexity and ambiguity inherent in real-world situations. Zadeh [1] introduced the concept of a fuzzy set, which grades possibilities within the range [0, 1]. Since then, fuzzy logic has been employed to describe imprecision, ambiguity, and vagueness across domains [2−4]. Decision-makers frequently encounter uncertainty-related challenges that are difficult to predict and manage due to the intricate modeling and control conditions associated with these uncertainties.

Atanassov [5] expanded on the concept of fuzzy sets (FS) by introducing intuitionistic fuzzy sets (IFSs), which incorporate membership degrees (MD) and non-membership degrees (NMD) summing to unity. Atanassov [6] used intuitionistic fuzzy components in a geometric context, while Xu [7] represented weighted geometric notations for intuitionistic fuzzy numbers (IFNs). Recently, Khan et al. [8] introduced the concept of diamond IFS and discussed some basic properties. Garg [9] applied Einstein's t-norm principles to IFS. Additionally, researchers have developed a practical method for determining OWA weights. Using this method, the aim is to mitigate the impact of biased arguments on the decision outcome by assigning them lower weights. Xue [10] explores this method by applying Choquet's integral, measure, and representative payoffs, effectively addressing problems in intuitionistic and uncertain contexts. Khalil [11] examines two novel distance metrics: The absolute normalized Euclidean distance and the square Hamming distance. Both metrics are employed as IFSs in decision-making processes. Yager [12] developed the Pythagorean fuzzy set ($PyFS$), an extension of the IFS concept, where the sum of the squares of the membership degree (MD) and non-membership degree (NMD) does not exceed one. For further study, see [13−15] and the references therein. Farhadinia [16] introduced a decision-making technique using $PyFS$ based on similarity measures. Yager [17] incorporated multiple aggregate operators (AOs) into the $PyFS$ framework, and Garg [33] enhanced $PyFS$s with more comprehensive operational rules and aggregation operators. In [18], various Pythagorean fuzzy Dombi aggregation operators were proposed and analyzed. Garg applied Einstein t-norm operating principles to $PyFN$s in [19, 20] and developed two symmetric $PyFAO$s in [21]. Zeng [22] provided information on ordered weighted averaging (OWA) and probabilistic averaging. Deqing [23] proposed several distance measures that consider the four parameters of $PyFS$s and $PyFN$s. Firozja [24] introduced a novel similarity measure for $PyFS$s (S-norm) using triangle conorms. For further details, please refer to [25−30].

The IFSs and $PyFS$ serve as the basis for various applications across many real-world industries. While the concepts of FSs, IFSs, and $PyFS$s have diverse applications, they each possess unique limitations related to membership degrees (MD) and non-membership degrees (NMD). To address these limitations, Riaz [31] introduced the concept of the Linear Diophantine Fuzzy Set ($LDFS$), which incorporates control factors (CFs). The inclusion of CFs causes the $LDFS$ model to be more comprehensive and effective than previous models (see [32−34]). This innovation adds control parameters (CPs) and fills gaps left by existing structures, thereby expanding the scope for MD and NMD applications. Additionally, $LDFS$s assign two grades to information, where the total of these grades, such as the product of control factors with MD and NMD, cannot exceed one. Numerous researchers have contributed to the study of $LDFS$s. For instance, Iampan [35] introduced linear Diophantine fuzzy Einstein aggregation operators for multi-criteria decision-making problems within the post-acute care (PAC) model network for patients with cerebrovascular disorders (CVDs). Ayub [36] was the first to define decision-making using linear Diophantine fuzzy relations and their algebraic properties. Mahmood [37] proposed generalized Hamacher aggregation operators based on linear Diophantine uncertain linguistic settings and explored their applications in decision-making scenarios. Additionally, Khan et al. [38−42] apply fuzzy theory in calculus to establish various types of fuzzy inequalities as well their applications in field of optimization. For further study see [43−45] and the references therein.

The q-rung linear Diophantine fuzzy set (q-$RLDFS$) is an advanced generalization of the $PyFS$, q-$ROFS$, and $LDFS$. Almagrabi [46] introduced this concept and highlighted its significant features. Aggregation operators are crucial for effectively aggregating uncertainty in decision-making scenarios using Yager approach [47]. The q-$LDFS$ incorporates MD (a) and NMD (F) with control factors (α, β), adhering to the constraints $0 \leq (\alpha)^q \Psi(x) + (\beta)^q \partial(x) \leq 1, \forall\, x \in E, q \geqslant 1$, with $0 \leq \alpha^q + \beta^q \leq 1$. This enables the flexible selection of MD and NMD values. Using q-$RLDFS$, Qiyas [48] proposed novel distance and similarity metrics. Even though the q-$RLDF$ handles NMD and MD, there remains a gap regarding the neutral degree (ND). To address this, the fuzzy set theory must incorporate new fuzzy numbers. Gulistan and Pedrycz [49] created a new version of fuzzy set, which is known as q-fractional fuzzy sets, and they found some application using the $MADM$ technique. For more information, see [50−54] and the references therein.

We ask the following question: Why do we need a (p, q)-fractional linear Diophantine fuzzy set?

In real-world scenarios, the total of membership grades (MG) and non-membership grades (NMG) in all forms of FS can occasionally exceed 1, as in the case of $0.9 + 0.7 > 1$, and the square sum can also exceed 1, as in the case of $(0.9)^2 + (0.7)^2 > 1$. IFS and $PyFS$ have failed in these situations. To address these shortcomings, the constraints on MG and NMG in the case of q-$ROFS$ are changed to $0 \leq \Psi(x)^q + \partial(x)^q \leq 1$. With extraordinarily high numbers, we can handle MG and NMG. If both MG $\partial(x)$ and NMG $\Psi(x)$ are equal to 1 (i.e., $\Psi(x) = \partial(x) = 1$), then in some actual problems we have $1^q + 1^q > 1$, which violates the restriction of q-$ROFS$. The notion of $LDFS$ was then proposed by Riaz and Hashmi (2019), who also discussed the importance of reference parameters (RPs). These parameters hold the requirement $0 \leq \alpha\Psi(x) + \beta\partial(x) \leq 1$ with $0 \leq \alpha + \beta \leq 1$. However, in this case as well, the decision makers' (DM) total of RPs could be greater than one, i.e., $\alpha + \beta > 1$, which goes against the $LDFS$ restriction. Thus, $LDFS$'s objective concerning RPs was not met. In ordered to overcome such condition, (Almagrabi et al. 2021) introduced q-$RLDFS$ using q-$ROFS$ approach over RPs which hold the condition $0 \leq \alpha^q\Psi(x) + \beta^q\partial(x) \leq 1$ with $0 \leq \alpha^q + \beta^q \leq 1$. In the case of q-$RLDFS$, some certain practical issues we obtain $1^q + 1^q > 1$ or $1^q + (0.7)^2 > 1$ or $(0.7)^2 + 1^q > 1$, which violates the restriction on RPs. This causes the multi attribute decision makings (MADM) to be limited and affects the optimum decision. To evaluate the optimal choice based on attribute records, we introduce a novel hybrid structure called the (p, q)-fractional linear Diophantine fuzzy set ((p, q)-$FLDFS$), which integrates both $LDFS$ and q-$LDFS$. This collection offers an overview of generic forms of $LDFS$. We also explore specific aggregation procedures for integrating fuzzy (p,q)-fractional linear Diophantine information in uncertain emergency situations. These operators are unique due to their ability to synthesize (p,q)-fractional linear Diophantine fuzzy information, which enhances the concept of (p,q)-fractional linear Diophantine fuzzy aggregation operators. Additionally, the proposed aggregation operators support multiple-attribute decision-making in the (p,q)- $FLDF$ context, serving as valuable tools for decision-makers. The major contributions of this study are highlighted in the following areas (see [55−58]):

(1) In order to close this knowledge gap, our first goal is to implement the new (p,q)-$FLDFS$ technique with $p,q > 1$.

(2) The implementation of the qth and pth fractional value of CFs capabilities in $LDFS$ is the second goal, as pth and qth values are not manageable by IFSs, $PyFS$s, q- $ROFS$ s, and $LDFS$s. The system as conceived is better than the existing methods, and DM has total control over grade selection. This model also describes the issue by altering the bodily sensation of connection. The corresponding assemblage is changed to $LDFS$ when $p = q = 1$. Moreover, the (p,q)-fractional Diophantine space expands as the pth and qth values rise, providing border boundaries with a bigger search space to convey a wider range of fuzzy data. Consequently, we could be able to characterize a wider variety of fuzzy data using (p,q)- $FLDFS$s.

(3) Establishing a direct link between the current studies and $MADM$ issues is our third goal. We developed parametric decision support strategies to address multi-attribute problems.

To accomplish these objectives, the primary framework of this document is presented below. In Section 2, the basic notions of FS and generalized FSs are presented. In Section 3, we develop a certain class of new FSs that is known as (p,q)-$FLDFS$s. Additionally, analysis over (p,q)-$FLDFS$s is also given as well as interpretation of sensitivity and comparison is also discussed. Some basic operations and relations are also proved. We build the (p,q)-$FLDFWAA$, (p,q)-$FLDFOWAA$, and (p,q)-$FLDFHWAA$ operators in Section 4 and thoroughly investigate several well-known and practicable properties and exceptional outcomes. In Section 5, We present the (p,q)-$FLDFWGA$, (p,q)-$FLDFOWGA$, and (p,q)-$FLDFHWGA$ operators and also characterize the properties of these operators. In Section 6, we suggest multi-attribute decision-making techniques for assessing green supply chain management in the context of the (p,q)-fractional linear Diophantine fuzzy information. The comparison analysis highlights the benefits of these techniques. In Section 7, we provide an explanation of our closing remarks.

2. Preliminaries

In this section, we first go over the fundamental idea and its understanding-related features before developing a new one. Now, we start with the basic definition of IFS such that:

Definition 1. Zadeh (1965) Suppose an arbitrary nonempty set E. A fuzzy set $\mathcal{L}$ is characterized on E as;

$$\mathcal{L} = \{(x, \Psi(x)) \mid x \in E\}.$$

In this case, function Ψ transforms E to [0,1], and function $\Psi(x)$ is considered the membership grade (MG) of x in E for any x that is in E such that $x \in E, 0 \leq \Psi(x) \leq 1$.

Definition 2. (Atanassov 1986) Let us have a fixed universe E and its sub-set $\mathcal{L}$. The set

$$\mathcal{L} = \{\langle x, \Psi(x), \partial(x)\rangle : \text{for all } x \in E\},$$

where $0 \leq \Psi(x) + \partial(x) \leq 1$, is called intuitionistic fuzzy set and functions $\Psi, \partial: E \to [0, 1]$ indicate the degree of membership (validity, etc.) and non-membership grades (NMG) (non-validity, etc.) of element $x \in E$ to a fixed set $\mathcal{L} \subseteq E$. Now, we can define also function $\pi: E \to [0, 1]$ by means of

$$\pi(x) = 1 - \Psi(x) - \partial(x).$$

and it corresponds to degree of indeterminacy (uncertainty, etc.), see Figure 1.

Definition 3. (Yager 2013a, b) Consider a fixed set E and the Pythagorean fuzzy set ($PyFS$) A_P on E have the following mathematical symbol:

$$A_P = \{(x, \Psi(x), \partial(x)) \mid x \in E\},$$

where $\Psi(x)$ and $\partial(x) \in [0,1]$ are MG and NMG functions with subject to $(\Psi(x))^2 + (\partial(x))^2 \leq 1$, see Figure 1. The hesitancy MG is characterized by

$$\pi(x) = \sqrt{1 - (\Psi(x))^2 - (\partial(x))^2}.$$

Definition 4. (Gulistan and Pedrycz) Let us have a fixed universe E and its sub-set $\mathcal{L}$. The set

$$\mathcal{L} = \{(x, \langle \Psi(x), \partial(x) \rangle_q): \text{for all } x \in E\},$$

where $0 \leq \frac{\Psi(x)}{q} + \frac{\partial(x)}{q} \leq 1$ with $2 \leq q$, is called q-fractional fuzzy set (q-FFS) and functions $\Psi, \partial : E \to [0, 1]$ indicate the degree of membership (validity, etc.) and non-membership (non-validity, etc.) of element $x \in E$ to a fixed set $\mathcal{L} \subseteq E$. Now, we can define also function $\pi: E \to [0, 1]$ by means of

$$\pi(x) = 1 - \frac{\Psi(x)}{p} + \frac{\partial(x)}{q},$$

and it corresponds to degree of indeterminacy (uncertainty, etc.).

