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Research article

Soft almost weakly continuous functions and soft Hausdorff spaces

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Abstract: Beyond the realm of soft topology, soft continuity can aid in the creation of digital images and computational topological applications. This paper investigates soft almost weakly continuous, a novel family of generalized soft continuous functions. The soft pre-continuous and soft weakly continuous function classes are included in this class. We obtain many characterizations of soft almost weakly continuous functions. Furthermore, we investigate the link between soft almost weakly continuous functions and their general topology counterparts. We present adequate conditions for a soft almost weakly continuous function to become soft weakly continuous (soft pre-continuous). We also present various results of soft composition, restriction, preservation, product, and soft graph theorems in terms of soft almost weakly continuous functions.

Keywords: soft weakly continuous functions; soft pre-continuous functions; soft separation axioms; generated soft topology
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1. Introduction and Preliminaries

Mathematical modeling of uncertainty in economics, engineering, social sciences, environmental science, and health is necessary to solve complex problems. Despite their shortcomings, other theories like fuzzy sets [1] and rough sets could be useful in managing ambiguity and uncertainty. Digital image processing combines stochastic and cognitive uncertainties. Stochastic uncertainty can be reduced using statistical models and deep learning algorithms, while cognitive uncertainty is handled via fuzzy logic and uncertainty propagation techniques. Uncertainty modeling improves the robustness and accuracy of image processing in complex applications such as medical imaging, autonomous driving, and remote sensing [2]. Particularly in numerical and computational topological applications, soft set theory is a more straightforward and comprehensive method than fuzzy or rough sets. In this framework, soft continuity is a useful idea for handling ambiguous, partial, or incomplete data. The parameters in a manner that deviates from the strict frameworks of rough or fuzzy set theory. The requirement for additional parametrization tools is one of the main areas where this mathematical technique needs to be improved. Molodtsov [3] created the soft set theory in response to criticisms of previous uncertainty management strategies. Soft sets, or parameterized universe possibilities, were considered. Uncertainty in set modeling was initially demonstrated in [4] and enhanced in [5]. This consistent construction also has a wide range of applications. Soft set theory has been used in finance [6], medicine [7–9], statistics [10, 11], decision-making [12–14], and forecasting [15] .... These real-world applications have proved the framework's problem-solving abilities while also confirming its applicability and efficacy. Several academics have examined and investigated the key concepts and principles of soft set theory [16–18].

To construct a soft topology for a given set of parameters, Shabir and Naz [19] defined a soft topology spanning a family of soft sets. More research in this field was prompted by their work, which clarified the relationships between concepts in soft topology and classical topology. Since the inception of soft topology, several contributions have been made to the study of topological concepts in soft contexts, including soft semi-compact spaces [20], soft nodec spaces [21], soft weakly quasi-continuous functions [22], generalized soft open sets [23–25], soft Q-sets [26, 27], soft separation axioms [28–30], sum of soft topological spaces [31], and generated soft topological spaces [32].

The authors [33] examined soft set mappings and their potential applications in medical diagnosis. The concept of soft mapping with characterizations was first presented in [34]. Soft continuity for soft mappings was introduced in [35]. Several modifications, such as soft α-continuity [36], soft β-continuity [37], soft continuity [38, 39], soft SD-continuity [40], soft ω_p-continuity [41], soft ω^0-continuity [42], soft bi-continuity [43], and soft semi ω-continuity [44] appeared. The smooth transition of a function between its values at adjacent places is characterized by this mathematical idea, which is explored in depth in these publications.

Numerous studies in soft topology and other branches of mathematics have focused on soft continuity. Soft continuity has an important role in many fields, such as engineering, science, business, economics, and soft topological models. Scientists have taken an interest in this subject. Applying soft set theory in fields like computational topology, where handling imprecision and uncertainty is a significant difficulty, can result from an understanding of its significance. In image processing, uncertainty is a natural challenge. Smooth transition patterns can be the framework, particularly in applications like edge detection and image extraction where blurred areas at white margins and smooth transitions are highly realistic. Applying models to handle stochastic and perceptual uncertainty can lead to more resilient systems for dealing with uncertainty. Basic simulation models enhance the accuracy and reliability of digital imaging and help bridge theoretical and practical applications, especially in areas such as medical imaging, self-driving cars, and surveillance. Soft topology provides a flexible way to model data by allowing partial membership in sets, which helps manage ambiguity in ambiguous or incomplete data. It is particularly useful in fields such as bioinformatics, machine learning, and image processing, where data is sparse or noisy. This topology enhances the flexibility of the model by handling ambiguity, making it valuable for applications such as sensor data analysis, pattern recognition, and clustering, further raising readers' interest in its benefits in managing uncertainty. This inspired us to write this paper.

This paper looks into soft almost weakly continuous functions, which are a new class of generalized soft continuous functions. This class includes both the soft pre-continuous and soft weakly continuous function classes. We find several characterizations of soft almost weakly continuous functions. Furthermore, we look at the relationship between soft nearly weakly continuous functions and their general topology equivalents. We provide sufficient requirements for a soft almost weakly continuous function to become soft weakly continuous (soft pre-continuous). We also offer findings from the soft composition, limitation, preservation, product, and soft graph theorems in terms of soft nearly weakly continuous functions.

Our findings have possible important implications for the development of digital image processing and computational topological applications, indicating soft topology's ability to expand beyond its traditional boundaries. As the subject of soft topology advances, this study sets the door for future investigation of soft continuous functions and their applications in a variety of disciplines.

Let $\mathcal{M}$ and T be two non-empty sets, with $\mathcal{M}$ being a set of parameters. A soft set over T relative to $\mathcal{M}$ is a function from $\mathcal{M}$ to T's powerset. $SS(T,\mathcal{M})$ refers to the collection of all soft sets over T relative to $\mathcal{M}$. Assume that K is in $SS(T,\mathcal{M})$. K is denoted by C_U if $K(a) = U$ for all $a \in \mathcal{M}$. $C_\emptyset$ will be denoted by $0_\mathcal{M}$, and C_T by $1_\mathcal{M}$. If $K(a) = U$ and $K(d) = \emptyset$ for every $d \in \mathcal{M} - \{a\}$, then a_U denotes K. For convenience, $a_{\{x\}}$ shall be denoted as a_x and will be referred to as a soft point for every $a \in \mathcal{M}$ and $x \in T$. The collection of all soft points over T with respect to $\mathcal{M}$ is denoted by $SP(T,\mathcal{M})$. $a_x \in SP(T,\mathcal{M})$ is considered to belong to $K \in SS(T,\mathcal{M})$ (notation: $a_x \widetilde{\in} K$) if $x \in K(a)$. Let $S, R \in SS(T,\mathcal{M})$. Then S is a soft subset of R, denoted by $S \widetilde{\subseteq} R$, if $S(a) \subseteq R(a)$ for each $a \in \mathcal{M}$. The soft union (resp. intersection, difference) of S and R is denoted by $S \widetilde{\cup} R$ (resp. $S \widetilde{\cap} R$, $S - R$) and defined by $\left(S \widetilde{\cup} R\right)(a) = S(a) \cup R(a)$ (resp. $\left(S \widetilde{\cap} R\right)(a) = S(a) \cap R(a)$, $(S - R)(a) = S(a) - R(a)$) for each $a \in \mathcal{M}$. For any sub-collection $\mathcal{R} \subseteq SS(T,\mathcal{M})$, the soft union (resp. soft intersection) of the members of $\mathcal{R}$ are denoted by $\widetilde{\cup}_{R\in\mathcal{R}} R$ (resp. $\widetilde{\cap}_{R\in\mathcal{R}} R$) and defined by $\left(\widetilde{\cup}_{R\in\mathcal{R}} R\right)(a) = \cup_{R\in\mathcal{R}} R(a)$ (resp. $\left(\widetilde{\cap}_{R\in\mathcal{R}} R\right)(a) = \cap_{R\in\mathcal{R}} R(a)$) for each $a \in \mathcal{M}$. Let $SS(T,\mathcal{M})$ and $SS(W,\mathcal{N})$ be two families of soft sets, and $s : T \longrightarrow W$, $v : \mathcal{M} \longrightarrow \mathcal{N}$ be two functions. Then a soft mapping $f_{sv} : SS(T,\mathcal{M}) \longrightarrow SS(W,\mathcal{N})$ is defined as follows: For each $H \in SS(T,\mathcal{M})$ and $K \in SS(W,\mathcal{N})$, $(f_{sv}(H))(b) = \emptyset$ if $v^{-1}(b) = \emptyset$, $(f_{sv}(H))(b) = \cup_{a\in v^{-1}(b)} s(H(a))$ if $v^{-1}(b) \neq \emptyset$, and $\left(f_{sv}^{-1}(K)\right)(a) = s^{-1}(K(v(a)))$. A sub-collection $\sigma \subseteq SS(T,\mathcal{M})$ is called a soft topology on T relative to $\mathcal{M}$, and the triplet $(T,\sigma,\mathcal{M})$ is called a soft topological space if $\{0_\mathcal{M}, 1_\mathcal{M}\} \subseteq \sigma$, $S \widetilde{\cap} R \in \sigma$ for any $\{S,R\} \subseteq \sigma$, and $\widetilde{\cup}_{R\in\mathcal{R}} R$ for any $\mathcal{R} \subseteq \sigma$. Let $(T,\sigma,\mathcal{M})$ be a soft topological space and let $R \in SS(T,\mathcal{M})$. Then R is called a soft open set in $(T,\sigma,\mathcal{M})$ if $R \in \sigma$ and R is called a soft closed set in $(T,\sigma,\mathcal{M})$ if $1_\mathcal{M} - R \in \sigma$.

