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Research article

Normalized ground state solutions for the Chern–Simons–Schrödinger equations with mixed Choquard-type nonlinearities

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Abstract: In this paper, we study the existence and the limit behavior of normalized solutions for the Chern–Simons–Schrödinger equations with mixed Choquard-type nonlinearities and $\frac{\alpha}{2} + 2 < q < p < +\infty$. Moreover, we also get the relationship between the minimizer and the ground state solution under the Pohožaev–Nehari manifold of the Chern–Simons–Schrödinger equations.

Keywords: normalized solution; Chern–Simons–Schrödinger equations; Choquard-type nonlinearities; variational method
Mathematics Subject Classification: 35J20, 35J60

1. Introduction

In this article, the gauged Schrödinger equations in $\mathbb{R}^2$ are mainly studied:

$$-\Delta u + \lambda u + \Big(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s)ds\Big)u = \mu(I_\alpha * |u|^q)|u|^{q-2}u + \gamma(I_\alpha * |u|^p)|u|^{p-2}u \tag{1.1}$$

under the constraint

$$\int_{\mathbb{R}^2} |u|^2 dx = c > 0, \tag{1.2}$$

where $u \in H_r^1(\mathbb{R}^2) = \{u \in H^1(\mathbb{R}^2) : u(x) = u(|x|)\}$, $\lambda \in \mathbb{R}$ is the Lagrange multiplier, μ, $\gamma \in \mathbb{R}$, $\frac{\alpha}{2} + 2 < q < p < +\infty$, $h(s) = \frac{1}{2}\int_0^s u^2(l)\, l dl$, and I_α is a Riesz potential (see [21]), $\alpha \in (0, 2)$.

Consider the following time-dependent Schrödinger system with the Chern–Simons gauge fields:

$$\begin{cases} iD_0\phi + (D_1D_1 + D_2D_2)\phi = f(\phi), \\ \partial_0 A_1 - \partial_1 A_0 = -\mathrm{Im}\left(\overline{\phi} D_2\phi\right), \\ \partial_0 A_2 - \partial_2 A_0 = \mathrm{Im}\left(\overline{\phi} D_1\phi\right), \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|\phi|^2, \end{cases} \tag{1.3}$$

where i denotes the imaginary unit, $\partial_0 = \frac{\partial}{\partial t}$, $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$, $\phi \in \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ is the complex scalar field, $(t, x) \in \mathbb{R} \times \mathbb{R}^2$, A_j: $\mathbb{R}^{1+2} \to \mathbb{R}$ is the gauge field, $D_j = \partial_j + iA_j$ is the covariant derivative for $j = 0, 1, 2$, and the function f denotes the nonlinearity. For the physical background, since the 19th century, the Chen–Simons theory has been applied in various fields of quantum physics, and this system is important in the study of the high-temperature superconductor and Aharovnov–Bohm scattering, for more details, we can refer the readers to [9, 15, 16] and the references therein.

The system (1.3) is invariant under the following gauge transformation:

$$\phi \to \phi e^{i\hat{\chi}},\ A_j \to A_j - \partial_j \hat{\chi},$$

where $\hat{\chi} : \mathbb{R}^{1+2} \to \mathbb{R}$ is an arbitrary C^∞ function. If we seek the standing wave solutions to (1.3) of the form

$$\begin{aligned} &\phi(t, x) = u(|x|)\, e^{i\lambda t},\ A_0(t, x) = k(|x|), \\ &A_1(t, x) = \tfrac{x_2}{|x|^2} h(|x|),\ A_2(t, x) = -\tfrac{x_1}{|x|^2} h(|x|), \end{aligned} \tag{1.4}$$

where $\lambda \in \mathbb{R}$ and u, k, h are real-valued functions on $[0, \infty)$ with $h(0) = 0$ and note the form of A_1 and A_2 satisfies the Coulomb gauge condition $\partial_1 A_1 + \partial_2 A_2 = 0$, then we obtain the corresponding elliptic equation for u

$$-\Delta u + \lambda u + \Big(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s) ds\Big) u = f(u), \quad x \in \mathbb{R}^2. \tag{1.5}$$

When $\lambda \in \mathbb{R}$ in (1.4) is a given and fixed frequency, many researchers have investigated the existence and multiplicity of nontrivial solutions for (1.5). Byeon et al. [2] have considered the case $f(u) = \omega|u|^{p-2}u$ when $\lambda > 0$, $\omega > 0$, $p \in (2, \infty)$ and $p \neq 4$, they proved the existence of standing wave solutions. Xiao et al. [27] have considered the existence of the positive energy solutions of (1.5) when $f(u) = a(|x|)|u|^{q-2}u + b(|x|)|u|^{p-2}u$. Chen et al. [7] have proved the existence of a class of ground-state solutions to (1.5) with $V(x) \in C\left(\mathbb{R}^2, \mathbb{R}\right)$ and $f \in C\left(\mathbb{R}^2, \mathbb{R}\right)$. When $\lambda \in \mathbb{R}$ is a given and fixed frequency, more research results in this area can be found in [1, 3, 8, 27, 28] and the references therein. Now, to obtain the research content of this article, we give the definition of the ground state solution of (1.1).