Definition 5. (Yager 2016) Suppose E be a fixed set. A q-rung orthopair fuzzy set (q- $ROFS$) B on E have the following mathematical symbol;

$$B = \{(x, \Psi(x), \partial(x)): x \in E\},$$

where $\Psi(x)$ and $\partial(x) \in [0,1]$ are MG and NMG functions with subject to $0 \leq (\Psi(x))^q + (\partial(x))^q \leq 1$; $q \geq 1$ (see Figure 1). The hesitancy part is characterized as

$$\pi(x) = \sqrt[q]{1 - (\Psi(x))^q - (\partial(x))^q},$$

see Figure 1.

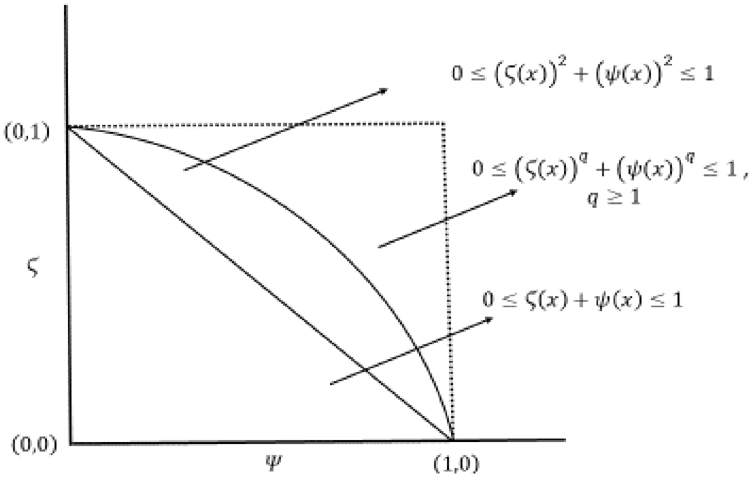

Figure 1. A comparison between intuitionistic fuzzy space, Pythagorean fuzzy space, and q-rung orthopair fuzzy space.

Definition 6. (Riaz and Hashmi 2019) Suppose E be a fixed non-empty reference set and the linear Diophantine fuzzy set ($LDFS$) is characterized by G_D and mathematical characterized as:

$$G_D = \{(x, \langle \Psi(x), \partial(x) \rangle, \langle \alpha, \beta \rangle) : x \in E\},$$

where $\Psi(x), \partial(x), \alpha, \beta \in [0,1]$ are MG, NMG and references parameters (RPs) respectively, and hold the condition $0 \le \alpha\Psi(x) + \beta\partial(x) \le 1$, $\forall\, x \in E$ with $0 \le \alpha + \beta \le 1$ (see Figure 2). These RPs could be useful in defining or characterizing a particular model. Indeterminacy degree can be characterized as

$$\Upsilon\pi(x) = 1 - (\alpha)\Psi(x) - (\beta)\partial(x),$$

where Υ is the RP of the indeterminacy degree.

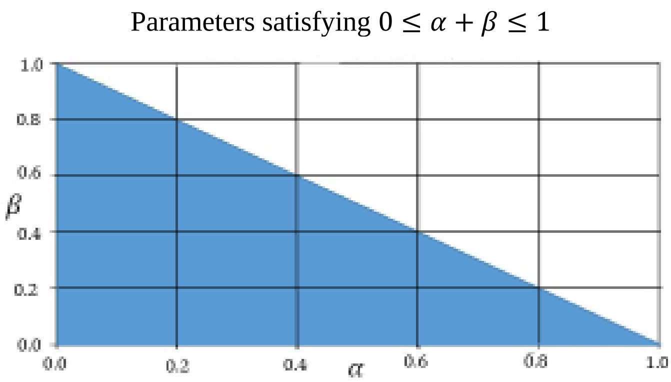

Figure 2. Graphical representation of parameters of linear Diophantine fuzzy set.

Definition 7. Suppose E be a fixed non-empty reference set and the q-rung linear Diophantine fuzzy set (q-$RLDFS$) is characterized by $\mathcal{L}_{Dq}$ and mathematical characterized as

$$\mathcal{L}_{Dq} = \{(x, \langle \Psi(x), \partial(x) \rangle, \langle \alpha, \beta \rangle) : x \in E\},$$

where $\Psi(x), \partial(x), \alpha, \beta \in [0,1]$. These functions fulfill the restriction $0 \leq (\alpha)^q \Psi(x) + (\beta)^q \partial(x) \leq 1, \forall\, x \in E, q \geqslant 1$,with $0 \leq \alpha^q + \beta^q \leq 1$, (see Figure 3). These RPs could be useful in defining or characterizing a particular model. The part of the hesitation may be calculated as

$$\curlyvee\pi(x) = \sqrt[q]{1 - \left((\alpha)^q \Psi(x) + (\beta)^q \partial(x)\right)},$$

where $\curlyvee$ stand for the RPs related to the level of uncertainty or hesitancy.

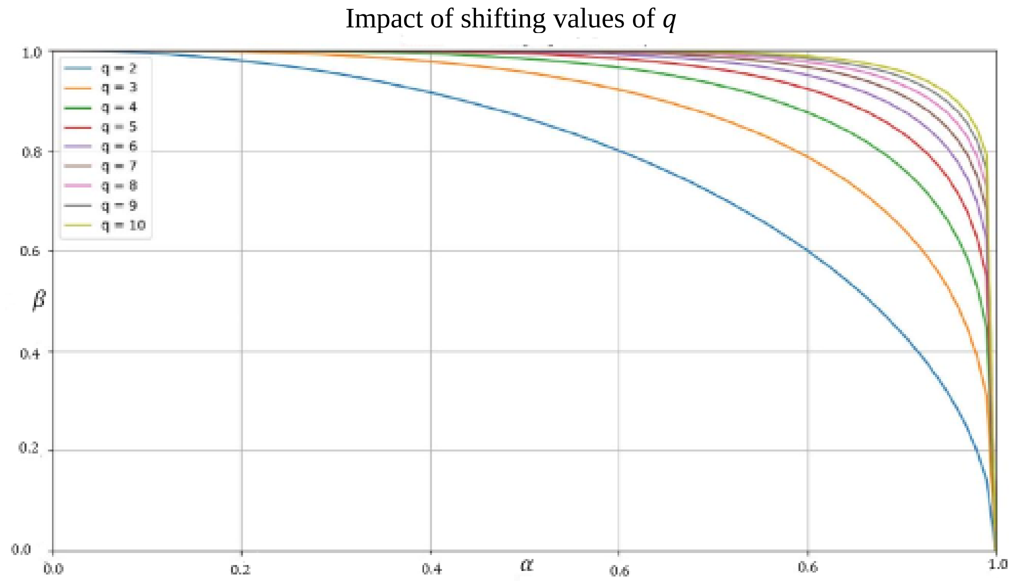

Figure 3. Graphical representation of parameters of q-rung linear Diophantine fuzzy set for different values of q.

3. The (p, q)-fractional linear Diophantine fuzzy set

In this section, we start with the novel idea of fuzzy set, which is known as (p, q)-fractional linear Diophantine fuzzy set.

Definition 8. Let us have a fixed universe E and its sub-set $\mathcal{L}$. The set

$$\mathcal{L} = \left\{\left(x, \langle \Psi(x), \partial(x)\rangle_{(p,q)}, \langle \alpha, \beta\rangle_{(p,q)}\right): \text{for all } x \in E\right\},$$

where $0 \leq \frac{\alpha\Psi(x)}{p} + \frac{\beta\partial(x)}{q} \leq 1$ with $0 \leq \frac{\alpha}{p} + \frac{\beta}{q} \leq 1$, see Figure 4 and $p, q \geq 2$, is called (p, q)-fractional linear Diophantine fuzzy set ((p, q)-$FLDFS$) and functions $\Psi, \partial : E \to [0, 1]$ indicate the degree of membership (validity, etc.) and non-membership (non-validity, etc.) of element $x \in E$ to a fixed set $\mathcal{L} \subseteq E$, where α and β are RPs that support in the identification or description of a particular model.

Now, we can define also function $\pi: E \to [0, 1]$ by means of

$$\curlyvee\pi(x) = 1 - \frac{\alpha\Psi(x)}{p} + \frac{\beta\partial(x)}{q}.$$

and it corresponds to degree of indeterminacy (uncertainty, etc.), where $\curlyvee$ is RP is a direct link with degree of indeterminacy.

3.1. Analysis

The RPs serve to define and classify a given system, and they also alter the system's physical meaning or sense. They do away with limitations and increase the grade space utilized in (p, q)-$FLDFS$. The proposed method of (p, q)-$FLDFS$ is more efficient and flexible than other approaches due to the addition of values (p, q) in the fraction of RPs (see Figure 4). This method constructs strong relation with multi-attribute decision making ($MADM$) problems.

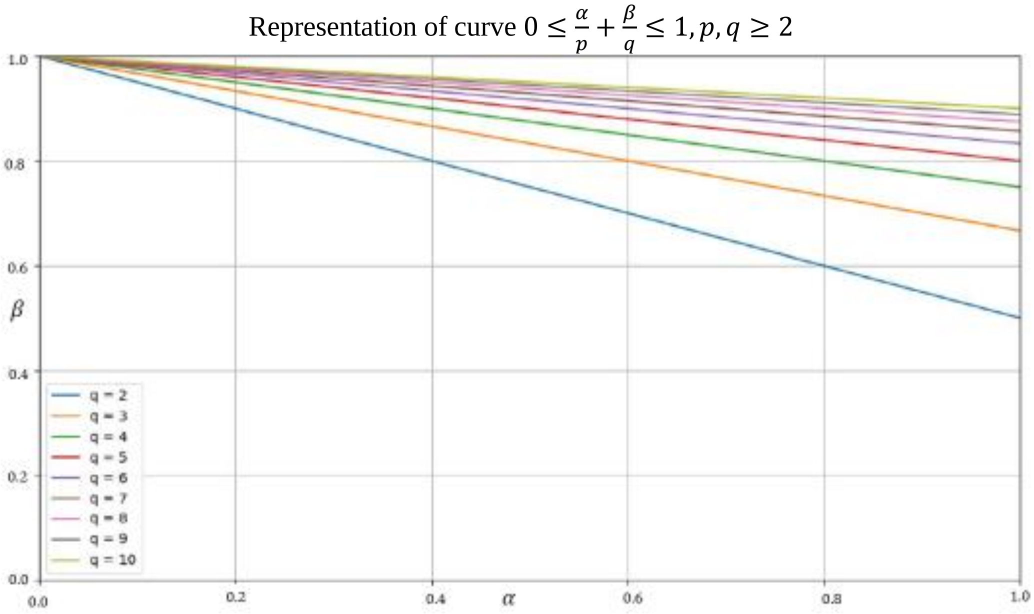

Figure 4. Curve representation of parameters of (p, q)-fractional linear Diophantine fuzzy set for different values of p and q.

For convenience, we can write $\mathcal{L} = \left(\langle \Psi, \partial \rangle_{(p,q)}, \langle \alpha, \beta \rangle_{(p,q)}\right)$ to represent a (p, q)-$FLDFS$.

3.2. Interpretation of sensitivity analysis and comparison

Definition 9. For each value of $p, q \geq 2$, the complete square with vertices $\left((\Psi, \partial), (0,0)\right)$, $\left((\Psi, \partial), (0,1)\right)$, $\left((\Psi, \partial), (1,1)\right)$ and $\left((\Psi, \partial), (1,0)\right)$ is achieved. Therefore, there is no sensitivity of the values in the RPs of (p, q)-$FLDFS$ to (p, q). All points within the square that meet the criterion $\frac{\alpha}{p} + \frac{\beta}{q} \leq 1$ and $p, q \geq 2$, as shown in Figure 5. Note that the dark blue shading covering the region inside the square. For $p, q \geq 2$, the graph is illustrated in Figures 3−5.