To be clear, throughout this work, we will make reference to concepts and terms from [45, 46]. The acronyms TS and STS stand for topological space and soft topological space, respectively.

Now, let us review some of the main concepts that will be applied in the follow-up.

Definition 1.1. [47] A function $p : (G,\mathfrak{I}) \longrightarrow (H,\aleph)$ between TSs is called almost weakly continuous (a.w.c) if $p^{-1}(V) \widetilde{\subseteq} Int_\mathfrak{I}\left(Cl_\mathfrak{I}\left(p^{-1}(Cl_\aleph(V))\right)\right)$ for each $V \in \aleph$.

In this paper, we would like mainly to extend the concept of almost weak continuity in classical topology to include STSs.

Definition 1.2. Let $(G,\Psi,\mathcal{L})$ be a STS and let $H \in SS(G,\mathcal{L})$. Then

(a) H is a soft semi-open [48] (resp. soft pre-open [49], soft α-open [44], soft regular-open [50]) set in $(G,\Psi,\mathcal{L})$ if $H \widetilde{\subseteq} Cl_\Psi(Int_\Psi(H))$

(resp. $H \widetilde{\subseteq} Int_\Psi(Int_\Psi(H))$, $H \widetilde{\subseteq} Int_\Psi(Cl_\Psi(Int_\Psi(H)))$, $H = Int_\Psi(Cl_\Psi(H))$). The family of all soft semi-open sets (resp. soft pre-open sets, soft α-open sets, soft regular-open sets) in $(G,\Psi,\mathcal{L})$ will be denoted by $SO(\Psi)$ (resp. $PO(\Psi)$, $\alpha(\Psi)$, $RO(\Psi)$).

(b) H is called a soft pre-closed [49] (resp. soft regular-closed [50]) set in $(G, \Psi, \mathcal{L})$ if $1_{\mathcal{L}} - H \in PO(\Psi)$ (resp. $1_{\mathcal{L}} - H \in RO(\Psi)$). The family of all soft pre-closed sets (resp. soft regular-closed sets) in $(G, \Psi, \mathcal{L})$ will be denoted by $PC(\Psi)$ (resp. $RC(\Psi)$).

Definition 1.3. [49] Let $(G, \Psi, \mathcal{L})$ be a STS and let $H \in SS(G, \mathcal{L})$. Then

(a) $pInt_{\Psi}(H)$ represents the soft pre-interior of H in $(G, \Psi, \mathcal{L})$ and is defined by

$$pInt_{\Psi}(H) = \widetilde{\cup}\left\{R : R \in PO(\Psi) \text{ and } R\widetilde{\subseteq}H\right\}.$$

(b) $pCl_{\Psi}(H)$ represents the soft pre-closure of H in $(G, \Psi, \mathcal{L})$ and is defined by

$$pCl_{\Psi}(H) = \widetilde{\cap}\left\{D : D \in PC(\Psi) \text{ and } H\widetilde{\subseteq}D\right\}.$$

Definition 1.4. [51] Let $(G, \Psi, \mathcal{L})$ be a STS and let $H \in SS(G, \mathcal{L})$. The soft θ-closure of H in $(G, \Psi, \mathcal{L})$ is denoted by $Cl^{\theta}_{\Psi}(H)$, where $\theta Cl_{\varphi}(K) \in SS(G, \mathcal{L})$ and defined as follows:

$a_x\widetilde{\in}Cl^{\theta}_{\Psi}(H)$ iff for each $K \in \Psi$ such that $a_x\widetilde{\in}K$, $H\widetilde{\cap}Cl_{\Psi}(K) \neq 0_{\mathcal{L}}$.

Definition 1.5. A soft function $f_{sv} : (G, \Psi, \mathcal{L}) \longrightarrow (H, \Phi, \mathcal{M})$ is called

(a) soft semi-continuous [52] if $f^{-1}_{sv}(Y) \in SO(\Psi)$ for every $Y \in \Phi$.

(b) soft pre-continuous [53] if $f^{-1}_{sv}(Y) \in PO(\Psi)$ for every $Y \in \Phi$.

(c) soft weakly continuous [54] if for every $d_x \in SP(G, \mathcal{L})$ and every $R \in \Phi$ such that $f_{sv}(d_x)\widetilde{\in}R$, there exists $T \in \Psi$ such that $d_x\widetilde{\in}T$ and $f_{sv}(T)\widetilde{\subseteq}Cl_{\Phi}(R)$.

(d) soft almost α-continuous [55] if for every $d_x \in SP(G, \mathcal{L})$ and every $R \in RO(\Phi)$ such that $f_{sv}(d_x)\widetilde{\in}R$, we find $T \in \alpha(\Psi)$ such that $d_x\widetilde{\in}T$ and $f_{sv}(T)\widetilde{\subseteq}R$.

Definition 1.6. A STS $(G, \Psi, \mathcal{L})$ is called

(a) [56] soft Hausdorff if for each $d_x, e_y \in SP(G, \mathcal{L})$ such that $d_x \neq e_y$, we find $T, R \in \Psi$ such that $d_x\widetilde{\in}T$, $e_y\widetilde{\in}R$, and $T\widetilde{\cap}R = 0_{\mathcal{L}}$.

(b) [56] soft regular if for each $d_x \in SP(G, \mathcal{L})$ and every $T \in \Psi$ such that $d_x\widetilde{\in}T$, we find $R \in \Psi$ such that $d_x\widetilde{\in}R\widetilde{\subseteq}Cl_{\Psi}(R)\widetilde{\subseteq}T$.

(c) [56] soft Urysohn if for each $d_x, e_y \in SP(G, \mathcal{L})$ such that $d_x \neq e_y$, we find $T, R \in \Psi$ such that $d_x\widetilde{\in}T$, $e_y\widetilde{\in}R$, and $Cl_{\Psi}(T)\widetilde{\cap}Cl_{\Psi}(R) = 0_{\mathcal{L}}$.

(d) [57] soft pre-T_2 if for each $d_x, e_y \in SP(G, \mathcal{L})$ such that $d_x \neq e_y$, we find $T, R \in PO(\Psi)$ such that $d_x\widetilde{\in}T$, $e_y\widetilde{\in}R$, and $T\widetilde{\cap}R = 0_{\mathcal{L}}$.

(e) [58] soft submaximal if $\{M : Cl_{\Psi}(M) = 1_{\mathcal{L}}\} \subseteq \Psi$.

For a soft function $f_{sv} : SP(G, \mathcal{A}) \longrightarrow SP(H, \mathcal{B})$, the soft set

$\widetilde{\cup}\left\{(a, v(a))_{(x,s(x))} : a \in \mathcal{A} \text{ and } x \in G\right\}$ is represented by $Graph(f_{sv})$ and is called the soft graph of f_{sv}. So, $(d, e)_{(x,y)}\widetilde{\in}Graph(f_{sv})$ iff $f_{sv}(d_x) = e_y$ iff $s(x) = y$ and $v(d) = e$.