For given $\lambda \in \mathbb{R}$, assuming $u_a \in H_r^1(\mathbb{R}^2)$ is a nontrivial solution of (1.1), it is said to be a ground state solution if it achieves the infimum of the C^1-energy functional $E_\lambda(u) : H_r^1(\mathbb{R}^2) \to \mathbb{R}$ given by

$$\begin{aligned} E_\lambda(u) :=& \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{\lambda}{2}\int_{\mathbb{R}^2} |u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\Big(\int_0^{|x|} \frac{s}{2} u^2(s) ds\Big)^2 dx \\ &- \frac{\mu}{2q}\int_{\mathbb{R}^2} (I_\alpha * |u|^q)|u|^q dx - \frac{\gamma}{2p}\int_{\mathbb{R}^2} (I_\alpha * |u|^p)|u|^p dx \end{aligned} \tag{1.6}$$

among all the nontrivial solutions, namely,

$$E_\lambda(u_a) := \inf_{u \in I_\lambda(u)} E_\lambda(u), \tag{1.7}$$

where

$$I_\lambda(u) := \{u \in H_r^1(\mathbb{R}^2) \setminus \{0\} : M_\lambda(u) = 0\}, \tag{1.8}$$

$$\begin{aligned} M_\lambda(u) :=&\beta \int_{\mathbb{R}^2} |\nabla u|^2 dx + (\beta-1)\lambda \int_{\mathbb{R}^2} |u|^2 dx + (3\beta-2)\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\Big(\int_0^{|x|} \frac{s}{2}u^2(s)ds\Big)^2 dx \\ &-\mu\frac{2\beta q-(2+\alpha)}{2q}\int_{\mathbb{R}^2}(I_\alpha * |u|^q)|u|^q dx - \gamma\frac{2\beta p-(2+\alpha)}{2p}\int_{\mathbb{R}^2}(I_\alpha * |u|^p)|u|^p dx, \end{aligned} \tag{1.9}$$

where $\beta > 0$ and $I_\lambda(u)$ are usually called the Pohožaev–Nehari manifold [2].

In recent years, many scholars considered that the frequency $\lambda \in \mathbb{R}$ in (1.4) is unknown and used as a Lagrange multiplier. In this case, the L^2-norm of solutions is prescribed, which is usually called the normalized solution problem. The normalized solutions seem to be more meaningful from the physical point of view, as it is often adopted to represent the power supply in nonlinear optics or the total number of atoms in Bose–Einstein condensation. The relevant articles are as follows:

In [32], Zuo et al. considered the following nonlinear Schrödinger equations:

$$(-\Delta)^s u + \mu u + \lambda V(x)u - |u|^{p-2}u = 0,\ x \in \mathbb{R}^N, \tag{1.10}$$

where $V(x)$ is a parametric potential term with some assumptions, they obtained the existence of normalized solution through establishing the minimization of the energy functional associated with the principal equation imposing basic assumptions on the potential. And there have been many mathematicians studying the normalized solutions of the Chern–Simons–Schrödinger equations. Among them, (1.1) and (1.2) can be viewed in the following form

$$\begin{cases} -\Delta u + \lambda u + \Big(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s}u^2(s)\mathrm{d}s\Big)u = f(u),\ u \in H_r^1(\mathbb{R}^2), \\ \int_{\mathbb{R}^2} |u|^2 \mathrm{d}x = c > 0. \end{cases} \tag{1.11}$$

Li et al. [17] have considered the nonlinearity $f(u) = |u|^{p-2}u$ for (1.11), they proved that the existence and multiplicity of constraint critical points: when $p = 4$, they proved a sufficient condition for the nonexistence of constraint critical points and obtain infinitely many minimizers of the corresponding energy functional; when $p > 4$, for suitable $c > 0$, they obtained the critical point. Yuan [30] obtained the diversity of normalized solutions for (1.11) with nonlinearity $f(u) = \omega|u|^{p-2}u$ using the minimax theorem. Huang et al. [14] have considered that nonlinearity $f \in C(\mathbb{R},\mathbb{R})$ enjoys critical exponential growth for (1.11), they investigated the existence of normalized solutions. When the frequency $\lambda \in \mathbb{R}$ is unknown and as a Lagrange multiplier, more research results in this area can be read from [4, 10, 13, 20, 31] and the references therein.

Then, motivated by [2, 5, 6, 10, 33], we study the existence of the solutions of (1.1) and (1.2). It is standard to show that the critical points of the following C^1-energy functional defined on $H_r^1(\mathbb{R}^2)$:

$$\begin{aligned} E(u) :=&\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\Big(\int_0^{|x|} \frac{s}{2}u^2(s)ds\Big)^2 dx - \frac{\mu}{2q}\int_{\mathbb{R}^2}(I_\alpha * |u|^q)|u|^q dx \\ &- \frac{\gamma}{2p}\int_{\mathbb{R}^2}(I_\alpha * |u|^p)|u|^p dx \end{aligned} \tag{1.12}$$

under the mass constraint

$$S_c := \Big\{u \in H_r^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} |u|^2 dx = c > 0\Big\}. \tag{1.13}$$

Whereupon, we can search for solutions to (1.1) possessing a given L^2-norm, that is, finding $(u_a, \lambda) \in (H_r^1(\mathbb{R}^2), \mathbb{R})$ solving (1.1) together with the normalized condition $\int_{\mathbb{R}^2} |u_a(x)|^2 dx = c > 0$. Furthermore, we show the definition of a normalized ground state solution to (1.1) on S_c: u_a is a ground state solution of (1.1) on S_c if $(u_a, \lambda) \in S_c \times \mathbb{R}$ is a solution to (1.1) that satisfies:

$$E|'_{S_c}(u_a) = 0 \text{ and } E(u_a) = \inf\{E(u) : u \in S_c, E|'_{S_c}(u_a) = 0\}.$$

We note

$$A(u) := \int_{\mathbb{R}^2} |\nabla u|^2 dx, \quad B(u) := \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\Big(\int_0^{|x|} \frac{s}{2} u^2(s) ds\Big)^2 dx,$$
$$C_q(u) := \int_{\mathbb{R}^2} (I_\alpha * |u|^q) |u|^q dx, \quad D_n(u) := \int_{\mathbb{R}^2} |u|^n dx,$$

where $\frac{\alpha}{2} + 2 < q < +\infty$ and $n \in \mathbb{R}^+$. Setting $u_t(x) := tu(tx)$ for $t > 0$, then $u_t \in S_c$, it holds that

$$D_n(u_t) = t^{n-2} D_n(u),\ A(u_t) = t^2 A(u),\ B(u_t) = t^2 B(u),\ C_q(u_t) = t^{2q-(2+\alpha)} C_q(u).$$