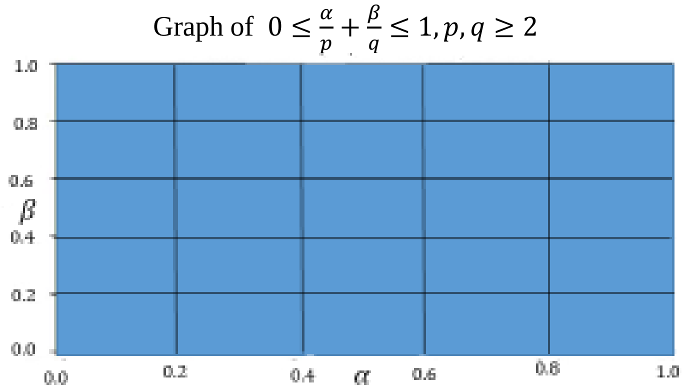

Figure 5. Sensitivity analysis of parameters of (p, q)-fractional linear Diophantine fuzzy set.

In Figure 6, the dotted lines depict the curves for the condition $\frac{\alpha}{p} + \frac{\beta}{q} \leq 1$ with varying $p = q$ values, while the solid lines represent the curves for the condition $\alpha^q + \beta^q \leq 1$ with constant q values.

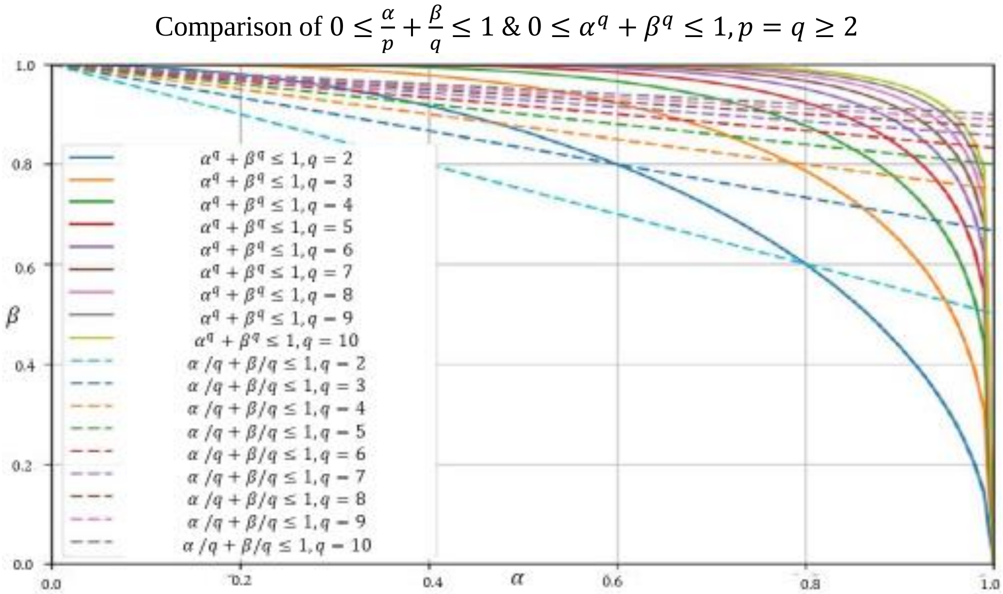

Figure 6. Comparison analysis between parameters of two fuzzy sets.

Definition 10. A (p, q)-fractional linear Diophantine fuzzy number ((p, q)-$FLDFN$) is denoted and defined as

$$\mathcal{L} = \{\langle \Psi, \partial \rangle_{(p,q)}, \langle \alpha, \beta \rangle_{(p,q)}\},$$

where $\mathcal{L}$ represent the (p, q)-$FLDFN$ with conditions;

(i) $0 \leq \frac{\alpha}{p} + \frac{\beta}{q} \leq 1, p, q \geqslant 2,$

(ii) $0 \leq \frac{\alpha \Psi(x)}{p} + \frac{\beta \partial(x)}{q} \leq 1,$

(iii) $0 \leq \alpha, \Psi(x), \beta, \partial(x) \leq 1$.
For the sake of simplicity, the set of (p, q) -fractional linear Diophantine fuzzy numbers ((p, q)-$FLDFNs$).
Next definition is about absolute (p, q)-$FLDFS$ and null or empty (p, q)-$FLDFS$.
Definition 11. A (p, q)-$FLDFS$ on E of the form

${}^{1}\mathcal{L} = \{(x, (1,1), (1,1)) : x \in E\}$ is called absolute (p, q)-$FLDFS$ and

${}^{0}\mathcal{L} = \{(x, (0,0), (0,0)) : x \in E\}$ is called empty or null (p, q)-$FLDFS$.
It is noteworthy to notice that these definitions differ from the one given in [Riaz and Hashim] for absolute and null (p, q)-$FLDFS$s.
It is important to remember that the (p, q)-fractional linear Diophantine space grows as the rung q increases. As a result, the boundary limits have a larger search space that can represent a wider range of the fuzzy data.

3.3. Basic operations on (p, q)-fractional linear Diophantine fuzz sets

In this section, we propose some of the basic operations on (p, q)-$FLDFS$s like inclusion, union, intersection, complement, and some compositions as well as some properties are also illustrated. For the sake of easy understanding, we will take the following three (p, q)-$FLDFS$s over fixed universe E:

$$\mathcal{L} = \left\{ \left(x, \langle \Psi_{\mathcal{L}}(x), \partial_{\mathcal{L}}(x) \rangle_{(p,q)}, \langle \alpha_{\mathcal{L}}, \beta_{\mathcal{L}} \rangle_{(p,q)} \right) : \text{for all } x \in E \right\},$$

$$Y = \left\{ \left(x, \langle \Psi_{Y}(x), \partial_{Y}(x) \rangle_{(p,q)}, \langle \alpha_{Y}, \beta_{Y} \rangle_{(p,q)} \right) : \text{for all } x \in E \right\},$$

$$Z = \left\{ \left(x, \langle \Psi_{Z}(x), \partial_{Z}(x) \rangle_{(p,q)}, \langle \alpha_{Z}, \beta_{Z} \rangle_{(p,q)} \right) : \text{for all } x \in E \right\}.$$

Definition 12. Let $\mathcal{L}$ and Y be two (p, q)-$FLDFS$s. Then,

- $\mathcal{L} \subseteq Y$ iff $\Psi_{\mathcal{L}}(x) \preccurlyeq \Psi_{Y}(x)$, $\partial_{\mathcal{L}}(x) \succcurlyeq \partial_{Y}(x)$, $\alpha_{\mathcal{L}} \preccurlyeq \alpha_{Y}$ and $\beta_{\mathcal{L}} \succcurlyeq \beta_{Y}$,
- $\mathcal{L} = Y$ iff $\mathcal{L} \subseteq Y$ and $\mathcal{L} \supseteq Y$,
- $\mathcal{L} \cup Y = \Big\{ \Big(x,\ \langle \vee \left(\Psi_{\mathcal{L}}(x), \Psi_{Y}(x)\right), \wedge \left(\partial_{\mathcal{L}}(x), \partial_{Y}(x)\right) \rangle_{(p,q)}, \langle \vee \left(\alpha_{\mathcal{L}}, \alpha_{Y}\right), \wedge \left(\beta_{\mathcal{L}}, \beta_{Y}\right) \rangle_{(p,q)} \Big) :$

  for all $x \in E \Big\}$,
- $\mathcal{L} \cap Y = \Big\{ \Big(x,\ \langle \wedge \left(\Psi_{\mathcal{L}}(x), \Psi_{Y}(x)\right), \vee \left(\partial_{\mathcal{L}}(x), \partial_{Y}(x)\right) \rangle_{(p,q)}, \langle \wedge \left(\alpha_{\mathcal{L}}, \alpha_{Y}\right), \vee \left(\beta_{\mathcal{L}}, \beta_{Y}\right) \rangle_{(p,q)} \Big) :$

  for all $x \in E \Big\}$,
- $\mathcal{L}^{c} = \left\{ \left(x,\ \langle \partial_{\mathcal{L}}(x), \Psi_{\mathcal{L}}(x) \rangle_{(p,q)}, \langle \beta_{\mathcal{L}}, \alpha_{\mathcal{L}} \rangle_{(p,q)} \right) : \text{for all } x \in E \right\}$.

Proposition 1. Let $\mathcal{L}$, Y and Z be three (p, q)-$FLDFS$s. Then, following properties holds such that

1) $\mathcal{L} \subseteq Y$ and $Y \subseteq Z$ implies $\mathcal{L} \subseteq Z$; (Inclusion property),
2) $\mathcal{L} \cup Y = Y \cup \mathcal{L}$ and $\mathcal{L} \cap Y = Y \cap \mathcal{L}$; (Commutative law),

3) $\mathcal{L} \cup (Y \cup Z) = (\mathcal{L} \cup Y) \cup Z$ and $\mathcal{L} \cap (Y \cap Z) = (\mathcal{L} \cap Y) \cap Z$; (Associative law)
4) $\mathcal{L} \cup (Y \cap Z) = (\mathcal{L} \cup Y) \cap (\mathcal{L} \cup Z)$ and $\mathcal{L} \cap (Y \cup Z) = (\mathcal{L} \cap Y) \cup (\mathcal{L} \cap Z)$; (Distributive laws),
5) De-Morgan's Laws holds for $\mathcal{L}$ and Y.

Proof. (1) Consider $\mathcal{L} \subseteq Y$ and $Y \subseteq Z$, then by Definition 12, we have

$$\begin{gathered}\Psi_{\mathcal{L}}(x) \preccurlyeq \Psi_Y(x)\,,\ \partial_{\mathcal{L}}(x) \succcurlyeq \partial_Y(x),\\ \alpha_{\mathcal{L}} \preccurlyeq \alpha_Y\,,\ \beta_{\mathcal{L}} \succcurlyeq \beta_Y,\end{gathered} \tag{1}$$

and

$$\begin{gathered}\Psi_Y(x) \preccurlyeq \Psi_Z(x),\ \partial_Y(x) \succcurlyeq \partial_Z(x),\\ \alpha_Y \preccurlyeq \alpha_Z,\ \beta_Y \succcurlyeq \beta_Z.\end{gathered} \tag{2}$$

Combining (1) and (2), we have

$$\begin{gathered}\Psi_{\mathcal{L}}(x) \preccurlyeq \Psi_Y(x) \preccurlyeq \Psi_Z(x),\ \partial_{\mathcal{L}}(x) \succcurlyeq \partial_Y(x) \succcurlyeq \partial_Z(x),\\ \alpha_{\mathcal{L}} \preccurlyeq \alpha_Y \preccurlyeq \alpha_Z,\ \beta_{\mathcal{L}} \succcurlyeq \beta_Y \succcurlyeq \beta_Z.\end{gathered} \tag{3}$$

From (3), we conclude that

$$\begin{gathered}\Psi_{\mathcal{L}}(x) \preccurlyeq \Psi_Z(x), \partial_{\mathcal{L}}(x) \succcurlyeq \partial_Z(x),\\ \alpha_{\mathcal{L}} \preccurlyeq \alpha_Z, \beta_{\mathcal{L}} \succcurlyeq \beta_Z.\end{gathered}$$

Hence, $\mathcal{L} \subseteq Z$.
Similarly, the remailing results 2)−5) can be proved easily.

3.4. Comparison between two (p, q)-$FLDFNs$

It is well known fact that comparison laws in fuzzy theory play a critical role, especially in the field of decision making and some other optimization problems. These laws enable us to differentiate the two (p, q)-$FLDFS$s as well as sometime these rules tell us the worth of the relation between these two (p, q)-$FLDFS$s that this relation is how much strong.

