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Research article

The discernibility approach for multi-granulation reduction of generalized neighborhood decision information systems

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Abstract: Attribute reduction of a decision information system (DIS) using multi-granulation rough sets is one of the important applications of granular computing. Constructing discernibility matrices by rough sets to get attribute reducts of a DIS is an important reduction method. By analyzing the commonalities between the multi-granulation reduction structure of decision multi-granulation spaces and that of incomplete DISs based on discernibility tool, this paper explored a general model for the multi-granulation reduction of DISs by the discernibility technique. First, the definition of the generalized neighborhood decision information system (GNDIS) was presented. Second, knowledge reduction of GNDISs by multi-granulation rough sets was discussed, and discernibility matrices and discernibility functions were constructed to characterize multi-granulation reduction structures of GNDISs. Third, the multi-granulation reduction structures of decision multi-granulation spaces and incomplete DISs were characterized by the reduction theory of GNDISs based on discernibility. Then, the multi-granulation reduction of GNDISs by the discernibility tool provided a theoretical foundation for designing algorithms of multi-granulation reduction of DISs.

Keywords: discernibility matrix; general model; generalized neighborhood information system; knowledge reduction; multi-granulation rough sets
Mathematics Subject Classification: 68T30, 68T37

1. Introduction

With the development of information technology, the datasets with lots of features have been collected in many application fields. However, datasets usually contain many redundant features, which may affect the classification ability of datasets and increase the complexity of learning algorithms. Feature selection is a data preprocessing technology which selects a subset from the original feature set to improve the performance of learning algorithms. So far, feature selection has been applied to rule extraction [1], decision-making [4, 24], and data mining [30].

Rough set theory [12] is a typical granular computing model and a meaningful mathematical tool for feature selection, which is also called attribute reduction in rough set theory. In rough set theory, the datasets are represented as information systems. Due to the diversity of the datasets, different types of information systems are discussed, whose attribute reduction structures are explored by different rough set models. For example, the attribute reduction of a complete decision information system (CDIS) was investigated based on the classical Pawlak rough set model, which is defined by equivalence (indiscernibility) relations or partitions [13, 16, 31]. The attribute reduction of an incomplete decision information system (IDIS) was discussed by relation rough sets [3, 9, 17, 23]. The neighborhood rough set model, defined on neighborhood granularies, was used to attribute reduction of neighborhood DISs [5, 6, 33]. The covering rough sets defined on coverings were utilized for reducing covering DISs [2, 22, 25, 26]. The rough set models mentioned above are constructed by a single granular knowledge. However, in real-world applications, more and more datasets should be described via multiple granular structures. Qian et al. proposed the concepts of multi-granulation rough sets in CDISs and discussed attribute reduction of CDISs based on multi-granulation rough sets [14, 18]. Kong et al. explored multi-granulation reduction of information systems [8]. Attribute reduction of IDISs based on multi-granulation rough sets was explored in [15]. Zhang et al. defined a generalized multi-granulation fuzzy neighborhood rough set model and discussed a feature selection method by the model [34].

The technique of constructing discernibility matrices and discernibility functions, proposed by Skowron and Rauszer [20] and Skowron [19], is an important attribute reduction method. Yao and Zhao defined a minimal family of discernibility sets based on the family of discernibility sets to compute the reducts of CDISs [29]. Zhao et al. constructed the relative discernibility matrix of a CDIS to get relative reducts [35]. Jiang and Yu proposed a compactness discernibility information tree to obtain the minimal attribute reduction of a CDIS [7]. Ma et al. introduced a compressed binary discernibility matrix and designed an incremental attribute reduction algorithm for getting an attribute reduction set of a dynamic CDIS [10]. A binary discernibility matrix was designed to get attribute relative reducts of an IDIS [11]. Chen et al. [2], Wang et al. [22] and Yang et al. [25] constructed discernibility matrices to obtain the reducts of covering DISs. The discernibility techniques are also used to achieve the knowledge reduction of information systems based on multi-granulation rough sets. Tan et al. constructed discenibility matrices and discernibility functions to calculate the attribute reducts of a decision multi-granulation space (DMS) [21] and verified the effectiveness of the reduction methods by numerical experiments. However, the optimistic lower reducts are not calculated by discenibility matrices in [21]. Zhang et al. explored the attribute reduction of an IDIS by the discernibility approach in multi-granulation rough set theory [32] and presented numerical experiments to show the feasibility and effectiveness of the algorithms to get reducts. However, the attribute reduction of IDISs based on optimistic multi-granulation rough sets is not considered in [32].

The purpose of this paper is to analyze and compare the multi-granulation reduction theory of DMSs [21] and IDISs [32] from discernibility, and to present a general model for the multi-granulation reduction of DISs by the discernibility technique, which can provide a theoretical basis for the multi-granulation reduction of DISs based on discernibility. The notation of a GNDIS is introduced in this paper, the multi-granulation reductions of GNDISs based on the multi-granulation rough set are discussed, and discernibility matrices are constructed to compute the multi-granulation reducts of GNDISs. Then, the pessimistic (or optimistic) approximations in DMSs and IDISs can be changed into the multi-granulation pessimistic (or optimistic) approximations in GNDISs. Moreover, the pessimistic multi-granulation reducts and the optimistic multi-granulation reducts of DMSs discussed in [21] can be computed by the discernibility matrices and discernibility functions based on the multi-granulation reduction theory of GNDISs. Additionally, the pessimistic multi-granulation reduction structures of IDISs discussed in [32] are characterized by the reduction theory of GNDISs based on discernibility technique.

The remaining structure of this paper is organized as follows. In Section 2, the definitions about multi-granulation rough sets are reviewed. In Section 3, we introduce the definition of the multi-granulation rough sets in a GNDIS and discuss knowledge reduction of a GNDIS based on the multi-granulation rough sets. The discernibility matrices and discernibility functions are constructed to characterize the multi-granulation reducts of a GNDIS. In Section 4, relationships between the multi-granulation reduction of DMSs and that of GNDISs are discussed. Moreover, the optimistic lower reducts of DMSs are discussed by the discernibility matrices in this section. Section 5 explores relationships between the multi-granulation reduction of IDISs and that of GNDISs. Then, the optimistic multi-granulation reductions of IDISs from the discernibility technique are presented in this section. Section 6 concludes this study.

2. Preliminary knowledge on multi-granulation rough sets

In this section, we review some basic concepts about multi-granulation rough sets, which were proposed by Qian et al. [18] to approximate a target concept using multiple binary relations.

2.1. Multi-granulation rough sets in DMSs

Suppose that (U, A, d) is a DIS, in which U is the universe, A is a family of condition attributes where $a : U \to V_a$ for any $a \in A$, V_a is the value set of a, and $d : U \to V_d$ is a decision attribute where V_d is the value set of d.

In a DIS (U, A, d), $\mathcal{A} = \{A_k | A_k \subseteq A, k = 1, 2, \cdots, m\}$ is a family of attribute subsets. Then, $(U, \mathcal{A}, d)$ is called an DMS [28]. Each $A_k \in \mathcal{A}$ induces an equivalent relation $R_{A_k} = \{(x, y) \in U \times U | \forall a \in A_k (a(x) = a(y))\}$ and generates a granular structure $U/R_{A_k} = \{[x]_{A_k} | x \in U\}$, in which $[x]_{A_k} = \{y \in U | (x, y) \in R_{A_k}\}$. The decision attribute d generates a partition $U/R_d = \{[x]_d | x \in U\} \triangleq \{X_1, X_2, \cdots, X_n\}$, each of which is the decision class with the same decision attribute values. The pessimistic multi-granulation lower and upper approximations and optimistic multi-granulation lower and upper approximations were discussed by Qian et al. [14, 18].

Definition 1. *[14] Given a DMS $(U, \mathcal{A}, d)$ with $\mathcal{A} = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \le k \le m\}$, let $X \subseteq U$. Define the pessimistic multi-granulation lower and upper approximations of X as*

$$\underline{\sum_{\mathcal{A}} A_k^P}(X) = \{x \in U | ([x]_{A_1} \subseteq X) \wedge ([x]_{A_2} \subseteq X) \wedge \cdots \wedge ([x]_{A_m} \subseteq X)\},$$

$$\overline{\overline{\sum_{\mathcal{A}} A_k^P}}(X) = \sim \underline{\sum_{\mathcal{A}} A_k^P}(\sim X).$$

One calls $(\underline{\sum_{\mathcal{A}} A_k^P}(X), \overline{\sum_{\mathcal{A}} A_k^P}(X))$ *a pessimistic multi-granulation rough set.*

For each $x \in U$, $\mathcal{H} \subseteq \mathcal{A}$, denote $\cup_{A \in \mathcal{H}}[x]_A$ by $N_{\mathcal{H}}(x)$. Then, it is easy to get that $\underline{\sum_{\mathcal{A}} A_k^P}(X) = \{x \in U | N_{\mathcal{A}}(x) \subseteq U\}$.

Definition 2. *[18] Given a DMS* $(U, \mathcal{A}, d)$ *with* $\mathcal{A} = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$, *let* $X \subseteq U$. *Define the optimistic multi-granulation lower and upper approximations of X as*

$$\underline{\sum_{\mathcal{A}} A_k^O}(X) = \{x \in U | ([x]_{A_1} \subseteq X) \vee ([x]_{A_2} \subseteq X) \vee \cdots \vee ([x]_{A_m} \subseteq X)\},$$

$$\overline{\sum_{\mathcal{A}} A_k^O}(X) = \sim \underline{\sum_{\mathcal{A}} A_k^O}(\sim X).$$

$(\underline{\sum_{\mathcal{A}} A_k^P}(X), \overline{\sum_{\mathcal{A}} A_k^P}(X))$ *is called an optimistic multi-granulation rough set.*

2.2. Multi-granulation rough sets in IDISs

If some attribute values of attributes in A of a DIS (U, A, d) are are missing or unknown, then the missing attribute value is expressed by special symbol '$*$' and (U, A, d) is termed as an IDIS. For any $A_k \subseteq A$, a tolerance relation is defined by Kryszkiewicz as [9]:

$$SIM(A_k) = \{(x, y) \in U \times U | \forall a \in A_k (a(x) = a(y) \vee a(x) = * \vee a(y) = *)\}.$$

A granular structure is induced by A_k as $U/SIM(A_k) = \{S_{A_k}(x) | x \in U\}$, where $S_{A_k}(x) = \{y \in U | (x, y) \in SIM(A_k)\}$.

The multi-granulation rough sets in IDISs were introduced by Qian et al. [15].

Definition 3. *[15] Given an IDIS* (U, A, d) *with* $\mathcal{A}^I = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$, *let* $X \subseteq U$. *Define the pessimistic multi-granulation lower and upper approximations of X as*

$$\underline{\sum_{\mathcal{A}^I} A_k^P}(X) = \{x \in U | (S_{A_1}(x) \subseteq X) \wedge (S_{A_2}(x) \subseteq X) \wedge \cdots \wedge (S_{A_m}(x) \subseteq X)\},$$

$$\overline{\sum_{\mathcal{A}^I} A_k^P}(X) = \sim \underline{\sum_{\mathcal{A}^I} A_k^P}(\sim X).$$

Define the optimistic multi-granulation lower and upper approximations of X

$$\underline{\sum_{\mathcal{A}^I} A_k^O}(X) = \{x \in U | (S_{A_1}(x) \subseteq X) \vee (S_{A_2}(x) \subseteq X) \vee \cdots \vee (S_{A_m}(x) \subseteq X)\},$$

$$\overline{\sum_{\mathcal{A}^I} A_k^O}(X) = \sim \underline{\sum_{\mathcal{A}^I} A_k^O}(\sim X).$$

$(\underline{\sum_{\mathcal{A}^I} A_k^P}(X), \overline{\sum_{\mathcal{A}^I} A_k^P}(X))$ *and* $(\underline{\sum_{\mathcal{A}^I} A_k^O}(X), \overline{\sum_{\mathcal{A}^I} A_k^O}(X))$ *are, respectively, the pessimistic multi-granulation rough set and optimistic multi-granulation rough set of X.*

For each $x \in U$, $\mathcal{H} \subseteq \mathcal{A}^I$, denote $\cup_{A \in \mathcal{H}} S_A(x)$ by $IN_{\mathcal{H}}(x)$. It is clear that $\underline{\sum_{\mathcal{A}^I} A_k^P}(X) = \{x \in U | IN_{\mathcal{A}^I}(x) \subseteq U\}$.

3. Multi-granulation reduction of GNDISs

In this section, we present the definitions of multi-granulation rough sets in GNDISs and explore multi-granulation reduction of GNDISs. Throughout this paper, the universe of discourse U is nonempty and finite. The family of all subsets of U is denoted by $\mathcal{P}(U)$. For $X \subseteq U$, $\sim X$ is the complementary set of X.

3.1. Multi-granulation rough sets in GNDISs

Some basic concepts about the neighborhood operator is presented in [27].

Definition 4. *[27] Let U be the universe. A mapping $N : U \to \mathcal{P}(U)$ is called a neighborhood operator. If $x \in N(x)$ for all $x \in U$, N is a reflexive neighborhood operator. If $x \in N(y) \Rightarrow y \in N(x)$ for all $x, y \in U$, N is a symmetric neighborhood operator. If $[y \in N(x), z \in N(y)] \Rightarrow z \in N(x)$ for all $x, y, z \in U$, N is a transitive neighborhood operator. If the neighborhood operator N is reflexive, symmetric and transitive, N is called a Pawlak neighborhood operator.*

Clearly, $\{N(x)|x \in U\}$ of a Pawlak neighborhood operator N forms a partition of U.

Definition 5. *Let N be a reflexive neighborhood operator on U. Denote $\{N(x)|x \in U\}$ by C_N. The ordered pair (U, N) is called a generalized neighborhood approximation space.*

Clearly, C_N from (U, N) is a covering. We introduce the generalized neighborhood multi-granulation rough sets now.

Definition 6. *Let $N_1, N_2, \cdots, N_m (m \geq 2)$ be reflexive neighborhood operators on U and $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$, $N_d : U \to P(U)$ be a Pawlak neighborhood operator, then $(U, \mathcal{N}, N_d)$ is called a GNDIS. For $X \subseteq U$, define the generalized neighborhood pessimistic lower approximation $\underline{\sum_{\mathcal{N}} N_k^P}(X)$ and pessimistic upper approximation $\overline{\sum_{\mathcal{N}} N_k^P}(X)$ by*

$$\underline{\sum_{\mathcal{N}} N_k^P}(X) = \{x \in U | (N_1(x) \subseteq X) \wedge (N_2(x) \subseteq X) \wedge \cdots \wedge (N_m(x) \subseteq X)\},$$

$$\overline{\sum_{\mathcal{N}} N_k^P}(X) = \sim \underline{\sum_{\mathcal{N}} N_k^P}(\sim X).$$

$(\underline{\sum_{\mathcal{N}} N_k^P}(X), \overline{\sum_{\mathcal{N}} N_k^P}(X))$ is the generalized neighborhood pessimistic multi-granulation rough set of X.

For $\mathcal{H} \subseteq \mathcal{N}$ and $x \in U$, denote $GN_{\mathcal{H}}(x) = \bigcup_{N \in \mathcal{H}} N(x)$. Then, it is clear that $\underline{\sum_{\mathcal{N}} N_k^P}(X) = \{x \in U | GN_{\mathcal{N}}(x) \subseteq X\}$.

Proposition 1. *Let $(U, \mathcal{N}, N_d)$ be a GNDIS, $X, Y \subseteq U$, $\mathcal{H} \subseteq \mathcal{N}$ and $\mathcal{H} \neq \emptyset$, then:*

(1) $\underline{\sum_{\mathcal{N}} N_k^P}(\emptyset) = \emptyset$, $\underline{\sum_{\mathcal{N}} N_k^P}(U) = U$, $\overline{\sum_{\mathcal{N}} N_k^P}(\emptyset) = \emptyset$, $\overline{\sum_{\mathcal{N}} N_k^P}(U) = U$.

(2) $\underline{\sum_{\mathcal{N}} N_k^P}(X) \subseteq X \subseteq \overline{\sum_{\mathcal{N}} N_k^P}(X)$.

(3) $X \subseteq Y \Rightarrow \underline{\sum_{\mathcal{N}} N_k^P}(X) \subseteq \underline{\sum_{\mathcal{N}} N_k^P}(Y)$, $\overline{\sum_{\mathcal{N}} N_k^P}(X) \subseteq \overline{\sum_{\mathcal{N}} N_k^P}(Y)$.

(4) $\underline{\sum_{\mathcal{N}} N_k^P}(X \cap Y) = \underline{\sum_{\mathcal{N}} N_k^P}(X) \cap \underline{\sum_{\mathcal{N}} N_k^P}(Y)$, $\overline{\sum_{\mathcal{N}} N_k^P}(X \cup Y) = \overline{\sum_{\mathcal{N}} N_k^P}(X) \cup \overline{\sum_{\mathcal{N}} N_k^P}(Y)$.

(5) $\underline{\sum_{\mathcal{N}} N_k^P}(\underline{\sum_{\mathcal{N}} N_k^P}(X)) \subseteq \underline{\sum_{\mathcal{N}} N_k^P}(X)$, $\overline{\sum_{\mathcal{N}} N_k^P}(X) \subseteq \overline{\sum_{\mathcal{N}} N_k^P}(\overline{\sum_{\mathcal{N}} N_k^P}(X))$.

(6) $\underline{\sum_{\mathcal{N}} N_k^P}(X) \subseteq \underline{\sum_{\mathcal{H}} N_k^P}(X)$, $\overline{\sum_{\mathcal{H}} N_k^P}(X) \subseteq \overline{\sum_{\mathcal{N}} N_k^P}(X)$.

Proof. It is verified by Definition 6. □

Definition 7. *Given a GNDIS* $(U, \mathcal{N}, N_d)$ *with* $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$, *let* $X \subseteq U$. *Define the generalized neighborhood optimistic lower approximation* $\underline{\sum_{\mathcal{N}} N_k^O}(X)$ *and optimistic upper approximation* $\overline{\sum_{\mathcal{N}} N_k^O}(X)$ *of* X *as*

$$\underline{\sum_{\mathcal{N}} N_k^O}(X) = \{x \in U | (N_1(x) \subseteq X) \vee (N_2(x) \subseteq X) \vee \cdots \vee (N_m(x) \subseteq X)\},$$

$$\overline{\sum_{\mathcal{N}} N_k^O}(X) = \sim \underline{\sum_{\mathcal{N}} N_k^O}(\sim X).$$

$(\underline{\sum_{\mathcal{N}} N_k^O}(X), \overline{\sum_{\mathcal{N}} N_k^O}(X))$ *is the generalized neighborhood optimistic multi-granulation rough set of* X.

Proposition 2. *Let* $(U, \mathcal{N}, N_d)$ *be a GNDIS,* $X, Y \subseteq U$, $\mathcal{H} \subseteq \mathcal{N}$ *and* $\mathcal{H} \neq \emptyset$, *then:*

(1) $\underline{\sum_{\mathcal{N}} N_k^O}(\emptyset) = \emptyset$, $\underline{\sum_{\mathcal{N}} N_k^O}(U) = U$, $\overline{\sum_{\mathcal{N}} N_k^O}(\emptyset) = \emptyset$, $\overline{\sum_{\mathcal{N}} N_k^O}(U) = U$.