Now, we define the fibering map $t \in (0, +\infty) \mapsto \Phi_u(t)$ given by

$$\Phi_u(t) := E(u_t) = \frac{t^2}{2} A(u) + \frac{t^2}{2} B(u) - \mu \frac{t^{2q-(2+\alpha)}}{2q} C_q(u) - \gamma \frac{t^{2p-(2+\alpha)}}{2p} C_p(u). \tag{1.14}$$

Through a similar discussion in [3], we get the Pohožaev–Nehari functional:

$$\frac{d}{dt}\Big|_{t=1} \Phi_u(t) = M(u) := A(u) + B(u) - \mu \frac{2q - (2+\alpha)}{2q} C_q(u) - \gamma \frac{2p - (2+\alpha)}{2p} C_p(u). \tag{1.15}$$

Hence, notice that

$$\begin{aligned}
\Phi'_u(t) =& \frac{M(u_t)}{t}, \\
\Phi''_u(t) =& A(u) + B(u) - \mu \frac{(2q-2-\alpha)(2q-3-\alpha) t^{2q-(4+\alpha)}}{2q} C_q(u) \\
& - \gamma \frac{(2p-2-\alpha)(2p-3-\alpha) t^{2p-(4+\alpha)}}{2p} C_p(u).
\end{aligned}$$

Following the idea of Soave [23, 24], we introduce a natural constraint Pohožaev–Nehari manifold:

$$I(c) := \{u \in S_c : M(u) = 0\},$$

and we denote

$$\begin{aligned}
I^+(c) &:= \{u \in I(c) : \Phi''_u(1) > 0\}, \\
I^0(c) &:= \{u \in I(c) : \Phi''_u(1) = 0\}, \\
I^-(c) &:= \{u \in I(c) : \Phi''_u(1) < 0\}.
\end{aligned}$$

Moreover, following the arguments in [23], if $I^0(c) = \emptyset$, $I(c)$ is a smooth submanifold of codimension 2 of $H_r^1(\mathbb{R}^2)$ and a submanifold of codimension 1 in S_c.

Next, the following theorems are our main results.

Theorem 1.1. *Let* $\frac{\alpha}{2}+2 < q < p < +\infty$, $\mu < 0$, $\gamma > 0$, *there exists a constant* c_* *such that for* $0 < c < c_*$, (1.2) *has a normalized ground state solution* $(u_a, \lambda) \in (H_r^1(\mathbb{R}^2), \mathbb{R}^+)$, *that is*

$$E(u_a) = \inf_{u \in I(c)} E(u) > 0. \tag{1.16}$$

Moreover, we get $\int_{\mathbb{R}^2} |\nabla u_a|^2 dx \to +\infty$ as $c \to 0$.

Theorem 1.2. *Let* $\lambda(u)$ *be the Lagrange multiplier corresponding to a minimizer* u *of* $\inf_{u\in I(c)} E(u)$, *then for given* $\lambda \in \{\lambda(u) : u$ *is a minimizer of* $\inf_{u\in I(c)} E(u)\}$, *any ground state solution* $w \in H_r^1(\mathbb{R}^2)\backslash\{0\}$ *of* (1.1) *is a minimizer of* $\inf_{u\in I(c)} E(u)$, *namely,*

$$\int_{\mathbb{R}^2} |w|^2 dx = c \text{ and } E(w) = \inf_{u\in I(c)} E(w).$$

And the minimizer of $\inf_{u\in I(c)} E(u)$ is unique if and only if the ground state solution of (1.1) is unique.

Remark 1.1. *For the nonlinearity* $f(u) = \mu(I_\alpha * |u|^q)|u|^{q-2}u + \gamma(I_\alpha * |u|^p)|u|^{p-2}u$ *in* (1.1), *this is derived from the Choquard equation. For some sources and research on the Choquard equations, we refer to [6, 21, 22, 25] and the references therein. For now, there are few studies on the properties of the solution to the Chern–Simons–Schrödinger equations with Choquard-type nonlinearity, which can be found in [29]. And this article aims to study the relationship between the ground state solution of* (1.1) *and the minimizer of* (1.1) *and* (1.2), *and through the variational methods, the ground state solution of* (1.1) *can be obtained. Therefore, we have provided our hypothesis.*

Remark 1.2. *(i) For Theorem 1.1: we have considered the existence of normalized solutions for Chern–Simons–Schrödinger equations with nonlinearity* $f(u) = \mu(I_\alpha*|u|^q)|u|^{q-2}u+\gamma(I_\alpha*|u|^p)|u|^{p-2}u$, *and compared to [31], Yao et al. have considered the existence of normalized solutions for Chern–Simons–Schrödinger systems with exponential critical growth* $f(u)$. *Our results are different, and my approach extends the existing [31] results. Furthermore, we also study the limit behavior of the ground state solutions. To the best of our knowledge, the results we obtained seem to be the first attention paid to the normalized solution problem of the Chern–Simons–Schrödinger equations with mixed Choquard–type nonlinearities. And to prove Theorem 1.1, we use the minimax theorem to prove the existence of a Palais–Smale sequence* $\{u_n\} \subset I(c)$ *for* $E(u)$. *Due to the presence of the Chern–Simons term* $\left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s) ds\right)u$, *it is difficult to prove that* u_a *on* S_c *at level* $\inf_{u\in I(c)} E(u)$ *is a normalized ground state solution.*

(ii) For Theorem 1.2: in [6], for small values of the parameter, Chen et al. have used the variational method to obtain the relationship between the number of solutions of Choquard equations and the profile of one of the continuous functions. Now, we consider the relationship between the ground state solution of (1.1) *and the minimizer of* (1.1) *and* (1.2), *which seems to be a new result for the Chern–Simons–Schrödinger equations with mixed Choquard–type nonlinearities. In order to prove Theorem 1.2, due to the presence of the Chern–Simons term* $\left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s) ds\right)u$, *we encounter difficulties in obtaining that any minimizer* u *of* $\inf_{u\in I(c)} E(u)$ *is a ground state solution of* (1.1).