Definition 1.7. [22] Let $f_{sv} : (G, \Psi, \mathcal{L}) \longrightarrow (H, \Phi, \mathcal{M})$ be a soft function. Then $Graph(f_{sv})$ is said to be soft strongly closed with respect to $(G \times H, pr(\Psi\times\Phi), \mathcal{L} \times \mathcal{M})$ if for each $(d, e)_{(x,y)}\widetilde{\in}1_{\mathcal{L}\times\mathcal{M}} - Graph(f_{sv})$, there exist $T \in \Psi$ and $R \in \Phi$ such that $d_x\widetilde{\in}T$, $e_y\widetilde{\in}R$, and $(T \times Cl_{\Phi}(R))\widetilde{\cap}Graph(f_{sv}) = 0_{\mathcal{L}\times\mathcal{M}}$.

2. Soft almost weakly continuous functions

Definition 2.1. A soft function $f_{sv} : (G, \Psi, \mathcal{L}) \longrightarrow (H, \Phi, \mathcal{M})$ is called soft almost weakly continuous (soft a.w.c) if $f^{-1}_{sv}(Y)\widetilde{\subseteq}Int_{\Psi}\left(Cl_{\Psi}\left(f^{-1}_{sv}(Cl_{\Phi}(Y))\right)\right)$ for each $Y \in \Phi$.

Theorem 2.2. Let $\{(G,\varphi_a) : a \in \mathcal{L}\}$ and $\{(H,\sigma_b) : b \in \mathcal{M}\}$ be two collections of TSs. Consider the functions $s : G \longrightarrow H$ and $v : \mathcal{L} \longrightarrow \mathcal{M}$, where v is a bijection. Then $f_{sv} : (G, \oplus_{a\in\mathcal{L}}\varphi_a, \mathcal{L}) \longrightarrow (H, \oplus_{b\in\mathcal{M}}\sigma_b, \mathcal{M})$ is soft a.w.c iff $s : (G,\varphi_a) \longrightarrow (H, \sigma_{v(a)})$ is a.w.c for all $a \in \mathcal{L}$.

Proof. Necessity. Let $f_{sv} : (G, \oplus_{a\in\mathcal{L}}\varphi_a, \mathcal{L}) \longrightarrow (H, \oplus_{b\in\mathcal{M}}\sigma_b, \mathcal{M})$ be soft a.w.c. Let $a \in \mathcal{L}$. Let $U \in \sigma_{w(a)}$. Then, $(v(a))_U \in \oplus_{b\in\mathcal{M}}\sigma_b$. So, $f_{sv}^{-1}((v(a))_U) \widetilde{\subseteq} Int_{\oplus_{a\in\mathcal{L}}\varphi_a}\left(Cl_{\oplus_{a\in\mathcal{L}}\varphi_a}\left(f_{sv}^{-1}\left(Cl_{\oplus_{b\in\mathcal{M}}\sigma_b}((v(a))_U)\right)\right)\right)$. Thus,

$\left(f_{sv}^{-1}((v(a))_U)\right)(a) \subseteq \left(Int_{\oplus_{a\in\mathcal{L}}\varphi_a}\left(Cl_{\oplus_{a\in\mathcal{P}}\varphi_a}\left(f_{sv}^{-1}\left(Cl_{\oplus_{b\in\mathcal{M}}\sigma_b}((v(a))_U)\right)\right)\right)\right)(a)$. Since $v : \mathcal{L} \longrightarrow \mathcal{M}$ is injective, $f_{sv}^{-1}((v(a))_U) = a_{s^{-1}(U)}$ and so, $\left(f_{sv}^{-1}((v(a))_U)\right)(a) = \left(a_{s^{-1}(U)}\right)(a) = s^{-1}(U)$. By Lemma 4.9 of [59],

$$\left(Int_{\oplus_{a\in\mathcal{L}}\varphi_a}\left(Cl_{\oplus_{a\in\mathcal{L}}\varphi_a}\left(f_{sv}^{-1}\left(Cl_{\oplus_{b\in\mathcal{M}}\sigma_b}((v(a))_U)\right)\right)\right)\right)(a) = Int_{\varphi_a}\left(Cl_{\varphi_a}\left(\left(f_{sv}^{-1}\left(Cl_{\oplus_{b\in\mathcal{M}}\sigma_b}((v(a))_U)\right)\right)(a)\right)\right).$$

Furthermore, it is not difficult to check that $\left(f_{sv}^{-1}\left(Cl_{\oplus_{b\in\mathcal{M}}\sigma_b}((v(a))_U)\right)\right)(a) = s^{-1}(Cl_{\sigma_{v(a)}}(U))$. Therefore, $s^{-1}(U) \subseteq Int_{\varphi_a}\left(Cl_{\varphi_a}\left(s^{-1}(Cl_{\sigma_{v(a)}}(U))\right)\right)$. This shows that $s : (G,\varphi_a) \longrightarrow (H,\sigma_{v(a)})$ is a.w.c.

Sufficiency. Let $s : (G,\varphi_a) \longrightarrow (H,\sigma_{v(a)})$ be a.w.c for all $a \in \mathcal{L}$. Let $K \in \oplus_{b\in\mathcal{M}}\sigma_b$. Then, $K(b) \in \sigma_b$ for all $b \in \mathcal{M}$. For every $b \in \mathcal{M}$, $s : \left(G, \varphi_{v^{-1}(b)}\right) \longrightarrow (H,\sigma_b)$ is a.w.c, and so

$$s^{-1}(K(b)) \subseteq Int_{\varphi_{v^{-1}(b)}}\left(Cl_{\varphi_{v^{-1}(b)}}\left(s^{-1}(Cl_{\sigma_b}(K(b)))\right)\right).$$

Claim. $f_{sv}^{-1}(K) \widetilde{\subseteq} Int_{\oplus_{a\in\mathcal{L}}\varphi_a}\left(Cl_{\oplus_{a\in\mathcal{L}}\varphi_a}\left(f_{sv}^{-1}\left(Cl_{\oplus_{b\in\mathcal{M}}\sigma_b}(K)\right)\right)\right)$ which ends the proof.

Proof of Claim. Let $a_x \widetilde{\in} f_{sv}^{-1}(K)$. Then $f_{sv}(a_x) = (v(a))_{s(x)} \widetilde{\in} K$. So, $s(x) \in K(v(a))$ and thus, $x \in s^{-1}(K(v(a))) \subseteq Int_{\varphi_a}\left(Cl_{\varphi_a}\left(s^{-1}(Cl_{\sigma_{v(a)}}(K(v(a))))\right)\right)$. It is not difficult to check that $s^{-1}(Cl_{\sigma_{v(a)}}(K(v(a))) = \left(f_{sv}^{-1}\left(Cl_{\oplus_{b\in\mathcal{M}}\sigma_b}(K)\right)\right)(a)$. Thus, $x \in Int_{\varphi_a}\left(Cl_{\varphi_a}\left(\left(f_{sv}^{-1}\left(Cl_{\oplus_{b\in\mathcal{M}}\sigma_b}(K)\right)\right)(a)\right)\right)$. Furthermore, by Lemma 4.9 of [59],

$$Int_{\varphi_a}\left(Cl_{\varphi_a}\left(\left(f_{sv}^{-1}\left(Cl_{\oplus_{b\in\mathcal{M}}\sigma_b}(K)\right)\right)(a)\right)\right) = \left(Int_{\oplus_{a\in\mathcal{L}}\varphi_a}\left(Cl_{\oplus_{a\in\mathcal{L}}\varphi_a}\left(f_{sv}^{-1}\left(Cl_{\oplus_{b\in\mathcal{M}}\sigma_b}(K)\right)\right)\right)\right)(a).$$

This shows that $a_x \widetilde{\in} Int_{\oplus_{a\in\mathcal{L}}\varphi_a}\left(Cl_{\oplus_{a\in\mathcal{L}}\varphi_a}\left(f_{sv}^{-1}\left(Cl_{\oplus_{b\in\mathcal{M}}\sigma_b}(K)\right)\right)\right)$.

Corollary 2.3. Consider the functions $s : (G,\mathfrak{I}) \longrightarrow (H,\aleph)$ and $v : \mathcal{L} \longrightarrow \mathcal{M}$, where v is a bijection. Then $s : (G,\mathfrak{I}) \longrightarrow (H,\aleph)$ is a.w.c iff $f_{sv} : (G,\tau(\mathfrak{I}),\mathcal{L}) \longrightarrow (H,\tau(\aleph),\mathcal{M})$ is soft a.w.c.