Definition 13. Let $\mathcal{L} = \{\langle \Psi_{\mathcal{L}}, \partial_{\mathcal{L}} \rangle_{(p,q)}, \langle \alpha_{\mathcal{L}}, \beta_{\mathcal{L}} \rangle_{(p,q)}\}$ be a (p, q)-$FLDFN$. Then, score function $\left(S_{(p,q)\text{-}FLDFN}(\mathcal{L})\right)$ and accuracy functions $\left(H_{(p,q)\text{-}FLDFN}(\mathcal{L})\right)$ of $\mathcal{L}$ are characterized and characterized as

$$S_{(p,q)\text{-}FLDFN}(\mathcal{L}) = \tfrac{1}{2}[\Psi_{\mathcal{L}}(x) - \partial_{\mathcal{L}}(x) + \alpha - \beta], \tag{4}$$

where, $-1 \leq S_{(p,q)\text{-}FLDFN}(\mathcal{L}) \leq 1$ and $p, q \geq 2$.

$$H_{(p,q)\text{-}FLDFN}(\mathcal{L}) = \tfrac{1}{2}\left[\tfrac{\Psi_{\mathcal{L}}(x) + \partial_{\mathcal{L}}(x)}{2} + \tfrac{\alpha+\beta}{2}\right], \tag{5}$$

where, $0 \leq H_{(p,q)\text{-}FLDFN}(\mathcal{L}) \leq 1$.
The following rules define the comparison between two (p, q)-$FLDFNs$ $\mathcal{L}_1$ and $\mathcal{L}_2$ such that

- $\mathcal{L}_1$ is higher ranked than $\mathcal{L}_2$ if $S_{(p,q)\text{-}FLDFN}(\mathcal{L}_1) > S_{(p,q)\text{-}FLDFN}(\mathcal{L}_2)$.
- $\mathcal{L}_1$ is lower ranked than $\mathcal{L}_2$ if $S_{(p,q)\text{-}FLDFN}(\mathcal{L}_1) < S_{(p,q)\text{-}FLDFN}(\mathcal{L}_2)$.

When $S_{(p,q)\text{-}FLDFN}(\mathcal{L}_1) = S_{(p,q)\text{-}FLDFN}(\mathcal{L}_2)$ for two (p, q)-$FLDFNs$, then

- $\mathcal{L}_1$ is higher ranked than $\mathcal{L}_2$ if $H_{(p,q)\text{-}FLDFN}(\mathcal{L}_1) > H_{(p,q)\text{-}FLDFN}(\mathcal{L}_2)$.
- $\mathcal{L}_1$ is lower ranked than $\mathcal{L}_2$ if $H_{(p,q)\text{-}FLDFN}(\mathcal{L}_1) < H_{(p,q)\text{-}FLDFN}(\mathcal{L}_2)$.
- $\mathcal{L}_1$ is similar $\mathcal{L}_2$ if $H_{(p,q)\text{-}FLDFN}(\mathcal{L}_1) = H_{(p,q)\text{-}FLDFN}(\mathcal{L}_2)$.

Example 1. Let $\mathcal{L}_1 = \left(\langle 1, .9\rangle_{(2,2)}, \langle 1,1\rangle_{(2,2)}\right)$ and $\mathcal{L}_2 = \left(\langle .4, .9\rangle_{(3,3)}, \langle 1,1\rangle_{(3,3)}\right)$ be two alternatives with $(p,q)\text{-}FLDFNs$. Then, score function is utilized to determine the preferred option such that

$$S_{(p,q)\text{-}FLDFN}(\mathcal{L}_1) = \frac{1}{2}[1 - .9 + 0] = .05,$$

$$S_{(p,q)\text{-}FLDFN}(\mathcal{L}_2) = \frac{1}{2}[.4 - .9 + 0] = -.25.$$

Hence, option $\mathcal{L}_2$ is preferable to option $\mathcal{L}_1$.

Example 2. If $(p,q)\text{-}FLDFNs$ for two alternative are $\mathcal{L}_2 = \left(\langle 1, .5\rangle_{(2,2)}, \langle 1,1\rangle_{(2,2)}\right)$ and $\mathcal{L}_2 = \left(\langle .9, .4\rangle_{(2,2)}, \langle 1,1\rangle_{(2,2)}\right)$, then score function is utilized to determine the preferred option such that

$$S_{(p,q)\text{-}FLDFN}(\mathcal{L}_1) = \frac{1-.5}{2} = .25,$$

$$S_{(p,q)\text{-}FLDFN}(\mathcal{L}_2) = \frac{.9-.4}{2} = .25.$$

Thus, we are unsure of which option is preferable in this situation. However, using Eq (5), we can get

$$H_{(p,q)\text{-}FLDFN}(\mathcal{L}_1) = \frac{1+.5}{4} + \frac{1}{2} = .88,$$

$$H_{(p,q)\text{-}FLDFN}(\mathcal{L}_2) = \frac{.9+.4}{4} + \frac{1}{2} = .83.$$

As a result, alternative $\mathcal{L}_1$ is superior to alternative $\mathcal{L}_2$.

4. The (p,q)-fractional linear Diophantine fuzzy weighted averaging aggregation operators

In this section, we propose some types of (p,q)-fractional linear Diophantine fuzzy weighted averaging aggregation operators. First, we define the (p,q)-fractional linear Diophantine fuzzy weighted averaging aggregation operator.

Definition 14. Let $\mathcal{L}_1 = \left\{\langle \Psi_1, \partial_1\rangle_{(p,q)}, \langle \alpha_1, \beta_1\rangle_{(p,q)}\right\}$ and $\mathcal{L}_2 = \left\{\langle \Psi_2, \partial_2\rangle_{(p,q)}, \langle \alpha_2, \beta_2\rangle_{(p,q)}\right\}$ be two $(p,q)\text{-}FLDFN$s and $\lambda > 0$

$\mathcal{L}_1 \oplus \mathcal{L}_2$

$$= \left(\langle\langle \sqrt[p]{(\Psi_1)^p + (\Psi_2)^p - (\Psi_1)^p(\Psi_2)^p}, \beta_1\beta_2 \rangle, \partial_1\partial_2\rangle_{(p,q)}, \langle \sqrt[p]{(\alpha_1)^p + (\alpha_2)^p - (\alpha_1)^p(\alpha_2)^p}, \beta_1\beta_2 \rangle_{(p,q)} \right),$$

$$\lambda\mathcal{L}_1 = \left(\langle \sqrt[p]{1 - (1 - (\Psi_1)^p)^\lambda}, {\partial_1}^\lambda\rangle_{(p,q)}, \langle \sqrt[p]{1 - (1 - (\alpha_1)^p)^\lambda}, {\beta_1}^\lambda\rangle_{(p,q)}\right).$$

For the sake of simplicity, the set of (p,q)-fractional linear Diophantine fuzzy numbers ($(p,q)\text{-}FLDFNs$) on E is characterized by $(p,q)\text{-}FLDFN(E)$.

Definition 15. The (p,q)-fractional linear Diophantine fuzzy weighted averaging aggregation ((p,q)-$FLDFWAA$) operator on "n" numbers of (p,q)-$FLDFNs$ on the set E is characterized with the help of this transformation Ω: (p,q)-$FLDFN(E) \rightarrow (p,q)$-$FLDFN(E)$ associated with weight vector $\omega = (\omega_1, \omega_2, \omega_3, \dots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows: When $\{\mathcal{L}_1 = \left(\langle \Psi_1, \partial_1 \rangle_{(p,q)}, \langle \alpha_1, \beta_1 \rangle_{(p,q)}\right), \mathcal{L}_2 = \left(\langle \Psi_2, \partial_2 \rangle_{(p,q)}, \langle \alpha_2, \beta_2 \rangle_{(p,q)}\right), \dots \dots \dots, \mathcal{L}_n = \left(\langle \Psi_n, \partial_n \rangle_{(p,q)}, \langle \alpha_n, \beta_n \rangle_{(p,q)}\right)\}$ are (p,q)-$FLDFNs$,

$$(p,q)\text{-}FLDFWAA_{\omega}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots \dots \dots, \mathcal{L}_n) = \prod_{j=1}^{n} \omega_j \mathcal{L}_j.$$

Theorem 1. The (p,q)-fractional linear Diophantine fuzzy weighted averaging aggregation ((p,q)-$FLDFWAA$) operator on "n" numbers of (p,q)-$FLDFNs$ on the set E is characterized with the help of this transformation Ω: (p,q)-$FLDFN(E) \rightarrow (p,q)$-$FLDFN(E)$ associated with weight vector $\omega = (\omega_1, \omega_2, \omega_3, \dots \dots \dots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows: When $\{\mathcal{L}_1 = \left(\langle \Psi_1, \partial_1 \rangle_{(p,q)}, \langle \alpha_1, \beta_1 \rangle_{(p,q)}\right), \mathcal{L}_2 = \left(\langle \Psi_2, \partial_2 \rangle_{(p,q)}, \langle \alpha_2, \beta_2 \rangle_{(p,q)}\right), \dots \dots \dots, \mathcal{L}_n = \left(\langle \Psi_n, \partial_n \rangle_{(p,q)}, \langle \alpha_n, \beta_n \rangle_{(p,q)}\right)\}$ are (p,q)-$FLDFNs$,

$$(p,q)\text{-}FLDFWAA_{\omega}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots \dots \dots, \mathcal{L}_n)$$

$$= \prod_{j=1}^{n} \omega_j \mathcal{L}_j$$

$$= \left(\langle \sqrt[p]{1 - \prod_{j=1}^{n} \left(1 - (\Psi_j)^p\right)^{\omega_j}}, \prod_{j=1}^{n} \partial_j^{\ \omega_j} \rangle_{(p,q)}, \langle \sqrt[p]{1 - \prod_{j=1}^{n} \left(1 - (\alpha_j)^p\right)^{\omega_j}}, \prod_{j=1}^{n} \beta_j^{\ \omega_j} \rangle_{(p,q)} \right).$$

This operator can easily be proved with support of (p,q)-$FLDFNs$ operations and mathematical -induction. Here, μ and ν are representing the membership and non-membership function. ω is called weight function, $\mathcal{L}_j$ are (p,q)-$FLDFNs$, where $j \in N$.

Proof. The demonstration of the proof is similar to the operators of intuitionistic fuzzy sets, so it is omitted.

Definition 16. The (p,q)-fractional linear Diophantine fuzzy ordered weighted averaging aggregation ((p,q)-$FLDFOWAA$) operator on "n" numbers of (p,q)-$FLDFNs$ is characterized with the help of this transformation $\Omega: (p,q)$-$FLDFN(E) \rightarrow (p,q)$-$FLDFN(E)$ associated with $\omega = (\omega_1, \omega_2, \omega_3, \dots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows: When $\{\mathcal{L}_1 = \left(\langle \Psi_1, \partial_1 \rangle_{(p,q)}, \langle \alpha_1, \beta_1 \rangle_{(p,q)}\right), \mathcal{L}_2 = \left(\langle \Psi_2, \partial_2 \rangle_{(p,q)}, \langle \alpha_2, \beta_2 \rangle_{(p,q)}\right), \dots \dots \dots, \mathcal{L}_n = \left(\langle \Psi_n, \partial_n \rangle_{(p,q)}, \langle \alpha_n, \beta_n \rangle_{(p,q)}\right)\}$ are (p,q)-$FLDFN$,

$$(p,q)\text{-}FLDFOWAA_{\omega}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots \dots, \mathcal{L}_n) = \prod_{j=1}^{n} \omega_j \mathcal{L}_{\sigma(j)},$$

where $\left(\sigma(1), \sigma(2), \sigma(3), \dots \dots \dots \dots, \sigma(n)\right)$ is the arrangement of $j \in N$, for which $\mathcal{L}_{\sigma(j-1)} \geq \mathcal{L}_{\sigma(j)}$, for all $j \in N$.

Theorem 5. The (p,q)-fractional linear Diophantine fuzzy ordered weighted averaging aggregation

((p,q)-$FLDFOWAA$) operator on "n" numbers of (p,q)-$FLDFNs$ is characterized with the help of this transformation Ω: (p,q)-$FLDFN(E) \rightarrow (p,q)$-$FLDFN(E)$ associated with $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows: When $\{\mathcal{L}_1 = (\langle \Psi_1, \partial_1 \rangle_{(p,q)}, \langle \alpha_1, \beta_1 \rangle_{(p,q)}), \mathcal{L}_2 = (\langle \Psi_2, \partial_2 \rangle_{(p,q)}, \langle \alpha_2, \beta_2 \rangle_{(p,q)}), \ldots \ldots \ldots, \mathcal{L}_n = (\langle \Psi_n, \partial_n \rangle_{(p,q)}, \langle \alpha_n, \beta_n \rangle_{(p,q)})\}$ are (p,q)-$FLDFNs$,

$$(p,q)\text{-}FLDFOWAA_{\omega}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots \ldots, \mathcal{L}_n)$$

$$= \prod_{j=1}^{n} \omega_j \, \mathcal{L}_{\sigma(j)}$$

$$= \left(\langle \sqrt[p]{1 - \prod_{j=1}^{n} \left(1 - \left(\Psi_{\sigma(j)}\right)^p\right)^{\omega_j}}, \prod_{j=1}^{n} \partial_{\sigma(j)}{}^{\omega_j} \rangle_{(p,q)}, \langle \sqrt[p]{1 - \prod_{j=1}^{n} \left(1 - \left(\alpha_{\sigma(j)}\right)^p\right)^{\omega_j}}, \prod_{j=1}^{n} \beta_{\sigma(j)}{}^{\omega_j} \rangle_{(p,q)} \right),$$

where $(\sigma(1), \sigma(2), \sigma(3), \ldots \ldots \ldots \ldots, \sigma(n))$ is the arrangement of $j \in N$, for which $\mathcal{L}_{\sigma(j-1)} \geq \mathcal{L}_{\sigma(j)}$, for all $j \in N$.