The following article is arranged as follows: Section 2 contains some required results, then proves Theorem 1.1. Section 3 gives the proof of Theorem 1.2. Section 4 gives a summary of this article.

We finish this introduction with some notation. Throughout this paper, the norm of Sobolev space $H^1(\mathbb{R}^2)$ is $\|u\| = \left(\int_{\mathbb{R}^2}(|u|^2 + |\nabla u|^2)dx\right)^{1/2}$. For $s \geq 1$, the norm of Lebesgue space $L^s(\mathbb{R}^2)$ is $\|v\|_s = \left(\int_{\mathbb{R}^2} |v|^s dx\right)^{1/s}$. The embedding $H^1(\mathbb{R}^2) \hookrightarrow L^s$ $(s \geq 2)$ is continuous; the embedding $H^1_r(\mathbb{R}^2) \hookrightarrow L^s$ $(s > 2)$ is compact. " $\to$ " and " $\rightharpoonup$ " are recorded as strong and weak convergence. Let $(X, \|\cdot\|_X)$ be a Banach space with dual space $(X^{-1}, \|\cdot\|_{X^{-1}})$. The tangent space S_c at $u \in H^1(\mathbb{R}^2)$ is defined as

$$\mathbb{T}_u = \{v \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} uvdx = 0\}.$$

The norm of the C^1 restriction function $E|'_{S_c}$ at $u \in H^1(\mathbb{R}^2)$ is defined by

$$\left\| E|'_{S_c} \right\|_{H^{-1}(\mathbb{R}^2)} = \sup_{v\in\mathbb{T}_u, \|v\|_{H^1(\mathbb{R}^2)}=1} E'(u)[v].$$

Various positive constants are represented by $C, C_0, C_1, C_2, \cdots, C(q)$.

2. Case $\mu < 0, \gamma > 0$

Lemma 2.1. *The functional $E(u)$ is bounded from below by a positive constant and coercive on $I(c) = I^-(c)$.*

Proof. Let $u \in I(c)$, we have

$$\begin{aligned}\Phi''_u(1) =& A(u) + B(u) - \mu\frac{(2q-2-\alpha)(2q-3-\alpha)}{2q}C_q(u) - \gamma\frac{(2p-2-\alpha)(2p-3-\alpha)}{2p}C_p(u)\\ =& (4+\alpha-2q)(A(u)+B(u)) + \gamma\frac{2(2p-2-\alpha)(q-p)}{2p}C_p(u)\\ <& 0.\end{aligned}$$

Then, $I(c) = I^-(c)$. And by Gagliardo–Nirenberg inequality of Hartree type [21], there exists a constant $N(\alpha, p) > 0$ such that

$$\begin{aligned}A(u) + B(u) =& \frac{\mu(2q-2-\alpha)}{2q}C_q(u) + \frac{\gamma(2p-2-\alpha)}{2p}C_p(u)\\ \leq& \frac{\gamma(2p-2-\alpha)N(\alpha,p)}{2p}c^{\frac{2+\alpha}{2}}(A(u)+B(u))^{\frac{2p-2-\alpha}{2}},\end{aligned}$$

which implies that

$$A(u) + B(u) \geq \left(\frac{2p}{\gamma(2p-2-\alpha)N(\alpha,p)c^{\frac{2+\alpha}{2}}}\right)^{\frac{2}{2p-4-\alpha}} > 0. \tag{2.1}$$

Therefore,

$$E(u) = \frac{1}{2}A(u) + \frac{1}{2}B(u) - \frac{\mu}{2q}C_q(u) - \frac{\gamma}{2p}C_p(u)$$

$$
\begin{aligned}
&=\frac{2q-4-\alpha}{2(2q-2-\alpha)}(A(u)+B(u))+\frac{\gamma(p-q)}{p(2q-2-\alpha)}C_p(u)\\
&\geq\frac{2q-4-\alpha}{2(2q-2-\alpha)}(A(u)+B(u))\\
&>0,
\end{aligned}
$$

the functional $E(u)$ is bounded from below by a positive constant. □

Lemma 2.2. *For any $u \in S_c$, there exists a unique $t_u > 0$ such that $u_{t_u} \in I(c)$.*

Proof. Let $u \in S_c$, we have

$$
\begin{aligned}
\Phi_u'(t) =&tA(u)+tB(u)-\mu\frac{2q-(2+\alpha)}{2q}t^{2q-(3+\alpha)}C_q(u)-\gamma\frac{2p-(2+\alpha)}{2p}t^{2p-(3+\alpha)}C_p(u)\\
=&t^{2q-(3+\alpha)}\Big(A(u)t^{\frac{1}{2q-4-\alpha}}+B(u)t^{\frac{1}{2q-4-\alpha}}-\mu\frac{2q-(2+\alpha)}{2q}C_q(u)-\gamma\frac{2p-(2+\alpha)}{2p}t^{2(p-q)}C_p(u)\Big)\\
:=&t^{2q-(3+\alpha)}\zeta(t).
\end{aligned}
$$

Since $\frac{\alpha}{2}+2<q<p<+\infty$, one has $\Phi_u'(0)=0$, $\Phi_u'(t)>0$ for t small, and $\Phi_u'(t)<0$ for t large. Then there exists $t_u>0$ such that $\Phi_u'(t_u)=0$ and $u_{t_u}\in I(c)$. Next, we claim that t_u is unique. For $t>0$, the exponents $2q-4-\alpha$ and $2(p-q)$ are positive, then $\zeta(t)$ is strictly decreasing. Since $\{t>0|\,\Phi_u'(t)=0\}=\{t>0|\,\zeta(t)=0\}$, t_u is unique for any $u\in S_c$. □

Next, we define $X: S_c \to \mathbb{R}$, $X(u) := E(u_{t_u})$, where $t_u > 0$ is given by Lemma 2.2. By a similar proof of [24, Proposition 2.9], we obtain the following lemmas.