Proof. For each $a \in \mathcal{L}$ and $b \in \mathcal{M}$, put $\varphi_a = \mathfrak{I}$ and $\sigma_b = \aleph$. Then $\tau(\mathfrak{I}) = \oplus_{a\in\mathcal{L}}\varphi_a$ and $\tau(\aleph) = \oplus_{b\in\mathcal{M}}\sigma_b$. Theorem 2.2 ends the proof.

Theorem 2.4. Soft weakly continuous functions are soft a.w.c.

Proof. Let $f_{sv} : (G,\Psi,\mathcal{L}) \longrightarrow (H,\Phi,\mathcal{M})$ be soft weakly continuous. Let $Y \in \Phi$. Then by Theorem 5.1 of [59],

$$\begin{aligned} f_{sv}^{-1}(Y) &\widetilde{\subseteq} Int_\Psi\left(f_{sv}^{-1}(Cl_\Phi(Y))\right) \\ &\widetilde{\subseteq} Int_\Psi\left(Cl_\Psi\left(f_{sv}^{-1}(Cl_\Phi(Y))\right)\right). \end{aligned}$$

It follows that f_{sv} is soft a.w.c.

Theorem 2.5. Soft pre-continuous functions are soft a.w.c.

Proof. Let $f_{sv} : (G,\Psi,\mathcal{L}) \longrightarrow (H,\Phi,\mathcal{M})$ be soft pre-continuous. Let $Y \in \Phi$. Then, $f_{sv}^{-1}(Y) \widetilde{\subseteq} Int_\Psi\left(Cl_\Psi\left(f_{sv}^{-1}(Y)\right)\right) \widetilde{\subseteq} Int_\Psi\left(Cl_\Psi\left(f_{sv}^{-1}(Cl_\Phi(Y))\right)\right)$. It follows that f_{sv} is soft a.w.c.

The following two examples show that the implications in Theorems 2.4 and 2.5 are not reversible in general:

Example 2.6. Let $\mathfrak{I}$ and $\aleph$ be the indiscrete and discrete topologies on $\mathbb{R}$. Consider the identity functions $s : (\mathbb{R},\mathfrak{I}) \longrightarrow (\mathbb{R},\aleph)$ and $v : \mathbb{N} \longrightarrow \mathbb{N}$. Then $f_{sv} : (\mathbb{R},\tau(\mathfrak{I}),\mathbb{N}) \longrightarrow (\mathbb{R},\tau(\aleph),\mathbb{N})$ is soft a.w.c but not soft weakly continuous.

Example 2.7. Let $G = \{1,2,3,4\}$, $\Im = \{\emptyset, G, \{2\}, \{3\}, \{2,3\}, \{1,2\}, \{1,2,3\}, \{2,3,4\}\}$, and $\mathcal{L} = (0,1)$. Define $s : (G,\Im) \longrightarrow (G,\Im)$ and $v : \mathcal{L} \longrightarrow \mathcal{L}$ by $s(1) = 3$, $s(2) = 4$, $s(3) = 2$, $s(4) = 1$, and $v(a) = a$ for all $a \in \mathcal{L}$. Then $f_{sv} : (\mathbb{R}, \tau(\Im), \mathcal{L}) \longrightarrow (\mathbb{R}, \tau(\aleph), \mathcal{L})$ is soft a.w.c but not soft pre-continuous.

Theorem 2.8. A soft function $f_{sv} : (G,\Psi,\mathcal{A}) \longrightarrow (H,\Phi,\mathcal{B})$ is soft a.w.c iff $f_{sv}(Cl_\Psi(K)) \widetilde{\subseteq} Cl^\theta_\Phi(f_{sv}(K))$ for every $K \in \Psi$.

Proof. Necessity. Suppose that f_{sv} is soft a.w.c. Let $K \in \Psi$ and suppose to the contrary that there exists $b_y \widetilde{\in} f_{sv}(Cl_\Psi(K)) - Cl^\theta_\Phi(f_{sv}(K))$. Since $b_y \widetilde{\notin} Cl^\theta_\Phi(f_{sv}(K))$, then there exists $T \in \Phi$ such that $b_y \widetilde{\in} T$ and $Cl_\Phi(T) \widetilde{\cap} f_{sv}(K) = 0_\mathcal{B}$. This implies that $f^{-1}_{sv}(Cl_\Phi(T)) \widetilde{\cap} K = 0_\mathcal{A}$ and consequently,

$$Int_\Psi\left(Cl_\Psi\left(f^{-1}_{sv}(Cl_\Phi(T))\right)\right) \widetilde{\cap} Cl_\Psi(K) = 0_\mathcal{A}.$$

Since f_{sv} is soft a.w.c, then $f^{-1}_{sv}(T) \widetilde{\subseteq} Int_\Psi\left(Cl_\Psi\left(f^{-1}_{sv}(Cl_\Phi(T))\right)\right)$. Thus, $f^{-1}_{sv}(T) \widetilde{\cap} Cl_\Psi(K) = 0_\mathcal{A}$ and hence, $T \widetilde{\cap} f_{sv}(Cl_\Psi(K)) = 0_\mathcal{B}$. This implies that $b_y \widetilde{\notin} f_{sv}(Cl_\Psi(K))$, which is a contradiction.

Sufficiency. Suppose that $f_{sv}(Cl_\Psi(K)) \widetilde{\subseteq} Cl^\theta_\Phi(f_{sv}(K))$ for every $K \in \Psi$. Let $K \in \Psi$. Then

$$\begin{aligned} & f_{sv}\left(1_\mathcal{A} - Int_\Psi\left(Cl_\Psi\left(f^{-1}_{sv}(Cl_\Phi(K))\right)\right)\right) = \\ & f_{sv}\left(Cl_\Psi\left(1_\mathcal{A} - Cl_\Psi\left(f^{-1}_{sv}(Cl_\Phi(K))\right)\right)\right) = \\ & f_{sv}\left(Cl_\Psi\left(Int_\Psi\left(1_\mathcal{A} - f^{-1}_{sv}(Cl_\Phi(K))\right)\right)\right) = \\ & f_{sv}\left(Cl_\Psi\left(Int_\Psi\left(f^{-1}_{sv}(1_\mathcal{B} - Cl_\Phi(K))\right)\right)\right). \end{aligned}$$

Since $Int_\Psi\left(f^{-1}_{sv}(1_\mathcal{B} - Cl_\Phi(K))\right) \in \Phi$, then by assumption,

$$\begin{aligned} f_{sv}\left(Cl_\Psi\left(Int_\Psi\left(f^{-1}_{sv}(1_\mathcal{B} - Cl_\Phi(K))\right)\right)\right) &\widetilde{\subseteq} Cl^\theta_\Phi\left(f_{sv}\left(Int_\Psi\left(f^{-1}_{sv}(1_\mathcal{B} - Cl_\Phi(K))\right)\right)\right) \\ &\widetilde{\subseteq} Cl^\theta_\Phi\left(f_{sv}\left(\left(f^{-1}_{sv}(1_\mathcal{B} - Cl_\Phi(K))\right)\right)\right) \\ &\widetilde{\subseteq} Cl^\theta_\Phi(1_\mathcal{B} - Cl_\Phi(K)). \end{aligned}$$

Since $1_\mathcal{B} - Cl_\Phi(K) \in \Phi$, then $Cl^\theta_\Phi(1_\mathcal{B} - Cl_\Phi(K)) = Cl_\Phi(1_\mathcal{B} - Cl_\Phi(K))$. Since $1_\mathcal{B} - Cl_\Phi(K) \widetilde{\subseteq} 1_\mathcal{B} - K$ and $1_\mathcal{B} - K \in \Phi^c$, then $Cl_\Phi(1_\mathcal{B} - Cl_\Phi(K)) \widetilde{\subseteq} 1_\mathcal{B} - K$. Therefore,

$$\begin{aligned} f_{sv}\left(1_\mathcal{A} - Int_\Psi\left(Cl_\Psi\left(f^{-1}_{sv}(Cl_\Phi(K))\right)\right)\right) &= f_{sv}\left(Cl_\Psi\left(Int_\Psi\left(f^{-1}_{sv}(1_\mathcal{B} - Cl_\Phi(K))\right)\right)\right) \\ &\widetilde{\subseteq} Cl^\theta_\Phi(1_\mathcal{B} - Cl_\Phi(K)) \\ &\widetilde{\subseteq} 1_\mathcal{B} - K, \end{aligned}$$

and thus,

$$\begin{aligned} 1_\mathcal{A} - Int_\Psi\left(Cl_\Psi\left(f^{-1}_{sv}(Cl_\Phi(K))\right)\right) &\widetilde{\subseteq} f^{-1}_{sv}\left(f_{sv}\left(1_\mathcal{A} - Int_\Psi\left(Cl_\Psi\left(f^{-1}_{sv}(Cl_\Phi(K))\right)\right)\right)\right) \\ &\widetilde{\subseteq} f^{-1}_{sv}(1_\mathcal{B} - K) \\ &= 1_\mathcal{A} - f^{-1}_{sv}(K). \end{aligned}$$

This implies that $f^{-1}_{sv}(K) \widetilde{\subseteq} Int_\Psi\left(Cl_\Psi\left(f^{-1}_{sv}(Cl_\Phi(K))\right)\right)$. Hence, f_{sv} is soft a.w.c.