Proof. The demonstration of the proof is similar to the operators of intuitionistic fuzzy sets, so it is omitted.

Definition 17. The (p,q)-fractional linear Diophantine fuzzy hybrid weighted averaging aggregation ((p,q)-$FLDFHWAA$) operator on "n" numbers of (p,q)-$FLDFNs$ is characterized with the help of this transformation Ω: (p,q)-$FLDFN(E) \rightarrow (p,q)$-$FLDFN(E)$ associated with $\omega = (\omega_1, \omega_2, \omega_3, \ldots \ldots \ldots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$, and it can be computed as follows: When $\{\mathcal{L}_1 = (\langle \Psi_1, \partial_1 \rangle_{(p,q)}, \langle \alpha_1, \beta_1 \rangle_{(p,q)}), \mathcal{L}_2 = (\langle \Psi_2, \partial_2 \rangle_{(p,q)}, \langle \alpha_2, \beta_2 \rangle_{(p,q)}), \ldots, \mathcal{L}_n = (\langle \Psi_n, \partial_n \rangle_{(p,q)}, \langle \alpha_n, \beta_n \rangle_{(p,q)})\}$ are (p,q)-$FLDFNs$,

$$(p,q)\text{-}FLDFHWAA_{\omega}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots \ldots \ldots, \mathcal{L}_n)$$

$$= \prod_{j=1}^{n} \omega_j \, \mathcal{L}^{\star}_{\sigma(j)},$$

where $\mathcal{L}^{\star}_{\sigma(j)}$ is biggest jth weighted (p,q)-fractional linear Diophantine fuzzy values $\mathcal{L}^{\star}_j$ $\left(\mathcal{L}^{\star}_j = \left(\mathcal{L}_j\right)^{n\omega_j}, j \in N\right)$ and $\omega = (\omega_1, \omega_2, \omega_3, \ldots \ldots \ldots, \omega_n)^T$ be the weights of $\mathcal{L}^{\star}_j$ by means of $\omega > 0$ with $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 10. The (p,q)-fractional linear Diophantine fuzzy hybrid weighted averaging aggregation ((p,q)-$FLDFHWAA$) operator on "n" numbers of (p,q)-$FLDFNs$ is characterized with the help of this transformation Ω: (p,q)-$FLDFN(E) \rightarrow (p,q)$-$FLDFN(E)$ associated with $\omega = (\omega_1, \omega_2, \omega_3, \ldots \ldots \ldots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows: When $\{\mathcal{L}_1 = (\langle \Psi_1, \partial_1 \rangle_{(p,q)}, \langle \alpha_1, \beta_1 \rangle_{(p,q)}), \mathcal{L}_2 = (\langle \Psi_2, \partial_2 \rangle_{(p,q)}, \langle \alpha_2, \beta_2 \rangle_{(p,q)}), \ldots, \mathcal{L}_n = (\langle \Psi_n, \partial_n \rangle_{(p,q)}, \langle \alpha_n, \beta_n \rangle_{(p,q)})\}$ are (p,q)-$FLDFNs$,

$$(p,q)\text{-}FLDFHWAA_{\omega}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots \ldots \ldots, \mathcal{L}_n)$$

$$= \prod_{j=1}^{n} \omega_j \, \mathcal{L}^{\star}_{\sigma(j)}$$

$$= \left(\langle \sqrt[p]{1 - \prod_{j=1}^{n} \left(1 - \left(\Psi^{\star}_{\sigma(j)}\right)^p\right)^{\omega_j}}, \prod_{j=1}^{n} \partial^{\star}_{\sigma(j)}{}^{\omega_j} \rangle_{(p,q)}, \langle \sqrt[p]{1 - \prod_{j=1}^{n} \left(1 - \left(\alpha^{\star}_{\sigma(j)}\right)^p\right)^{\omega_j}}, \prod_{j=1}^{n} \beta^{\star}_{\sigma(j)}{}^{\omega_j} \rangle_{(p,q)} \right),$$

where $\mathcal{L}^{\star}_{\sigma(j)}$ is biggest jth weighted (p,q)-fractional linear Diophantine fuzzy values $\mathcal{L}^{\star}_{j}$ $\left(\mathcal{L}^{\star}_{j} = n\omega_j\mathcal{L}_j, j \in N\right)$ and $\omega = (\omega_1, \omega_2, \omega_3, \ldots\ldots\ldots, \omega_n)^T$ be the weights of $\mathcal{L}^{\star}_{j}$ by means of $\omega > 0$ with $\sum_{j=1}^{n} \omega_j = 1$.

Proof. The demonstration of the proof is similar to the operators of intuitionistic fuzzy sets, so it is omitted.

It is interesting to note that if $\omega = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots\ldots\ldots, \frac{1}{n}\right)$, then (p,q)-$FLDFWAA$ and (p,q)-$FLDFOWAA$ operators are considered to be exceptional cases of(p,q)-$FLDFHWAA$ operator. Thus, it concludes that (p,q)-$FLDFHWAA$ operators are the extension of (p,q)-$FLDFWAA$ and (p,q)-$FLDFOWAA$ operators.

5. The (p, q)-fractional linear Diophantine fuzzy weighted geometric aggregation operators

In this section, we propose some (p,q)-fractional linear Diophantine fuzzy weighted geometric aggregation operators. First, we define the (p,q)-fractional linear Diophantine fuzzy weighted geometric aggregation operator.

Definition 18. Let $\mathcal{L}_1 = \left\{\langle \Psi_1, \partial_1\rangle_{(p,q)}, \langle \alpha_1, \beta_1\rangle_{(p,q)}\right\}$ and $\mathcal{L}_2 = \left\{\langle \Psi_2, \partial_2\rangle_{(p,q)}, \langle \alpha_2, \beta_2\rangle_{(p,q)}\right\}$ be two (p,q)-fractional linear Diophantine numbers and $\lambda > 0$

$$\mathcal{L}_1 \otimes \mathcal{L}_2 = \left(\langle \Psi_1\Psi_2, \sqrt[q]{(\partial_1)^q + (\partial_2)^q - (\partial_1)^q(\partial_2)^q}\rangle_{(p,q)}, \langle \alpha_1\alpha_2, \sqrt[q]{(\beta_1)^q + (\beta_2)^q - (\beta_1)^q(\beta_2)^q}\rangle_{(p,q)}\right),$$

$$\mathcal{L}_1{}^{\lambda} = \left(\langle {\Psi_1}^{\lambda}, \sqrt[q]{1 - (1 - (\partial_1)^q)^{\lambda}}\rangle_{(p,q)}, \langle (\alpha_1)^{\lambda}, \sqrt[q]{1 - (1 - (\beta_1)^q)^{\lambda}}\rangle_{(p,q)}\right).$$

Definition 19. The (p,q)-fractional linear Diophantine fuzzy weighted geometric aggregation ((p,q)-$FLDFWGA$) operator on "n" numbers of (p,q)-$FLDFNs$ on the set E is characterized with the help of this transformation Ω: (p,q)-$FLDFN(E) \rightarrow (p,q)$-$FLDFN(E)$ associated with weight vector $\omega = (\omega_1, \omega_2, \omega_3, \ldots\ldots\ldots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows: When $\left\{\mathcal{L}_1 = \left(\langle \Psi_1, \partial_1\rangle_{(p,q)}, \langle \alpha_1, \beta_1\rangle_{(p,q)}\right), \mathcal{L}_2 = \left(\langle \Psi_2, \partial_2\rangle_{(p,q)}, \langle \alpha_2, \beta_2\rangle_{(p,q)}\right), \ldots\ldots\ldots, \mathcal{L}_n = \left(\langle \Psi_n, \partial_n\rangle_{(p,q)}, \langle \alpha_n, \beta_n\rangle_{(p,q)}\right)\right\}$ are (p,q)-$FLDFNs$,

$$(p,q)\text{-}FLDFWGA_{\omega}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots\ldots\ldots, \mathcal{L}_n) = \prod_{j=1}^{n} \mathcal{L}_j{}^{\omega_j}.$$

Theorem 11. The (p,q)-fractional linear Diophantine fuzzy weighted geometric aggregation ((p,q)-$FLDFWGA$) operator on "n" numbers of (p,q)-$FLDFNs$ on the set E is characterized with the help of this transformation Ω: (p,q)-$FLDFN(E) \rightarrow (p,q)$-$FLDFN(E)$ associated with weight vector $\omega = (\omega_1, \omega_2, \omega_3, \ldots\ldots\ldots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows: When $\left\{\mathcal{L}_1 = \left(\langle \Psi_1, \partial_1\rangle_{(p,q)}, \langle \alpha_1, \beta_1\rangle_{(p,q)}\right), \mathcal{L}_2 = \left(\langle \Psi_2, \partial_2\rangle_{(p,q)}, \langle \alpha_2, \beta_2\rangle_{(p,q)}\right), \ldots\ldots\ldots, \mathcal{L}_n = \left(\langle \Psi_n, \partial_n\rangle_{(p,q)}, \langle \alpha_n, \beta_n\rangle_{(p,q)}\right)\right\}$ are (p,q)-$FLDFNs$,

$$(p,q)\text{-}FLDFWGA_{\omega}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots\ldots\ldots, \mathcal{L}_n)$$

$$= \prod_{j=1}^{n} \mathcal{L}_j{}^{\omega_j}$$

$$= \left(\langle \prod_{j=1}^{n} \Psi_j^{\omega_j}, \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - (\partial_j)^q\right)^{\omega_j}} \rangle_{(p,q)}, \langle \prod_{j=1}^{n} \alpha_j^{\omega_j}, \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - (\beta_j)^q\right)^{\omega_j}} \rangle_{(p,q)} \right).$$

This operator can be proven with support of (p,q)-$FLDFNs$ operations and mathematical induction. Here, μ and ν are representing the membership and non-membership function. ω is called weight function and $\mathcal{L}_j$ is (p,q)-$FLDFNs$, where $j \in N$.

Proof. The demonstration of the proof is similar to the operators of intuitionistic fuzzy sets, so it is omitted.

Definition 20. The (p,q)-fractional linear Diophantine fuzzy ordered weighted geometric aggregation ((p,q)-$FLDOWGA$) operator on "n" numbers of (p,q)-$FLDFNs$ is characterized with the help of this transformation Ω: (p,q)-$FLDFN(E) \rightarrow (p,q)$-$FLDFN(E)$ associated with $\omega = (\omega_1, \omega_2, \omega_3, \dots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows: When $\{\mathcal{L}_1 = (\langle \Psi_1, \partial_1 \rangle_{(p,q)}, \langle \alpha_1, \beta_1 \rangle_{(p,q)}), \mathcal{L}_2 = (\langle \Psi_2, \partial_2 \rangle_{(p,q)}, \langle \alpha_2, \beta_2 \rangle_{(p,q)}), \dots \dots \dots, \mathcal{L}_n = (\langle \Psi_n, \partial_n \rangle_{(p,q)}, \langle \alpha_n, \beta_n \rangle_{(p,q)})\}$ are (p,q)-$FLDFN$,

$$(p,q)\text{-}FLDFOWGA_{\omega}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n)$$

$$= \prod_{j=1}^{n} \mathcal{L}_{\sigma(j)}^{\omega_j}$$

where $(\sigma(1), \sigma(2), \sigma(3), \dots \dots \dots, \sigma(n))$ is the arrangement of $j \in N$, for which $\mathcal{L}_{\sigma(j-1)} \geq \mathcal{L}_{\sigma(j)}$, for all $j \in N$.