Lemma 2.3. *For any $u \in S_c$ and $v \in \mathbb{T}_u$, we get*

$$X'(u)[v]=E'(u_{t_u})[v_{t_u}]. \tag{2.2}$$

Lemma 2.4. *Let F be a homotopy-stable family of compact subsets of S_c with closed boundary B, and let $e(c) := \inf_{H\in F}\max_{u\in H} X(u)$. Suppose that B is contained in a connected component of $I(c)$ and that $\max\{\sup X(B), 0\} < e(c) < +\infty$. Then, there exists a Palais–Smale sequence $\{u_n\} \subset I(c)$ for E restricted on S_c at level $\inf_{u\in I(c)} E(u)$.*

Proof. From [12, Definition 3.1], let $\{D_n\} \subset F$ be a minimizing sequence satisfying:

$$\max_{u\in D_n} X(u) < e(c) + \frac{1}{n},\ \forall\, n \in \mathbb{N}, \tag{2.3}$$

and define the homotopy map $\xi : [0,1]\times S_c \to S_c$ by $\xi(t,u) := (1-t+tt_u)\,u((1-t+tt_u)x)$. Since $t_u = 1$ for any $B \subset I(c)$, it is clear that $\xi(t,u) = u$ for $(t,u) \in (\{0\}\times S_c)\cup([0,1]\times B)$, and it is easy to verify its continuity. Then, using the definition of $e(c)$, we have

$$A_n := \xi(\{1\}\times D_n) = \{u_{t_u} : u \in D_n\} \in F,\ \forall\, n \in \mathbb{N}. \tag{2.4}$$

It follows from Lemma 2.1 that A_n is a subset of $I^-(c)$ for every $n \in \mathbb{N}$, and there is $u \in D_n$ such that $v = u_{t_u} \in A_n$. Moreover, from Lemma 2.2, we have $X(v) = X(u_{t_u}) = E(u_{t_u}) = X(u)$, it holds that $\max_{u\in D_n} X(u) = \max_{u\in A_n} X(u)$. By the Minimax theorem [12, Theorem 3.2]; there exists a Palais–Smale sequence $\{v_n\}$ at level $e(c)$ in S_c with $\text{dist}(v_n, A_n) \to 0$ as $n \to +\infty$. If $\{v_n\} \subset I(c)$, this concludes the proof. If not, we put $t_n := t_{v_n}$ for every $n \in \mathbb{N}$ due to Lemma 2.2 and consider the sequence $\{u_n := t_{v_n} v_n(t_{v_n} x)\} \subset I(c)$. It is enough to prove that $\{u_n\}$ is a Palais–Smale sequence at level $e(c)$ in S_c. It follows that $A(u_n)$ is bounded from below and above. Then, there exists a constant $C > 0$ such that $C^{-1} < t_n^2 < C$. Indeed, it holds that $t_n^2 = \frac{A(u_n)}{A(v_n)}$. Consequently, we have

$$\begin{aligned}\left\| E|'_{S_c}(u_n)\right\|_{H^{-1}(\mathbb{R}^2)} &= \sup_{\|\psi\|=1,\psi\in\mathbb{T}_u} E'(u_n)[\psi]\\ &= \sup_{\|\psi\|=1,\psi\in\mathbb{T}_u} E'((v_n)_{t_n})[(\psi_{t_n^{-1}})_{t_n}]\\ &= \sup_{\|\psi\|=1,\psi\in\mathbb{T}_u} X'(v_n)[\psi_{t_n^{-1}}]\\ &\leq C \sup_{\|\psi\|=1,\psi\in\mathbb{T}_u} \|X'(v_n)\|_{H^{-1}(\mathbb{R}^2)}\|\psi\|\\ &\leq C\|X'(v_n)\|_{H^{-1}(\mathbb{R}^2)}.\end{aligned}$$

It follows that $\{u_n\}$ is a Palais–Smale sequence at level $e(c)$ in S_c. Then, we obtain

$$e(c) = \inf_{H\in F} \max_{u\in H} X(u) = \inf_{u\in S_c} X(u) = \inf_{u\in I(c)} E(u), \tag{2.5}$$

that is, there exists a Palais–Smale sequence $\{u_n\} \subset I(c)$ restricted on S_c at level $\inf_{u\in I(c)} E(u)$. □

Lemma 2.5. *Let $u \in I(c)$ be a nontrivial solution to* (1.1)*, then there exists a c_*, such that for* $0 < c < c_*$*,* $\lambda > 0$*.*

