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Research article

Energy solutions to the bi-harmonic parabolic equations

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Abstract: This study explores the threshold of global existence and exponential decay versus finite-time blow-up for solutions to an inhomogeneous nonlinear bi-harmonic heat problem. The novelty is to consider the inhomogeneous source term. The method uses some standard stable sets under the flow of the fourth-order parabolic problem, due to Payne-Sattynger.

Keywords: inhomogeneous fourth-order parabolic problem; nonlinear equations; global/non-global solutions
Mathematics Subject Classification: 35K30, 35K25

1. Introduction

This note investigates the initial value problem for the inhomogeneous non-linear fourth-order parabolic equation

$$\begin{cases} \partial_t u + \Delta^2 u + u = |x|^{-\varrho}|u|^{p-1}u; \\ u(0,\cdot) = u_0. \end{cases} \tag{IBNLH}$$

The wave function is $u : (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}$ for some integer number $N \geq 3$. The inhomogeneous nonlinear source term satisfies $p > 1$ and $\varrho > 0$.

The fourth-order parabolic problem models a variety of physical processes, such as phase transition, thin-film theory, and lubrication theory. In particular, it can be used to describe the evolution process of nanoscale thin films, with epitaxial growth; see, for instance, [7, 10, 13, 20].

In recent years, fourth-order parabolic equations have been studied extensively. We refer the reader to the survey paper [2], where Section 14 includes some higher-order parabolic problems. The global well-posedness and finite-time blow-up properties of solutions have been investigated by many authors. See [4, 5, 8, 14, 15, 17, 21] and the references therein for the background for the study of bi-harmonic parabolic problems.

This note aims to obtain a threshold of global existence and exponential decay versus finite time blow-up of energy solutions to the inhomogeneous nonlinear bi-harmonic parabolic problem (IBNLH). The novelty is to consider the inhomogeneous regime $\varrho \neq 0$, which complements the results in [19]. The method uses the standard stable sets under the flow of (IBNLH), due to Payne-Sattynger [12].

The plan of this note is as follows: Section 2 contains the main result and some standard estimates needed in the sequel. Section 3 proves the main result.

Let us recall the standard Lebesgue space

$$\begin{aligned} L^r &:= L^r(\mathbb{R}^N) \\ &:= \{u : \mathbb{R}^N \to \mathbb{C},\ \text{measurable function, such that } \int_{\mathbb{R}^N} |u(x)|^r\, dx < \infty\}. \end{aligned}$$

For $r \geq 1$, the usual Lebesgue norm reads

$$\|u\|_r := \|u\|_{L^r} := \Big(\int_{\mathbb{R}^N} |u(x)|^r\, dx\Big)^{\frac{1}{r}}.$$

Finally, letting the standard Laplacian operator $\Delta := \sum_{k=1}^N \frac{\partial^2}{\partial x_k^2}$, we denote the following Sobolev space and its usual norm

$$H^2 := \{f \in L^2, \quad \Delta f \in L^2\};$$
$$\|\cdot\|_{H^2} := \Big(\|\cdot\|^2 + \|\Delta\cdot\|^2\Big)^{\frac{1}{2}}.$$

2. Background and main result

This section contains the main contribution of this note and some useful standard estimates.

2.1. Preliminary

Let us denote the free bi-harmonic heat kernel

$$e^{-t\Delta^2}u := \mathcal{F}^{-1}\Big(e^{-t|\cdot|^4}\mathcal{F}u\Big), \tag{2.1}$$

where $\mathcal{F}$ is the Fourrier transform. Thanks to the Duhamel formula, solutions to (IBNLH) satisfy the integral equation

$$u = e^{-\cdot\Delta^2}u_0 + \int_0^{\cdot} e^{-(\cdot-s)\Delta^2}\Big(|x|^{-\varrho}|u|^{p-1}u\Big)\, ds. \tag{2.2}$$

If u resolves the equation (IBNLH), then so does the family $u_\kappa := \kappa^{\frac{4-\varrho}{p-1}} u(\kappa^4\cdot, \kappa\cdot), \kappa > 0$. Moreover, there is only one invariant Sobolev norm under the above dilatation, precisely

$$\|u_\kappa(t)\|_{\dot H^{s_c}} = \|u(\kappa^4 t)\|_{\dot H^{s_c}}, \quad s_c := \frac{N}{2} - \frac{4-\varrho}{p-1}.$$