Theorem 15. The (p,q)-fractional linear Diophantine fuzzy ordered weighted geometric aggregation ((p,q)-$FLDOWGA$) operator on "n" numbers of (p,q)-$FLDFNs$ is characterized with the help of this transformation Ω: (p,q)-$FLDFN(E) \rightarrow (p,q)$-$FLDFN(E)$ associated with $\omega = (\omega_1, \omega_2, \omega_3, \dots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows: When $\{\mathcal{L}_1 = (\langle \Psi_1, \partial_1 \rangle_{(p,q)}, \langle \alpha_1, \beta_1 \rangle_{(p,q)}), \mathcal{L}_2 = (\langle \Psi_2, \partial_2 \rangle_{(p,q)}, \langle \alpha_2, \beta_2 \rangle_{(p,q)}), \dots \dots \dots, \mathcal{L}_n = (\langle \Psi_n, \partial_n \rangle_{(p,q)}, \langle \alpha_n, \beta_n \rangle_{(p,q)})\}$ are (p,q)-$FLDFN$,

$$(p,q)\text{-}FLDFOWGA_{\omega}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n)$$

$$= \prod_{j=1}^{n} \mathcal{L}_{\sigma(j)}^{\omega_j}$$

$$= \left(\langle \prod_{j=1}^{n} \Psi_{\sigma(j)}^{\omega_j}, \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - (\partial_{\sigma(j)})^q\right)^{\omega_j}} \rangle_{(p,q)}, \langle \prod_{j=1}^{n} \alpha_{\sigma(j)}^{\omega_j}, \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - (\beta_{\sigma(j)})^q\right)^{\omega_j}} \rangle_{(p,q)} \right),$$

where $(\sigma(1), \sigma(2), \sigma(3), \dots \dots \dots, \sigma(n))$ is the arrangement of $j \in N$, for which $\mathcal{L}_{\sigma(j-1)} \geq \mathcal{L}_{\sigma(j)}$, for all $j \in N$.

Proof. The demonstration of the proof is similar to the operators of intuitionistic fuzzy sets, so it is omitted.

Definition 21. The (p,q)-fractional linear Diophantine fuzzy hybrid weighted geometric averaging aggregation ((p,q)-$FLDFHWGA$) operator on "n" numbers of (p,q)-$FLDFNs$ is characterized with the help of this transformation Ω: (p,q)-$FLDFN(E) \rightarrow (p,q)$-$FLDFN(E)$ associated with $\omega = (\omega_1, \omega_2, \omega_3, \dots, \omega_n)^T$ with $\sum_{j=1}^{n} \omega_j = 1$ and it can be computed as follows: When $\{\mathcal{L}_1 = (\langle \Psi_1, \partial_1 \rangle_{(p,q)}, \langle \alpha_1, \beta_1 \rangle_{(p,q)}), \mathcal{L}_2 = (\langle \Psi_2, \partial_2 \rangle_{(p,q)}, \langle \alpha_2, \beta_2 \rangle_{(p,q)}), \dots \dots \dots, \mathcal{L}_n = (\langle \Psi_n, \partial_n \rangle_{(p,q)}, \langle \alpha_n, \beta_n \rangle_{(p,q)})\}$ are (p,q)-$FLDFNs$,

$$(p,q)\text{-}FLDFHWGA_{\omega}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots\dots\dots,\mathcal{L}_n)=\prod_{j=1}^{n}{\mathcal{L}^{\star}_{\sigma(j)}}^{\omega_j},$$

where $\mathcal{L}^{\star}_{\sigma(j)}$ is biggest jth weighted (p,q)-fractional linear Diophantine fuzzy values $\mathcal{L}^{\star}_j$ $\left(\mathcal{L}^{\star}_j=\left(\mathcal{L}_j\right)^{n\omega_j}, j\in N\right)$ and $\omega=(\omega_1,\omega_2,\omega_3,\dots\dots\dots,\omega_n)^T$ be the weights of $\mathcal{L}^{\star}_j$ by means of $\omega>0$ with $\sum_{j=1}^{n}\omega_j=1$.

Theorem 20. The $(p,q)\text{-}FLDFHWGA$ operator on "n" numbers of $(p,q)\text{-}FLDFNs$ is characterized with the help of this transformation $\Omega{:}\,(p,q)\text{-}FLDFN(E)\rightarrow(p,q)\text{-}FLDFN(E)$ associated with $\omega=(\omega_1,\omega_2,\omega_3,\dots\dots\dots,\omega_n)^T$ with $\sum_{j=1}^{n}\omega_j=1$ and it can be computed as follows: When $\{\mathcal{L}_1=\left(\langle\Psi_1,\partial_1\rangle_{(p,q)},\langle\alpha_1,\beta_1\rangle_{(p,q)}\right),\mathcal{L}_2=\left(\langle\Psi_2,\partial_2\rangle_{(p,q)},\langle\alpha_2,\beta_2\rangle_{(p,q)}\right),\dots\dots\dots,\mathcal{L}_n=\left(\langle\Psi_n,\partial_n\rangle_{(p,q)},\langle\alpha_n,\beta_n\rangle_{(p,q)}\right)\}$ are $(p,q)\text{-}FLDFNs$,

$$(p,q)\text{-}FLDFHWGA_{\omega}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots\dots\dots,\mathcal{L}_n)$$

$$=\prod_{j=1}^{n}{\mathcal{L}^{\star}_{\sigma(j)}}^{\omega_j}$$

$$=\left(\langle\textstyle\prod_{j=1}^{n}{\Psi^{\star}_{\sigma(j)}}^{\omega_j},\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\left(\partial_{\sigma(j)}\Psi^{\star}_{\sigma(j)}\right)^{q}\right)^{\omega_j}}\rangle_{(p,q)},\langle\prod_{j=1}^{n}{\alpha^{\star}_{\sigma(j)}}^{\omega_j},\sqrt[q]{1-\prod_{j=1}^{n}\left(1-\left(\beta^{\star}_{\sigma(j)}\right)^{q}\right)^{\omega_j}}\rangle_{(p,q)}\right),$$

where $\mathcal{L}^{\star}_{\sigma(j)}$ is biggest jth weighted (p,q)-fractional linear Diophantine fuzzy values $\mathcal{L}^{\star}_j$ $\left(\mathcal{L}^{\star}_j=\left(\mathcal{L}_j\right)^{n\omega_j}, j\in N\right)$ and $\omega=(\omega_1,\omega_2,\omega_3,\dots\dots\dots,\omega_n)^T$ be the weights of $\mathcal{L}^{\star}_j$ by means of $\omega>0$ with $\sum_{j=1}^{n}\omega_j=1$.

It is interesting to note that if $\omega=\left(\frac{1}{n},\frac{1}{n},\frac{1}{n},\dots\dots\dots,\frac{1}{n}\right)$, then $(p,q)\text{-}FLDFWGA$ and $(p,q)\text{-}FLDFOWGA$ operators are considered to be exceptional cases of $(p,q)\text{-}FLDFHWGA$ operator. Thus, it concludes that $(p,q)\text{-}FLDFHWGA$ operators are the extension of $(p,q)\text{-}FLDFWGA$ and $(p,q)\text{-}FLDFOWGA$ operators.

6. $MADM$ approach using suggested techniques

The $MADM$ technique is highly effective and well-suited for selecting the optimal choice from a limited set of possibilities due to its structure. To enhance the effectiveness and quality of previously proposed methods, we introduce a section on the $MADM$ technique procedure incorporating four appropriate operators: The $(p,q)\text{-}FLDFWAA$ operator, $(p,q)\text{-}FLDFOWAA$ operator, and $(p,q)\text{-}FLDFHWAA$ operator. To assess some real-world issues, our objective is to calculate the decision-making process.

As a collection of finite values of alternatives, we take into consideration $E=\{E_1,E_2,\dots,E_m\}$. In addition, we choose a finite set of attributes, including, $\breve{\mathcal{D}}=\{\breve{\mathcal{D}}_1,\breve{\mathcal{D}}_2,\dots,\breve{\mathcal{D}}_n\}$ are chosen along with a weight vector $\omega=(\omega_1,\omega_2,\dots,\omega_n)^T$ such that $\omega_j>0$ with $\sum_{j=1}^{n}\omega_j=1$, for every alternative. Additionally, in order to calculate the matrix that assesses the optimal choice after taking the decision-making process into account, we hope to assign the $(p,q)\text{-}FLDF$ values to each alternative, observed that Ψ_j and ∂_j denote the positive and negative grades, where α_j and β_j are reference parameters corresponding to alternative (E_j) that satisfy the attribute $(\breve{\mathcal{D}}_j)$ provided by the decision makers, where

$0 \leq \frac{\alpha_j \Psi_j(x)}{p} + \frac{\beta_j \partial_j(x)}{q} \leq 1$ and $0 \leq \frac{\alpha_j}{p} + \frac{\beta_j}{q} \leq 1$ with $p, q \geq 2$. Additionally, we stated the refusal degree $\Upsilon_j \pi(x) = 1 - \frac{\alpha_j \Psi_j}{p} + \frac{\beta_j \partial_j}{q}$. As a result, in order to accomplish the aforementioned approach, we take into account a few real-world applications and attempt to assess them using theoretical frameworks.

6.1. The suggested algorithm

The primary impact of this subsection is to assess a process for illustrating the problem that will be addressed in the following section. The primary steps of the decision-making approach are outlined below:

Step 1. Determine a team matrix by incorporating their values into the (p,q)-$FLDFN$ form. Additionally, while we assign the values, we have two opinions "profit and cost", such as if we have cost-type data, then our first priority is to normalize it otherwise not.

Step 2. Using the six various types of operators "(p,q)-$FLDFWAA$ operator, (p,q)-$FLDFOWAA$ operator, (p,q)-$FLDFHWAA$ operator, (p,q)-$FLDFWGA$ operator, (p,q)-$FLDFOWGA$ operator, and (p,q)-$FLDFHWGA$ operator" aggregate the collection of data into a singleton set such that

$$(p,q)\text{-}FLDFWAA_{\omega}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots\dots\dots, \mathcal{L}_n)$$

$$= \prod_{j=1}^{n} \omega_j \mathcal{L}_j$$

$$= \left(\langle 1 - \prod_{j=1}^{n} \left(1 - \Psi_j\right)^{\omega_j}, \prod_{j=1}^{n} \partial_j^{\ \omega_j} \rangle_{(p,q)}, \langle \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left(\alpha_j\right)^q\right)^{\omega_j}}, \prod_{j=1}^{n} \beta_j^{\ \omega_j} \rangle_{(p,q)} \right).$$

$$(p,q)\text{-}FLDFOWAA_{\omega}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots\dots, \mathcal{L}_n)$$

$$= \prod_{j=1}^{n} \omega_j \mathcal{L}_{\sigma(j)}$$

$$= \left(\langle 1 - \prod_{j=1}^{n} \left(1 - \Psi_{\sigma(j)}\right)^{\omega_j}, \prod_{j=1}^{n} \partial_{\sigma(j)}^{\ \omega_j} \rangle_{(p,q)}, \langle \sqrt[q]{1 - \prod_{j=1}^{n} \left(1 - \left(\alpha_{\sigma(j)}\right)^q\right)^{\omega_j}}, \prod_{j=1}^{n} \beta_{\sigma(j)}^{\ \omega_j} \rangle_{(p,q)} \right).$$

$$(p,q)\text{-}FLDFHWAA_{\omega}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots\dots\dots, \mathcal{L}_n)$$

$$= \prod_{j=1}^{n} \omega_j \mathcal{L}_{\sigma(j)}^{\star}$$

$$= \left(\langle \sqrt[p]{1 - \prod_{j=1}^{n} \left(1 - \left(\Psi_{\sigma(j)}^{\star}\right)^p\right)^{\omega_j}}, \prod_{j=1}^{n} \partial_{\sigma(j)}^{\star\ \omega_j} \rangle_{(p,q)}, \langle \sqrt[p]{1 - \prod_{j=1}^{n} \left(1 - \left(\alpha_{\sigma(j)}^{\star}\right)^p\right)^{\omega_j}}, \prod_{j=1}^{n} \beta_{\sigma(j)}^{\star\ \omega_j} \rangle_{(p,q)} \right).$$

$$(p,q)\text{-}FLDFWGA_{\omega}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots\dots\dots, \mathcal{L}_n)$$