Proof. Testing (1.1) by u, we obtain that

$$A(u) + 3B(u) + \lambda D_2(u) = \mu C_q(u) + \gamma C_p(u). \tag{2.6}$$

From [3], there holds

$$B(u) \leq \frac{1}{16\pi^2} A(u) D_2^2(u), \tag{2.7}$$

then, for $u \in I(c)$ and by Gagliardo–Nirenberg inequality of Hartree type [21], we have

$$\begin{aligned}\lambda c &= \mu C_q(u) + \gamma C_p(u) - A(u) - 3B(u)\\ &= \frac{2+\alpha}{2p-2-\alpha} A(u) - \frac{4p-6-3\alpha}{2p-2-\alpha} B(u) + \frac{\mu(p-q)(2+\alpha)}{q(2p-2-\alpha)} C_p(u)\\ &\geq \frac{2+\alpha}{2p-2-\alpha} A(u) - \frac{4p-6-3\alpha}{(2p-2-\alpha)16\pi^2} c^2 A(u) + \frac{\mu(p-q)(2+\alpha)}{q(2p-2-\alpha)} N(\alpha,p) c^{\frac{2+\alpha}{2}} A(u)^{\frac{2p-2-\alpha}{2}}\\ &= \frac{(2+\alpha)16\pi^2 - (4p-6-3\alpha)c^2}{(2p-2-\alpha)16\pi^2} A(u) + \frac{\mu(p-q)(2+\alpha)}{q(2p-2-\alpha)} N(\alpha,p) c^{\frac{2+\alpha}{2}} A(u)^{\frac{2p-2-\alpha}{2}}.\end{aligned}$$

For $0 < c < c_1 := \left(\frac{(2+\alpha)16\pi^2}{4p-6-3\alpha}\right)^{\frac{1}{2}}$, let

$$f(t) := C_1 t - C_2 t^{\frac{2p-2-\alpha}{2}}, \tag{2.8}$$

where

$$C_1 = \frac{(2+\alpha)16\pi^2 - (4p-6-3\alpha)c^2}{(2p-2-\alpha)16\pi^2},\ C_2 = -\frac{\mu(p-q)(2+\alpha)}{q(2p-2-\alpha)}N(\alpha,p)c^{\frac{2+\alpha}{2}}. \tag{2.9}$$

When $t \in \Big(0, (\frac{C_1}{C_2})^{\frac{2}{2p-4-\alpha}}\Big)$, $f(t) > 0$, and

$$(\frac{C_1}{C_2})^{\frac{2}{2p-4-\alpha}} \to +\infty \text{ as } c \to 0. \tag{2.10}$$

There exists a bounded sequence $\{u_n\} \subset I(c)$ and a positive constant C_3 such that

$$A(u) \le \liminf_{n\to\infty} A(u_n) \le C_3. \tag{2.11}$$

Then, there exists a positive constant c_2 such that for $0 < c < c_2$, $(\frac{C_1}{C_2})^{\frac{2}{2p-4-\alpha}} > C_3$. Hence, when $0 < c < c_* := \min\{c_1, c_2\}$, we get $\lambda > 0$. □

Lemma 2.6. *Let $\{u_n\} \subset I(c)$ be a bounded Palais–Smale sequence for E restricted on S_c at level $\inf_{u\in I(c)} E(u)$, up to a subsequence, $u_n \to u_a$ in $H_r^1(\mathbb{R}^2)\setminus\{0\}$. In particular, $u_a \in S_c$ is a radial normalized solution to* (1.1) *for some $\lambda > 0$.*

Proof. Since $\{u_n\} \subset I(c)$ is a bounded Palais–Smale sequence, there exists a $u_a \in H_r^1(\mathbb{R}^2)$ such that, up to a subsequence, $u_n \rightharpoonup u_a$ in $H_r^1(\mathbb{R}^2)$, $u_n \to u_a$ in $L^t(\mathbb{R}^2)(t > 2)$, and a.e. in $\mathbb{R}^2$. Next, we claim that $u_a \ne 0$. Otherwise, according to the Hardy–Littlewood–Sobolev inequality [18] and Lions' concentration compactness principle [19, Lemma I.1], it is clear that, for any $q \in (\frac{\alpha}{2}+2, +\infty)$,

$$0 \le C_q(u_n) \le C(q) D^{2q}_{\frac{4q}{2+\alpha}}(u) \to 0 \text{ as } n \to +\infty. \tag{2.12}$$

For $\{u_n\} \subset I(c)$,

$$\lim_{n\to\infty} E(u_n) = \lim_{n\to\infty}\Big(\mu\frac{2q-(4+\alpha)}{4q}C_q(u_n) + \gamma\frac{2p-(4+\alpha)}{4p}C_p(u_n)\Big) = 0,$$

it is impossible. Then $u_a \ne 0$. And by the Lagrange multipliers theory, there exists $\lambda_n \in \mathbb{R}$ such that for any $\varphi \in H_r^1(\mathbb{R}^2)$, we have

$$\begin{aligned}&\int_{\mathbb{R}^2} \nabla u_n \nabla\varphi \mathrm{d}x + \lambda_n \int_{\mathbb{R}^2} u_n\varphi \mathrm{d}x + \omega\int_{\mathbb{R}^2}\Big(\frac{h_n^2(|x|)}{|x|^2}u_n\varphi + (\int_{|x|}^{+\infty}\frac{h_n(s)}{s}u_n^2(s)\mathrm{d}s)u_n\varphi\Big)\mathrm{d}x \\ &- \mu\int_{\mathbb{R}^2}(I_\alpha * |u_n|^q)|u_n|^{q-1}\varphi dx - \gamma\int_{\mathbb{R}^2}(I_\alpha * |u_n|^p)|u_n|^{p-1}\varphi dx = o(1).\end{aligned} \tag{2.13}$$