So, the heat problem (IBNLH) is said to be energy-sub-critical if

$$s_c < 2 \Leftrightarrow p < p^c := 1 + \frac{2(4-\varrho)}{N-4}, \tag{2.3}$$

where, we take $p^c = \infty$ if $1 \le N \le 4$. Let us denote the so-called action and constraint

$$S(u) := \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|u\|^2 - \frac{1}{1+p}\int_{\mathbb{R}^N} |x|^{-\varrho}|u|^{1+p}\,dx; \tag{2.4}$$

$$K(u) := \|\Delta u\|^2 + \|u\|^2 - \int_{\mathbb{R}^N} |x|^{-\varrho}|u|^{1+p}\,dx. \tag{2.5}$$

A solution to (IBNLH) formally satisfies

$$\partial_t S(u(t)) = -\|\partial_t u\|^2; \tag{2.6}$$

$$-2K(u(t)) = \partial_t \|u(t)\|^2. \tag{2.7}$$

Let us denote the minimization problem

$$m := \inf_{0 \neq u \in H^2} \Big\{ S(u) \quad \mathrm{s.t} \quad K(u) = 0 \Big\}. \tag{2.8}$$

Then, it is known [18, Theorem 2.17] that $m > 0$ is reached in a so-called ground state

$$Q + \Delta^2 Q - |x|^{-\varrho}|Q|^{p-1}Q = 0, \quad 0 \neq Q \in H^2. \tag{2.9}$$

In the spirit of [12], one defines some stable sets under the flow of (IBNLH).

$$\mathcal{PS}^+ := \Big\{ u \in H^2 \quad \text{s. t} \quad K(u) > 0 \quad \text{and} \quad S(u) < m \Big\}; \tag{2.10}$$

$$\mathcal{PS}^- := \Big\{ u \in H^2 \quad \text{s. t} \quad K(u) < 0 \quad \text{and} \quad S(u) < m \Big\}. \tag{2.11}$$

The so-called Strichartz estimates will be useful.

Definition 2.1. *A couple of real numbers (q, r) is said to be admissible if*

$$2 \le r < \frac{2N}{N-4}, \quad 2 \le q, r \le \infty \quad \text{and} \quad N\Big(\frac{1}{2} - \frac{1}{r}\Big) = \frac{4}{q}.$$

Denote the set of admissible pairs by Λ. If I is a time slab, one denotes the Strichartz spaces

$$\Omega(I) := \bigcap_{(q,r)\in\Lambda} L^q(I, L^r).$$

The Strichartz estimates read as follows.

Proposition 2.1. *Let $N \ge 1$ and $T > 0$. Then,*

$$\sup_{(q,r)\in\Lambda} \|e^{-\cdot\Delta^2} f\|_{L^q_T(L^r)} \lesssim \|f\|; \tag{2.12}$$

$$\sup_{(q,r)\in\Lambda} \|u - e^{-\cdot\Delta^2} u_0\|_{L^q_T(L^r)} \lesssim \inf_{(\tilde q,\tilde r)\in\Lambda} \|\partial_t u + \Delta^2 u\|_{L^{\tilde q'}_T(L^{\tilde r'})}; \tag{2.13}$$

$$\sup_{(q,r)\in\Lambda} \|\Delta u\|_{L^q_T(L^r)} \lesssim \|\Delta u_0\| + \|\partial_t u + \Delta^2 u\|_{L^2_T(\dot W^{1,\frac{2N}{2+N}})}, \quad \forall N \ge 3. \tag{2.14}$$

Proof. Let the free fourth order heat equation

$$(\partial_t + \Delta^2)u = 0, \quad u(0, \cdot) = u_0.$$

Taking the Fourrier part of u, yields

$$u(t, x) = \mathcal{F}^{-1}\Big(y \mapsto e^{-t|y|^4}\Big) * u_0 := e^{-t\Delta^2}u_0.$$

It's known [1] that $\mathcal{F}^{-1}\Big(y \mapsto e^{-t|y|^4}\Big)(x) = \frac{1}{t^{\frac{N}{4}}}h(\frac{x}{t^{\frac{1}{4}}})$ for a certain function h satisfying $|h(y)| \lesssim e^{-d|y|^{\frac{4}{3}}}$ for some $d > 0$. This implies that

$$\|e^{-t\Delta^2}u_0\|_{L^\infty} \lesssim t^{-\frac{N}{4}}\|u_0\|_{L^1} \quad \text{and} \quad \|e^{-t\Delta^2}u_0\|_{L^2} \lesssim \|u_0\|_{L^2}.$$

By interpolation, yields $\|e^{-t\Delta^2}u_0\|_{L^r} \lesssim t^{-\frac{N}{4}(1-\frac{2}{p})}\|u_0\|_{L^{r'}}$ for all $r \geq 2$. Thus, applying [6, Theorem 1.2], we get (2.12) and (2.13). Finally, (2.14) follows arguing as in [11, (3.19)]. □

Using a contraction argument via Proposition 2.1 and following lines in [3, Theorem 1.2], we obtain the existence of energy solutions to (IBNLH).