$$= \prod_{j=1}^{n} \mathcal{L}_j^{\ \omega_j}$$

$$= \left(\langle \prod_{j=1}^{n} \Psi_j^{\ \omega_j}, 1 - \prod_{j=1}^{n} \left(1 - \partial_j\right)^{\omega_j} \rangle_{(p,q)}, \langle \prod_{j=1}^{n} \alpha_j^{\ \omega_j}, \sqrt[p]{1 - \prod_{j=1}^{n} \left(1 - \left(\beta_j\right)^p\right)^{\omega_j}} \rangle_{(p,q)} \right).$$

$$(p,q)\text{-}FLDFOWGA_{\omega}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots,\mathcal{L}_n)$$

$$=\prod_{j=1}^{n}\mathcal{L}_{\sigma(j)}{}^{\omega_j}$$

$$=\Bigg(\langle\prod_{j=1}^{n}\Psi_{\sigma(j)}{}^{\omega_j},1-\prod_{j=1}^{n}\big(1-$$

$$\partial_{\sigma(j)}\big)^{\omega_j}\rangle_{(p,q)},\langle\prod_{j=1}^{n}\alpha_{\sigma(j)}{}^{\omega_j},\sqrt[p]{1-\prod_{j=1}^{n}\big(1-\big(\beta_{\sigma(j)}\big)^{p}\big)^{\omega_j}}\rangle_{(p,q)}\Bigg).$$

$$(p,q)\text{-}FLDFHWGA_{\omega}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots\dots\dots,\mathcal{L}_n)$$

$$=\prod_{j=1}^{n}\mathcal{L}^{\star}_{\sigma(j)}{}^{\omega_j}$$

$$=\Bigg(\langle\prod_{j=1}^{n}\Psi^{\star}_{\sigma(j)}{}^{\omega_j},\sqrt[q]{1-\prod_{j=1}^{n}\big(1-\big(\partial_{\sigma(j)}\Psi^{\star}_{\sigma(j)}\big)^{q}\big)^{\omega_j}}\rangle_{(p,q)},\langle\prod_{j=1}^{n}\alpha^{\star}_{\sigma(j)}{}^{\omega_j},\sqrt[q]{1-\prod_{j=1}^{n}\big(1-\big(\beta^{\star}_{\sigma(j)}\big)^{q}\big)^{\omega_j}}\rangle_{(p,q)}\Bigg).$$

Step 3. Determine the aggregated theory's score values, such as

$$S_{(p,q)\text{-}FLDFN}(\mathcal{L})=\frac{1}{2}[\Psi(x)-\partial(x)+\alpha-\beta],$$

where, $-1\leq S_{(p,q)\text{-}FLDFN}(\mathcal{L})\leq 1$ and $p,q\geq 2$.
In the event that the score function is not successful, then the accuracy function will be used like

$$H_{(p,q)\text{-}FLDFN}(\mathcal{L})=\frac{1}{2}\left[\frac{\Psi(x)+\partial(x)}{2}+\frac{\alpha+\beta}{2}\right],$$

where, $0\leq H_{(p,q)\text{-}FLDFN}(\mathcal{L})\leq 1$.
Step 4. Try to identify the standout among the alternatives by analyzing the ranking values based on the score values.

To improve the value of the assessed techniques and enable the practical application of the aforementioned procedure, we take into consideration a number of numerical examples that demonstrate the superiority and validity of the invented operators. The suggested algorithm's geometrical interpretation is presented in the form of Figure 7.

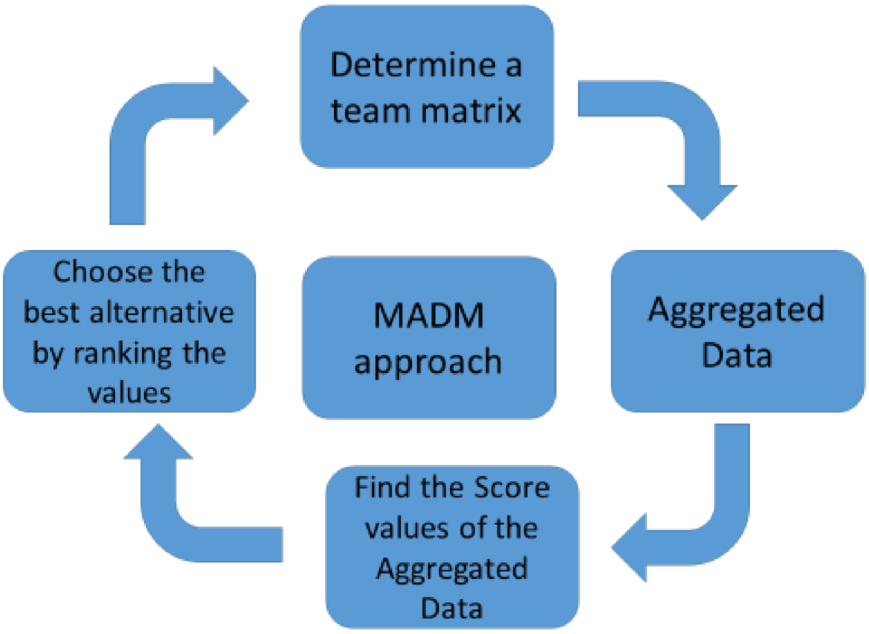

Figure 7. Geometrical interpretation of the proposed algorithm.

6.2. Numerical example

In this section, we examine the green supply chain, also known as the sustainable supply chain or eco-friendly supply chain. This method is utilized to determine the optimal approach for import or export while considering environmental impact. In this example, we consider four green supply chains and evaluate the best one based on the proposed theory, for instance:

E_1: A **Closed-Loop Supply Chain** is a supply chain management system that integrates forward logistics with reverse logistics. It focuses on the lifecycle of a product, from design and manufacturing to consumption, disposal, and recycling. The aim is to create a sustainable and efficient process that minimizes waste, reduces environmental impact, and maximizes resource utilization.

E_2: A **Lean and Green Supply Chain** combines the principles of lean manufacturing and supply chain management with sustainable and eco-friendly practices. The goal is to enhance efficiency, reduce waste, and minimize the environmental impact of supply chain operations.

E_3: A **Low-Carbon Supply Chain** focuses on reducing greenhouse gas emissions throughout the entire supply chain process. This approach aligns with global efforts to combat climate change by minimizing carbon footprints from production, transportation, and distribution activities.

E_4: A **Biomimicry-Inspired Supply Chain** leverages the principles and strategies observed in nature to optimize supply chain processes. Biomimicry, the practice of learning from and emulating nature's designs and processes, can lead to more efficient, sustainable, and resilient supply chains.

We consider the four alternatives above, and to select the best one, we use the following attributes/criteria:

$\breve{\mathcal{D}}_1$: **Growth analysis** typically refers to the examination and evaluation of trends and patterns in various aspects of business or economic growth over time. It involves assessing factors that contribute to growth, understanding their impacts, and forecasting future trends.

$\breve{\mathcal{D}}_2$: **Social impact** refers to the effect an organization's actions and activities have on the surrounding community and society at large. It goes beyond financial outcomes to include the positive or negative changes experienced by individuals, groups, or communities as a result of these actions. Social impact encompasses a wide range of dimensions, including economic, cultural, environmental, and health-related aspects.

$\breve{\mathcal{D}}_3$: **Environmental impact** refers to the effect of human activities on the environment, encompassing both the positive and negative consequences on ecosystems, natural resources, and overall ecological balance. It is crucial to assess and mitigate environmental impact to ensure sustainable development and the preservation of natural resources for future generations.

$\breve{\mathcal{D}}_4$: **Political impact** refers to the influence and consequences of political decisions, actions, and events on individuals, communities, organizations, and societies at large. It encompasses a wide range of effects on governance, policies, public opinion, and socio-economic conditions.

$\breve{\mathcal{D}}_5$: **Economic impact** refers to the effect that an event, policy, decision, or action has on the economy of a region, country, or the global economy. It encompasses a wide range of consequences, both direct and indirect, on various economic indicators, sectors, and stakeholders.

Furthermore, we apply the weight vector as $(.32,.27,.17,.14,.1)^T$ with $\sum_{j=1}^{n} \omega_j = 1$; hence, we utilize the aforementioned procedure to evaluate the optimal choice. The primary steps of the decision-making technique are outlined below:

Step 1. Determine a team matrix by incorporating their values into the (p,q)-$FLDFN$ form, see Table 1. Additionally, while we assign the values, we have two opinions "same type of data and different type of data", such as if we have different type of data, then our first priority is to normalize such that

$$\mathcal{L}_j = \begin{cases} \left(\langle \Psi_j, \partial_j \rangle_{(p,q)}, \langle \alpha_j, \beta_j \rangle_{(p,q)}\right), & same\ type\ input\ data \\ \left(\langle \partial_j, \Psi_j \rangle_{(p,q)}, \langle \beta_j, \alpha_j \rangle_{(p,q)}\right), & different\ type\ input\ data. \end{cases}$$

In this case, since the input data for all attributes is identical, there is no need to normalize the data. All alternatives and criteria in our specific problem are of the same nature.

Table 1. Decision matrix of (p,q)-$FLDF$ information.

$p = q = 3$	E_1	E_2	E_3	E_4
$\breve{\mathcal{D}}_1$	$(\langle 1,0.959\rangle, \langle 0.798,0.733\rangle)$	$(\langle 1,0.959\rangle, \langle 0.798,0.733\rangle)$	$(\langle 1,0.959\rangle, \langle 0.798,0.733\rangle)$	$(\langle 1,0.959\rangle, \langle 0.798,0.733\rangle)$
$\breve{\mathcal{D}}_2$	$(\langle 0.895,0.925\rangle, \langle 0.767,0.745\rangle)$	$(\langle 0.895,0.925\rangle, \langle 0.767,0.745\rangle)$	$(\langle 0.895,0.925\rangle, \langle 0.767,0.745\rangle)$	$(\langle 0.895,0.925\rangle, \langle 0.767,0.745\rangle)$
$\breve{\mathcal{D}}_3$	$(\langle 1,0.92\rangle, \langle 0.7,0.812\rangle)$	$(\langle 1,0.92\rangle, \langle 0.7,0.812\rangle)$	$(\langle 1,0.92\rangle, \langle 0.7,0.812\rangle)$	$(\langle 1,0.92\rangle, \langle 0.7,0.812\rangle)$
$\breve{\mathcal{D}}_4$	$(\langle 0.882,0.949\rangle, \langle 0.836,0.691\rangle)$	$(\langle 0.882,0.949\rangle, \langle 0.836,0.691\rangle)$	$(\langle 0.882,0.949\rangle, \langle 0.836,0.691\rangle)$	$(\langle 0.882,0.949\rangle, \langle 0.836,0.691\rangle)$
$\breve{\mathcal{D}}_5$	$(\langle 0.862,0.9\rangle, \langle 0.7,0.837\rangle)$	$(\langle 0.862,0.9\rangle, \langle 0.7,0.837\rangle)$	$(\langle 0.862,0.9\rangle, \langle 0.7,0.837\rangle)$	$(\langle 0.862,0.9\rangle, \langle 0.7,0.837\rangle)$

Step 2. Using the six operators " (p,q)-$FLDFWAA$ operator, (p,q)-$FLDFOWAA$ operator, (p,q)-$FLDFHWAA$ operator, (p,q)-$FLDFWGA$ operator, (p,q)-$FLDFOWGA$ operator, and (p,q)-$FLDFHWGA$ operator" aggregate the collection of data into a singleton set (Tables 2–7).

Table 2. (p,q)-$FLDFWAA$.

$(\langle 1,0.937735\rangle,\langle 0.774703461,0.752904\rangle)$
$(\langle 1,0.923953\rangle,\langle 0.746329917,0.772672\rangle)$
$(\langle 1,0.907838\rangle,\langle 0.758168815,0.76754\rangle)$
$(\langle 1,0.936066\rangle,\langle 0.781357244,0.750463\rangle)$

Table 3. (p,q)-$FLDFOWAA$.

$(\langle 1,0.940863\rangle,\langle 0.785099,0.745575\rangle)$
$(\langle 1,0.91523\rangle,\langle 0.748987,0.769711\rangle)$
$(\langle 1,0.900818\rangle,\langle 0.777094,0.745447\rangle)$
$(\langle 1,0.940793\rangle,\langle 0.800037,0.72855\rangle)$

Table 4. (p,q)-$FLDFHWAA$.