(2) $\underline{\sum_{\mathcal{N}} N_k^O}(X) \subseteq X \subseteq \overline{\sum_{\mathcal{N}} N_k^O}(X)$.

(3) $X \subseteq Y \Rightarrow \underline{\sum_{\mathcal{N}} N_k^O}(X) \subseteq \underline{\sum_{\mathcal{N}} N_k^O}(Y)$, $\overline{\sum_{\mathcal{N}} N_k^O}(X) \subseteq \overline{\sum_{\mathcal{N}} N_k^O}(Y)$.

(4) $\underline{\sum_{\mathcal{N}} N_k^O}(X \cap Y) \subseteq \underline{\sum_{\mathcal{N}} N_k^O}(X) \cap \underline{\sum_{\mathcal{N}} N_k^O}(Y)$, $\overline{\sum_{\mathcal{N}} N_k^O}(X) \cup \overline{\sum_{\mathcal{N}} N_k^O}(Y) \subseteq \overline{\sum_{\mathcal{N}} N_k^O}(X \cup Y)$.

(5) $\underline{\sum_{\mathcal{N}} N_k^O}(\underline{\sum_{\mathcal{N}} N_k^O}(X)) \subseteq \underline{\sum_{\mathcal{N}} N_k^O}(X)$, $\overline{\sum_{\mathcal{N}} N_k^O}(X) \subseteq \overline{\sum_{\mathcal{N}} N_k^O}(\overline{\sum_{\mathcal{N}} N_k^O}(X))$.

(6) $\underline{\sum_{\mathcal{H}} N_k^O}(X) \subseteq \underline{\sum_{\mathcal{N}} N_k^O}(X)$, $\overline{\sum_{\mathcal{N}} N_k^O}(X) \subseteq \overline{\sum_{\mathcal{H}} N_k^O}(X)$.

Proof. It is easy to obtain the conclusion by Definition 7. □

3.2. Pessimistic multi-granulation reduction of GNDISs

In a GNDIS $(U, \mathcal{N}, N_d)$, $C_{N_d} = \{N_d(y) | y \in U\}$ is a partition of U. In the following, the GNDIS mentioned satisfies $|C_{N_d}| \geq 2$. In this subsection, we present pessimistic multi-granulation reduction of GNDISs.

Definition 8. *Given a GNDIS* $(U, \mathcal{N}, N_d)$, *let* $\mathcal{H} \subseteq \mathcal{N}$.

(1) If $\underline{\sum_{\mathcal{N}} N_k^P}(N_d(y)) = \underline{\sum_{\mathcal{H}} N_k^P}(N_d(y))$ *for all* $y \in U$, *then we say that* $\mathcal{H}$ *is a generalized neighborhood pessimistic lower consistent set (GNPL-consistent set). Denote the family of all GNPL-consistent sets by* $Cons_L^P(\mathcal{N})$. *If* $\mathcal{H} \in Cons_L^P(\mathcal{N})$, *and* $\mathcal{H}' \notin Cons_L^P(\mathcal{N})$ *whenever* $\mathcal{H}' \subset \mathcal{H}$, *then* $\mathcal{H}$ *is called a GNPL-reduct. Denote the set of all GNPL-reducts by* $Red_L^P(\mathcal{N})$, *the core w.r.t. GNPL-reducts is defined as* $Core_L^P(\mathcal{N}) = \bigcap\{\mathcal{H} | \mathcal{H} \in Red_L^P(\mathcal{N})\}$.

(2) If $\overline{\sum_{\mathcal{N}} N_k^P}(N_d(y)) = \overline{\sum_{\mathcal{H}} N_k^P}(N_d(y))$ *for all* $y \in U$, *then we say that* $\mathcal{H}$ *is a generalized neighborhood pessimistic upper consistent set (GNPU-consistent set). Denote the family of all GNPU-consistent sets by* $Cons_U^P(\mathcal{N})$. *If* $\mathcal{H} \in Cons_U^P(\mathcal{N})$, *and* $\mathcal{H}' \notin Cons_U^P(\mathcal{N})$ *whenever* $\mathcal{H}' \subset \mathcal{H}$, *then* $\mathcal{H}$ *is called a GNPU-reduct. Denote the set of all GNPU-reducts as* $Red_U^P(\mathcal{N})$, *the core w.r.t. GNPU-reducts is defined by* $Core_U^P(\mathcal{N}) = \bigcap\{\mathcal{H} | \mathcal{H} \in Red_U^P(\mathcal{N})\}$.

From Definition 8, we can see that a GNPL-reduct (or a GNPU-reduct) is a minimal subset of $\mathcal{N}$, which preserves the pessimistic lower approximations (or the pessimistic upper approximations) of all sets in C_{N_d}. The pessimistic lower and upper reducts of a GNDIS are different, which are illustrated by an example in the following.

Example 1. *(1) A GNDIS* $(U, \mathcal{N}, N_d)$ *is presented in Table 1, where* $U = \{x_1, x_2, \cdots, x_6\}$ *and* $\mathcal{N} = \{N_1, N_2, \cdots, N_5\}$*. The generalized neighborhood granules of* $x \in U$ *are presented in Table 2.*

Table 1. A GNDIS.

$*$	x_1	x_2	x_3	x_4	x_5	x_6
$N_1(x_i)$	$\{x_1, x_2\}$	$\{x_2, x_3, x_4\}$	$\{x_3, x_5\}$	$\{x_1, x_4\}$	$\{x_2, x_4, x_5\}$	$\{x_2, x_6\}$
$N_2(x_i)$	$\{x_1, x_3\}$	$\{x_2, x_5\}$	$\{x_2, x_3\}$	$\{x_1, x_3, x_4\}$	$\{x_3, x_5\}$	$\{x_1, x_6\}$
$N_3(x_i)$	$\{x_1, x_2\}$	$\{x_2, x_4\}$	$\{x_3, x_6\}$	$\{x_1, x_2, x_4\}$	$\{x_4, x_5\}$	$\{x_3, x_6\}$
$N_4(x_i)$	$\{x_1, x_2\}$	$\{x_2, x_3\}$	$\{x_2, x_3\}$	$\{x_1, x_2, x_4\}$	$\{x_2, x_5\}$	$\{x_1, x_6\}$
$N_5(x_i)$	$\{x_1, x_3\}$	$\{x_2, x_4\}$	$\{x_3, x_6\}$	$\{x_1, x_4\}$	$\{x_3, x_5\}$	$\{x_3, x_6\}$
$N_d(x_i)$	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_3\}$	$\{x_4, x_5\}$	$\{x_4, x_5\}$	$\{x_6\}$

Table 2. The generalized neighborhood granules of $x \in U$ w.r.t. $\mathcal{N}$ in Example 1.

$*$	x_1	x_2	x_3	x_4	x_5	x_6
$GN_{\mathcal{N}}(x_i)$	$\{x_1, x_2, x_3\}$	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3, x_5, x_6\}$	$\{x_1, x_2, x_3, x_4\}$	$\{x_2, x_3, x_4, x_5\}$	$\{x_1, x_2, x_3, x_6\}$

By Definition 6, we get that $\underline{\sum_{\mathcal{N}} N_k^P}(\{x_1, x_2, x_3\}) = \{x_1\}$, $\underline{\sum_{\mathcal{N}} N_k^P}(\{x_4, x_5\}) = \emptyset$, $\underline{\sum_{\mathcal{N}} N_k^P}(\{x_6\}) = \emptyset$, $\overline{\sum_{\mathcal{N}} N_k^P}(\{x_1, x_2, x_3\}) = U$, $\overline{\sum_{\mathcal{N}} N_k^P}(\{x_4, x_5\}) = \{x_2, x_3, x_4, x_5\}$, $\overline{\sum_{\mathcal{N}} N_k^P}(\{x_6\}) = \{x_3, x_6\}$.

Let $\mathcal{H} = \{N_3, N_4\}$*. We compute that* $\underline{\sum_{\mathcal{H}} N_k^P}(N_d(x_i)) = \underline{\sum_{\mathcal{N}} N_k^P}(N_d(x_i))$ *for* $i = 1, 2, \cdots, 6$*. Thus,* $\mathcal{H}$ *is a GNPL-consistent set. Let* $\mathcal{H}_1 = \{N_3\}$*. We get that* $\underline{\sum_{\mathcal{H}_1} N_k^P}(\{x_4, x_5\}) = \{x_5\} \neq \underline{\sum_{\mathcal{N}} N_k^P}(\{x_4, x_5\})$*, which follows that* $\mathcal{H}_1$ *is not a GNPL-consistent set. Let* $\mathcal{H}_2 = \{N_4\}$*. We obtain that* $\underline{\sum_{\mathcal{H}_2} N_k^P}(\{x_1, x_2, x_3\}) = \{x_1, x_2, x_3\} \neq \underline{\sum_{\mathcal{N}} N_k^P}(\{x_1, x_2, x_3\})$*, which implies that* $\mathcal{H}_2$ *is not a GNPL-consistent set. So* $\mathcal{H}$ *is a GNPL-reduct. Due to* $\overline{\sum_{\mathcal{H}} N_k^P}(\{x_4, x_5\}) = \{x_2, x_4, x_5\} \neq \overline{\sum_{\mathcal{N}} N_k^P}(\{x_4, x_5\})$*,* $\mathcal{H}$ *is not a GNPU-consistent set, then* $\mathcal{H}$ *is not a GNPU-reduct. At the same time, we get a conclusion: A GNPL-reduct is not necessarily a GNPU-reduct.*

(2) The operator $N_d : U \to P(U)$ *in (1) is changed by:* $N_d(x_1) = N_d(x_2) = N_d(x_3) = \{x_1, x_2, x_3\}$, $N_d(x_4) = N_d(x_5) = N_d(x_6) = \{x_4, x_5, x_6\}$*. Then, we get another GNDIS* $(U, \mathcal{N}, N_d)$*. By Definition 6, we have that* $\underline{\sum_{\mathcal{N}} N_k^P}(\{x_1, x_2, x_3\}) = \{x_1\}$, $\underline{\sum_{\mathcal{N}} N_k^P}(\{x_4, x_5, x_6\}) = \emptyset$, $\overline{\sum_{\mathcal{N}} N_k^P}(\{x_1, x_2, x_3\}) = U$, $\overline{\sum_{\mathcal{N}} N_k^P}(\{x_4, x_5, x_6\}) = \{x_2, x_3, x_4, x_5, x_6\}$.

Let $\mathcal{H} = \{N_2\}$*. We obtain that* $\overline{\sum_{\mathcal{H}} N_k^P}(N_d(x_i)) = \overline{\sum_{\mathcal{N}} N_k^P}(N_d(x_i))$ *for* $i = 1, 2, \cdots, 6$*. Thus,* $\mathcal{H}$ *is a GNPU-reduct. It follows from* $\underline{\sum_{\mathcal{H}} N_k^P}(\{x_1, x_2, x_3\}) = \{x_1, x_3\} \neq \overline{\sum_{\mathcal{N}} N_k^P}(\{x_1, x_2, x_3\})$ *that* $\mathcal{H}$ *is not a GNPL-*

consistent set, then $\mathcal{H}$ is not a GNPL-reduct. Hence, we obtain a conclusion: A GNPU-reduct is not necessarily a GNPL-reduct.

A matrix is constructed to compute all the GNPL-reducts.

Definition 9. *Consider that $(U, \mathcal{N}, N_d)$ is a GNDIS, where $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$. Letting $x \in U, N_d(y) \in C_{N_d}$, define*

$$GD_L^P(x, N_d(y)) = \begin{cases} \{N_k \in \mathcal{N} | N_k(x) \not\subseteq N_d(y)\}, & GN_{\mathcal{N}}(x) \not\subseteq N_d(y), \\ \mathcal{N}, & else. \end{cases}$$

$\mathcal{GD}_L^P = \{GD_L^P(x, N_d(y)) | x \in U, N_d(y) \in C_{N_d}\}$ *is called a GNPL-discernibility matrix (GNPL-D matrix) of $(U, \mathcal{N}, N_d)$.*

Proposition 3. *Let $(U, \mathcal{N}, N_d)$ be a GNDIS, whose GNPL-D matrix is $\mathcal{GD}_L^P = \{GD_L^P(x, N_d(y)) | x \in U, N_d(y) \in C_{N_d}\}$. Then,*

(1) $\forall x \in U$, $GD_L^P(x, N_d(x)) \neq \emptyset$.

(2) $\forall x \in U$, $N_d(y) \in C_{N_d}$ with $x \notin N_d(y)$, $GD_L^P(x, N_d(y)) = \mathcal{N}$.

Proof. (1) $\forall x \in U$, if $GN_{\mathcal{N}}(x) \not\subseteq N_d(x)$, then there is an $N_k \in \mathcal{N}$ satisfying $N_k(x) \not\subseteq N_d(x)$. Hence $N_k \in GD_L^P(x, N_d(x))$, which implies that $GD_L^P(x, N_d(x)) \neq \emptyset$. If $GN_{\mathcal{N}}(x) \subseteq N_d(x)$, by Definition 9, then $GD_L^P(x, N_d(x)) = \mathcal{N} \neq \emptyset$.

(2) For any $x \in U$, $N_d(y) \in C_{N_d}$ with $x \notin N_d(y)$, we get that $GN_{\mathcal{N}}(x) \not\subseteq N_d(y)$ and $N_k(x) \not\subseteq N_d(y)$ for all $N_k \in \mathcal{N}$. Thus, $GD_L^P(x, N_d(y)) = \{N_k \in \mathcal{N} | N_k(x) \not\subseteq N_d(y)\} = \mathcal{N}$. □

By Proposition 3, $\forall x \in U$, $N_d(y) \in C_{N_d}$, $GD_L^P(x, N_d(y)) \neq \emptyset$. Utilizing the GNPL-D matrix, the GNPL-reducts can be characterized.

Theorem 1. *Suppose that $(U, \mathcal{N}, N_d)$ is a GNDIS, where $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$. Letting $\mathcal{H} \subseteq \mathcal{N}$ and $N_k \in \mathcal{N}$,*

(1) $\mathcal{H} \in Cons_L^P(\mathcal{N}) \Leftrightarrow \mathcal{H} \cap GD_L^P(x, N_d(y)) \neq \emptyset$ for all $x \in U, N_d(y) \in C_{N_d}$.

(2) $\mathcal{H} \in Red_L^P(\mathcal{N}) \Leftrightarrow \mathcal{H} \cap GD_L^P(x, N_d(y)) \neq \emptyset$ for all $x \in U, N_d(y) \in C_{N_d}$, and for any $\mathcal{H}_0 \subset \mathcal{H}$, there exists a $GD_L^P(x, N_d(y))$ such that $GD_L^P(x, N_d(y)) \cap \mathcal{H}_0 = \emptyset$.

(3) $N_k \in Core_L^P(\mathcal{N}) \Leftrightarrow \exists x \in U, N_d(y) \in C_{N_d}$, $GD_L^P(x, N_d(y)) = \{N_k\}$.

Proof. (1) "⇒". $\forall x \in U$, $N_d(y) \in C_{N_d}$, if $GN_{\mathcal{N}}(x) \not\subseteq N_d(y)$, then $x \notin \underline{\sum_{\mathcal{N}} N_k^P(N_d(y))}$. Since $\mathcal{H}$ is a GNPL-consistent set, $\underline{\sum_{\mathcal{N}} N_k^P(N_d(y))} = \underline{\sum_{\mathcal{H}} N_k^P(N_d(y))}$. Hence, $x \notin \underline{\sum_{\mathcal{H}} N_k^P(N_d(y))}$. It implies that $GN_{\mathcal{H}}(x) = \cup_{N_k \in \mathcal{H}} N_k(x) \not\subseteq N_d(y)$. Then, we can find an $N_k \in \mathcal{H}$ such that $N_k(x) \not\subseteq N_d(y)$. Therefore, $\mathcal{H} \cap GD_L^P(x, N_d(y)) \neq \emptyset$. If $GN_{\mathcal{H}}(x) \subseteq N_d(y)$, then $GD_L^P(x, N_d(y)) = \mathcal{N}$. It is clear that $\mathcal{H} \cap GD_L^P(x, N_d(y)) \neq \emptyset$.

"⇐". $\forall y \in U$, by Proposition 1(6), $\underline{\sum_{\mathcal{N}} N_k^P(N_d(y))} \subseteq \underline{\sum_{\mathcal{H}} N_k^P(N_d(y))}$. $\forall x \notin \underline{\sum_{\mathcal{N}} N_k^P(N_d(y))}$, we get that $GN_{\mathcal{N}}(x) \not\subseteq N_d(y)$. Since $\mathcal{H} \cap GD_L^P(x, N_d(y)) \neq \emptyset$, let $N_k \in \mathcal{H} \cap GD_L^P(x, N_d(y))$. Thus, according to Definition 9, $N_k(x) \not\subseteq N_d(y)$. It follows that $GN_{\mathcal{H}}(x) = \cup_{N_k \in \mathcal{H}} N_k(x) \subseteq N_d(y)$. Therefore, $x \notin \underline{\sum_{\mathcal{H}} N_k^P(N_d(y))}$, which implies that $\underline{\sum_{\mathcal{H}} N_k^P(N_d(y))} \subseteq \underline{\sum_{\mathcal{N}} N_k^P(N_d(y))}$. We can get that $\underline{\sum_{\mathcal{N}} N_k^P(N_d(y))} = \underline{\sum_{\mathcal{H}} N_k^P(N_d(y))}$.

(2) It is verified from (1).

(3) "⇒". If not, for every $GD_L^P(x, N_d(y)) \in \mathcal{GD}_L^P$ satisfying $N_k \in GD_L^P(x, N_d(y))$, we have $|GD_L^P(x, N_d(y))| \geq 2$. Let $\mathcal{H} = \cup\{GD_L^P(x, N_d(y)) - \{N_k\} | x, y \in U\}$, then $\mathcal{H}$ is a GNPL-consistent set. Thus, there exists a GNPL-reduct $\mathcal{H}_0 \subseteq \mathcal{H}$ and $N_k \notin \mathcal{H}_0$, which contradicts the fact that $N_k \in \mathrm{Core}_L^P(\mathcal{N})$.

"⇐". If not, we can find a reduct $\mathcal{H}$ such that $N_k \notin \mathcal{H}$. Since $GD_L^P(x, N_d(y)) = \{N_k\}$, we obtain that $y \in GN_{\mathcal{N}}(x)$, $y \in N_k(x)$ and $y \notin N_l(x)$ for all $l \neq k (l \in \{1, 2, \cdots, m\})$. Then, $y \notin \cup_{l \neq k} N_l(x)$. Since $N_k \notin \mathcal{H}$, $GN_{\mathcal{H}}(x) \subseteq \cup_{l \neq k} N_l(x)$. It implies that $y \notin GN_{\mathcal{H}}(x)$. Hence $GN_{\mathcal{N}}(x) \neq GN_{\mathcal{H}}(x)$, which contradicts the fact that $\mathcal{H}$ is a GNPL-consistent set. □

Definition 10. *Let $(U, \mathcal{N}, N_d)$ be a GNDIS, whose GNPL-D matrix is $\mathcal{GD}_L^P = \{\mathcal{GD}_L^P(x, N_d(y)) | x \in U, N_d(y) \in C_{N_d}\}$. Define $f(\mathcal{GD}_L^P) = \wedge\{\vee GD_L^P(x, N_d(y)) | \, GD_L^P(x, N_d(y)) \in \mathcal{GD}_L^P\}$.*

$\vee GD_L^P(x, N_d(y))$ is the disjunction of all neighborhood operators in $GD_L^P(x, N_d(y))$, and $\wedge\{\vee GD_L^P(x, N_d(y)) | GD_L^P(x, N_d(y)) \in \mathcal{GD}_L^P\}$ is the conjunction of $\vee GD_L^P(x, N_d(y))$.