Proposition 2.2. *Let $N \geq 3$, $0 < \varrho < \min\{4, \frac{N}{2}\}$, $\max\{1, \frac{2(1-\varrho)}{N}\} < p < p^c$ and $u_0 \in H^2$. Then, there exist $T := T_{N,\varrho,p,\|u_0\|_{H^2}} > 0$, and a unique local solution of* (IBNLH)*, in the space*

$$C([0, T], H^2) \bigcap_{(q,r)\in\Lambda} L^q_T(W^{2,r}).$$

We end this sub-section with a useful ordinary differential inequality result [9, Lemma 4.2].

Lemma 2.1. *Letting a real decreasing function on* $[0, \infty)$ *such that*

$$(g')^2 \geq A + Bg^{2+\frac{1}{\epsilon}}, \tag{2.15}$$

for certain $A > 0, B > 0$. Then, there exists $T > 0$ such that

$$\lim_{t\to T^-} g(t) = 0; \tag{2.16}$$

$$T \leq \epsilon 2^{\frac{1+3\epsilon}{2\epsilon}} A^{-\frac{1}{2}}(AB^{-1})^{2+\frac{1}{\epsilon}}\Big(1 - (1 + (AB^{-1})^{2+\frac{1}{\epsilon}}g(0))^{-\frac{1}{2\epsilon}}\Big). \tag{2.17}$$

From now on, we hide the time variable t for simplicity, spreading it out only when necessary.

2.2. Main result

The contribution of this note is the next threshold of global existence and exponential decay versus finite time blow-up of solutions to (IBNLH).

Theorem 2.1. *Let $N \geq 3$, $0 < \varrho <$ min $\{4, \frac{N}{2}\}$, max $\{1, \frac{2(1-\varrho)}{N}\} < p < p^c$ and $u_0 \in H^2$. Take the maximal solution of* (IBNLH)*, denoted by $u \in C([0, T^+), H^2)$.*

1. If $u_0 \in \mathcal{PS}^-$, then $T^+ < \infty$ and

$$\lim_{t\to T^+} \int_0^t \|u(s)\|^2 \, ds = \infty. \tag{2.18}$$

2. *If* $u_0 \in \mathcal{PS}^+$, *then* $T^+ = \infty$ *and there is* $\alpha > 0$ *such that*

$$\|u(t)\| \leq \|u_0\| e^{-\alpha t}, \quad \forall t \geq 0. \tag{2.19}$$

In view of the results stated in the above theorem, some comments are in order.

- The existence of the energy solution to (IBNLH) is given by Proposition 2.2.
- The global solution with data in $\mathcal{PS}^+$ decays exponentially.
- Arguing as in [16, Lemma 5.1], it follows that $\mathcal{PS}^\pm$ are stable sets under the flow of (IBNLH).
- The above result complements [19] in the inhomogeneous regime, namely $\varrho \neq 0$.

3. Global/non global existence of energy solutions

In this section, we prove Theorem 2.1. Let us define, for $\lambda > 0, \tau > 0$, the real function on $t \in [0, T^+)$,

$$\varphi(t) := \int_0^t \|u(s)\|^2 \, ds + (T^+ - t)\|u_0\|^2 + \lambda(\tau + t)^2. \tag{3.1}$$

Taking account of (2.7), we compute the derivatives

$$\varphi'(t) = \|u(t)\|^2 - \|u_0\|^2 + 2\lambda(\tau + t); \tag{3.2}$$

$$\varphi''(t) = -2K(u(t)) + 2\lambda. \tag{3.3}$$

Thus, by (2.4), (2.6), and (3.3), we obtain for $\lambda > (1 + p)S(u_0)$,

$$\begin{aligned} \varphi''(t) &= -2(\|u\|_{H^2}^2 - \int_{\mathbb{R}^N} |x|^{-\varrho}|u|^{1+p} \, dx) + 2\lambda \\ &= -2(\|u\|_{H^2}^2 + (1 + p)(S(u) - \frac{1}{2}\|u\|_{H^2}^2)) + 2\lambda \\ &= 2(\frac{p-1}{2}\|u\|_{H^2}^2 - (1 + p)S(u)) + 2\lambda \\ &\geq -2(1 + p)S(u_0) + 2(1 + p)(\int_0^t \|u_t(s)\|^2 \, ds + \lambda) - 2p\lambda \\ &> 0. \end{aligned} \tag{3.4}$$