Therefore, we obtain

$$\lambda_n c = \mu C_q(u_n) + \gamma C_p(u_n) - A(u_n) - 3B(u_n) + o(1), \tag{2.14}$$

which implies that $\{\lambda_n\}$ is bounded as well, and then there exists $\lambda \in \mathbb{R}$ such that $\lambda_n \to \lambda$ as $n \to +\infty$. Moreover, by [10, 11] and $\{u_n\} \subset I(c)$, we obtain

$$\lambda_n c = \mu\frac{2+\alpha}{2q}C_q(u_n) + \gamma\frac{2+\alpha}{2p}C_p(u_n) - 4B(u_n)$$

$$\to \mu\frac{2+\alpha}{2q}C_q(u_a)+\gamma\frac{2+\alpha}{2p}C_p(u_a)-4B(u_a)=:\lambda c \text{ as } n\to+\infty,$$

and by weak convergence, for some $\lambda\in\mathbb{R}$,

$$E'(u_a)[\varphi]+\lambda\int_{\mathbb{R}^2}u_a\varphi\mathrm{d}x=0 \tag{2.15}$$

for any $\varphi\in H_r^1(\mathbb{R}^2)$. Therefore, we obtain that

$$\begin{aligned}-\Delta u_a+\lambda u_a+\Big(\frac{h^2(|x|)}{|x|^2}+\int_{|x|}^{+\infty}\frac{h(s)}{s}u_a^2(s)ds\Big)u_a&=\mu(I_\alpha*|u_a|^q)|u_a|^{q-2}u_a\\ +\gamma(I_\alpha*|u_a|^p)|u_a|^{p-2}u_a \text{ in } \mathbb{R}^2.\end{aligned} \tag{2.16}$$

Choosing $\varphi=u_n$ in (2.13) and (2.15), by Lemma 2.5 and [10, 11], we holds

$$A(u_n)+\lambda D_2(u_n)\to A(u_a)+\lambda D_2(u_a) \text{ as } n\to+\infty, \tag{2.17}$$

which implies that $u_n\to u_a$ in $H_r^1(\mathbb{R}^2)\setminus\{0\}$. □

Proof of Theorem 1.1. It follows from Lemma 2.1 that the Palais–Smale sequence obtained in Lemma 2.4 is bounded. So, in view of Lemmas 2.4–2.6, there exists a $u_a\in I(c)$ and $0<c<c_*$ such that $E(u_a)=\inf_{u\in I(c)}E(u)>0$, $E|'_{S_c}(u_a)=0$ and $\lambda>0$. Moreover, from (2.1), we have $\int_{\mathbb{R}^2}|\nabla u_a|^2dx\to+\infty$ as $c\to 0$. So, we get Theorem 1.1.

3. Relationship between minimizer and ground state solution

For any $u\in H_r^1(\mathbb{R}^2)\setminus\{0\}$, let $u_t^\beta(x):=t^\beta u(tx)$ by some positive β. Define the fibering map $t\in(0,+\infty)\mapsto\Phi_u^\beta(t)$ given by

$$\begin{aligned}\Phi_u^\beta(t):&=E_\lambda(u_t^\beta)\\ &=\frac{t^{2\beta}}{2}A(u)+\frac{\lambda t^{2\beta-2}}{2}D_2(u)+\frac{t^{6\beta-4}}{2}B(u)-\mu\frac{t^{2\beta q-(2+\alpha)}}{2q}C_q(u)-\gamma\frac{t^{2\beta p-(2+\alpha)}}{2p}C_p(u).\end{aligned} \tag{3.1}$$

Lemma 3.1. *For any $u\in H_r^1(\mathbb{R}^2)\setminus\{0\}$, there exists a unique $t_u^*>0$ such that $u^\beta_{t_u^*}\in I_\lambda(u)$.*

Proof. Let $u\in H_r^1(\mathbb{R}^2)\setminus\{0\}$ be fixed; by (3.1), we obtain

$$\begin{aligned}\Phi_u^{\beta\prime}(t)=0 \Leftrightarrow&\beta t^{2\beta}A(u)+(\beta-1)t^{2(\beta-1)}\lambda D_2(u)+(3\beta-2)t^{6\beta-4}B(u)\\ &-\frac{\mu(2\beta q-(2+\alpha))}{2q}t^{2\beta q-(2+\alpha)}C_q(u)-\frac{\gamma(2\beta p-(2+\alpha))}{2p}t^{2\beta p-(2+\alpha)}C_p(u)=0\\ \Leftrightarrow&M_\lambda(u_t^\beta)=0\\ \Leftrightarrow&u_t^\beta\in I_\lambda(u).\end{aligned}$$

For $\frac{\alpha}{2}+2<q<p<+\infty$, one has $\Phi_u^{\beta\prime}(0)=0$, $\Phi_u^{\beta\prime}(t)>0$ for $t>0$ small, and $\Phi_u^{\beta\prime}(t)<0$ for $t>0$ large. Then, there exists $t_u^*>0$ such that $\Phi_u^{\beta\prime}(t_u^*)=0$ and $u^\beta_{t_u^*}\in I_\lambda(u)$. Next, we claim that t_u^* is unique for any $u\in H_r^1(\mathbb{R}^2)\setminus\{0\}$. For $\frac{\alpha}{2}+2<q<p<+\infty$,

$$\Phi_u^{\beta\prime}(t)=\beta A(u)t^{2\beta-1}+(\beta-1)\lambda D_2(u)t^{2\beta-3}+(3\beta-2)B(u)t^{6\beta-5}$$

$$
\begin{aligned}
&-\mu\frac{2\beta q-(2+\alpha)}{2q}t^{2\beta q-(3+\alpha)}C_q(u)-\gamma\frac{2\beta p-(2+\alpha)}{2p}t^{2\beta p-(3+\alpha)}C_p(u)\\
=&t^{2\beta q-(3+\alpha)}\Big(\beta A(u)t^{\frac{1}{2\beta(q-1)-2-\alpha}}+(\beta-1)\lambda D_2(u)t^{\frac{1}{2\beta(q-1)-\alpha}}+(3\beta-2)B(u)t^{\frac{1}{2\beta(q-3)+2-\alpha}}\\
&-\mu\frac{2\beta q-(2+\alpha)}{2q}C_q(u)-\gamma\frac{2\beta p-(2+\alpha)}{2p}t^{2(p-q)\beta}C_p(u)\Big)\\
:=&t^{2\beta q-(3+\alpha)}\zeta^{\beta}(t).
\end{aligned}
$$