Theorem 2. *Let $\mathcal{H} = \{N_1, N_2, \cdots, N_k\} \subseteq \mathcal{N}$. $\mathcal{H} \in Red_L^P(\mathcal{N}) \Leftrightarrow N_1 \wedge N_2 \wedge \cdots \wedge N_k$ is a prime implicant of $f(\mathcal{GD}_L^P)$.*

Proof. It is trivial based on Definition 10. □

Remark 1. *The $\mathcal{GD}_L^P$ in Definition 9 can be simplified as $(\mathcal{GD}_L^P)^* = \{GD_L^P(x, N_d(x)) | x \in U\}$.*

In fact, due to Proposition 3, for any $x \in U$ and $N_d(y) \in C_{N_d}$ with $x \notin N_d(y)$, $\emptyset \neq GD_L^P(x, N_d(x)) \subseteq GD_L^P(x, N_d(y)) = \mathcal{N}$. Then, by Definition 10, $f((\mathcal{GD}_L^P)^*) = f(\mathcal{GD}_L^P)$.

We employ Example 2 below to explain the discernibility method for calculating all the GNPL-reducts of a GNDIS.

Example 2. *Continued from Example 1(1). By Definition 9, we obtain that*

$$\mathcal{GD}_L^P = \left(\begin{array}{c|ccc} GD_L^P(x_i, N_d(x_j)) & N_d(x_1) = N_d(x_2) = N_d(x_3) & N_d(x_4) = N_d(x_5) & N_d(x_6) \\ \hline x_1 & \mathcal{N} & \mathcal{N} & \mathcal{N} \\ x_2 & \{N_1, N_2, N_3, N_5\} & \mathcal{N} & \mathcal{N} \\ x_3 & \{N_1, N_3, N_5\} & \mathcal{N} & \mathcal{N} \\ x_4 & \mathcal{N} & \mathcal{N} & \mathcal{N} \\ x_5 & \mathcal{N} & \{N_1, N_2, N_4, N_5\} & \mathcal{N} \\ x_6 & \mathcal{N} & \mathcal{N} & \mathcal{N} \end{array}\right).$$

From Remark 1, we have that

$$(\mathcal{GD}_L^P)^* = \left(\begin{array}{c|c} x_i & GD_L^P(x_i, N_d(x_i)) \\ \hline x_1 & \mathcal{N} \\ x_2 & \{N_1, N_2, N_3, N_5\} \\ x_3 & \{N_1, N_3, N_5\} \\ x_4 & \mathcal{N} \\ x_5 & \{N_1, N_2, N_4, N_5\} \\ x_6 & \mathcal{N} \end{array}\right).$$

Algorithm 1 A logic algorithm for calculating all the GNPL-reducts of a GNDIS

Input: A GNDIS $(U, \mathcal{N}, N_d)$ with $U = \{x_1, x_2, \cdots, x_n\}$ and $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$

Output: All the GNPL-reducts $Red_L^P(\mathcal{N})$

1: **for** $i = 1 : n$ **do**
2: Initialize $GD_L^P(x_i, N_d(x_i)) \leftarrow \emptyset$;
3: **for** $k = 1 : m$ **do**
4: **if** $N_k(x_i) \nsubseteq N_d(x_i)$ **then**
5: $GD_L^P(x_i, N_d(x_i)) \leftarrow GD_L^P(x_i, N_d(x_i)) \cup \{N_k\}$
6: **end if**
7: **end for**
8: **if** $GD_L^P(x_i, N_d(x_i)) = \emptyset$ **then**
9: $GD_L^P(x_i, N_d(x_i)) \leftarrow \mathcal{N}$
10: **end if**
11: **end for**
12: Initialize $Red_L^P(\mathcal{N}) \leftarrow \emptyset$;
13: **for** $i = 1 : n$ **do**
14: $Red_L^P(\mathcal{N}) \leftarrow Red_L^P(\mathcal{N}) \wedge (\vee GD_L^P(x_i, N_d(x_i)))$
15: **end for**
16: Compute $Red_L^P(\mathcal{N}) \leftarrow \vee_{l=1}^{t}(\wedge_{k=1}^{s_l})N_k$;
17: **Return** $Red_L^P(\mathcal{N})$.

By Theorem 1(3), $Core_L^P(\mathcal{N}) = \emptyset$. According to Definition 10, $f(\mathcal{GD}_L^P) = f((\mathcal{GD}_L^P)^*) = (N_1 \vee N_2 \vee N_3 \vee N_5) \wedge (N_1 \vee N_3 \vee N_5) \wedge (N_1 \vee N_2 \vee N_4 \vee N_5) \wedge (N_1 \vee N_2 \vee N_3 \vee N_4 \vee N_5) = (N_1) \vee (N_2 \wedge N_3) \vee (N_3 \wedge N_4) \vee (N_5)$. Then, $\{N_1\}$, $\{N_2, N_3\}$, $\{N_3, N_4\}$ and $\{N_5\}$ are GNPL-reducts.

By the analysis above, we present Algorithm 1 to calculate all the GNPL-reducts of a GNDIS. In Algorithm 1, the time complexity of Steps 1–11 is $O(|U|^2|\mathcal{N}|)$, and the time complexity of Steps 2–17 is $O(\prod_{GD \in \mathcal{GD}_L^P} |GD|)$. The total time complexity of Algorithm 1 is $O(|U|^2|\mathcal{N}| + \prod_{GD \in \mathcal{GD}_L^P} |GD|)$.

We construct a discernibility matrix to get all the GNPU-reducts as follows:

Definition 11. *Suppose that $(U, \mathcal{N}, N_d)$ is a GNDIS, where $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$. Letting $x \in U$ and $N_d(y) \in C_{N_d}$, define*

$$GD_U^P(x, N_d(y)) = \begin{cases} \{N_k \in \mathcal{N} | N_k(x) \cap N_d(y) \neq \emptyset\}, & GN_{\mathcal{N}}(x) \cap N_d(y) \neq \emptyset, \\ \emptyset, & else. \end{cases}$$

$\mathcal{GD}_U^P = \{GD_U^P(x, N_d(y)) | x \in U, N_d(y) \in C_{N_d}\}$ is called a GNPU-discernibility matrix (GNPU-D matrix) of $(U, \mathcal{N}, N_d)$.

Proposition 4. *$\forall x \in U$, $GD_U^P(x, N_d(x)) = \mathcal{N}$.*

Proof. $\forall x \in U$, $GN_{\mathcal{N}}(x) \cap N_d(x) \neq \emptyset$, then $GD_U^P(x, N_d(x)) = \{N_k \in \mathcal{N} | N_k(x) \cap N_d(x) \neq \emptyset\} = \mathcal{N}$. □

Theorem 3. *Consider that $(U, \mathcal{N}, N_d)$ is a GNDIS, where $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$. Letting $\mathcal{H} \subseteq \mathcal{N}$ and $N_k \in \mathcal{N}$,* *(1)* $\mathcal{H} \in Cons_U^P(\mathcal{N}) \Leftrightarrow \mathcal{H} \cap GD_U^P(x, N_d(y)) \neq \emptyset$ *for all* $GD_U^P(x, N_d(y)) \neq \emptyset$.

(2) $\mathcal{H} \in Red_U^P(\mathcal{N}) \Leftrightarrow \mathcal{H} \cap GD_U^P(x, N_d(y)) \neq \emptyset$ *for all* $GD_U^P(x, N_d(y)) \neq \emptyset$, *and for any* $\mathcal{H}_0 \subset \mathcal{H}$, *there exists a* $GD_U^P(x, N_d(y)) \neq \emptyset$ *such that* $GD_U^P(x, N_d(y)) \cap \mathcal{H}_0 = \emptyset$.

(3) $N_k \in Core_U^P(\mathcal{N}) \Leftrightarrow \exists x \in U, N_d(y) \in C_{N_d}$, $GD_U^P(x, N_d(y)) = \{N_k\}$.

Proof. (1) "⇒". $\forall x \in U, N_d(y) \in C_{N_d}$, if $GD_U^P(x, N_d(y)) \neq \emptyset$, then $GN_{\mathcal{N}}(x) \cap N_d(y) \neq \emptyset$, which follows that $x \in \overline{\sum_{\mathcal{N}} N_k^P}(N_d(y))$. Since $\mathcal{H}$ is a GNPU-consistent set, $\overline{\sum_{\mathcal{N}} N_k^P}(N_d(y)) = \overline{\sum_{\mathcal{H}} N_k^P}(N_d(y))$. It implies that $x \in \overline{\sum_{\mathcal{H}} N_k^P}(N_d(y))$. Hence, there is an $N_k \in \mathcal{H}$ satisfying $N_k(x) \cap N_d(y) \neq \emptyset$. By Definition 11, $N_k \in GD_U^P(x, N_d(y))$. Thus, $\mathcal{H} \cap GD_U^P(x, N_d(y)) \neq \emptyset$.

"⇐". $\forall y \in U$, by Proposition 1(6), $\overline{\sum_{\mathcal{H}} N_k^P}(N_d(y)) \subseteq \overline{\sum_{\mathcal{N}} N_k^P}(N_d(y))$. $\forall x \in \overline{\sum_{\mathcal{N}} N_k^P}(N_d(y))$, $GN_{\mathcal{N}}(x) \cap N_d(y) \neq \emptyset$. It follows from $\mathcal{H} \cap GD_U^P(x, N_d(y)) \neq \emptyset$ that there exists an $N_k \in \mathcal{N}$ such that $N_k \in \mathcal{H} \cap GD_U^P(x, N_d(y))$. Hence $N_k(x) \cap N_d(y) \neq \emptyset$, which implies that $GN_{\mathcal{H}}(x) \cap N_d(y) \neq \emptyset$. Then, $x \in \overline{\sum_{\mathcal{H}} N_k^P}(N_d(y))$. So $\overline{\sum_{\mathcal{N}} N_k^P}(N_d(y)) \subseteq \overline{\sum_{\mathcal{H}} N_k^P}(N_d(y))$.

(2) It is easy to obtain (2) by (1).

(3) Similar to the proof of (3) in Theorem 1. □

Definition 12. *Let* $(U, \mathcal{N}, N_d)$ *be a GNDIS, whose GNPU-D matrix is* $\mathcal{GD}_U^P = \{GD_U^P(x, N_d(y)) | x \in U, N_d(y) \in C_{N_d}\}$. *Define* $f(\mathcal{GD}_U^P) = \wedge\{\vee GD_U^P(x, N_d(y)) | \; GD_U^P(x, N_d(y)) \in \mathcal{GD}_U^P, GD_U^P(x, N_d(y)) \neq \emptyset\}$.

Theorem 4. *Let* $\mathcal{H} = \{N_1, N_2, \cdots, N_k\} \subseteq \mathcal{N}$. $\mathcal{H} \in Cons_U^P(\mathcal{N}) \Leftrightarrow N_1 \wedge N_2 \wedge \cdots \wedge N_k$ *is a prime implicant of* $f(\mathcal{GD}_U^P)$.

Proof. It is clear based on Definition 12. □

By Theorem 4, the set of all GNPU-reducts in a GNDIS and the set of all prime implicants of $f(\mathcal{GD}_U^P)$ are the one-to-one correspondence. Example 3 is employed to illustrate the above theorems.

Example 3. *Continued from Example 1(1). By Definition 11, we get that*

$\mathcal{GD}_U^P =$

$GD_U^P(x_i, N_d(x_j))$	$N_d(x_1) = N_d(x_2) = N_d(x_3)$	$N_d(x_4) = N_d(x_5)$	$N_d(x_6)$
x_1	$\mathcal{N}$	$\emptyset$	$\emptyset$
x_2	$\mathcal{N}$	$\{N_1, N_2, N_3, N_5\}$	$\emptyset$
x_3	$\mathcal{N}$	$\{N_1\}$	$\{N_3, N_5\}$
x_4	$\mathcal{N}$	$\mathcal{N}$	$\emptyset$
x_5	$\{N_1, N_2, N_4, N_5\}$	$\mathcal{N}$	$\emptyset$
x_6	$\mathcal{N}$	$\emptyset$	$\mathcal{N}$

According to Theorem 3, $Core_U^P(\mathcal{N}) = \{N_1\}$. *By Definition 12,* $f(\mathcal{GD}_U^P) = (N_1) \wedge (N_3 \vee N_5) \wedge (N_1 \vee N_2 \vee N_4 \vee N_5) \wedge (N_1 \vee N_2 \vee N_3 \vee N_5) \wedge (N_1 \vee N_2 \vee N_3 \vee N_4 \vee N_5) = (N_1 \wedge N_3) \vee (N_1 \wedge N_5)$.

Then, $\{N_1, N_3\}$ *and* $\{N_1, N_5\}$ *are GNPU-reducts.*

3.3. Optimistic multi-granulation reduction of GNDISs

In this subsection, we discuss optimistic multi-granulation reduction of GNDISs.

Definition 13. *Let $(U, \mathcal{N}, N_d)$ be a GNDIS.*

(1) $\mathcal{H} \subseteq \mathcal{N}$ is a generalized neighborhood optimistic lower consistent set (GNOL-consistent set) if $\underline{\sum_{\mathcal{N}} N_k^O}(N_d(y)) = \underline{\sum_{\mathcal{H}} N_k^O}(N_d(y))$ for every $y \in U$. Denote the family of all GNOL-consistent sets by $Cons_L^O(\mathcal{N})$. If $\mathcal{H} \in Cons_L^O(\mathcal{N})$, and $\mathcal{H}' \notin Cons_L^O(\mathcal{N})$ whenever $\mathcal{H}' \subset \mathcal{H}$, then $\mathcal{H}$ is said to be a GNOL-reduct. Denote the set of all GNOL-reducts as $Red_L^O(\mathcal{N})$, the core w.r.t. GNOL-reducts is defined by $Core_L^O(\mathcal{N}) = \bigcap\{\mathcal{H} | \mathcal{H} \in Red_L^O(\mathcal{N})\}$.

(2) $\mathcal{H} \subseteq \mathcal{N}$ is a generalized neighborhood optimistic upper consistent set (GNOU-consistent set) if $\overline{\sum_{\mathcal{N}} N_k^O}(N_d(y)) = \overline{\sum_{\mathcal{H}} N_k^O}(N_d(y))$ for all $y \in U$. Denote the family of all GNOU-consistent sets by $Cons_U^O(\mathcal{N})$. If $\mathcal{H} \in Cons_U^O(\mathcal{N})$, and $\mathcal{H}' \notin Cons_U^O(\mathcal{N})$ whenever $\mathcal{H}' \subset \mathcal{H}$, then $\mathcal{H}$ is said to be a GNOU-reduct. Denote the set of all GNOU-reducts by $Red_U^O(\mathcal{N})$, the core w.r.t. GNOU-reducts is defined as $Core_U^O(\mathcal{N}) = \bigcap\{\mathcal{H} | \mathcal{H} \in Red_U^O(\mathcal{N})\}$.

By Definition 13, a GNOL-reduct (or GNOU-reduct) is a minimal subset of $\mathcal{N}$ that maintains the optimistic lower approximations (or optimistic upper approximations) of all $N_d(y) \in C_{N_d}$. The GNOL-reduct and GNOU-reduct are different as illustrated by the next example.

Example 4. *Continued from Example 1(1). Change the Pawlak neighborhood operator $N_d : U \to P(U)$ in Example 1(1) by: $N_d(x_1) = N_d(x_3) = \{x_1, x_3\}$, $N_d(x_2) = N_d(x_4) = N_d(x_5) = N_d(x_6) = \{x_2, x_4, x_5, x_6\}$. Then, we get a new GNDIS $(U, \mathcal{N}, N_d)$. According to Definition 7, $\underline{\sum_{\mathcal{N}} N_k^O}(\{x_1, x_3\}) = \{x_1\}$, $\underline{\sum_{\mathcal{N}} N_k^O}(\{x_2, x_4, x_5, x_6\}) = \{x_2, x_5, x_6\}$, $\overline{\sum_{\mathcal{N}} N_k^O}(\{x_1, x_3\}) = \{x_1, x_3, x_4, x_6\}$, $\overline{\sum_{\mathcal{N}} N_k^O}(\{x_2, x_4, x_5, x_6\}) = \{x_2, x_4, x_5, x_6\}$.*

Let $\mathcal{H} = \{N_1, N_5\}$. We compute that $\underline{\sum_{\mathcal{H}} N_k^O}(N_d(x_i)) = \underline{\sum_{\mathcal{N}} N_k^O}(N_d(x_i))$ for $i = 1, 2, \cdots, 6$. Thus, $\mathcal{H}$ is a GNOL-consistent set. Let $\mathcal{H}_1 = \{N_1\}$. We get that $\underline{\sum_{\mathcal{H}_1} N_k^O}(\{x_1, x_3\}) = \emptyset \neq \underline{\sum_{\mathcal{N}} N_k^O}(\{x_1, x_3\})$, which follows that $\mathcal{H}_1$ is not a GNOL-consistent set. Let $\mathcal{H}_2 = \{N_5\}$. We obtain that $\underline{\sum_{\mathcal{H}_2} N_k^O}(\{x_2, x_4, x_5, x_6\}) = \{x_2\} \neq \underline{\sum_{\mathcal{N}} N_k^O}(\{x_2, x_4, x_5, x_6\})$, which implies that $\mathcal{H}_2$ is not a GNOL-consistent set. So, $\mathcal{H}$ is a GNOL-reduct. Due to $\overline{\sum_{\mathcal{H}} N_k^P}(\{x_4, x_5\}) = \{x_2, x_4, x_5\} \neq \overline{\sum_{\mathcal{N}} N_k^P}(\{x_4, x_5\})$, $\mathcal{H}$ is not a GNOU-consistent set, then $\mathcal{H}$ is not a GNOU-reduct.

Let $\mathcal{H} = \{N_2, N_3\}$. We obtain that $\overline{\sum_{\mathcal{H}} N_k^O}(N_d(x_i)) = \overline{\sum_{\mathcal{N}} N_k^O}(N_d(x_i))$ for $i = 1, 2, \cdots, 6$. Thus, $\mathcal{H}$ is a GNOU-consistent set. Let $\mathcal{H}_1 = \{N_2\}$. Then, $\overline{\sum_{\mathcal{H}_1} N_k^O}(\{x_1, x_3\}) = \{x_1, x_3, x_4, x_5, x_6\} \neq \overline{\sum_{\mathcal{N}} N_k^O}(\{x_1, x_3\})$, which implies that $\mathcal{H}_1$ is not a GNOU-consistent set. Let $\mathcal{H}_2 = \{N_3\}$. Then, $\overline{\sum_{\mathcal{H}_2} N_k^O}(\{x_2, x_4, x_5, x_6\}) = U \neq \overline{\sum_{\mathcal{N}} N_k^O}(\{x_1, x_3\})$, which follows that $\mathcal{H}_2$ is not a GNOU-consistent set. Hence, $\mathcal{H}$ is a GNOU-reduct. It follows from $\underline{\sum_{\mathcal{H}} N_k^O}(\{x_2, x_4, x_5, x_6\}) = \{x_2, x_5\} \neq \underline{\sum_{\mathcal{N}} N_k^O}(\{x_2, x_4, x_5, x_6\})$ that $\mathcal{H}$ is not a GNOL-consistent set, then $\mathcal{H}$ is not a GNOL-reduct.