Theorem 2.9. If $f_{sv} : (G,\Psi,\mathcal{A}) \longrightarrow (H,\Phi,\mathcal{B})$ is soft a.w.c and $(H,\Phi,\mathcal{B})$ is soft regular, then f_{sv} is soft pre-continuous.

Proof. Let $f_{sv} : (G,\Psi,\mathcal{L}) \longrightarrow (H,\Phi,\mathcal{M})$ be soft a.w.c. Let $K \in \Psi$. Then, by Theorem 2.8, $f_{sv}(Cl_\Psi(K)) \widetilde{\subseteq} Cl^\theta_\Phi(f_{sv}(K))$. Since $(H,\Phi,\mathcal{B})$ is soft regular, then $Cl^\theta_\Phi(f_{sv}(K)) = Cl_\Phi(f_{sv}(K))$. Thus, $f_{sv}(Cl_\Psi(K)) \widetilde{\subseteq} Cl_\Phi(f_{sv}(K))$. Therefore, by Theorem 18 (d) of [61], f_{sv} is soft pre-continuous.

Theorem 2.10. For a soft function $f_{sv} : (G,\Psi,\mathcal{A}) \longrightarrow (H,\Phi,\mathcal{B})$, the next are equivalent:

(a) f_{sv} is soft a.w.c.

(b) $Cl_\Psi\left(Int_\Psi\left(f^{-1}_{sv}(K)\right)\right) \widetilde{\subseteq} f^{-1}_{sv}(Cl_\Phi(K))$ for every $K \in \Phi$.

(c) For each $a_x \in SP(G,\mathcal{A})$ and each $K \in \Phi$ such that $f_{sv}(a_x) \widetilde{\in} K$, $Cl_\Psi\left(f^{-1}_{sv}(Cl_\Phi(K))\right)$ is a soft neighborhood of a_x.

Proof. (a) $\longrightarrow$ (b): Let $K \in \Phi$. Then $1_{\mathcal{B}} - Cl_{\Phi}(K) \in \Phi$, and by (a),

$$\begin{aligned} 1_{\mathcal{A}} - f_{sv}^{-1}(Cl_{\Phi}(K)) &= f_{sv}^{-1}(1_{\mathcal{B}} - Cl_{\Phi}(K)) \\ &\widetilde{\subseteq} Int_{\Psi}\left(Cl_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(1_{\mathcal{B}} - Cl_{\Phi}(K)))\right)\right) \\ &= Int_{\Psi}\left(Cl_{\Psi}\left(f_{sv}^{-1}(1_{\mathcal{B}} - Int_{\Phi}(Cl_{\Phi}(K)))\right)\right) \\ &= Int_{\Psi}\left(Cl_{\Psi}\left(1_{\mathcal{A}} - f_{sv}^{-1}(Int_{\Phi}(Cl_{\Phi}(K)))\right)\right) \\ &= Int_{\Psi}\left(1_{\mathcal{A}} - Int_{\Psi}\left(f_{sv}^{-1}(Int_{\Phi}(Cl_{\Phi}(K)))\right)\right) \\ &= 1_{\mathcal{A}} - Cl_{\Psi}\left(Int_{\Psi}\left(f_{sv}^{-1}(Int_{\Phi}(Cl_{\Phi}(K)))\right)\right) \\ &\widetilde{\subseteq} 1_{\mathcal{A}} - Cl_{\Psi}\left(Int_{\Psi}\left(f_{sv}^{-1}((K))\right)\right). \end{aligned}$$

It follows that $Cl_{\Psi}\left(Int_{\Psi}\left(f_{sv}^{-1}(K)\right)\right)\widetilde{\subseteq} f_{sv}^{-1}(Cl_{\Phi}(K))$.

(b) $\longrightarrow$ (c): Let $a_x \in SP(G, \mathcal{A})$ and let $K \in \Phi$ such that $f_{sv}(a_x)\widetilde{\in} K$. Since $1_{\mathcal{B}} - Cl_{\Phi}(K) \in \Phi$, then by (b),

$$Cl_{\Psi}\left(Int_{\Psi}\left(f_{sv}^{-1}(1_{\mathcal{B}} - Cl_{\Phi}(K))\right)\right) \widetilde{\subseteq} f_{sv}^{-1}(Cl_{\Phi}(1_{\mathcal{B}} - Cl_{\Phi}(K))).$$

Thus,

$$\begin{aligned} 1_{\mathcal{A}} - Int_{\Psi}\left(Cl_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(K))\right)\right) &= Cl_{\Psi}\left(1_{\mathcal{A}} - Cl_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(K))\right)\right) \\ &= Cl_{\Psi}\left(Int_{\Psi}\left(1_{\mathcal{A}} - f_{sv}^{-1}(Cl_{\Phi}(K))\right)\right) \\ &= Cl_{\Psi}\left(Int_{\Psi}\left(f_{sv}^{-1}(1_{\mathcal{B}} - Cl_{\Phi}(K))\right)\right) \\ &\widetilde{\subseteq} f_{sv}^{-1}(Cl_{\Phi}(1_{\mathcal{B}} - Cl_{\Phi}(K))) \\ &= f_{sv}^{-1}(1_{\mathcal{B}} - Int_{\Phi}(Cl_{\Phi}(K))) \\ &\widetilde{\subseteq} f_{sv}^{-1}(1_{\mathcal{B}} - K) \\ &= 1_{\mathcal{A}} - f_{sv}^{-1}(K). \end{aligned}$$

Therefore, we obtain $a_x\widetilde{\in} f_{sv}^{-1}(K)\widetilde{\subseteq} Int_{\Psi}\left(Cl_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(K))\right)\right)$, and hence $Cl_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(K))\right)$ is a soft neighborhood of a_x.

(c) $\longrightarrow$ (a): Let $K \in \Phi$. To show that $f_{sv}^{-1}(K)\widetilde{\subseteq} Int_{\Psi}\left(Cl_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(K))\right)\right)$, let $a_x\widetilde{\in} f_{sv}^{-1}(K)$. Then $f_{sv}(a_x)\widetilde{\in} K$ and by (c), $Cl_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(K))\right)$ is a soft neighborhood of a_x. Therefore, $a_x\widetilde{\in} Int_{\Psi}\left(Cl_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(K))\right)\right)$.

Definition 2.11. A soft function $f_{sv} : (G, \Psi, \mathcal{L}) \longrightarrow (H, \Phi, \mathcal{M})$ is called soft pre-open if $f_{sv}(K)\widetilde{\subseteq} Int_{\Phi}(Cl_{\Phi}(f_{sv}(K)))$ for each $K \in \Psi$.

Theorem 2.12. Every soft open function is soft pre-open.

Proof. Let $f_{sv} : (G, \Psi, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$ be soft open. Let $K \in \Psi$. Then $f_{sv}(K) \in \Phi$ and so, $f_{sv}(K)\widetilde{\subseteq} Int_{\Phi}(Cl_{\Phi}(f_{sv}(K)))$. It follows that f_{sv} is soft pre-open.

Soft pre-open functions are not soft open in general.

Example 2.13. Let $G = \mathbb{R}$, $\mathfrak{I} = \{\emptyset, G, \mathbb{Q}\}$, and $\aleph$ be the usual topology on $\mathbb{R}$. Let $\mathcal{L} = \{a, b\}$. Consider the identity functions $s : (G, \mathfrak{I}) \longrightarrow (G, \aleph)$ and $v : \mathcal{L} \longrightarrow \mathcal{L}$. Then $f_{sv} : (G, \tau(\mathfrak{I}), \mathcal{L}) \longrightarrow (G, \tau(\aleph), \mathcal{L})$ is soft pre-open but not soft open.