So, (3.4) implies that

$$\min\{\varphi, \varphi', \varphi''\} > 0, \quad \text{on} \quad [0, T^+). \tag{3.5}$$

Let us denote the quantities

$$a := \int_0^t \|u(s)\|^2 \, ds + \lambda(\tau + t)^2; \tag{3.6}$$

$$b := \frac{1}{2}\varphi'(t) = \frac{1}{2}\int_0^t \partial_s\|u(s)\|^2 \, ds + \lambda(\tau + t); \tag{3.7}$$

$$c := \int_0^t \|\partial_t u(s)\|^2 \, ds + \lambda. \tag{3.8}$$

Compute for $X \in \mathbb{R}$, the polynomial

$$\begin{aligned} aX^2 - 2bX + c &= \int_0^t \|Xu(s)\|^2 \, ds + \lambda(X\tau + tX)^2 - X\Big(\int_0^t \partial_s \|u(s)\|^2 \, ds + 2\lambda(\tau + t)\Big) \\ &\quad + \int_0^t \|\partial_t u(s)\|^2 \, ds + \lambda \\ &\geq \int_0^t \big(\|Xu(s)\| - \|\partial_t u(s)\|\big)^2 \, ds + \lambda(X(\tau + t) - 1)^2 \\ &\geq 0. \end{aligned} \tag{3.9}$$

So, (3.9) implies that

$$b^2 - ac \leq 0. \tag{3.10}$$

Moreover, taking account of (3.4), we write

$$\begin{aligned} \varphi\varphi'' - \frac{1+p}{2}(\varphi')^2 &\geq a\big(-2(1+p)S(u_0) + 2(1+p)c - 2p\lambda\big) - 2(1+p)b^2 \\ &= 2(1+p)(ac - b^2) - 2a\big((1+p)S(u_0) + p\lambda\big). \end{aligned} \tag{3.11}$$

Take the real function

$$g := \varphi^{-\frac{p-1}{2}}, \tag{3.12}$$

with a derivative

$$g' = -\frac{p-1}{2}\varphi'\varphi^{-\frac{1+p}{2}} < 0. \tag{3.13}$$

Moreover, by (3.11), we have

$$\begin{aligned} g'' &= -\frac{p-1}{2}\Big(\varphi''\varphi^{-\frac{1+p}{2}} - \frac{p+1}{2}(\varphi')^2\varphi^{-\frac{3+p}{2}}\Big) \\ &= -\frac{p-1}{2} g^{\frac{3+p}{p-1}}\Big(\varphi''\varphi - \frac{p+1}{2}(\varphi')^2\Big) \\ &\leq -(p-1)g^{\frac{3+p}{p-1}}\Big((1+p)(ac - b^2) - a\big((1+p)S(u_0) + p\lambda\big)\Big). \end{aligned} \tag{3.14}$$

Integrating (3.14) in time after testing with g', it follows that

$$\begin{aligned} (g')^2 &\geq (g'(0))^2 - \frac{(p-1)^2}{1+p}\Big(g^{\frac{2(1+p)}{p-1}} - g^{\frac{2(1+p)}{p-1}}(0)\Big)\Big((1+p)(ac - b^2) - a\big((1+p)S(u_0) + p\lambda\big)\Big) \\ &= (g'(0))^2 + g^{\frac{2(1+p)}{p-1}}(0)\frac{(p-1)^2}{1+p}\big((1+p)(ac - b^2) - a((1+p)S(u_0) + p\lambda\big) \\ &\quad - \frac{(p-1)^2}{1+p}\Big((1+p)(ac - b^2) - a\big((1+p)S(u_0) + p\lambda\big)\Big)g^{\frac{2(1+p)}{p-1}} \end{aligned}$$

$$:= A + Bg^{\frac{2(1+p)}{p-1}}. \tag{3.15}$$

Moreover,

$$\begin{aligned} A &= (g'(0))^2 + g^{\frac{2(1+p)}{p-1}}(0)\frac{(p-1)^2}{1+p}\Big((1+p)(ac-b^2) - a((1+p)S(u_0) + p\lambda)\Big) \\ &\geq \lambda^2\tau^2(p-1)^2(T^+\|u_0\|^2 + \lambda\tau)^{-(1+p)} - a\frac{(p-1)^2}{1+p}(T^+\|u_0\|^2 + \lambda\tau)^{-(1+p)}((1+p)S(u_0) + p\lambda) \\ &= (p-1)^2(T^+\|u_0\|^2 + \lambda\tau)^{-(1+p)}\Big(\lambda^2\tau^2 - \frac{a}{1+p}((1+p)S(u_0) + p\lambda)\Big). \end{aligned} \tag{3.16}$$