For some $\beta>1$ and $t>0$, the exponents $2\beta q-(3+\alpha)$, $2\beta(q-1)-2-\alpha$, $2\beta(q-1)-\alpha$, $2\beta(q-3)+2-\alpha$ and $2(p-q)\beta$ are positive, then $\zeta^{\beta}(t)$ is strictly decreasing. Since $\left\{t>0|\, \Phi_u^{\beta'}(t)=0\right\}=\left\{t>0|\, \zeta^{\beta}(t)=0\right\}$, t_u^* is unique for any $u\in H_r^1(\mathbb{R}^2)\backslash\{0\}$. □

Corollary 3.1. *For $u\in I_\lambda(u)$,*

$$E_\lambda(u)=\Phi_u^{\beta}(t_u^*)=\max_{t>0}\Phi_u^{\beta}(t). \tag{3.2}$$

Lemma 3.2. *For $u\in I_\lambda(u)$, we have*

$$E_\lambda(u)\geq\inf_{u\in I(c)}E(u)+\frac{1}{2}\lambda c. \tag{3.3}$$

Especially, the equality holds if and only if u is a minimizer of $\inf_{u\in I(c)}E(u)$, and u is a ground state solution of (1.1). Moreover, any minimizer u of $\inf_{u\in I(c)}E(u)$ is a ground state solution of (1.1).

Proof. For $u\in I_\lambda(u)$ and Corollary 3.1, we get

$$E_\lambda(u)\geq E_\lambda(u_t^{\beta}). \tag{3.4}$$

and $E_\lambda(u)=E_\lambda(u_t^{\beta})$ if and only if $t=1$. Then, by (1.16), one has

$$E_\lambda(u)\geq E_\lambda(u_{t_c}^{\beta})=E(u_{t_c}^{\beta})+\frac{1}{2}\lambda c\geq\inf_{u\in I(c)}E(u)+\frac{1}{2}\lambda c, \tag{3.5}$$

where $t_c=(\frac{c}{D_2(u)})^{\frac{1}{2\beta-2}}$. On the one hand, if the equality holds, then by (3.5), one has $E(u_{t_c}^{\beta})=\inf_{u\in I(c)}E(u)$ and $E_\lambda(u)=E_\lambda(u_{t_c}^{\beta})$. By Corollary 3.1, it implies that $t_c=1$, i.e., $D_2(u)=c$, leading to $E(u)=\inf_{u\in I(c)}E(u)$. Hence, u is a minimizer of $\inf_{u\in I(c)}E(u)$. Otherwise, by (3.3), there exists $v\in I_\lambda(u)$ such that

$$E_\lambda(v)\geq\inf_{u\in I(c)}E(u)+\frac{1}{2}\lambda c=E(u)+\frac{1}{2}\lambda c=E_\lambda(u), \tag{3.6}$$

this contradiction shows that u is a ground state solution of (1.1). On the other hand, if u is a minimizer of $\inf_{u\in I(c)}E(u)$, then we obtain

$$E_\lambda(u)=E(u)+\frac{1}{2}\lambda D_2(u)=\inf_{u\in I(c)}E(u)+\frac{1}{2}\lambda c, \tag{3.7}$$

which implies that the equality holds. □

Proof of Theorem 1.2. By Lemmas 3.1 and 3.2, let $w \in H_r^1(\mathbb{R}^2)\backslash\{0\}$ be any ground state solution of (1.1), for given $\lambda \in \{\lambda(u) : u$ is a minimizer of $\inf\limits_{u\in I(c)} E(u)\}$, we have

$$E_\lambda(w) \leq \inf_{u\in I(c)} E(u) + \frac{1}{2}\lambda c. \tag{3.8}$$

Then, combing (3.3) and (3.8) such that $E_\lambda(w) = \inf\limits_{u\in I(c)} E(u) + \frac{1}{2}\lambda c$. By Lemma 3.2, we obtain that u is a minimizer of $\inf\limits_{u\in I(c)} E(u)$. Let the minimizer u of $\inf\limits_{u\in I(c)} E(u)$ be unique. Then, u is a ground state solution of (1.1) with given λ. Otherwise, there exists $v \in E_\lambda(u)$ such that v is another ground state solution of (1.1). Then, by Lemma 3.2, we have

$$E_\lambda(u) = E_\lambda(v) = \inf_{u\in I(c)} E(u) + \frac{1}{2}\lambda c, \tag{3.9}$$

which shows v is a minimizer of $\inf\limits_{u\in I(c)} E(u)$. This is a contradiction. Similarly, we can prove that the minimizer of $\inf\limits_{u\in I(c)} E(u)$ is unique if the ground state solution of (1.1) with the given λ is unique. This concludes the proof of Theorem 1.2.

4. Conclusions

In this article, we obtain the existence of ground state solutions for Chern–Simons–Schrödinger equations with mixed Choquard-type nonlinearities under L^2-norm constraints. By controlling the size of c and the parameters $\mu < 0$ and $\gamma > 0$, make it possible to find these solutions on Pohožaev–Nehari manifold and consider the limit behavior of these solutions. Furthermore, by assuming the existence of the ground state solutions of the equations, we have found the relationship between the minimizer and the ground state solution under the Pohožaev–Nehari manifold of the Chern–Simons–Schrödinger equations, which greatly enriches the research content on solutions in the Chern–Simons–Schrödinger equations. We hope that the research results of this article can provide new ideas and directions for further research in this field.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

Yipeng Qiu: conceptualization, methodology, mriting-original draft; Yingying Xiao: supervision, writing-review and editing; Yan Zhao: writing-original draft, validation; Shengyue Xu: validation. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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