Theorem 2.14. A soft function $f_{sv} : (G, \Psi, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$ is soft pre-open iff $f_{sv}^{-1}(Cl_{\Phi}(T))\widetilde{\subseteq} Cl_{\Psi}\left(f_{sv}^{-1}(T)\right)$ for every $T \in \Phi$.

Proof. Necessity. Let f_{sv} be soft pre-open. Let $T \in \Phi$. Let $a_x\widetilde{\in} f_{sv}^{-1}(Cl_{\Phi}(T))$ and let $K \in \Psi$ such that $a_x\widetilde{\in} K$. Since f_{sv} is soft pre-open, then $f_{sv}(K)\widetilde{\subseteq} Int_{\Phi}(Cl_{\Phi}(f_{sv}(K)))$. Since $a_x\widetilde{\in} K$, then $f_{sv}(a_x)\widetilde{\in} f_{sv}(K)\widetilde{\subseteq} Int_{\Phi}(Cl_{\Phi}(f_{sv}(K)))$ and so, $f_{sv}(a_x)\widetilde{\in} Int_{\Phi}(Cl_{\Phi}(f_{sv}(K))) \in \Phi$. Since $a_x\widetilde{\in} f_{sv}^{-1}(Cl_{\Phi}(T))$, then $f_{sv}(a_x)\widetilde{\in} Cl_{\Phi}(T)$. Thus, $T\widetilde{\cap} Int_{\Phi}(Cl_{\Phi}(f_{sv}(K))) \neq 0_{\mathcal{B}}$ and hence, $T\widetilde{\cap} Cl_{\Phi}(f_{sv}(K)) \neq 0_{\mathcal{B}}$. Consequently, $T\widetilde{\cap} f_{sv}(K) \neq 0_{\mathcal{B}}$. Choose $b_y\widetilde{\in} K$ such that $f_{sv}\left(b_y\right)\widetilde{\in} T$. Then, $b_y\widetilde{\in} K\widetilde{\cap} f_{sv}^{-1}(T)$. Hence,

$a_x \widetilde{\in} Cl_\Psi\left(f_{sv}^{-1}(T)\right)$.

Sufficiency. Suppose that $f_{sv}^{-1}(Cl_\Phi(T)) \widetilde{\subseteq} Cl_\Psi\left(f_{sv}^{-1}(T)\right)$ for every $T \in \Phi$. Let $K \in \Psi$. Suppose to the contrary that there exists $a_x \widetilde{\in} K$ such that $f_{sv}(a_x) \widetilde{\notin} Int_\Phi(Cl_\Phi(f_{sv}(K)))$. Let $T = 1_{\mathcal{B}} - Cl_\Phi(f_{sv}(K))$. Then, $T \in \Phi$ and by assumption, $f_{sv}^{-1}(Cl_\Phi(T)) \widetilde{\subseteq} Cl_\Psi\left(f_{sv}^{-1}(T)\right)$. Since

$$\begin{aligned} f_{sv}^{-1}(Cl_\Phi(T)) &= f_{sv}^{-1}(Cl_\Phi(1_{\mathcal{B}} - Cl_\Phi(f_{sv}(K)))) \\ &= f_{sv}^{-1}(1_{\mathcal{B}} - Int_\Phi(Cl_\Phi(f_{sv}(K)))) \\ &= 1_{\mathcal{A}} - f_{sv}^{-1}(Int_\Phi(Cl_\Phi(f_{sv}(K)))), \end{aligned}$$

and $a_x \widetilde{\in} 1_{\mathcal{A}} - f_{sv}^{-1}(Int_\Phi(Cl_\Phi(f_{sv}(K))))$, then $a_x \widetilde{\in} f_{sv}^{-1}(Cl_\Phi(T)) \widetilde{\subseteq} Cl_\Psi\left(f_{sv}^{-1}(T)\right)$ and so,

$$\begin{aligned} a_x \widetilde{\in} \quad & Cl_\Psi\left(f_{sv}^{-1}(T)\right) = \\ & Cl_\Psi\left(f_{sv}^{-1}(1_{\mathcal{B}} - Cl_\Phi(f_{sv}(K)))\right) = \\ & Cl_\Psi\left(1_{\mathcal{A}} - f_{sv}^{-1}(Cl_\Phi(f_{sv}(K)))\right). \end{aligned}$$

Since $a_x \widetilde{\in} K \in \Psi$, then $K \widetilde{\cap} \left(1_{\mathcal{A}} - f_{sv}^{-1}(Cl_\Phi(f_{sv}(K)))\right) \neq 0_{\mathcal{A}}$.

Since $K \widetilde{\subseteq} f_{sv}^{-1}(f_{sv}(T)) \widetilde{\subseteq} f_{sv}^{-1}(Cl_\Phi(f_{sv}(K)))$, then $1_{\mathcal{A}} - f_{sv}^{-1}(Cl_\Phi(f_{sv}(K))) \widetilde{\subseteq} 1_{\mathcal{A}} - K$.

Thus, $K \widetilde{\cap} (1_{\mathcal{A}} - K) \neq 0$. This is a contradiction.

Theorem 2.15. Soft a.w.c soft pre-open functions are soft pre-continuous.

Proof. Let $f_{sv} : (G, \Psi, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$ be soft a.w.c and soft pre-open. Let $T \in \Phi$. Since f_{sv} is soft a.w.c, then $f_{sv}^{-1}(T) \widetilde{\subseteq} Int_\Psi\left(Cl_\Psi\left(f_{sv}^{-1}(Cl_\Phi(T))\right)\right)$. Since f_{sv} is soft pre-open, then by Theorem 2.14, $f_{sv}^{-1}(Cl_\Phi(T)) \widetilde{\subseteq} Cl_\Psi\left(f_{sv}^{-1}(T)\right)$. Therefore, we have $f_{sv}^{-1}(T) \widetilde{\subseteq} Int_\Psi\left(Cl_\Psi\left(f_{sv}^{-1}(Cl_\Phi(T))\right)\right) \widetilde{\subseteq} Int_\Psi\left(Cl_\Psi\left(f_{sv}^{-1}(T)\right)\right)$. This shows that f_{sv} is soft pre-continuous.

Corollary 2.16. Soft weakly continuous soft pre-open functions are soft pre-continuous.

Theorem 2.17. Soft a.w.c soft semi-continuous functions are soft weakly continuous.

Proof. Let $f_{sv} : (G, \Psi, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$ be soft a.w.c and soft semi-continuous. Let $T \in \Phi$. Since f_{sv} is soft semi-continuous, then $f_{sv}^{-1}(T) \in SO(\Psi)$. So, by Lemma 1 of [22], $Cl_\Psi\left(f_{sv}^{-1}(T)\right) = Cl_\Psi\left(Int_\Psi\left(\left(f_{sv}^{-1}(T)\right)\right)\right)$. Furthermore, by Theorem 2.10, $Cl_\Psi\left(Int_\Psi\left(f_{sv}^{-1}(T)\right)\right) \widetilde{\subseteq} f_{sv}^{-1}(Cl_\Phi(T))$. Hence, $Cl_\Psi\left(f_{sv}^{-1}(T)\right) \widetilde{\subseteq} f_{sv}^{-1}(Cl_\Phi(T))$. It follows from Theorem 5.1 of [59] that f_{sv} is soft weakly continuous.

Lemma 2.18. Let $(G, \Psi, \mathcal{A})$ be a STS and let $D \in SS(G, \mathcal{A})$. Then

(a) If $K \in \Psi$, then $K \widetilde{\cap} Cl_\Psi(D) \widetilde{\subseteq} Cl_\Psi\left(K \widetilde{\cap} D\right)$.

(b) If $S \in \Psi^c$, then $Int_\Psi\left(D \widetilde{\cup} S\right) \widetilde{\subseteq} Int_\Psi(D) \widetilde{\cup} S$.

Proof. (a) Let $a_x \widetilde{\in} K \widetilde{\cap} Cl_\Psi(D)$ and let $H \in \Psi$ such that $a_x \widetilde{\in} H$. Then, we have $a_x \widetilde{\in} K \widetilde{\cap} H \in \Psi$. Since $a_x \widetilde{\in} Cl_\Psi(D)$, then $\left(K \widetilde{\cap} D\right) \widetilde{\cap} H = \left(K \widetilde{\cap} H\right) \widetilde{\cap} D \neq 0_{\mathcal{A}}$. Thus, $a_x \widetilde{\in} Cl_\Psi\left(K \widetilde{\cap} D\right)$.