So, (3.16) implies that

$$A > 0, \quad \text{for} \quad \lambda >> 1. \tag{3.17}$$

Thus, applying (2.16), we get $T^+ < \infty$ and $\lim_{t\to T^+}\int_0^t \|u(s)\|^2\,ds = \infty$. This proves the finite time blow-up (2.18). Now, if $u_0 \in \mathcal{PS}^+$, then,

$$\begin{aligned} 2m &> \|u\|_{H^2}^2 - \frac{2}{1+p}\int_{\mathbb{R}^N} |x|^{-\varrho}|u|^{1+p}\,dx \\ &> (1 - \frac{2}{1+p})\|u\|_{H^2}^2. \end{aligned} \tag{3.18}$$

So, (3.18) implies that $\sup_{t\in[0,T^+)}\|u(t)\|_{H^2} < \sqrt{\frac{2m(1+p)}{p-1}}$ and u is global. Thus, by the stability of $\mathcal{PS}^+$ under the flow of (IBNLH) we get

$$u(t) \in \mathcal{PS}^+, \quad \forall t \geq 0. \tag{3.19}$$

Let us define for $\gamma > 0$ some modified functional and sets as follows:

$$K_\gamma(u) := \gamma\|u\|_{H^2}^2 - \int_{\mathbb{R}^N} |x|^{-\varrho}|u|^{1+p}\,dx; \tag{3.20}$$

$$m_\gamma := \inf_{0\neq u\in H^2}\{S(u), \quad K_\gamma(u) = 0\}; \tag{3.21}$$

$$\mathcal{PS}_\gamma^+ := \Big\{u \in H^2 \quad \text{s. t} \quad K_\gamma(u) > 0 \quad \text{and} \quad S(u) < m_\gamma\Big\}; \tag{3.22}$$

$$\mathcal{PS}_\gamma^- := \Big\{u \in H^2 \quad \text{s. t} \quad K_\gamma(u) \leq 0 \quad \text{and} \quad S(u) < m\Big\}. \tag{3.23}$$

The next auxiliary result follows lines in [8, Preliminaries].

Lemma 3.1. *The next properties hold.*

1. $\lim_{\gamma\to 0^+} m_\gamma = 0, \quad \lim_{\gamma\to+\infty} m_\gamma = -\infty$*;*
2. $\gamma \to m_\gamma$ *is increasing on* $[0, 1]$ *and decreasing otherwise, and* $m_1 = m$*;*
3. *Let* $u \in H^2$ *satisfy* $S(u) < m$ *and* $\gamma_1 < 1 < \gamma_2$ *be roots of* $m_\gamma = S(u)$*; then,* $K_\gamma(u)$ *has a constant sign in* (γ_1, γ_2)*.*

Now, by (2.7) via the last point in Lemma 3.1, we write for $\gamma \in (\gamma_1, 1)$,

$$\begin{aligned}
\frac{1}{2}\partial_t\|u\|^2 &= -K(u) \\
&= -\|u\|_{H^2}^2 + \int_{\mathbb{R}^N} |x|^{-\varrho}|u|^{1+p}\,dx \\
&= -(1-\gamma)\|u\|_{H^2}^2 - \gamma\|u\|_{H^2}^2 + \int_{\mathbb{R}^N} |x|^{-\varrho}|u|^{1+p}\,dx \\
&= -(1-\gamma)\|u\|_{H^2}^2 - K_\gamma(u) \\
&< -(1-\gamma)\|u\|^2. \qquad (3.24)
\end{aligned}$$

Finally, (3.24) gives the requested estimate (2.19). This ends the proof of Theorem 2.1.

4. Conclusions

This note gives a threshold of global existence and exponential decay versus finite time blow-up of energy solutions to the inhomogeneous nonlinear bi-harmonic parabolic problem (IBNLH). The novelty is to consider the inhomogeneous regime $\varrho \neq 0$, which complements the results in [19]. The method uses the standard stable sets under the flow of (IBNLH), due to Payne-Sattynger [12].

Author contributions

Saleh Almuthaybiri: Formal analysis, funding acquisition; Tarek Saanouni: Project administration, resources, supervision, validation, review. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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