Hence, we get that a GNOU-reduct is not necessarily a GNOL-reduct, and a GNOL-reduct is not necessarily a GNOU-reduct. In the following, we calculate GNOL-reducts and GNOL-reducts of a

GNDIS by the discernibility technique.

Definition 14. *Consider a GNDIS* $(U, \mathcal{N}, N_d)$*, where* $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$*.* $\forall x \in U, N_d(y) \in C_{N_d}$*, and define*

$$GD_L^O(x, N_d(y)) = \begin{cases} \{N_k \in \mathcal{N} | N_k(x) \subseteq N_d(y)\}, & x \in \underline{\sum_{\mathcal{N}} N_k^O}(N_d(y)), \\ \mathcal{N}, & else. \end{cases}$$

$\mathcal{GD}_L^O = \{GD_L^O(x, N_d(y)) | x \in U, N_d(y) \in C_{N_d}\}$ *is called a GNOL-discernibility matrix (GNOL-D matrix) of* $(U, \mathcal{N}, N_d)$*.*

Theorem 5. *In a GNDIS* $(U, \mathcal{N}, N_d)$ *with* $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$*, let* $\emptyset \neq \mathcal{H} \subseteq \mathcal{N}$*,* $N_k \in \mathcal{N}$*, then*

(1) $\mathcal{H} \in Cons_L^O(\mathcal{N}) \Leftrightarrow \mathcal{H} \cap GD_L^O(x, N_d(y)) \neq \emptyset$ *for each* $GD_L^O(x, N_d(y)) \in \mathcal{GD}_L^O$*.*

(2) $\mathcal{H} \in Red_L^O(\mathcal{N}) \Leftrightarrow \mathcal{H} \cap GD_L^P(x, N_d(y)) \neq \emptyset$ *for all* $GD_L^O(x, N_d(y)) \in \mathcal{GD}_L^O$*, and for any* $\mathcal{H}_0 \subset \mathcal{H}$*, there exist some* $GD_L^O(x, N_d(y)) \in \mathcal{GD}_L^O$ *such that* $GD_L^O(x, N_d(y)) \cap \mathcal{H}_0 = \emptyset$*.*

(3) $N_k \in Core_L^O(\mathcal{N}) \Leftrightarrow \exists x \in U, N_d(y) \in C_{N_d}$*,* $GD_L^O(x, N_d(y)) = \{N_k\}$*.*

Proof. (1) "⇒". $\forall x \in U, N_d(y) \in C_{N_d}$, if $x \in \underline{\sum_{\mathcal{N}} N_k^O}(N_d(y)) = \underline{\sum_{\mathcal{H}} N_k^O}(N_d(y))$, according to Definition 7, we can find an $N_k \in \mathcal{H}$ such that $N_k(x) \subseteq N_d(y)$. Thus, $N_k \in GD_L^O(x, N_d(y))$. It follows that $\mathcal{H} \cap GD_L^O(x, N_d(y)) \neq \emptyset$. If $x \notin \underline{\sum_{\mathcal{N}} N_k^O}(N_d(y))$, then $GD_L^O(x, N_d(y)) = \mathcal{N}$. It is clear that $\mathcal{H} \cap GD_L^P(x, N_d(y)) \neq \emptyset$.

"⇐". $\forall y \in U$, by Proposition 2(6), $\underline{\sum_{\mathcal{H}} N_k^O}(N_d(y)) \subseteq \underline{\sum_{\mathcal{N}} N_k^O}(N_d(y))$. $\forall x \in \underline{\sum_{\mathcal{N}} N_k^O}(N_d(y))$, we obtain that $GD_L^O(x, N_d(y)) \neq \emptyset$. Since $\mathcal{H} \cap GD_L^O(x, N_d(y)) \neq \emptyset$, let $N_k \in \mathcal{H} \cap GD_L^O(x, N_d(y))$, then $N_k(x) \subseteq N_d(y)$. Hence, $x \in \underline{\sum_{\mathcal{N}} N_k^O}(N_d(y))$. It implies that $\underline{\sum_{\mathcal{N}} N_k^O}(N_d(y)) \subseteq \underline{\sum_{\mathcal{H}} N_k^O}(N_d(y))$.

(2) It is verified by (1).

(3) With the reference to the proof of (3) in Theorem 1. □

Proposition 5. $\forall x \in U, N_d(y) \in C_{N_d}$*, if* $x \notin N_d(y)$*, then* $GD_L^O(x, N_d(y)) = \mathcal{N}$*.*

Proof. $\forall N_d(y) \in C_{N_d}$, $\underline{\sum_{\mathcal{N}} N_k^O}(N_d(y)) \subseteq N_d(y)$. If $x \notin N_d(y)$, then $x \notin \underline{\sum_{\mathcal{N}} N_k^O}(N_d(y))$. Hence, according to Definition 14, $GD_L^O(x, N_d(y)) = \mathcal{N}$. □

Remark 2. *The* $\mathcal{GD}_L^O$ *in Definition 14 can be simplified as* $(\mathcal{GD}_L^O)^* = \{GD_L^O(x, N_d(x)) | x \in U\}$*.*

In fact, by Proposition 5, for every $x \in U$ *and* $N_d(y) \in C_{N_d}$ *with* $x \notin N_d(y)$*,* $GD_L^P(x, N_d(y)) = \mathcal{N}$*. Then, the* $\mathcal{GD}_L^O$ *in Theorem 5 can be changed into* $(\mathcal{GD}_L^O)^*$*.*

Definition 15. *Assume that* $(U, \mathcal{N}, N_d)$ *is a GNDIS, whose GNOL-D matrix is* $(\mathcal{GD}_L^O)^* = \{GD_L^O(x, N_d(x)) | x \in U\}$*. Define* $f((\mathcal{GD}_L^O)^*) = \wedge\{\vee GD_L^O(x, N_d(x)) | x \in U\}$*.*

Theorem 6. *Let* $\mathcal{H} = \{N_1, N_2, \cdots, N_k\} \subseteq \mathcal{N}$*.* $\mathcal{H} \in Red_L^O(\mathcal{N}) \Leftrightarrow N_1 \wedge N_2 \wedge \cdots \wedge N_k$ *is a prime implicant of* $f((\mathcal{GD}_L^O)^*)$*.*

Proof. It can be obtained by Definition 15. □

Algorithm 2 A logic algorithm for computing all the GNOL-reducts of a GNDIS

Input: A GNDIS $(U, \mathcal{N}, N_d)$ with $U = \{x_1, x_2, \cdots, x_n\}$ and $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$

Output: All the GNPL-reducts $Red_L^O(\mathcal{N})$

for $i = 1 : n$ **do**
  Initialize $GD_L^O(x_i, N_d(x_i)) \leftarrow \emptyset$;
  for $k = 1 : m$ **do**
    if $N_k(x_i) \subseteq N_d(x_i)$ **then**
      $GD_L^O(x_i, N_d(x_i)) \leftarrow GD_L^O(x_i, N_d(x_i)) \cup \{N_k\}$
    end if
  end for
  if $GD_L^O(x_i, N_d(x_i)) = \emptyset$ **then**
    $GD_L^O(x_i, N_d(x_i)) \leftarrow \mathcal{N}$
  end if
end for
Initialize $Red_L^O(\mathcal{N}) \leftarrow \emptyset$;
for $i = 1 : n$ **do**
  $Red_L^O(\mathcal{N}) \leftarrow Red_L^O(\mathcal{N}) \wedge (\vee GD_L^O(x_i, N_d(x_i)))$
end for
Compute $Red_L^O(\mathcal{N}) \leftarrow \vee_{l=1}^{t} (\wedge_{k=1}^{s_l}) N_k$;
Return $Red_L^O(\mathcal{N})$.

By means of Theorem 6, all GNOL-reducts of a GNDIS can be obtained by $f((\mathcal{GD}_L^O)^*)$. An algorithm for calculating all the GNOL-reducts of a GNDIS is presented as Algorithm 2. The total time complexity of Algorithm 2 is $O(|U|^2|\mathcal{N}| + \prod_{GD\in(\mathcal{GD}_L^O)^*} |GD|)$. Example 5 is employed to state the calculation process.

Example 5. *Continued from Example 1(1). We have that* $\underline{\sum_{\mathcal{N}} N_k^O}(\{x_1, x_2, x_3\}) = \{x_1, x_2, x_3\}$, $\underline{\sum_{\mathcal{N}} N_k^O}(\{x_4, x_5\}) = \{x_5\}$, $\underline{\sum_{\mathcal{N}} N_k^O}(\{x_6\}) = \emptyset$. *According to Definition 14, we get that*

$$\mathcal{GD}_L^O = \left(\begin{array}{c|ccc} GD_L^O(x_i, N_d(x_j)) & N_d(x_1) = N_d(x_2) = N_d(x_3) & N_d(x_4) = N_d(x_5) & N_d(x_6) \\ \hline x_1 & \mathcal{N} & \mathcal{N} & \mathcal{N} \\ x_2 & \{N_4\} & \mathcal{N} & \mathcal{N} \\ x_3 & \{N_2, N_4\} & \mathcal{N} & \mathcal{N} \\ x_4 & \mathcal{N} & \mathcal{N} & \mathcal{N} \\ x_5 & \mathcal{N} & \{N_3\} & \mathcal{N} \\ x_6 & \mathcal{N} & \mathcal{N} & \mathcal{N} \end{array} \right),$$

and

$$(\mathcal{GD}_L^O)^* = \begin{pmatrix} x_i & GD_L^O(x_i, N_d(x_i)) \\ \hline x_1 & \mathcal{N} \\ x_2 & \{N_4\} \\ x_3 & \{N_2, N_4\} \\ x_4 & \mathcal{N} \\ x_5 & \{N_3\} \\ x_6 & \mathcal{N} \end{pmatrix}.$$

By Definition 15, $f((\mathcal{GD}_L^O)^*) = (N_2 \vee N_4) \wedge (N_4) \wedge (N_3) \wedge (N_1 \vee N_2 \vee N_3 \vee N_4 \vee N_5) = N_3 \wedge N_4$. *Hence,* $\{N_3, N_4\}$ *is the GNOL-reduct. We also get that* $Core_L^O(\mathcal{N}) = \{N_3, N_4\}$.

Now, we construct a discernibility matrix to calculate GNOU-reducts.

Definition 16. *Let* $(U, \mathcal{N}, N_d)$ *be a GNDIS, where* $\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$. $\forall x \in U, N_d(y) \in C_{N_d}$*, define*

$$GD_U^O(x, N_d(y)) = \begin{cases} \{N_k \in \mathcal{N} | N_k(x) \cap N_d(y) = \emptyset\}, & x \notin \overline{\sum_{\mathcal{N}} N_k^O}(N_d(y)), \\ \mathcal{N}, & else. \end{cases}$$

$\mathcal{GD}_U^O = \{GD_U^O(x, N_d(y)) | x \in U, N_d(y) \in C_{N_d}\}$ *is called a GNOU-discernibility matrix (GNOU-D matrix) of* $(U, \mathcal{N}, N_d)$.

It is easy to get that $GD_U^O(x, N_d(y)) \neq \emptyset$ for all $x \in U, N_d(y) \in C_{N_d}$.

Theorem 7. *Given a GNDIS* $(U, \mathcal{N}, N_d)$*, let* $\mathcal{H} \subseteq \mathcal{N}$ *and* $N_k \in \mathcal{N}$*, then*

(1) $\mathcal{H} \in Cons_U^O(\mathcal{N}) \Leftrightarrow \mathcal{H} \cap GD_U^O(x, N_d(y)) \neq \emptyset$ *for all* $GD_U^O(x, N_d(y)) \in \mathcal{GD}_U^O$.

(2) $\mathcal{H} \in Red_U^O(\mathcal{N}) \Leftrightarrow \mathcal{H} \cap GD_U^O(x, N_d(y)) \neq \emptyset$ *for all* $GD_U^O(x, N_d(y)) \neq \emptyset$*, and for any* $\mathcal{H}_0 \subset \mathcal{H}$*, there exists a* $GD_U^O(x, N_d(y)) \in \mathcal{GD}_U^O$ *such that* $GD_U^O(x, N_d(y)) \cap \mathcal{H}_0 = \emptyset$.

(3) $N_k \in Core_U^O(\mathcal{N}) \Leftrightarrow \exists x \in U, N_d(y) \in C_{N_d}, GD_U^O(x, N_d(y)) = \{N_k\}$.

Proof. (1) "⇒". $\forall x \in U$, $N_d(y) \in C_{N_d}$, if $x \notin \overline{\sum_{\mathcal{N}} N_k^O}(N_d(y))$, we get that $x \notin \overline{\sum_{\mathcal{H}} N_k^O}(N_d(y))$. Then, $\exists N_k \in \mathcal{H}$, $N_k(x) \cap N_d(y) = \emptyset$. It implies that $N_k \in GD_U^O(x, N_d(y))$. Therefore, $\mathcal{H} \cap GD_U^O(x, N_d(y)) \neq \emptyset$. If $x \in \overline{\sum_{\mathcal{N}} N_k^O}(N_d(y))$, then $GD_U^O(x, N_d(y)) = \mathcal{N}$. It is verified that $\mathcal{H} \cap GD_U^O(x, N_d(y)) \neq \emptyset$.

"⇐". $\forall y \in U$, by Proposition 2(6), $\overline{\sum_{\mathcal{N}} N_k^O}(N_d(y)) \subseteq \overline{\sum_{\mathcal{H}} N_k^O}(N_d(y))$. For any $x \notin \overline{\sum_{\mathcal{N}} N_k^O}(N_d(y))$, $GD_U^O(x, N_d(y)) \neq \emptyset$. Due to $\mathcal{H} \cap GD_U^O(x, N_d(y)) \neq \emptyset$, let $N_k \in \mathcal{H} \cap GD_U^O(x, N_d(y))$. Then, $N_k(x) \cap N_d(y) = \emptyset$. Thus, $x \notin \overline{\sum_{\mathcal{H}} N_k^O}(N_d(y))$. It follows that $\overline{\sum_{\mathcal{H}} N_k^O}(N_d(y)) \subseteq \overline{\sum_{\mathcal{N}} N_k^O}(N_d(y))$. We conclude that $\overline{\sum_{\mathcal{H}} N_k^O}(N_d(y)) = \overline{\sum_{\mathcal{N}} N_k^O}(N_d(y))$.

(2) It is easy to obtain (2) by (1).

(3) By reference to the proof of (3) in Theorem 1. □

Definition 17. *Let* $(U, \mathcal{N}, N_d)$ *be a GNDIS, whose GNOU-D matrix is* $\mathcal{GD}_U^O = \{GD_U^O(x, N_d(y)) | x \in U, N_d(y) \in C_{N_d}\}$. *Define* $f(\mathcal{GD}_U^O) = \wedge\{\vee GD_U^O(x, N_d(y)) | \, GD_U^O(x, N_d(y)) \in \mathcal{GD}_U^O\}$.

Theorem 8. *Let* $\mathcal{H} = \{N_1, N_2, \cdots, N_k\} \subseteq \mathcal{N}$. $\mathcal{H} \in Red_U^O(\mathcal{N}) \Leftrightarrow N_1 \wedge N_2 \wedge \cdots \wedge N_k$ *is a prime implicant of* $f(\mathcal{GD}_U^O)$.

Proof. It is trivial based on Definition 17. □

By Theorem 8, all the GNOU-reducts can be obtained by the conjunctive and disjunctive operations of $\mathcal{GD}_U^O$.

Example 6. *Continued from Example 2. We have that* $\overline{\sum_{\mathcal{N}} N_k^O}(\{x_1, x_2, x_3\}) = \{x_1, x_2, x_3, x_4, x_6\}$, $\overline{\sum_{\mathcal{N}} N_k^O}(\{x_4, x_5\}) = \{x_4, x_5\}$, $\overline{\sum_{\mathcal{N}} N_k^O}(\{x_6\}) = \{x_6\}$. *By Definition 16, we deduce that*

$\mathcal{GD}_U^O =$

$GD_U^O(x_i, N_d(x_j))$	$N_d(x_1) = N_d(x_2) = N_d(x_3)$	$N_d(x_4) = N_d(x_5)$	$N_d(x_6)$
x_1	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
x_2	$\mathcal{N}$	$\{N_4\}$	$\mathcal{N}$
x_3	$\mathcal{N}$	$\{N_2, N_3, N_4, N_5\}$	$\{N_1, N_2, N_4\}$
x_4	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
x_5	$\{N_3\}$	$\mathcal{N}$	$\mathcal{N}$
x_6	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$

Then, $f(\mathcal{GD}_U^O) = (N_2 \vee N_3 \vee N_4 \vee N_5) \wedge (N_4) \wedge (N_1 \vee N_2 \vee N_4) \wedge (N_3) \wedge (N_1 \vee N_2 \vee N_3 \vee N_4 \vee N_5) = N_3 \wedge N_4$.

It follows from Theorem 8 that $\{N_3, N_4\}$ is the one and only one GNOU-reduct.

Remark 3. *(1) There is no necessary association between the GNPL-reduct and GNOL-reduct.*

From Examples 2 and 5, $\{N_5\}$ *is a GNPL-reduct. However,* $\{N_5\}$ *is not a GNOL-reduct.*

Continued from Example 1(1). A Pawlak neighborhood operator $N_d : U \to P(U)$ *is defined by:* $N_d(x_1) = N_d(x_6) = \{x_1, x_6\}$, $N_d(x_2) = N_d(x_3) = N_d(x_4) = N_d(x_5) = \{x_2, x_3, x_4, x_5\}$. *Then, we get a GNDIS* $(U, \mathcal{N}, N_d)$. *By Definition 7,* $\underline{\sum_{\mathcal{N}} N_k^O}(\{x_1, x_6\}) = \{x_6\}$, $\underline{\sum_{\mathcal{N}} N_k^O}(\{x_2, x_3, x_4, x_5\}) = \{x_2, x_3, x_5\}$. *Let* $\mathcal{H} = \{N_2\}$. *Then* $\underline{\sum_{\mathcal{N}} N_k^O}(N_d(x_i)) = \underline{\sum_{\mathcal{H}} N_k^O}(N_d(x_i))(i = 1, 2, \cdots, 6)$, *which follows that* $\mathcal{H}$ *is a GNOL-reduct. Since* $\underline{\sum_{\mathcal{N}} N_k^P}(\{x_2, x_3, x_4, x_5\}) = \{x_2, x_5\}$ *and* $\underline{\sum_{\mathcal{N}} N_k^P}(\{x_2, x_3, x_4, x_5\}) = \{x_2, x_3, x_5\}$, $\mathcal{H}$ *is not a GNPL-reduct.*

(2) There is no necessary association between the GNPU-reduct and GNOU-reduct.