(b) Since $S \in \Psi^c$, then $1_{\mathcal{A}} - S \in \Psi$ and by (a),

$$(1_{\mathcal{A}} - S) \widetilde{\cap} Cl_\Psi(1_{\mathcal{A}} - D) \ \widetilde{\subseteq}\ Cl_\Psi\left((1_{\mathcal{A}} - S) \widetilde{\cap} (1_{\mathcal{A}} - D)\right).$$

Since $Cl_\Psi(1_{\mathcal{A}} - D) = 1_{\mathcal{A}} - Int_\Psi(D)$, then

$$\begin{aligned} (1_{\mathcal{A}} - S) \widetilde{\cap} Cl_\Psi(1_{\mathcal{A}} - D) &= (1_{\mathcal{A}} - S) \widetilde{\cap} (1_{\mathcal{A}} - Int_\Psi(D)) \\ &= 1_{\mathcal{A}} - \left(S \widetilde{\cup} Int_\Psi(D)\right). \end{aligned}$$

Furthermore,

$$\begin{aligned} Cl_\Psi\left((1_{\mathcal{A}} - S) \widetilde{\cap} (1_{\mathcal{A}} - D)\right) &= Cl_\Psi\left(1_{\mathcal{A}} - \left(S \widetilde{\cup} D\right)\right) \\ &= 1_{\mathcal{A}} - \left(Int_\Psi\left(S \widetilde{\cup} D\right)\right). \end{aligned}$$

Therefore, $1_{\mathcal{A}} - \left(S \widetilde{\cup} Int_\Psi(D)\right) \widetilde{\subseteq} 1_{\mathcal{A}} - \left(Int_\Psi\left(S \widetilde{\cup} D\right)\right)$ and hence, $Int_\Psi\left(D \widetilde{\cup} S\right) \widetilde{\subseteq} Int_\Psi(D) \widetilde{\cup} S$.

Theorem 2.19. Let $(G, \Psi, \mathcal{A})$ be a STS and let $D \in SS(G, \mathcal{A})$. Then $pCl_\Psi(D) = D\widetilde{\cup}Cl_\Psi(Int_\Psi(D))$.

Proof. Since $Cl_\Psi(Int_\Psi(D)) \in \Psi^c$, then by Lemma 2.18 (b),

$$Int_\Psi\left(D\widetilde{\cup}Cl_\Psi(Int_\Psi(D))\right) \widetilde{\subseteq} Int_\Psi(D)\widetilde{\cup}Cl_\Psi(Int_\Psi(D)).$$

So,

$$\begin{aligned} Cl_\Psi\left(Int_\Psi\left(D\widetilde{\cup}Cl_\Psi(Int_\Psi(D))\right)\right) &\widetilde{\subseteq} Cl_\Psi\left(Int_\Psi(D)\widetilde{\cup}Cl_\Psi(Int_\Psi(D))\right) \\ &\widetilde{\subseteq} Cl_\Psi(Int_\Psi(D)) \\ &\widetilde{\subseteq} D\widetilde{\cup}Cl_\Psi(Int_\Psi(D)). \end{aligned}$$

Hence, $D\widetilde{\cup}Cl_\Psi(Int_\Psi(D)) \in PC(\Psi)$ and thus, $pCl_\Psi(D)\widetilde{\subseteq}D\widetilde{\cup}Cl_\Psi(Int_\Psi(D))$. Furthermore, since $pCl_\Psi(D) \in PC(\Psi)$, then $Cl_\Psi(Int_\Psi(D))\widetilde{\subseteq}Cl_\Psi(Int_\Psi(pCl_\Psi(D)))\widetilde{\subseteq}pCl_\Psi(D)$ and hence, $D\widetilde{\cup}Cl_\Psi(Int_\Psi(D))\widetilde{\subseteq}pCl_\Psi(D)$.

Theorem 2.20. For a soft function $f_{sv} : (G, \Psi, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$, the following are equivalent:

(a) f_{sv} is soft a.w.c.

(b) $pCl_\Psi\left(f_{sv}^{-1}(T)\right)\widetilde{\subseteq}f_{sv}^{-1}(Cl_\Phi(T))$ for every $T \in \Phi$.

(c) $f_{sv}^{-1}(T)\widetilde{\subseteq}pInt_\Psi\left(f_{sv}^{-1}(Cl_\Phi(T))\right)$ for every $T \in \Phi$.

(d) For each $a_x \in SP(G, \mathcal{A})$ and each $T \in \Phi$ such that $f_{sv}(a_x)\widetilde{\in}T$, there exists $K \in PO(\Psi)$ such that $a_x\widetilde{\in}K$ and $f_{sv}(K)\widetilde{\subseteq}Cl_\Phi(T)$.

Proof. (a) $\longrightarrow$ (b): Let $T \in \Phi$. Then by Theorem 2.10, $Cl_\Psi\left(Int_\Psi\left(f_{sv}^{-1}(T)\right)\right)\widetilde{\subseteq}f_{sv}^{-1}(Cl_\Phi(T))$. So,

$$\begin{aligned} f_{sv}^{-1}(T)\widetilde{\cup}Cl_\Psi\left(Int_\Psi\left(f_{sv}^{-1}(T)\right)\right) &\widetilde{\subseteq} f_{sv}^{-1}(T)\widetilde{\cup}f_{sv}^{-1}(Cl_\Phi(T)) \\ &= f_{sv}^{-1}(Cl_\Phi(T)). \end{aligned}$$

Furthermore, by Lemma 2.19, $pCl_\Psi(f_{sv}^{-1}(T)) = f_{sv}^{-1}(T)\widetilde{\cup}Cl_\Psi\left(Int_\Psi\left(f_{sv}^{-1}(T)\right)\right)$. It follows that $pCl_\Psi\left(f_{sv}^{-1}(T)\right)\widetilde{\subseteq}f_{sv}^{-1}(Cl_\Phi(T))$.

(b) $\longrightarrow$ (c): Let $T \in \Phi$. Then $1_\mathcal{B} - Cl_\Phi(T) \in \Phi$ and by (b),

$$pCl_\Psi\left(f_{sv}^{-1}(1_\mathcal{B} - Cl_\Phi(T))\right) \widetilde{\subseteq} f_{sv}^{-1}(Cl_\Phi(1_\mathcal{B} - Cl_\Phi(T))).$$

Now,

$$\begin{aligned} 1_\mathcal{A} - pInt_\Psi\left(f_{sv}^{-1}(Cl_\Phi(T))\right) &= pCl_\Psi\left(1_\mathcal{A} - f_{sv}^{-1}(Cl_\Phi(T))\right) \\ &= pCl_\Psi\left(f_{sv}^{-1}(1_\mathcal{B} - Cl_\Phi(T))\right). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} f_{sv}^{-1}(1_\mathcal{B} - Int_\Phi(Cl_\Phi(T))) &= 1_\mathcal{A} - f_{sv}^{-1}(Int_\Phi(Cl_\Phi(T))) \\ &\widetilde{\subseteq} 1_\mathcal{A} - f_{sv}^{-1}(T). \end{aligned}$$

Therefore, $1_\mathcal{A} - pInt_\Psi\left(f_{sv}^{-1}(Cl_\Phi(T))\right)\widetilde{\subseteq}1_\mathcal{A} - f_{sv}^{-1}(T)$. Hence, $f_{sv}^{-1}(T)\widetilde{\subseteq}pInt_\Psi\left(f_{sv}^{-1}(Cl_\Phi(T))\right)$.