$(\langle 1,0.773252\rangle,\langle 0.77068,0.321335\rangle)$
$(\langle 1,0.728785\rangle,\langle 0.738936,0.356436\rangle)$
$(\langle 1,0.679256\rangle,\langle 0.724187,0.347059\rangle)$
$(\langle 1,0.767761\rangle,\langle 0.767278,0.317188\rangle)$

Table 5. (p,q)-$FLDFWGA$.

$(\langle 0.939525,0.940481\rangle,\langle 0.767045,0.759433\rangle)$
$(\langle 0.971579,0.933439\rangle,\langle 0.734517,0.785833\rangle)$
$(\langle 0.915158,0.926687\rangle,\langle 0.745576,0.779970\rangle)$
$(\langle 0.989593,0.944710\rangle,\langle 0.77555,0.755742\rangle)$

Table 6. (p,q)-$FLDFOWGA$.

$(\langle 0.928775,0.942870\rangle,\langle 0.779625,0.749449\rangle)$
$(\langle 0.971579,0.929937\rangle,\langle 0.745711,0.773024\rangle)$
$(\langle 0.919911,0.952039\rangle,\langle 0.78401,0.737912\rangle)$
$(\langle 0.989593,0.94452\rangle,\langle 0.791188,0.735606\rangle)$

Table 7. (p,q)-$FLDFHWGA$.

$(\langle 0.348342,0.948012\rangle,\langle 0.77068,0.321335\rangle)$
$(\langle 0.717301,0.93901\rangle,\langle 0.738936,0.356436\rangle)$
$(\langle 0.31178,0.936147\rangle,\langle 0.724187,0.347059\rangle)$
$(\langle 0.770857,0.955213\rangle,\langle 0.767278,0.317188\rangle)$

Step 3. Refer to Tables 8 and 9 to find the aggregated theory's score values.

Table 8. (p,q)-$FLDFWAA$ score values.

$S_{(p,q)\text{-}FLDFN}$				
$p=3, q=2$	$S_{(p,q)\text{-}FLDFN}(E_1)$	$S_{(p,q)\text{-}FLDFN}(E_2)$	$S_{(p,q)\text{-}FLDFN}(E_3)$	$S_{(p,q)\text{-}FLDFN}(E_4)$
(p,q)-$FLDFWAA$	-0.01982	-0.05263	-0.03057	-0.01111
(p,q)-$FLDFOWAA$	-0.01704	-0.04376	-0.00636	0.23004
(p,q)-$FLDFHWAA$	0.290618	0.273823	0.290046	0.29167

Table 9. (p,q)-$FLDFWGA$ score values.

$S_{(p,q)\text{-}FLDFN}$				
$p=3, q=2$	$S_{(p,q)\text{-}FLDFN}(E_1)$	$S_{(p,q)\text{-}FLDFN}(E_2)$	$S_{(p,q)\text{-}FLDFN}(E_3)$	$S_{(p,q)\text{-}FLDFN}(E_4)$
(p,q)-$FLDFWGA$	-0.0632	-0.09156	-0.10271	-0.02989
(p,q)-$FLDFOWGA$	-0.05095	-0.07062	-0.04737	-0.00039
(p,q)-$FLDFHWGA$	-0.12259	0.027361	-0.18251	0.083372

Step 4. Analyze the ranking values based on the score values and look for the standout alternative among the four; refer to Tables 10 and 11.

Table 10. Ranking of (p,q)-$FLDFWAA$ operator.

$p=3, q=2$	$S_{(p,q)\text{-}FLDFN}$
(p,q)-$FLDFWAA$	$E_4 > E_1 > E_3 > E_2$
(p,q)-$FLDFOWAA$	$E_4 > E_3 > E_1 > E_2$
(p,q)-$FLDFHWAA$	$E_4 > E_1 > E_3 > E_2$

Table 11. Ranking of (p,q)-$FLDFWGA$ operator.

$p=3, q=2$	$S_{(p,q)\text{-}FLDFN}$
(p,q)-$FLDFWGA$	$E_4 > E_1 > E_2 > E_3$
(p,q)-$FLDFOWGA$	$E_4 > E_3 > E_1 > E_2$
(p,q)-$FLDFHWGA$	$E_4 > E_2 > E_1 > E_3$

The geometrically representation of Table 8 with respect to Table 10 (see Figure 8), we have,

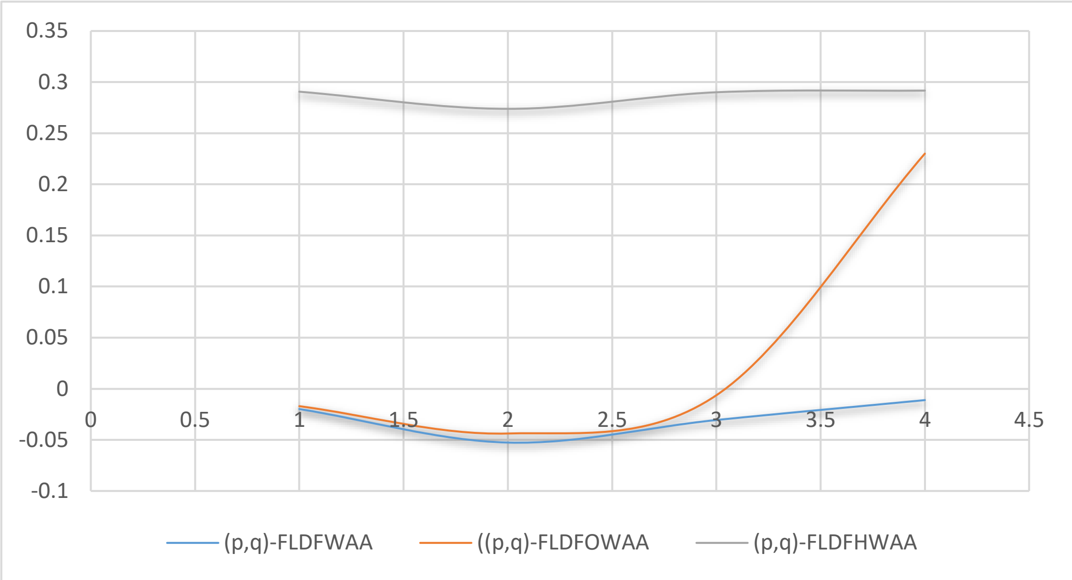

Figure 8. Scores of alternatives based on the three (p,q)-$FLDFWAA$ operators.

The geometrically representation of Table 9 with respect to Table 11 (see Figure 9), we have

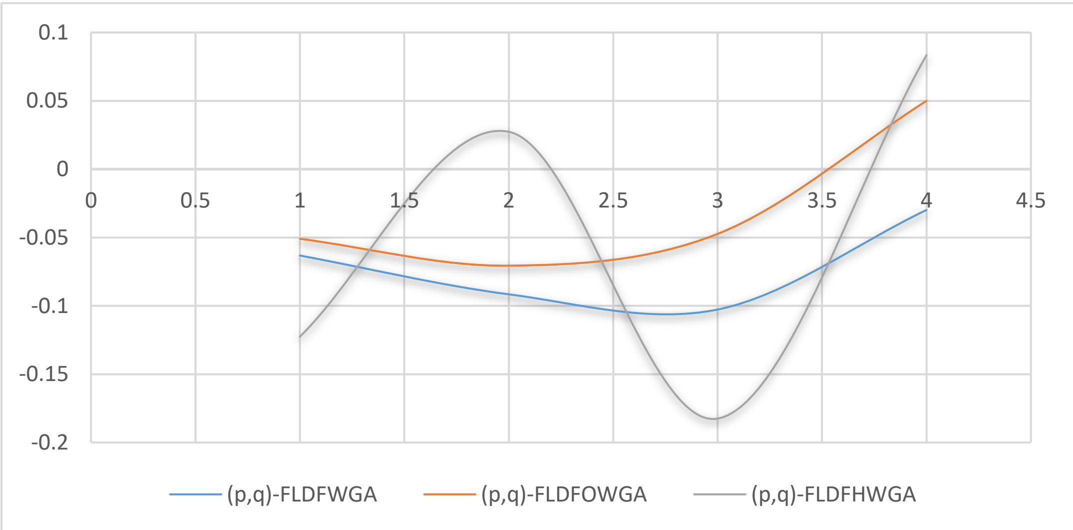

Figure 9. Scores of alternatives based on the three (p,q)-$FLDFWGA$.

By taking into account the theories of the (p,q)-$FLDFWAA$ operator, (p,q)-$FLDFOWAA$ operator, (p,q)-$FLDFHWAA$ operator, (p,q)-$FLDFWGA$ operator, (p,q)-$FLDFOWGA$ operator, and (p,q)-$FLDFHWGA$ operator we found that the most desirable decision is E_4. Note that each operator receives the same rating results, these operators are also steady.

In Table 12, the comparative study of (p,q)-$FLDFS$ is discussed with classical fuzzy sets.

Table 12. The comparative study of (p,q)-$FLDFS$'s with fuzzy approaches.

Collections	Remarks	Parameterization	Fractional Property
FS (Zadeh 1965)	Unable to handle non-membership $\partial(x)$	No	Not satisfied
IFS (Atanassov 1986)	cannot deal with the condition, $\Psi(x)+\partial(x)>1$	No	Not satisfied
$PyFS$ (Yager 2013a, b)	cannot deal with the condition, $\left(\Psi(x)\right)^2+\left(\partial(x)\right)^2>1$	No	Not satisfied
q-$ROFS$ (Yager 2016)	Unable to deal with smaller "q" values with the condition, $\left(\Psi(x)\right)^q+\left(\partial(x)\right)^q>1$ and for $\Psi(x)=1,\partial(x)=1$	No	Not satisfied
$LDFS$ (Riaz 2019)	This collection covers upon this situation, $0\le(\alpha)\Psi(x)+(\beta)\partial(x)\le 1$, and don't work under the influence of reference parameters (α,β). $\alpha+\beta>1$	Yes	Not satisfied
q-$ROLDFS$ (Almagrabi 2021)	This collection covers upon this situation, $0\le\alpha^q\Psi(x)+\beta^q\partial(x)\le 1$, and don't work under the influence of reference parameters $\langle\alpha,\beta\rangle$. $\alpha^q+\beta^q>1$ and for $\alpha=1,\beta=1$	Yes	Not satisfied
(p,q)-$FLDFS$ (Present work)	This collection covers upon this situation, $0\le\alpha^q\Psi(x)+\beta^q\partial(x)\le 1$, and work under the influence of reference parameters $\langle\alpha,\beta\rangle$. $\frac{\alpha}{p}+\frac{\beta}{q}\le 1$ and for $\alpha=1,\beta=1$	Yes	Satisfied

These fuzzy sets affect the optimal choice and restricts the $MADM$. We provide the novel concept of the $FLDFS$, which can handle these situations and resolve these contradictions.

7. Conclusions

In this work, we propose q-$FLDFS$ to remove the restriction on section of values of parameters. Some basic operations on q-$FLDFS$ are discussed as well as characterize some properties. A new version of score and accuracy functions are presented with the support of classical definition and elaborated with nontrivial examples. Some new aggregation operators over q-fractional linear Diophantine fuzzy information are introduced over average and geometric mean. Additionally, the order and hybrid concepts are also used between (p, q)-fractional linear Diophantine fuzzy information to extended the applications of aggregation operators and to get more valuable results. We apply the proposed aggregation operators in $MADM$ problem to evaluate the problem of green supply chain management based on the invented operators for (p, q)-fractional linear Diophantine fuzzy information. Finally, we use some numerical examples to show the supremacy and validity of the developed techniques by comparing their ranking results with the obtained ranking results of the techniques. In the future, we will extend this approach to aggregation operators and apply this idea in soft set theory. Moreover, we will attempt to modify the main concept to acquire more nontrivial results.

Author contributions

Muhammad Bilal Khan: Conceptualization, writing—original draft preparation, writing—review and editing, visualization, supervision, project administration; Jorge E. Macías-Díaz: Formal analysis; Hanan Alohali, Muhammad Bilal Khan, and Fahad Sikander: Methodology; Hanan Alohali and Fahad Sikander: Software, investigation, resources, data curation; Hanan Alohali, Jorge E. Macías-Díaz and Fahad Sikander: Validation; Hanan Alohali and Jorge E. Macías-Díaz: Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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