By Examples 3 and 6, $\{N_1, N_3\}$ *is a GNPU-reduct but not a GNOU-reduct, and* $\{N_3, N_4\}$ *is a GNOU-reduct instead of a GNPU-reduct.*

4. Relationships between the multi-granulation reduction of DMSs and that of GNDISs

In a DMS $(U, \mathcal{A}, d)$ with $\mathcal{A} = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$ and $U/R_d = \{[y]_d | y \in U\}$, define a mapping $N_k : U \to \mathcal{P}(U)$ by $N_k(x) = [x]_{A_k}$ for all $x \in U$ $(k = 1, 2, \cdots, m)$ and a mapping $N_d : U \to \mathcal{P}(U)$ by $N_d(y) = [y]_d$ for all $y \in U$. Thus, $N_k (k = 1, 2, \cdots, m)$ is a reflexive neighborhood operator on U and N_d is a Pawlak neighborhood operator, and we get a GNDIS $(U, \mathcal{N}^C, N_d)$ with $\mathcal{N}^C = \{N_1, N_2, \cdots, N_m\}$, which is a GNDIS induced by the DMS $(U, \mathcal{A}, d)$. Define a mapping $I : \mathcal{A} \to \mathcal{N}^C$ by $I(A_k) = N_k$ for all $A_k \in \mathcal{A}$. From the definition of $\mathcal{N}^C$, it is easy to get that I is a bijection.

Proposition 6. *Suppose that* $(U, \mathcal{A}, d)$ *is a DMS and* $\mathcal{A} = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$, *which induces the GNDIS* $(U, \mathcal{N}^C, N_d)$ *with* $\mathcal{N}^C = \{N_1, N_2, \cdots, N_m\}$. *Then, for each* $X \subseteq U$,

$\underline{\sum_{\mathcal{A}} A_k^P}(X) = \underline{\sum_{\mathcal{N}^C} N_k^P}(X)$, $\overline{\sum_{\mathcal{A}} A_k^P}(X) = \overline{\sum_{\mathcal{N}^C} N_k^P}(X)$,

$\underline{\sum_{\mathcal{A}} A_k^O}(X) = \underline{\sum_{\mathcal{N}^C} N_k^O}(X)$, $\overline{\sum_{\mathcal{A}} A_k^O}(X) = \overline{\sum_{\mathcal{N}^C} N_k^O}(X)$.

Proof. Since $N_k(x) = [x]_{A_k}$ for all $x \in U$ $(k = 1, 2, \cdots, m)$, it is directly according to Definitions 6 and 7, Definitions 1 and 2. □

From Proposition 6, the generalized neighborhood pessimistic rough set model in Definition 6 is a general model of the pessimistic multi-granulation rough set model defined in [14], and the generalized neighborhood optimistic approximations in Definition 7 is an expansion of the optimistic multi-granulation approximations proposed in [18].

4.1. Pessimistic multi-granulation reduction of DMSs

Pessimistic multi-granulation reduction of DMSs are explored in [14, 18, 21].

Definition 18. *[14, 18, 21] Assume that $(U, \mathcal{A}, d)$ is a DMS, where $\mathcal{A} = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$. Let $\mathcal{H} \subseteq \mathcal{A}$ and $\mathcal{H} \neq \emptyset$.*

(1) $\mathcal{H}$ is called a complete pessimistic lower consistent set (CPL-consistent set) if $\underline{\sum_{\mathcal{A}} A_k^P}([y]_d) = \underline{\sum_{\mathcal{H}} A_k^P}([y]_d)$ for all $y \in U$. Denote the family of all CPL-consistent sets by $Cons_L^P(\mathcal{A})$. Moreover, if $\mathcal{H} \in Cons_L^P(\mathcal{A})$, and $\mathcal{H}' \notin Cons_L^P(\mathcal{A})$ whenever $\mathcal{H}' \subset \mathcal{H}$, then $\mathcal{H}$ is a CPL-reduct of $(U, \mathcal{A}, d)$. Denote the family of all CPL-reducts of $(U, \mathcal{A}, d)$ by $Red_L^P(\mathcal{A})$, and $Core_L^P(\mathcal{A}) = \bigcap_{\mathcal{H} \in Red_L^P(\mathcal{A})} \mathcal{H}$ is said to be a CPL-core.

(2) $\mathcal{H}$ is called a complete pessimistic upper consistent set (CPU-consistent set) if $\overline{\sum_{\mathcal{A}} A_k^P}([y]_d) = \overline{\sum_{\mathcal{H}} A_k^P}([y]_d)$ for all $y \in U$. Denote the family of all CPU-consistent sets by $Cons_U^P(\mathcal{A})$. Moreover, if $\mathcal{H} \in Cons_U^P(\mathcal{A})$, and $\mathcal{H}' \notin Cons_U^P(\mathcal{A})$ whenever $\mathcal{H}' \subset \mathcal{H}$, then $\mathcal{H}$ is a CPU-reduct of $(U, \mathcal{A}, d)$. Denote the family of all CPU-reducts of $(U, \mathcal{A}, d)$ by $Red_U^P(\mathcal{A})$, and $Core_U^P(\mathcal{A}) = \bigcap_{\mathcal{H} \in Red_U^P(\mathcal{A})} \mathcal{H}$ is said to be a CPU-core.

The relationships between the pessimistic multi-granulation reduction of $(U, \mathcal{A}, d)$ and the pessimistic multi-granulation reduction of the GNDIS $(U, \mathcal{N}^C, N_d)$ induced by $(U, \mathcal{A}, d)$ are presented as follows:

Theorem 9. *Let $(U, \mathcal{A}, d)$ be a DMS with $\mathcal{A} = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$, which induces the GNDIS $(U, \mathcal{N}^C, N_d)$ with $\mathcal{N}^C = \{N_1, N_2, \cdots, N_m\}$. Then, for $\mathcal{H} \subseteq \mathcal{A}$, $A_k \in \mathcal{A}$,*
(1) $\mathcal{H} \in Cons_L^P(\mathcal{A}) \Leftrightarrow I(\mathcal{H}) \in Cons_L^P(\mathcal{N}^C)$.
(2) $\mathcal{H} \in Red_L^P(\mathcal{A}) \Leftrightarrow I(\mathcal{H}) \in Red_L^P(\mathcal{N}^C)$.
(3) $A_k \in Core_L^P(\mathcal{A}) \Leftrightarrow I(A_k) \in Core_L^P(\mathcal{N}^C)$.
(4) $\mathcal{H} \in Cons_U^P(\mathcal{A}) \Leftrightarrow I(\mathcal{H}) \in Cons_U^P(\mathcal{N}^C)$.
(5) $\mathcal{H} \in Red_U^P(\mathcal{A}) \Leftrightarrow I(\mathcal{H}) \in Red_U^P(\mathcal{N}^C)$.
(6) $A_k \in Core_U^P(\mathcal{A}) \Leftrightarrow I(A_k) \in Core_U^P(\mathcal{N}^C)$.

Proof. (1) Due to Definitions 8 and 18, and Proposition 6,

$$\begin{aligned}\mathcal{H} \in Cons_L^P(\mathcal{A}) &\Leftrightarrow \sum_{\mathcal{A}} A_k^P([y]_d) = \sum_{\mathcal{H}} A_k^P([y]_d) \text{ for all } y \in U\\ &\Leftrightarrow \overline{\sum_{\mathcal{N}^C} N_k^P(N_d(y))} = \overline{\sum_{I(\mathcal{H})} N_k^P(N_d(y))} \text{ for all } y \in U\\ &\Leftrightarrow \overline{I(\mathcal{H})} \in Cons_L^P(\overline{\mathcal{N}^C}).\end{aligned}$$

(2) and (3). According to (1), Definitions 8 and 18, the conclusions are obtained.

(4)–(6). Similar to the proof of (1)–(3), the conclusions can be obtained by Definitions 8 and 18, and Proposition 6. □

To characterize the knowledge reduction of DMSs, discernibility matrices are designed by Tan el al. [21]. For any $\mathcal{H} \subseteq \mathcal{A}$, define the decision function by $f_{\mathcal{H}}(x_i) = \{d(x_j)|x_j \in N_{\mathcal{H}}(x_i)\}$. For each $x \in U$, define

$$P(x) = \begin{cases} \{A_k \in \mathcal{A}||f_{\{A_k\}}(x)| > 1\}, & |f_{\mathcal{A}}(x)| > 1,\\ \mathcal{A}, & |f_{\mathcal{A}}(x)| = 1.\end{cases}$$

$\mathcal{P} = \{P(x)|x \in U\}$ is called a CPL-discernibility matrix. For any $(x, y) \in U \times U$, define

$$Q(x, y) = \begin{cases} \{A_k \in \mathcal{A}|d(x) \in f_{\{A_k\}}(y)\}, & |f_{\mathcal{A}}(y)| > 1,\\ \mathcal{A}, & |f_{\mathcal{A}}(y)| = 1.\end{cases}$$

$\mathcal{Q} = \{Q(x, y)|x \in U\}$ is called a CPU-discernibility matrix.

Due to Definitions 9 and 11, and Theorems 1 and 3, we obtain

Corollary 1. *[21] For any $\mathcal{H} \subseteq \mathcal{A}$, $A_k \in \mathcal{A}$,*

(1) $\mathcal{H} \in Cons_L^P(\mathcal{A}) \Leftrightarrow \mathcal{H} \cap P(x) \neq \emptyset$ for all $x \in U$.

(2) $\mathcal{H} \in Red_L^P(\mathcal{A}) \Leftrightarrow \mathcal{H} \cap P(x) \neq \emptyset$ for all $x \in U$, and for every $\mathcal{H}_0 \subset \mathcal{H}$, there exists an $x \in U$ such that $P(x) \cap \mathcal{H}_0 = \emptyset$.

(3) $A_k \in Core_L^P(\mathcal{A}) \Leftrightarrow \exists x \in U,\ P(x) = \{A_k\}$.

(4) $\mathcal{H} \in Cons_U^P(\mathcal{A}) \Leftrightarrow \mathcal{H} \cap Q(x, y) \neq \emptyset$ for all $Q(x, y) \neq \emptyset$.

(5)$\mathcal{H} \in Red_U^P(\mathcal{A}) \Leftrightarrow \mathcal{H} \cap Q(x, y) \neq \emptyset$ for all $Q(x, y) \neq \emptyset$, and for every $\mathcal{H}_0 \subset \mathcal{H}$, there exists a $Q(x, y) \in \mathcal{Q}$ such that $Q(x, y) \cap \mathcal{H}_0 = \emptyset$.

(6) $A_k \in Core_U^P(\mathcal{A}) \Leftrightarrow$ there exist some $(x, y) \in U \times U$ such that $Q(x, y) = \{A_k\}$.

Proof. (1)–(3). According to Theorem 9, $\mathcal{H}$ is a CPL-consistent set (CPL-reduct) of $(U, \mathcal{A}, d) \Leftrightarrow I(\mathcal{H})$ is a GNPL-consistent set (GNPL-reduct) of $(U, \mathcal{N}^C, N_d)$. By Property 4 in [21], for any $x \in U$, $\mathcal{H} \subseteq \mathcal{A}$, $|f_{\mathcal{H}}(x)| > 1$ if $N_{\mathcal{H}}(x) \nsubseteq [x]_d$. Then, by Definition 9, $I(P(x)) = GD_L^P(x, N_d(x))$ for all $x \in U$.

According to Theorem 1, $\mathcal{H} \in Cons_L^P(\mathcal{A}) \Leftrightarrow I(\mathcal{H}) \in Cons_L^P(\mathcal{N}^C) \Leftrightarrow I(\mathcal{H}) \cap GD_L^P(x, N_d(x)) \neq \emptyset$ for each $x \in U \Leftrightarrow \mathcal{H} \cap P(x) \neq \emptyset$ for each $x \in U$. Then, we get (1). We can obtain (2) and (3) similarly.

(4)–(6). By Property 4 in [21], for any $x \in U$, $|f_{\mathcal{A}}(x)| > 1 \Leftrightarrow N_{\mathcal{A}}(x) \nsubseteq [x]_d \Leftrightarrow GN_{\mathcal{N}^C}(x) \nsubseteq N_d(x)$.

For any $x, y \in U$, $d(y) \in f_{\{A_k\}}(x) = \{d(z)|z \in [x]_{A_k}\}$

$$\begin{aligned}&\Leftrightarrow \exists z \in [x]_{A_k}, d(y) = d(z)\\ &\Leftrightarrow \exists z \in [x]_{A_k}, [y]_d = [z]_d\\ &\Leftrightarrow \exists z \in U, z \in [x]_{A_k} \cap [y]_d\\ &\Leftrightarrow [x]_{A_k} \cap [y]_d \neq \emptyset,\end{aligned}$$

then $\{A_k \in \mathcal{A}|d(y) \in f_{\{A_k\}}(x)\} = \{A_k \in \mathcal{A}|[x]_{A_k} \cap [y]_d \neq \emptyset\}$. Hence,

$$Q(y, x) = \begin{cases} \{A_k \in \mathcal{A}|[x]_{A_k} \cap [y]_d \neq \emptyset\}, & N_{\mathcal{A}}(x) \nsubseteq [x]_d,\\ \mathcal{A}, & N_{\mathcal{A}}(x) \subseteq [x]_d.\end{cases}$$

For any $x, y \in U$, there are four cases. (a) $GN_{\mathcal{N}^C}(x) \nsubseteq N_d(x)$ and $GN_{\mathcal{N}^C}(x) \cap N_d(y) \neq \emptyset$, that is, $N_{\mathcal{A}}(x) \nsubseteq [x]_d$ and $N_{\mathcal{A}}(x) \cap [y]_d \neq \emptyset$. By Definition 11, $I(Q(y,x)) = GD_U^P(x, N_d(y))$. (b) $GN_{\mathcal{N}^C}(x) \nsubseteq N_d(x)$ and $GN_{\mathcal{N}^C}(x) \cap N_d(y) = \emptyset$, namely, $N_{\mathcal{A}}(x) \nsubseteq [x]_d$ and $N_{\mathcal{A}}(x) \cap [y]_d = \emptyset$. From Definition 11, $I(Q(y,x)) = \emptyset = GD_U^P(x, N_d(y))$. (c) $GN_{\mathcal{N}^C}(x) \subseteq N_d(x)$ and $GN_{\mathcal{N}^C}(x) \cap N_d(y) \neq \emptyset$, that is, $N_{\mathcal{A}}(x) \subseteq [x]_d$ and $N_{\mathcal{A}}(x) \cap [y]_d \neq \emptyset$. Then $N_d(x) \cap N_d(y) \neq \emptyset$, which follows that $N_d(x) = N_d(y)$. According to Definition 11, $I(Q(y,x)) = I(\mathcal{A}) = \mathcal{N}^C = GD_U^P(x, N_d(x)) = GD_U^P(x, N_d(y))$. (d) $GN_{\mathcal{N}^C}(x) \subseteq N_d(x)$ and $GN_{\mathcal{N}^C}(x) \cap N_d(y) = \emptyset$, i.e., $N_{\mathcal{A}}(x) \subseteq [x]_d$ and $N_{\mathcal{A}}(x) \cap [y]_d = \emptyset$. By Definition 11, $I(Q(y,x)) = I(\mathcal{A}) = \mathcal{N}^C$ and $GD_U^P(x, N_d(y)) = \emptyset$. In conclusion, $\{I(Q(y,x)) | x, y \in U\} = \{GD_U^P(x, N_d(y)) | x, y \in U\}$.

Due to Theorem 3, $\mathcal{H} \in Cons_U^P(\mathcal{A}) \Leftrightarrow I(\mathcal{H}) \in Cons_U^P(\mathcal{N}^C) \Leftrightarrow I(\mathcal{H}) \cap GD_U^P(x, N_d(y)) \neq \emptyset$ for each $GD_U^P(x, N_d(y)) \neq \emptyset \Leftrightarrow I(\mathcal{H}) \cap I(Q(x,y)) \neq \emptyset$ for each $I(Q(x,y)) \neq \emptyset \Leftrightarrow \mathcal{H} \cap Q(x,y) \neq \emptyset$ for each $Q(x,y) \neq \emptyset$. Hence, (4) is found, and (5) and (6) are also obtained by Theorem 3. □

Remark 4. *Let $(U, \mathcal{A}, d)$ be a DMS with $\mathcal{A} = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$, which induces a GNDIS $(U, \mathcal{N}^C, N_d)$ with $\mathcal{N}^C = \{N_1, N_2, \cdots, N_m\}$. From the proof of Corollary 1, $\{I(Q(y,x)) | x, y \in U\} = \{GD_U^P(x, N_d(y)) | x \in U, N_d(y) \in C_d\}$. However, the matrix $\{GD_U^P(x, N_d(y)) | x \in U, N_d(y) \in C_d\}$ merges the same elements and has more empty sets in comparison with $\{I(Q(y,x)) | x, y \in U\}$.*

4.2. Optimistic multi-granulation reduction of DMSs

Optimistic multi-granulation reduction of DMSs is also discussed in [14, 18, 21].

Definition 19. *[14, 18, 21] Assume that $(U, \mathcal{A}, d)$ is a DMS, where $\mathcal{A} = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$. Let $\mathcal{H} \subseteq \mathcal{A}$ and $\mathcal{H} \neq \emptyset$.*

(1) $\mathcal{H}$ is called a complete optimistic lower consistent set (COL-consistent set) if $\underline{\sum_{\mathcal{A}} A_k^O}([y]_d) = \underline{\sum_{\mathcal{H}} A_k^O}([y]_d)$ for all $y \in U$. Denote the family of all COL-consistent sets by $Cons_L^O(\mathcal{A})$. Moreover, if $\mathcal{H} \in Cons_L^O(\mathcal{A})$, and $\mathcal{H}' \notin Cons_L^O(\mathcal{A})$ whenever $\mathcal{H}' \subset \mathcal{H}$, then $\mathcal{H}$ is a COL-reduct of $(U, \mathcal{A}, d)$. Denote the family of all COL-reducts of $(U, \mathcal{A}, d)$ by $Red_L^O(\mathcal{A})$, and $Core_L^O(\mathcal{A}) = \bigcap_{\mathcal{H} \in Red_L^O(\mathcal{A})} \mathcal{H}$ is called a COL-core.

(2) $\mathcal{H}$ is called a complete optimistic upper consistent set (COU-consistent set) if $\overline{\sum_{\mathcal{A}} A_k^O}([y]_d) = \overline{\sum_{\mathcal{H}} A_k^O}([y]_d)$ for all $y \in U$. Denote the family of all COU-consistent sets by $Cons_U^O(\mathcal{A})$. Moreover, if $\mathcal{H} \in Cons_U^O(\mathcal{A})$, and $\mathcal{H}' \notin Cons_U^O(\mathcal{A})$ whenever $\mathcal{H}' \subset \mathcal{H}$, then $\mathcal{H}$ is a COU-reduct of $(U, \mathcal{A}, d)$. Denote the family of all COU-reducts of $(U, \mathcal{A}, d)$ by $Red_U^O(\mathcal{A})$, and $Core_U^O(\mathcal{A}) = \bigcap_{\mathcal{H} \in Red_U^O(\mathcal{A})} \mathcal{H}$ is called a COU-core.

The optimistic multi-granulation reduction of $(U, \mathcal{A}, d)$ is closely associated with the optimistic multi-granulation reduction of the GNDIS $(U, \mathcal{N}^C, N_d)$ induced by $(U, \mathcal{A}, d)$.

Theorem 10. *Let $(U, \mathcal{A}, d)$ be a DMS with $\mathcal{A} = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$, which induces the GNDIS $(U, \mathcal{N}^C, N_d)$ with $\mathcal{N}^C = \{N_1, N_2, \cdots, N_m\}$. Then, for $\mathcal{H} \subseteq \mathcal{A}$, $A_k \in \mathcal{A}$,*
(1) $\mathcal{H} \in Cons_L^O(\mathcal{A}) \Leftrightarrow I(\mathcal{H}) \in Cons_L^O(\mathcal{N}^C)$.
(2) $\mathcal{H} \in Red_L^O(\mathcal{A}) \Leftrightarrow I(\mathcal{H}) \in Red_L^O(\mathcal{N}^C)$.
(3) $A_k \in Core_L^O(\mathcal{A}) \Leftrightarrow I(A_k) \in Core_L^O(\mathcal{N}^C)$.
(4) $\mathcal{H} \in Cons_U^O(\mathcal{A}) \Leftrightarrow I(\mathcal{H}) \in Cons_U^O(\mathcal{N}^C)$.

(5) $\mathcal{H} \in Red_U^O(\mathcal{A}) \Leftrightarrow I(\mathcal{H}) \in Red_U^O(\mathcal{N}^C)$.
(6) $A_k \in Core_U^O(\mathcal{A}) \Leftrightarrow I(A_k) \in Core_U^O(\mathcal{N}^C)$.

Proof. (1) By Definitions 13 and 19, and Proposition 6,

$$\begin{aligned}\mathcal{H} \in Cons_L^O(\mathcal{A}) &\Leftrightarrow \underline{\sum_{\mathcal{A}} A_k^O}([y]_d) = \underline{\sum_{\mathcal{H}} A_k^O}([y]_d) \text{ for all } y \in U \\ &\Leftrightarrow \underline{\sum_{\mathcal{N}^C} N_k^O}(N_d(y)) = \underline{\sum_{I(\mathcal{H})} N_k^O}(N_d(y)) \text{ for all } y \in U \\ &\Leftrightarrow I(\mathcal{H}) \in Cons_L^O(\mathcal{N}^C).\end{aligned}$$

(2) and (3). Due to (1) and Definitions 13 and 19, the conclusions are obtained.

(4)–(6). With reference to the proof of (1)–(3), the conclusions can be proved by Definitions 13 and 19, and Proposition 6. □

In [21], Tan et al. also presented a discenibility matrix to get COU-reducts. However, the optimistic lower reduction was not characterized by discenibility matrices in [21]. We present a discernibility matrix to compute COL-reducts.

Definition 20. *Let $(U, \mathcal{A}, d)$ be a DMS. For each $x \in U$, define*

$$MD(x) = \begin{cases} \{A_k \in \mathcal{A} \mid |f_{\{A_k\}}(x)| = 1\}, & x \in \underline{\sum_{\mathcal{A}} A_k^O}([x]_d), \\ \mathcal{A}, & else. \end{cases}$$

$\mathcal{MD} = \{MD(x) | x \in U\}$ is called a COL-discernibility matrix.

For any $x, y \in U$, define $G(x, y) = \{A_k \in \mathcal{A} | d(y) \notin f_{\{A_k\}}(x)\}$ [21]. $\mathcal{G} = \{G(x, y)|(x, y) \in U \times U\}$ is called a COU-discernibility matrix.

Remark 5. *From the proof of Corollary 1, $d(y) \notin f_{\{A_k\}}(x) \Leftrightarrow [x]_{A_k} \cap [y]_d = \emptyset$. If $\overline{\sum_{\mathcal{A}} A_k^O}([y]_d) = U$ for all $y \in U$, then $[x]_{A_k} \cap [y]_d \neq \emptyset$ for all $x \in U$ and $A_k \in \mathcal{A}$. It follows that $G(x, y) = \emptyset$ for all $x, y \in U$. Hence, we cannot get the COU-reducts from $\mathcal{G}$. Then, $G(x, y)$ is defined by*

$$G(x, y) = \begin{cases} \{A_k \in \mathcal{A} | d(y) \notin f_{\{A_k\}}(x)\}, & x \notin \overline{\sum_{\mathcal{A}} A_k^O}([y]_d), \\ \mathcal{A}, & else. \end{cases}$$

Corollary 2. *For any $\mathcal{H} \subseteq \mathcal{A}$, $A_k \in \mathcal{A}$,*

(1) $\mathcal{H} \in Cons_L^O(\mathcal{A}) \Leftrightarrow \mathcal{H} \cap MD(x) \neq \emptyset$ *for all* $MD(x) \in \mathcal{MD}$.

(2) $\mathcal{H} \in Red_L^O(\mathcal{A}) \Leftrightarrow \mathcal{H} \cap MD(x) \neq \emptyset$ *for all* $MD(x) \in \mathcal{MD}$, *and for every* $\mathcal{H}_0 \subset \mathcal{H}$, *there exists an* $MD(x) \in \mathcal{MD}$ *such that* $MD(x) \cap \mathcal{H}_0 = \emptyset$.

(3) $A_k \in Core_L^O(\mathcal{A}) \Leftrightarrow \exists x \in U$, $MD(x) = \{A_k\}$.

(4) $\mathcal{H} \in Cons_U^O(\mathcal{A}) \Leftrightarrow \mathcal{H} \cap G(x, y) \neq \emptyset$ *for all* $G(x, y) \in \mathcal{G}$ *[21].*

(5) $\mathcal{H} \in Red_U^O(\mathcal{A}) \Leftrightarrow \mathcal{H} \cap G(x, y) \neq \emptyset$ *for all* $G(x, y) \in \mathcal{G}$, *and for every* $\mathcal{H}_0 \subset \mathcal{H}$, *there exists a* $G(x, y) \in \mathcal{G}$ *such that* $G(x, y) \cap \mathcal{H}_0 = \emptyset$ *[21].*

(6) $A_k \in Core_U^O(\mathcal{A}) \Leftrightarrow$ *there exists a* $(x, y) \in U \times U$ *such that* $G(x, y) = \{A_k\}$ *[21].*

Proof. (1)–(3). By Theorem 10, $\mathcal{H}$ is a COL-consistent set (or COL-reduct) of $(U, \mathcal{A}, d) \Leftrightarrow I(\mathcal{H})$ is a GNOL-consistent set (or GNOL-reduct) of $(U, \mathcal{N}^C, N_d)$. For any $x \in U, A_k \in \mathcal{A}$, $|f_{\{A_k\}}(x)| = 1$ if $N_{\{A_k\}}(x) \subseteq [x]_d$. It follows from Definitions 14 and 20 that $I(MD(x)) = GD_L^O(x, N_d(x))$ for all $x \in U$.

Due to Theorem 5, $\mathcal{H} \in Cons_L^O(\mathcal{A}) \Leftrightarrow I(\mathcal{H}) \in Cons_L^O(\mathcal{N}^C) \Leftrightarrow I(\mathcal{H}) \cap GD_L^O(x, N_d(x)) \neq \emptyset$ for every $x \in U \Leftrightarrow \mathcal{H} \cap MD(x) \neq \emptyset$ for every $MD(x) \in \mathcal{MD}$. Hence (1) is obtained. (2) and (3) can be deduced from Theorem 5 analogously.

(4)–(6). According to Theorem 10, $\mathcal{H}$ is a COU-consistent set (or COU-reduct) of $(U, \mathcal{A}, d) \Leftrightarrow I(\mathcal{H})$ is a GNOU-consistent set (or GNOU-reduct) of $(U, \mathcal{N}^C, N_d)$.

For any $x, y \in U$, if $x \notin \overline{\sum_{\mathcal{A}} A_k^O}([y]_d) = \overline{\sum_{\mathcal{N}^C} N_k^O}(N_d(y))$, then

$$\begin{aligned} I(G(x,y)) &= I(\{A_k \in \mathcal{A} | d(y) \notin f_{\{A_k\}}(x)\}) \\ &= I(\{A_k \in \mathcal{A} | [y]_d \cap [x]_{A_k} = \emptyset\}) \\ &= \{N_k \in \mathcal{N}^C | N_d(y) \cap N_k(x) = \emptyset\} \\ &= GD_U^O(x, N_d(y)). \end{aligned}$$

If $x \in \overline{\sum_{\mathcal{A}} A_k^O}([y]_d) = \overline{\sum_{\mathcal{N}^C} N_k^O}(N_d(y))$, $I(G(x,y)) = \mathcal{N}^C = GD_U^O(x, N_d(y))$. We can conclude that $I(G(x,y)) = GD_U^O(x, N_d(y))$ for all $x, y \in U$.

By Theorem 7, $\mathcal{H} \in Cons_U^O(\mathcal{A}) \Leftrightarrow I(\mathcal{H}) \in Cons_U^O(\mathcal{N}^C) \Leftrightarrow I(\mathcal{H}) \cap GD_U^O(x, N_d(y)) \neq \emptyset$ for all $x, y \in U$ $\Leftrightarrow I(\mathcal{H}) \cap I(G(x,y)) \neq \emptyset$ for each $G(x,y) \in \mathcal{G} \Leftrightarrow \mathcal{H} \cap G(x,y) \neq \emptyset$ for each $G(x,y) \in \mathcal{G}$. Hence, (4) is proved. (5) and (6) are also proved by Theorem 7. □

We can see that a DMS is a GNDIS. Furthermore, due to Definition 10 and Theorem 2, a CPL-reduct can be obtained by a prime implicant of $f(\mathcal{P})$. According to Definition 12 and Theorem 4, a CPU-reduct can be obtained by a prime implicant of $f(Q)$. By Definition 15 and Theorem 6, the COL-reducts can be found from the prime implicants of $f(\mathcal{MD})$. By Definition 17 and Theorem 8, the COU-reducts can be found from the prime implicants of $f(\mathcal{G})$.

Example 7. *A DMS $(U, \mathcal{A}, d)$ is present in Table 3, where $\mathcal{A} = \{A_1 = \{a_1, a_2, a_3\}, A_2 = \{a_4, a_5\}, A_3 = \{a_6, a_7\}, A_4 = \{a_8, a_9, a_{10}\}\}$. The granulars $[x_i]_{A_k}, [x_i]_d$, and $N_{\mathcal{A}}(x_i)$ $(i = 1, \cdots, 6; k = 1, \cdots, 4)$ are presented in Table 4. We obtain*

$$Q = \left(\begin{array}{c|cccccc} Q(x_i, x_j) & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline x_1 & \mathcal{A} & \{A_1, A_2, A_3\} & \{A_3\} & \mathcal{A} & \mathcal{A} & \{A_3, A_4\} \\ x_2 & \{A_3\} & \mathcal{A} & \mathcal{A} & \{A_3\} & \{A_1, A_2, A_3\} & \{A_1, A_2, A_3\} \\ x_3 & \{A_3\} & \mathcal{A} & \mathcal{A} & \{A_3\} & \{A_1, A_2, A_3\} & \{A_1, A_2, A_3\} \\ x_4 & \mathcal{A} & \{A_1, A_2, A_3\} & \{A_3\} & \mathcal{A} & \mathcal{A} & \{A_3, A_4\} \\ x_5 & \mathcal{A} & \{A_1, A_2, A_3\} & \{A_3\} & \mathcal{A} & \mathcal{A} & \{A_3, A_4\} \\ x_6 & \{A_3\} & \emptyset & \{A_1, A_2, A_3\} & \emptyset & \{A_4\} & \mathcal{A} \end{array} \right).$$

Table 3. A DMS

	A_1			A_2		A_3		A_4			d
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	
x_1	2	2	1	2	1	3	1	0	2	1	1
x_2	1	2	2	1	1	1	2	0	3	2	2
x_3	2	0	3	1	2	3	1	1	3	2	2
x_4	2	1	1	2	1	1	2	0	2	1	1
x_5	1	2	2	1	1	1	2	1	2	1	1
x_6	2	0	3	1	2	3	1	1	2	1	3

Table 4. The granulars of elements in Example 6.

*	x_1	x_2	x_3	x_4	x_5	x_6
$[x_i]_{A_1}$	$\{x_1\}$	$\{x_2, x_5\}$	$\{x_3, x_6\}$	$\{x_4\}$	$\{x_2, x_5\}$	$\{x_3, x_6\}$
$[x_i]_{A_2}$	$\{x_1, x_4\}$	$\{x_2, x_5\}$	$\{x_3, x_6\}$	$\{x_1, x_4\}$	$\{x_2, x_5\}$	$\{x_3, x_6\}$
$[x_i]_{A_3}$	$\{x_1, x_3, x_6\}$	$\{x_2, x_4, x_5\}$	$\{x_1, x_3, x_6\}$	$\{x_2, x_4, x_5\}$	$\{x_2, x_4, x_5\}$	$\{x_1, x_3, x_6\}$
$[x_i]_{A_4}$	$\{x_1, x_4\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_4\}$	$\{x_5, x_6\}$	$\{x_5, x_6\}$
$[x_i]_d$	$\{x_1, x_4, x_5\}$	$\{x_2, x_3\}$	$\{x_2, x_3\}$	$\{x_1, x_4, x_5\}$	$\{x_1, x_4, x_5\}$	$\{x_6\}$
$N_{\mathcal{A}}(x_i)$	$\{x_1, x_3, x_4, x_6\}$	$\{x_2, x_4, x_5\}$	$\{x_1, x_3, x_6\}$	$\{x_1, x_2, x_4, x_5\}$	$\{x_2, x_4, x_5, x_6\}$	$\{x_1, x_3, x_5, x_6\}$

By Corollary 1, $Core_U^P(\mathcal{A}) = \{A_3, A_4\}$, and $\mathcal{A}_0 = \{A_3, A_4\}$ is the one and only one CPU-reduct.

The DMS $(U, \mathcal{A}, d)$ induces a GNDIS $(U, \mathcal{N}^C, N_d)$ with $\mathcal{N}^C = \{N_1, N_2, N_3, N_4\}$, where $N_i(x) = [x]_{A_i} (i = 1, 2, 3, 4)$ and $N_d(x) = [x]_d$ for all $x \in U$. By Definition 11, the GNPU-D matrix of $(U, \mathcal{N}^C, N_d)$ is

$$\mathcal{GD}_U^P = \left(\begin{array}{c|ccc} GD_U^P(x_i, N_d(x_j)) & N_d(x_1) = N_d(x_4) = N_d(x_5) & N_d(x_2) = N_d(x_3) & N_d(x_6) \\ \hline x_1 & \mathcal{N}^C & \{N_3\} & \{N_3\} \\ x_2 & \{N_1, N_2, N_3\} & \mathcal{N}^C & \emptyset \\ x_3 & \{N_3\} & \mathcal{N}^C & \{N_1, N_2, N_3\} \\ x_4 & \mathcal{N}^C & \{N_3\} & \emptyset \\ x_5 & \mathcal{N}^C & \{N_1, N_2, N_3\} & \{N_4\} \\ x_6 & \{N_3, N_4\} & \{N_1, N_2, N_3\} & \mathcal{N}^C \end{array} \right).$$

By Theorem 1, the GNPU-reduct of $(U, \mathcal{N}^C, N_d)$ is $\mathcal{N}_0 = \{N_3, N_4\}$, and $Core_L^P(\mathcal{N}^C) = \{N_3, N_4\}$. It is easy to get that $I(Q(x_j, x_i)) = GD_U^P(x_i, N_d(x_j))$. Moreover, $I(\mathcal{A}_0) = \mathcal{N}_0$ and $I(Core_U^P(\mathcal{A})) = Core_U^P(\mathcal{N}^C)$.

By Definition 2, $\underline{\sum_{\mathcal{A}} A_k^O}(\{x_1, x_4, x_5\}) = \{x_1, x_4\}$, $\underline{\sum_{\mathcal{A}} A_k^O}(\{x_2, x_3\}) = \{x_2, x_3\}$, $\underline{\sum_{\mathcal{A}} A_k^O}(\{x_6\}) = \emptyset$. According to Definition 20, we get

$$\mathcal{MD} = \left(\begin{array}{c|c} x_i & MD(x_i) \\ \hline x_1 & \{A_1, A_2, A_4\} \\ x_2 & \{A_4\} \\ x_3 & \{A_4\} \\ x_4 & \{A_1, A_2, A_4\} \\ x_5 & \emptyset \\ x_6 & \emptyset \end{array} \right).$$

Due to Corollary 2, $Core^O_L(\mathcal{A}) = \{A_4\}$*, and* $\{A_4\}$ *is the only COL-reduct of* $(U, \mathcal{A}, d)$*.*

5. Relationships between multi-granulation reduction of IDISs and that of GNDISs

In an IDIS (U, A, d), $\mathcal{A}^I = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$, a mapping $N_k : U \to \mathcal{P}(U)$ by $N_k(x) = S_{A_k}(x)$ for all $x \in U$ $(k = 1, 2, \cdots, m)$ and a mapping $N_d : U \to \mathcal{P}(U)$ by $N_d(y) = [y]_d$ for all $y \in U$. Then, $N_k (k = 1, 2, \cdots, m)$ is a reflexive neighborhood operator on U and N_d is a Pawlak neighborhood operator, and we get a GNDIS $(U, \mathcal{N}^I, N_d)$ with $\mathcal{N}^I = \{N_1, N_2, \cdots, N_m\}$, which is a GNDIS induced by the IDIS (U, A, d). Define a mapping $I : \mathcal{A}^I \to \mathcal{N}^I$ by $I(A_k) = N_k$ for all $A_k \in \mathcal{A}^I$. From the definition of $\mathcal{N}^I$, it is easy to get that I is a bijection.

Proposition 7. *Suppose that* (U, A, d) *is an IDIS and* $\mathcal{A}^I = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$*, which induces the GNDIS* $(U, \mathcal{N}^I, N_d)$ *with* $\mathcal{N}^I = \{N_1, N_2, \cdots, N_m\}$*. For each* $X \subseteq U$*,*

$$\underline{\sum_{\mathcal{A}^I} A^P_k}(X) = \underline{\sum_{\mathcal{N}^I} N^P_k}(X),\ \overline{\sum_{\mathcal{A}^I} A^P_k}(X) = \overline{\sum_{\mathcal{N}^I} N^P_k}(X),$$

$$\underline{\sum_{\mathcal{A}^I} A^O_k}(X) = \underline{\sum_{\mathcal{N}^I} N^O_k}(X),\ \overline{\sum_{\mathcal{A}^I} A^O_k}(X) = \overline{\sum_{\mathcal{N}^I} N^O_k}(X).$$

Proof. It is directly by Definitions 3, 6 and 7. □

By Proposition 7, the generalized neighborhood multi-granulation rough set models are extension models of the multi-granulation rough set models proposed in [15].

5.1. Pessimistic multi-granulation reduction of IDISs

Pessimistic multi-granulation reduction of an IDIS was discussed by Qian et. al. [14, 18, 32].

Definition 21. *[14, 18, 32] Given an IDIS* (U, A, d)*, let* $\mathcal{A}^I = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$ *and* $\mathcal{H} \subseteq \mathcal{A}^I$*.*

(1) $\mathcal{A}^I$ *is called an incomplete pessimistic lower consistent set (IPL-consistent set) if* $\underline{\sum_{\mathcal{A}^I} A^P_k}([y]_d) = \underline{\sum_{\mathcal{H}} A^P_k}([y]_d)$ *for all* $y \in U$*. Denote the family of all IPL-consistent sets as* $Cons^P_L(\mathcal{A}^I)$*. Moreover, if* $\mathcal{H} \in Cons^P_L(\mathcal{A}^I)$*, and* $\mathcal{H}' \notin Cons^P_L(\mathcal{A}^I)$ *whenever* $\mathcal{H}' \subset \mathcal{H}$*, then* $\mathcal{H}$ *is an IPL-reduct. Denote the family of all IPL-reducts of* (U, A, d) *by* $Red^P_L(\mathcal{A}^I)$*, and* $Core^P_L(\mathcal{A}^I) = \bigcap_{\mathcal{H} \in Red^P_L(\mathcal{A}^I)} \mathcal{H}$ *is said to be an IPL-core.*

(2) $\mathcal{A}^I$ *is called an incomplete pessimistic upper consistent set (IPU-consistent set) if* $\overline{\sum_{\mathcal{A}^I} A^P_k}([y]_d) = \overline{\sum_{\mathcal{H}} A^P_k}([y]_d)$ *for all* $y \in U$*. Denote the family of all IPU-consistent sets by* $Cons^P_U(\mathcal{A}^I)$*. Moreover, if* $\mathcal{H} \in Cons^P_U(\mathcal{A}^I)$*, and* $\mathcal{H}' \notin Cons^P_U(\mathcal{A}^I)$ *whenever* $\mathcal{H}' \subset \mathcal{H}$*, then* $\mathcal{H}$ *is an IPU-reduct. Denote the family of all IPU-reducts of* (U, A, d) *by* $Red^P_U(\mathcal{A}^I)$*, and* $Core^P_U(\mathcal{A}^I) = \bigcap_{\mathcal{H} \in Red^P_U(\mathcal{A}^I)} \mathcal{H}$ *is said to be an IPU-core.*

The multi-granulation reduction of an IDIS (U, A, d) can be changed into the multi-granulation reduction of the GNDIS $(U, \mathcal{N}^I, N_d)$ induced by (U, A, d).

Theorem 11. *Consider an IDIS* (U, A, d) *with* $\mathcal{A}^I = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$*, which induces a GNDIS* $(U, \mathcal{N}^I, N_d)$ *with* $\mathcal{N}^I = \{N_1, N_2, \cdots, N_m\}$*. Then, for* $\mathcal{H} \subseteq \mathcal{A}$*,* $A_k \in \mathcal{A}$*,*
(1) $\mathcal{H} \in Cons^P_L(\mathcal{A}^I) \Leftrightarrow I(\mathcal{H}) \in Cons^P_L(\mathcal{N}^I)$*.*

(2) $\mathcal{H} \in Red_L^P(\mathcal{A}^I) \Leftrightarrow I(\mathcal{H}) \in Red_L^P(\mathcal{N}^I)$.
(3) $A_k \in Core_L^P(\mathcal{A}^I) \Leftrightarrow I(A_k) \in Core_L^P(\mathcal{N}^I)$.
(4) $\mathcal{H} \in Cons_U^P(\mathcal{A}^I) \Leftrightarrow I(\mathcal{H}) \in Cons_U^P(\mathcal{N}^I)$.
(5) $\mathcal{H} \in Red_U^P(\mathcal{A}^I) \Leftrightarrow I(\mathcal{H}) \in Red_U^P(\mathcal{N}^I)$.
(6) $A_k \in Core_U^P(\mathcal{A}^I) \Leftrightarrow I(A_k) \in Core_U^P(\mathcal{N}^I)$.

Proof. (1) Due to Definitions 8 and 21, and Proposition 7,

$$\begin{aligned}\mathcal{H} \in Cons_L^P(\mathcal{A}^I) &\Leftrightarrow \underline{\sum_{\mathcal{A}^I} A_k^P}([y]_d) = \underline{\sum_{\mathcal{H}} A_k^P}([y]_d) \text{ for all } y \in U \\ &\Leftrightarrow \underline{\sum_{\mathcal{N}^I} N_k^P}(N_d(y)) = \underline{\sum_{I(\mathcal{H})} N_k^P}(N_d(y)) \text{ for all } y \in U \\ &\Leftrightarrow I(\mathcal{H}) \in Cons_L^P(\mathcal{N}^I).\end{aligned}$$

(2) and (3). By (1) and Definitions 8 and 21, the conclusions are proved.

(4)–(6). Similar to the proof of (1)–(3), the conclusions can be obtained by Definitions 13 and 19, and Proposition 6. □

Discernibility matrices were defined by Zhang et al. [32] to compute the IPL-reducts and IPU-reducts of an IDIS. For any $\mathcal{H} \subseteq \mathcal{A}^I$, define the decision function by $h_{\mathcal{H}}(x_i) = \{d(x_j)|x_j \in IN_{\mathcal{H}}(x_i)\}$. For any $x \in U$, define

$$IP(x) = \begin{cases} \{A_k \in \mathcal{A}^I \| h_{\{A_k\}}(x)| > 1\}, & |h_{\mathcal{A}^I}(x)| > 1, \\ \emptyset, & |h_{\mathcal{A}^I}(x)| = 1. \end{cases}$$

$\mathcal{IP} = \{IP(x)|x \in U\}$ is called an IPL-discernibility matrix. For $(x, y) \in U \times U$, define

$$IQ(x,y) = \begin{cases} \{A_k \in \mathcal{A}^I | d(y) \in h_{\{A_k\}}(x)\}, & d(y) \in h_{\mathcal{A}^I}(x), \\ \emptyset, & d(y) \notin h_{\mathcal{A}^I}(x), \end{cases}$$

$\mathcal{IQ} = \{IQ(x,y)|(x,y) \in U \times U\}$ is called an IPU-discernibility matrix.

Remark 6. *If* $|h_{\mathcal{A}^I}(x)| = 1$ *for each* $x \in U$ *in an IDIS* (U, A, d) *with* $\mathcal{A}^I = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$, *then each* $A_k \in \mathcal{A}^I$ *is an IPL-reduct. However,* $IP(x) = \emptyset$ *for all* $x \in U$. *In the following,* $IP(x)$ *is defined by*

$$IP(x) = \begin{cases} \{A_k \in \mathcal{A}^I \| h_{\{A_k\}}(x)| > 1\}, & |h_{\mathcal{A}^I}(x)| > 1, \\ \mathcal{A}^I, & |h_{\mathcal{A}^I}(x)| = 1. \end{cases}$$

By Definition 9 and Theorem 1, as well as Definition 11 and Theorem 3, we obtain

Corollary 3. *[32] Assume that* (U, A, d) *is an IDIS and* $\mathcal{A}^I = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$. *For any* $\mathcal{H} \subseteq \mathcal{A}^I$, $A_k \in \mathcal{A}^I$,

(1) $\mathcal{H} \in Cons_L^P(\mathcal{A}^I) \Leftrightarrow \mathcal{H} \cap IP(x) \neq \emptyset$ *for all* $IP(x) \in \mathcal{IP}$.

(2) $\mathcal{H} \in Red_L^P(\mathcal{A}^I) \Leftrightarrow \mathcal{H} \cap IP(x) \neq \emptyset$ *for all* $IP(x) \in \mathcal{IP}$, *and for each* $\mathcal{H}_0 \subset \mathcal{H}$, *there exists an* $IP(x) \in \mathcal{IP}$ *such that* $IP(x) \cap \mathcal{H}_0 = \emptyset$.

(3) $A_k \in Core_L^P(\mathcal{A}^I) \Leftrightarrow \exists x \in U,\ IP(x) = \{A_k\}$.

(4) $\mathcal{H} \in Cons_U^P(\mathcal{A}^I) \Leftrightarrow \mathcal{H} \cap IQ(x,y) \neq \emptyset$ *for all* $IQ(x,y) \neq \emptyset$.

(5)$\mathcal{H} \in Red_U^P(\mathcal{A}^I) \Leftrightarrow \mathcal{H} \cap IQ(x,y) \neq \emptyset$ *for all* $IQ(x,y) \neq \emptyset$, *and for each* $\mathcal{H}_0 \subset \mathcal{H}$, *there exists an* $IQ(x,y)$ *such that* $IQ(x,y) \cap \mathcal{H}_0 = \emptyset$.

(6) $A_k \in Core_U^P(\mathcal{A}^I) \Leftrightarrow$ *there exist some* $(x,y) \in U \times U$ *such that* $IQ(x,y) = \{A_k\}$.

Proof. (1)–(3). By Theorem 11, $\mathcal{H}$ is an IPL-consistent set (or an IPL-reduct) of $(U, A, d) \Leftrightarrow I(\mathcal{H})$ is a GNPL-consistent set (or a GNPL-reduct) of $(U, \mathcal{N}^I, N_d)$. According to Property 3 in [32], for any $x \in U, \mathcal{H} \subseteq \mathcal{A}^I$, $|h_{\mathcal{H}}(x)| > 1$ if $IN_{\mathcal{H}}(x) \nsubseteq [x]_d$. Then, $I(IP(x)) = GD^P_L(x, N_d(x))$.

By Theorem 1, $\mathcal{H} \in Cons^P_L(\mathcal{A}^I) \Leftrightarrow I(\mathcal{H}) \in Cons^P_L(\mathcal{N}^I) \Leftrightarrow I(\mathcal{H}) \cap I(IP(x)) \neq \emptyset$ for each $x \in U \Leftrightarrow \mathcal{H} \cap IP(x) \neq \emptyset$ for each $IP(x) \in \mathcal{IP}$.

Similar to the proof of (1), we can get (2) and (3) by Theorem 1.

(4)–(6). According to Theorem 11, $\mathcal{H}$ is an IPU-consistent set (or an IPU-reduct) of $(U, A, d) \Leftrightarrow I(\mathcal{H})$ is a GNPU-consistent set (or a GNPU-reduct) of $(U, \mathcal{N}^I, N_d)$.

For $x, y \in U$, $d(y) \in h_{\{A_k\}}(x) \Leftrightarrow S_{A_k}(x) \cap [y]_d \neq \emptyset$, and $d(y) \in h_{\mathcal{A}^I}(x) \Leftrightarrow IN_{\mathcal{A}^I}(x) \cap [y]_d \neq \emptyset$. Then,

$$IQ(x, y) = \begin{cases} \{A_k \in \mathcal{A}^I | S_{A_k}(x) \cap [y]_d \neq \emptyset\}, & IN_{\mathcal{A}^I}(x) \cap [y]_d \neq \emptyset, \\ \emptyset, & IN_{\mathcal{A}^I}(x) \cap [y]_d = \emptyset. \end{cases}$$

According to Definition 11, $I(IQ(x, y)) = GD^P_U(x, N_d(y))$. By Theorem 3, (4)–(6) hold. □

5.2. Optimistic multi-granulation reduction of IDISs

Optimistic multi-granulation reduction of IDIS was discussed by Qian et. al. [14, 18, 32].

Definition 22. *[14, 18, 32] Given an IDIS (U, A, d), let $\mathcal{A}^I = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$ and $\mathcal{H} \subseteq \mathcal{A}^I$.*

(1) $\mathcal{A}^I$ is called an incomplete optimistic lower consistent set (IOL-consistent set) if $\underline{\sum_{\mathcal{A}^I} A^O_k}([y]_d) = \underline{\sum_{\mathcal{H}} A^O_k}([y]_d)$ for all $y \in U$. Denote the family of all IOL-consistent sets as $Cons^O_L(\mathcal{A}^I)$. Moreover, if $\mathcal{H} \in Cons^O_L(\mathcal{A}^I)$, and $\mathcal{H}' \notin Cons^O_L(\mathcal{A}^I)$ whenever $\mathcal{H}' \subset \mathcal{H}$, then $\mathcal{H}$ is an IOL-reduct. Denote the family of all IOL-reducts of (U, A, d) by $Red^O_L(\mathcal{A}^I)$, and $Core^O_L(\mathcal{A}^I) = \bigcap_{\mathcal{H} \in Red^O_L(\mathcal{A}^I)} \mathcal{H}$ is said to be an IOL-core.

(2) $\mathcal{A}^I$ is called an incomplete optimistic upper consistent set (IOU-consistent set) if $\overline{\sum_{\mathcal{A}^I} A^O_k}([y]_d) = \overline{\sum_{\mathcal{H}} A^O_k}([y]_d)$ for all $x \in U$. Denote the family of all IOU-consistent sets as $Cons^O_U(\mathcal{A}^I)$. Moreover, if $\mathcal{H} \in Cons^O_U(\mathcal{A}^I)$, and $\mathcal{H}' \notin Cons^O_U(\mathcal{A}^I)$ whenever $\mathcal{H}' \subset \mathcal{H}$, then $\mathcal{H}$ is an IOU-reduct. Denote the family of all IOU-reducts of (U, A, d) by $Red^O_U(\mathcal{A}^I)$, and $Core^O_U(\mathcal{A}^I) = \bigcap_{\mathcal{H} \in Red^O_U(\mathcal{A}^I)} \mathcal{H}$ is said to be an IOU-core.

The multi-granulation reduction of an IDIS (U, A, d) can be changed into the multi-granulation reduction of the GNDIS $(U, \mathcal{N}^I, N_d)$ induced by (U, A, d).

Theorem 12. *Consider an IDIS (U, A, d) with $\mathcal{A}^I = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$, which induces a GNDIS $(U, \mathcal{N}^I, N_d)$ with $\mathcal{N}^I = \{N_1, N_2, \cdots, N_m\}$. Then, for $\mathcal{H} \subseteq \mathcal{A}$, $A_k \in \mathcal{A}$,*
(1) $\mathcal{H} \in Cons^O_L(\mathcal{A}^I) \Leftrightarrow I(\mathcal{H}) \in Cons^O_L(\mathcal{N}^I)$.
(2) $\mathcal{H} \in Red^O_L(\mathcal{A}^I) \Leftrightarrow I(\mathcal{H}) \in Red^O_L(\mathcal{N}^I)$.
(3) $A_k \in Core^O_L(\mathcal{A}^I) \Leftrightarrow I(A_k) \in Core^O_L(\mathcal{N}^I)$.
(4) $\mathcal{H} \in Cons^O_U(\mathcal{A}^I) \Leftrightarrow I(\mathcal{H}) \in Cons^O_U(\mathcal{N}^I)$.
(5) $\mathcal{H} \in Red^O_U(\mathcal{A}^I) \Leftrightarrow I(\mathcal{H}) \in Red^O_U(\mathcal{N}^I)$.
(6) $A_k \in Core^O_U(\mathcal{A}^I) \Leftrightarrow I(A_k) \in Core^O_U(\mathcal{N}^I)$.

Proof. It is verified according to Proposition 7, Definitions 13 and 22. □

However, the optimistic multi-granulation reduction of IDISs was not considered in [32]. In the following, we present two discernibility matrices to characterize the IOL-reducts and IOU-reducts of an IDIS.

Definition 23. *Given an IDIS (U, A, d), let $\mathcal{A}^I = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$. For any $x \in U$, define*

$$ID_L^O(x, [x]_d) = \begin{cases} \{A_k \in \mathcal{A}^I | S_{A_k}(x) \subseteq [x]_d\}, & x \in \underline{\sum_{\mathcal{A}^I} A_k^O([x]_d)}, \\ \mathcal{A}^I, & else. \end{cases}$$

$\mathcal{ID}_L^O = \{ID_L^O(x, [x]_d) | x \in U\}$ is called an IOL-discernibility matrix. For any $x \in U, [y]_d \in U/R_d$, define

$$ID_U^O(x, [y]_d) = \begin{cases} \{A_k \in \mathcal{A}^I | S_{A_k}(x) \cap [y]_d = \emptyset\}, & x \notin \overline{\sum_{\mathcal{A}^I} A_k^O([y]_d)}, \\ \mathcal{A}^I, & else. \end{cases}$$

$\mathcal{ID}_U^O = \{ID_U^O(x, [y]_d) | x \in U, [y]_d \in U/R_d\}$ is called an IOU-discernibility matrix.

Corollary 4. *Suppose that (U, A, d) is an IDIS and $\mathcal{A}^I = \{A_k \subseteq A | k \in \mathbb{Z}, 1 \leq k \leq m\}$. For any $\mathcal{H} \subseteq \mathcal{A}^I$, $A_k \in \mathcal{A}^I$,*

(1) $\mathcal{H} \in Cons_L^O(\mathcal{A}^I) \Leftrightarrow \mathcal{H} \cap ID_L^O(x, [x]_d) \neq \emptyset$ for all $ID_L^O(x, [x]_d) \in \mathcal{ID}_L^O$.

(2) $\mathcal{H} \in Red_L^O(\mathcal{A}^I) \Leftrightarrow \mathcal{H} \cap ID_L^O(x, [x]_d) \neq \emptyset$ for all $ID_L^O(x, [x]_d) \in \mathcal{ID}_L^O$, and for any $\mathcal{H}_0 \subset \mathcal{H}$, there exists an $ID_L^O(x, [x]_d) \in \mathcal{ID}_L^O$ such that $ID_L^O(x, [x]_d) \cap \mathcal{H}_0 = \emptyset$.

(3) $A_k \in Core_L^O(\mathcal{A}^I) \Leftrightarrow$ there is some $x \in U$ such that $ID_L^O(x, [x]_d) = \{A_k\}$.

(4) $\mathcal{H} \in Cons_U^O(\mathcal{A}^I) \Leftrightarrow \mathcal{H} \cap ID_U^O(x, [y]_d) \neq \emptyset$ for any $ID_U^O(x, [y]_d) \in \mathcal{ID}_U^O$.

(5) $\mathcal{H} \in Red_U^O(\mathcal{A}^I) \Leftrightarrow \mathcal{H} \cap ID_U^O(x, [y]_d) \neq \emptyset$ for any $ID_U^O(x, [y]_d) \in \mathcal{ID}_U^O$, and for each $\mathcal{H}_0 \subset \mathcal{H}$, there exists an $ID_U^O(x, [y]_d) \in \mathcal{ID}_U^O$ such that $ID_U^O(x, y) \cap \mathcal{H}_0 = \emptyset$.

(6) $A_k \in Core_U^O(\mathcal{A}^I) \Leftrightarrow$ there exist $x \in U, [y]_d \in U/R_d$ such that $ID_U^O(x, [y]_d) = \{A_k\}$.

Proof. By Theorem 12, $\mathcal{H}$ is an IOL-consistent set (or IOL-reduct, IOU-consistent set, IOU-reduct, respectively) of an IDIS $(U, A, d) \Leftrightarrow I(\mathcal{H})$ is a GNOL-consistent set (or GNOL-reduct, GNOU-consistent set, GNOU-reduct, respectively) of the GNDIS $(U, \mathcal{N}^I, N_d)$.

By Definitions 14 and 23, $I(ID_L^O(x, [x]_d)) = GD_L^O(x, N_d(x))$ for all $x \in U$. From Remark 2 and Theorem 5, (1)–(3) are obtained.

Due to Definitions 16 and 23, $I(ID_U^O(x, [y]_d)) = GD_U^O(x, N_d(y))$ for all $[y]_d \in U/R_d, x \in U$. According to Theorem 7, (4)–(6) hold. □

By Definition 10 and Theorem 2, an IPL-reduct can be obtained by a prime implicant of $f(\mathcal{IP})$. According to Definition 12 and Theorem 4, an IPU-reduct can be obtained by a prime implicant of $f(\mathcal{IQ})$. By Definition 15 and Theorem 6, the IOL-reducts can be found from the prime implicants of $f((\mathcal{ID}_L^O)^*)$. Due to Definition 17 and Theorem 8, the IOU-reducts can be found from the prime implicants of $f(\mathcal{ID}_U^O)$. We employ an example to illustrate the calculation method mentioned above.

Example 8. *An IDIS (U, A, d) is presented in Table 5, and $\mathcal{A}^I = \{A_1 = \{a_1, a_2, a_3\}, A_2 = \{a_4, a_5\}, A_3 = \{a_6, a_7\}, A_4 = \{a_8, a_9, a_{10}\}\}$. The granulars of elements are presented in Table 6.*

Table 5. An IDIS.

	A_1			A_2		A_3		A_4			d
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	
x_1	2	1	2	1	1	3	1	1	2	1	1
x_2	*	1	3	2	1	1	2	*	3	2	2
x_3	1	0	3	1	2	1	2	1	*	2	2
x_4	2	2	*	2	0	1	*	0	2	1	2
x_5	1	*	*	1	*	3	1	1	2	*	1
x_6	2	0	3	1	2	*	1	1	2	1	3

Table 6. The granulars of elements in Example 8.

*	x_1	x_2	x_3	x_4	x_5	x_6
$S_{A_1}(x_i)$	$\{x_1\}$	$\{x_2, x_5\}$	$\{x_3, x_5\}$	$\{x_4\}$	$\{x_2, x_3, x_5\}$	$\{x_6\}$
$S_{A_2}(x_i)$	$\{x_1, x_5\}$	$\{x_2\}$	$\{x_3, x_5, x_6\}$	$\{x_4\}$	$\{x_1, x_3, x_5, x_6\}$	$\{x_3, x_5, x_6\}$
$S_{A_3}(x_i)$	$\{x_1, x_5, x_6\}$	$\{x_2, x_3, x_4\}$	$\{x_2, x_3, x_4\}$	$\{x_2, x_3, x_4, x_6\}$	$\{x_1, x_5, x_6\}$	$\{x_1, x_4, x_5, x_6\}$
$S_{A_4}(x_i)$	$\{x_1, x_5, x_6\}$	$\{x_2, x_3\}$	$\{x_2, x_3, x_5\}$	$\{x_4\}$	$\{x_1, x_3, x_5, x_6\}$	$\{x_1, x_5, x_6\}$
$IN_{\mathcal{A}^I}(x_i)$	$\{x_1, x_5, x_6\}$	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3, x_4, x_5, x_6\}$	$\{x_2, x_3, x_4, x_6\}$	$\{x_1, x_2, x_3, x_5, x_6\}$	$\{x_1, x_3, x_4, x_5, x_6\}$

We get that $U/R_d = \{\{x_1, x_5\}, \{x_2, x_3, x_4\}, \{x_6\}\}$. *By Definition 3,* $\underline{\sum_{\mathcal{A}^I} A_k^O}(\{x_1, x_5\}) = \{x_1\}$, $\underline{\sum_{\mathcal{A}^I} A_k^O}(\{x_2, x_3, x_4\}) = \{x_2, x_3, x_4\}$, $\underline{\sum_{\mathcal{A}^I} A_k^O}(\{x_6\}) = \{x_6\}$, $\overline{\sum_{\mathcal{A}^I} A_k^O}(\{x_1, x_5\}) = \{x_1, x_5\}$, $\overline{\sum_{\mathcal{A}^I} A_k^O}(\{x_2, x_3, x_4\}) = \{x_2, x_3, x_4\}$, $\overline{\sum_{\mathcal{A}^I} A_k^O}(\{x_6\}) = \{x_6\}$.

According to Definition 23, we have

$$\mathcal{ID}_L^O = \left(\begin{array}{c|c} x_i & ID_L^O(x_i, N_d(x_i)) \\ \hline x_1 & \{A_1, A_2\} \\ x_2 & \{A_2, A_3, A_4\} \\ x_3 & \{A_3\} \\ x_4 & \{A_1, A_2, A_4\} \\ x_5 & \mathcal{A}^I \\ x_6 & \{A_1\} \end{array} \right).$$

Hence $f(\mathcal{ID}_L^O) = (A_1 \vee A_2) \wedge (A_2 \vee A_3 \vee A_4) \wedge (A_3) \wedge (A_1 \vee A_2 \vee A_4) \wedge (A_1) \wedge (A_1 \vee A_2 \vee A_3 \vee A_4) = A_1 \wedge A_3$, *then* $\{A_1, A_3\}$ *is the IOL-reduct.*

By Definition 23, we get

$$\mathcal{ID}_U^O = \left(\begin{array}{c|ccc} \mathcal{ID}_U^O(x_i, [x_j]_d) & [x_1]_d = [x_5]_d & [x_2]_d = [x_3]_d = [x_4]_d & [x_6]_d \\ \hline x_1 & \mathcal{A}^I & \mathcal{A}^I & \{A_1, A_2\} \\ x_2 & \{A_2, A_3, A_4\} & \mathcal{A}^I & \mathcal{A}^I \\ x_3 & \{A_3\} & \mathcal{A}^I & \{A_1, A_3, A_4\} \\ x_4 & \mathcal{A}^I & \mathcal{A}^I & \{A_1, A_2, A_4\} \\ x_5 & \mathcal{A}^I & \{A_3\} & \{A_1\} \\ x_6 & \{A_1\} & \{A_1, A_4\} & \mathcal{A}^I \end{array} \right).$$

Then, $f(\mathcal{ID}_U^O) = (A_2 \vee A_3 \vee A_4) \wedge (A_3) \wedge (A_1) \wedge (A_1 \vee A_4) \wedge (A_1 \vee A_2) \wedge (A_1 \vee A_3 \vee A_4) \wedge (A_1 \vee A_2 \vee A_4) \wedge (A_1) \wedge (A_1 \vee A_2 \vee A_3 \vee A_4) = A_1 \wedge A_3$*. Thus,* $\{A_1, A_3\}$ *is the IOU-reduct.*

6. Conclusions

The multi-granulation reduction structures of GNDISs based on multi-granulation rough sets have been discussed in this paper, and the discernibility matrices and discernibility functions have been constructed to calculate the multi-granulation reducts of GNDISs. Furthermore, the multi-granulation reductions of DMSs and IDISs have been characterized by the discernibility matrices and discernibility functions based on the reduction theory of GNDISs. Then, the multi-granulation reduction of GNDISs could be a general model for the multi-granulation reduction of DISs by discernibility technique, which provides a theoretical foundation for designing algorithms of multi-granulation reduction of DISs. We summarize the multi-granulation reducts of three kinds of DISs in Table 7. The discernibility method is a theoretical method for computing all the reducts, and the time consumption of the algorithm designed by computing the discernibility matrices and discernibility functions to get all the reducts of a high dimensional information system is high. Then, some heuristic reduction algorithms by discernibility matrices can be designed to get a reduct. Matrix computation or dynamic redution algorithms based on discernibility matrices could also be used to improve computational efficiency of reduction algorithms. In our further work, we will explore the multi-granulation reduction of partially labelled DISs by the discernibility technique.

Table 7. The multi-granulation reductions of GNDISs, DMSs and IDISs.

Information system	Granular structure	Granularity transform	GNIS or GNDIS	Reduction	Discernibility set
GNDIS $(U, \mathcal{N}, N_d)$	$\mathcal{N} = \{N_1, N_2, \cdots, N_m\}$	$*$	$*$	GNPL-reduct	$GD^P_L(x, N_d(y))$
	$C_{N_d} = \{N_d(y) \mid y \in U\}$	$*$	$*$	GNPU-reduct	$GD^P_U(x, N_d(y))$
		$*$	$*$	GNOL-reduct	$GD^O_L(x, N_d(y))$
		$*$	$*$	GNOU-reduct	$GD^O_U(x, N_d(y))$
DMS $(U, \mathcal{A}, d)$	$\mathcal{A} = \{A_k \subseteq A \mid k = 1, \cdots, m\}$	$N_k(x) = [x]_{A_k}$,	$(U, \mathcal{N}^C, N_d)$	CPL-reduct	$P(x)$ [21]
	$U/R_d = \{[y]_d \mid y \in U\}$	$N_d(y) = [y]_d$	$\mathcal{N}^C = \{N_k \mid k = 1, \cdots, m\}$		$I(P(x)) = GD^P_L(x, N_d(x))$
		$I(A_k) = N_k$		CPU-reduct	$Q(x, y)$ [21]
					$\{I(Q(y, x)) \mid x, y \in U\} =$
					$\{GD^P_U(x, N_d(y)) \mid x, y \in U\}$
				COL-reduct	$MD(x)$
					$I(MD(x)) = GD^O_L(x, N_d(x))$
				COU-reduct	$G(x, y)$ [21]
					$I(G(x, y)) = GD^O_U(x, N_d(y))$
IDIS (U, A, d)	$\mathcal{A}^I = \{A_k \subseteq A \mid k = 1, \cdots, m\}$	$N_k(x) = S_{A_k}(x)$,	$(U, \mathcal{N}^I, N_d)$	IPL-reduct	$IP(x)$ [32]
	$U/R_d = \{[y]_d \mid y \in U\}$	$N_d(y) = [y]_d$	$\mathcal{N}^I = \{N_k \mid k = 1, \cdots, m\}$		$I(IP(x)) = GD^P_L(x, N_d(x))$
		$I(A_k) = N_k$		IPU-reduct	$IQ(x, y)$ [32]
					$I(IQ(x, y)) = GD^P_U(x, N_d(y))$
				IOL-reduct	$ID^O_L(x, [x]_d)$
					$I(ID^O_L(x, [x]_d)) = GD^O_L(x, N_d(x))$
				IOU-reduct	$ID^O_U(x, [y]_d)$
					$I(ID^O_U(x, [y]_d)) = GD^O_U(x, N_d(y))$

Author contributions

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Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interests.

References

1. J. C. R. Alcantud, The semantics of N-soft sets, their applications, and a coda about three-way decision, *Inform. Sciences*, **606** (2022), 837–852. https://doi.org/10.1016/j.ins.2022.05.084
2. D. G. Chen, C. Z. Wang, Q. H. Hu, A new approach to attribute reduction of consistent and inconsistent covering decision systems with covering rough sets, *Inform. Sciences*, **177** (2007), 3500–3518. https://doi.org/10.1016/j.ins.2007.02.041

3. J. H. Dai, Z. Y. Wang, W. Y. Huang, Interval-valued fuzzy discernibility pair approach for attribute reduction in incomplete interval-valued information systems, *Inform. Sciences*, **642** (2023), 119215. https://doi.org/10.1016/j.ins.2023.119215

4. W. P. Ding, J. Nayak, B. Naik, D. Pelusi, M. Mishra, Fuzzy and real-coded chemical reaction optimization for intrusion detection in industrial big data environment, *IEEE T. Ind. Inform.*, **17** (2021), 4298–4307. https://doi.org/10.1109/tii.2020.3007419

5. H. Y. Gou, X. Y. Zhang, J. L. Yang , Z. Y. Lv, Three-way fusion measures and three-level feature selections based on neighborhood decision systems, *Appl. Soft Comput.*, **148** (2023), 110842. https://doi.org/10.1016/j.asoc.2023.110842

6. Q. H. Hu, D. R. Yu, J. F. Liu, C. X. Wu, Neighborhood rough set based heterogeneous feature subset selection, *Inform. Sciences*, **178** (2008), 3577–3594. https://doi.org/10.1016/j.ins.2008.05.024

7. Y. Jiang, Y. Yu, Minimal attribute reduction with rough set based on compactness discernibility information tree, *Soft Comput.*, **20** (2016), 2233–2243. https://doi.org/10.1007/s00500-015-1638-0

8. Q. Z. Kong, X. W. Zhang, W. H. Xu, S. T. Xie, Attribute reduction of multi-granulation information system, *Artif. Intell. Rev.*, **53** (2020), 1353–1371. https://doi.org/10.1007/s10462-019-09699-3

9. M. Kryszkiewicz, Rough set approach to incomplete information systems, *Inform. Sciences*, **112** (1998), 39–49. https://doi.org/10.1016/S0020-0255(98)10019-1

10. F. M. Ma, M. W. Ding, T. F. Zhang, J. Cao, Compressed binary discernibility matrix based incremental attribute reduction algorithm for group dynamic data, *Neurocomputing*, **344** (2019), 20–27. https://doi.org/10.1016/j.neucom.2018.01.094

11. F. M. Ma, T. F. Zhang, Generalized binary discernibility matrix for attribute reduction in incomplete information systems, *J. China Univ. Posts Telecommun.,* **24** (2017), 57–68. https://doi.org/10.1016/s1005-8885(17)60224-3

12. Z. Pawlak, Rough sets, *Int. J. Comput. Inf. Sci.*, **11** (1982), 341–356. https://doi.org/10.1007/bf01001956

13. Z. Pawlak, *Rough sets: Theoretical aspects of reasoning about data*, Boston: Kluwer Academic Publishers, 2012.

14. Y. H. Qian, S. Y. Li, J. Y. Liang, Z. Z. Shi, F. Wang, Pessimistic rough set based decisions: A multi-granulation fusion strategy, *Inform. Sciences*, **264** (2014), 196–210. https://doi.org/10.1016/j.ins.2013.12.014

15. Y. H. Qian, J. Y. Liang, C. Y. Dang, Incomplete multi-granulation rough set, *IEEE T. Syst. Man Cybern.*, **40** (2010), 420–431. https://doi.org/10.1109/tsmca.2009.2035436

16. Y. H. Qian, J. Y. Liang, W. Pedrycz, C. Y. Dang, Positive approximation: An accelerator for attribute reduction in rough set theory, *Artif. Intell.*, **174** (2010), 597–618. https://doi.org/10.1016/j.artint.2010.04.018

17. Y. H. Qian, J. Y. Liang, W. Pedrycz, C. Y. Dang, An efficient accelerator for attribute reduction from incomplete data in rough set framework, *Pattern Recogn.*, **44** (2011), 1658–1670. https://doi.org/10.1016/j.patcog.2011.02.020

18. Y. H. Qian, J. Y. Liang, Y. Y. Yao, C. Y. Dang, MGRS: A multi-granulation rough set, *Inform. Sciences*, **180** (2010), 949–970. https://doi.org/10.1016/j.ins.2009.11.023

19. A. Skowron, Boolean reasoning for decision rules generation, In: *Proceedings of the international symposium on methodologies for intelligent systems*, 1993, 295–305. https://doi.org/10.1007/3-540-56804-2_28

20. A. Skowron, C. Rauszer, The discernibility matrices and functions in information systems, In: *R. Slowinski (ed), Intelligent decision support, Handbook of applications and advances of the rough sets theory*, Kluwer Academic Publishers, Dordrecht, 1992. https://doi.org/10.1007/978-94-015-7975-9_21

21. A. H. Tan, W. Z. Wu, J. J. Li, T. J. Li, Reduction foundation with multi-granulation rough sets using discernibility, *Artif. Intell. Rev.*, **53** (2020), 2425–2452. https://doi.org/10.1007/s10462-019-09737-0

22. C. Z. Wang, Q. He, D. G. Chen, Q. H. Hu, A novel method for attribute reduction of covering decision systems, *Inform. Sciences*, **254** (2014), 181–196. https://doi.org/10.1016/j.ins.2013.08.057

23. W. Z. Wu, Knowledge reduction in random incomplete decision tables via evidence theory, *Fund. Inform.*, **115** (2012), 203–218. https://doi.org/10.3233/fi-2012-650

24. W. H. Xu, D. D. Guo, J. S. Mi, Y. H. Qian, K. Y. Zheng, W. P. Ding, Two-way concept-cognitive learning via concept movement viewpoint, *IEEE T. Neur. Netw. Lear. Syst.*, **34** (2023), 6798–6812. https://doi.org/10.1109/tnnls.2023.3235800

25. T. Yang, Y. F. Deng, B. Yu, Y. H. Qian, J. H. Dai, Local feature selection for large-scale data sets limited labels, *IEEE T. Knowl. Data En.*, **35** (2023), 7152–7163. https://doi.org/10.1109/tkde.2022.3181208

26. Y. Y. Yang, D. G. Chen, X. Zhang, Z. Y. Ji, Covering rough set-based incremental feature selection for mixed decision system, *Soft Comput.*, **26** (2022), 2651–2669. https://doi.org/10.1007/s00500-021-06687-0

27. Y. Y. Yao, Constructive and algebraic methods of theory of rough sets, *Inform. Sciences*, **109** (1998), 21–47. https://doi.org/10.1016/S0020-0255(98)00012-7

28. Y. Y. Yao, Y. H. She, Rough set models in multi-granulation spaces, *Inform. Sciences*, **327** (2016), 40–56. https://doi.org/10.1016/j.ins.2015.08.011

29. Y. Y. Yao, Y. Zhao, Discernibility matrix simplification for constructing attribute reducts, *Inform. Sciences*, **179** (2009), 867–882. https://doi.org/10.1016/j.ins.2008.11.020

30. E. A. K. Zaman, A. Mohamed, A. Ahmad, Feature selection for online streaming high-dimensional data: A state-of-the-art review, *Appl. Soft Comput.*, **127** (2022), 109355. https://doi.org/10.1016/j.asoc.2022.109355

31. C. C. Zhang, H. Liu, Z. X. Lu, J. H. Dai, Fast attribute reduction by neighbor inconsistent pair selection for dynamic decision tables, *Int. J. Mach. Learn. Cyber.*, **15** (2024), 739–756. https://doi.org/10.1007/s13042-023-01931-5

32. C. L. Zhang, J. J. Li, Y. D. Lin, Knowledge reduction of pessimistic multi-granulation rough sets in incomplete information systems, *Soft Comput.*, **25** (2021), 12825–12838. https://doi.org/10.1007/s00500-021-06081-w

33. J. Zhang, G. Q. Zhang , Z. W. Li, L. D. Qu, C. F. Wen, Feature selection in a neighborhood decision information system with application to single cell RNA data classification, *Appl. Soft Comput.*, **113** (2021), 107876. https://doi.org/10.1016/j.asoc.2021.107876

34. X. Y. Zhang, W. C. Zhao, Uncertainty measures and feature selection based on composite entropy for generalized multi-granulation fuzzy neighborhood rough set, *Fuzzy Set. Syst.*, **486** (2024), 108971. https://doi.org/10.1016/j.fss.2024.108971

35. Y. Zhao, Y. Y. Yao, F. Luo, Data analysis based on discernibility and indiscernibility, *Inform. Sciences*, **177** (2007), 4959–4976. https://doi.org/10.1016/j.ins.2007.06.031