(c) $\longrightarrow$ (d): Let $a_x \in SP(G, \mathcal{A})$ and let $T \in \Phi$ such that $f_{sv}(a_x)\widetilde{\in}T$. Then by (c), $a_x\widetilde{\in}f_{sv}^{-1}(T)\widetilde{\subseteq}pInt_\Psi\left(f_{sv}^{-1}(Cl_\Phi(T))\right)$. Let $K = pInt_\Psi\left(f_{sv}^{-1}(Cl_\Phi(T))\right)$. Then $a_x\widetilde{\in}K \in PO(\Psi)$ and

$$\begin{aligned} f_{sv}(K) &= f_{sv}\left(pInt_\Psi\left(f_{sv}^{-1}(Cl_\Phi(T))\right)\right) \\ &\widetilde{\subseteq} f_{sv}\left(f_{sv}^{-1}(Cl_\Phi(T))\right) \\ &\widetilde{\subseteq} Cl_\Phi(T). \end{aligned}$$

(d) $\longrightarrow$ (a): Let $T \in \Phi$. To show that $f_{sv}^{-1}(T)\widetilde{\subseteq}Int_\Psi\left(Cl_\Psi\left(f_{sv}^{-1}(Cl_\Phi(T))\right)\right)$, let $a_x\widetilde{\in}f_{sv}^{-1}(T)$. Then $f_{sv}(a_x)\widetilde{\in}T \in \Phi$. So, by (d), there exists $K \in PO(\Psi)$ such that $a_x\widetilde{\in}K$ and $f_{sv}(K)\widetilde{\subseteq}Cl_\Phi(T)$. Thus, $K\widetilde{\subseteq}f_{sv}^{-1}(Cl_\Phi(T))$, and so, $Int_\Psi(Cl_\Psi(K))\widetilde{\subseteq}Int_\Psi\left(Cl_\Psi\left(f_{sv}^{-1}(Cl_\Phi(T))\right)\right)$. Furthermore, since $K \in PO(\Psi)$, then $K\widetilde{\subseteq}Int_\Psi(Cl_\Psi(K))$. It follows that $a_x\widetilde{\in}Int_\Psi\left(Cl_\Psi\left(f_{sv}^{-1}(Cl_\Phi(T))\right)\right)$.

Lemma 2.21. Let $(G, \Psi, \mathcal{A})$ be a STS and let $T \in PO(\Psi)$. Then $Cl_\Phi(T) = Cl_\Phi(Int_\Phi(Cl_\Phi(T)))$.

Proof. Since $T \in PO(\Psi)$, then $T\widetilde{\subseteq}Int_{\Phi}(Cl_{\Phi}(T))$ and so $Cl_{\Phi}(T)\widetilde{\subseteq}Int_{\Phi}(Cl_{\Phi}(T))$. Furthermore, since $Int_{\Phi}(Cl_{\Phi}(T))\widetilde{\subseteq}Cl_{\Phi}(T)$, then $Cl_{\Phi}(Int_{\Phi}(Cl_{\Phi}(T)))\widetilde{\subseteq}Cl_{\Phi}(Cl_{\Phi}(T)) = Cl_{\Phi}(T)$.

Theorem 2.22. For a soft function $f_{sv} : (G, \Psi, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$, the following are equivalent:

(a) f_{sv} is soft a.w.c.

(b) $f_{sv}(pCl_{\Psi}(D))\widetilde{\subseteq}Cl^{\theta}_{\Phi}(f_{sv}(D))$ for every $D \in SS(G, \mathcal{A})$.

(c) $pCl_{\Psi}\left(f^{-1}_{sv}(M)\right)\widetilde{\subseteq}f^{-1}_{sv}\left(Cl^{\theta}_{\Phi}(M)\right)$ for every $M \in SS(H, \mathcal{B})$.

(d) $pCl_{\Psi}(f^{-1}_{sv}\left(Int_{\Phi}\left(Cl^{\theta}_{\Phi}(M)\right)\right))\widetilde{\subseteq}f^{-1}_{sv}\left(Cl^{\theta}_{\Phi}(M)\right)$ for every $M \in SS(H, \mathcal{B})$.

(e) $pCl_{\Psi}(f^{-1}_{sv}(Int_{\Phi}(Cl_{\Phi}(T))))\widetilde{\subseteq}f^{-1}_{sv}(Cl_{\Phi}(T))$ for every $T \in \Phi$.

(f) $pCl_{\Psi}(f^{-1}_{sv}(Int_{\Phi}(Cl_{\Phi}(T))))\widetilde{\subseteq}f^{-1}_{sv}(Cl_{\Phi}(T))$ for every $T \in PO(\Phi)$.

(g) $pCl_{\Psi}(f^{-1}_{sv}(Int_{\Phi}(R)))\widetilde{\subseteq}f^{-1}_{sv}(R)$ for every $R \in RC(\Phi)$.

Proof. (a) $\longrightarrow$ (b): Let $D \in SS(G, \mathcal{A})$. To see that $f_{sv}(pCl_{\Psi}(D))\widetilde{\subseteq}Cl^{\theta}_{\Phi}(f_{sv}(D))$, let $a_x\widetilde{\in}pCl_{\Psi}(D)$ and let $T \in \Phi$ such that $f_{sv}(a_x)\widetilde{\in}T$. Since f_{sv} is soft a.w.c, then by Theorem 2.20 (d), there exists $K \in PO(\Psi)$ such that $a_x\widetilde{\in}K$ and $f_{sv}(K)\widetilde{\subseteq}Cl_{\Phi}(T)$. Since $a_x\widetilde{\in}pCl_{\Psi}(D)$ and $a_x\widetilde{\in}K \in PO(\Psi)$, then $K\widetilde{\cap}D \neq 0_{\mathcal{A}}$ and hence, $0_{\mathcal{B}} \neq f_{sv}\left(K\widetilde{\cap}D\right)\widetilde{\subseteq}f_{sv}(K)\widetilde{\cap}f_{sv}(D)\widetilde{\subseteq}Cl_{\Phi}(T)\widetilde{\cap}f_{sv}(D)$. Therefore, we obtain $f_{sv}(a_x)\widetilde{\in}Cl^{\theta}_{\Phi}(f_{sv}(D))$.

(b) $\longrightarrow$ (c): Let $M \in SS(H, \mathcal{B})$. Then by (b),

$$\begin{aligned} f_{sv}\left(pCl_{\Psi}\left(f^{-1}_{sv}(M)\right)\right) &\widetilde{\subseteq} Cl^{\theta}_{\Phi}\left(f_{sv}\left(f^{-1}_{sv}(M)\right)\right) \\ &\widetilde{\subseteq} Cl^{\theta}_{\Phi}(M) \end{aligned}$$

And so,

$$\begin{aligned} pCl_{\Psi}\left(f^{-1}_{sv}(M)\right) &\widetilde{\subseteq} f^{-1}_{sv}\left(f_{sv}\left(pCl_{\Psi}\left(f^{-1}_{sv}(M)\right)\right)\right) \\ &\widetilde{\subseteq} f^{-1}_{sv}\left(Cl^{\theta}_{\Phi}(M)\right). \end{aligned}$$

(c) $\longrightarrow$ (d): Let $M \in SS(H, \mathcal{B})$. Then by (c),

$$pCl_{\Psi}(f^{-1}_{sv}\left(Int_{\Phi}\left(Cl^{\theta}_{\Phi}(M)\right)\right)) \widetilde{\subseteq} f^{-1}_{sv}\left(Cl^{\theta}_{\Phi}\left(Int_{\Phi}\left(Cl^{\theta}_{\Phi}(M)\right)\right)\right).$$

Since $Int_{\Phi}\left(Cl^{\theta}_{\Phi}(M)\right) \in \Phi$, then $Cl^{\theta}_{\Phi}\left(Int_{\Phi}\left(Cl^{\theta}_{\Phi}(M)\right)\right) = Cl_{\Phi}\left(Int_{\Phi}\left(Cl^{\theta}_{\Phi}(M)\right)\right)$. Since $Cl_{\Phi}\left(Int_{\Phi}\left(Cl^{\theta}_{\Phi}(M)\right)\right)\widetilde{\subseteq}Cl_{\Phi}\left(Cl^{\theta}_{\Phi}(M)\right)$ and $Cl^{\theta}_{\Phi}(M) \in \Phi^c$, then $Cl_{\Phi}\left(Cl^{\theta}_{\Phi}(M)\right) = Cl^{\theta}_{\Phi}(M)$. Therefore, we have

$$\begin{aligned} pCl_{\Psi}(f^{-1}_{sv}\left(Int_{\Phi}\left(Cl^{\theta}_{\Phi}(M)\right)\right) &\widetilde{\subseteq} f^{-1}_{sv}\left(Cl^{\theta}_{\Phi}\left(Int_{\Phi}\left(Cl^{\theta}_{\Phi}(M)\right)\right)\right) \\ &= f^{-1}_{sv}\left(Cl_{\Phi}\left(Int_{\Phi}\left(Cl^{\theta}_{\Phi}(M)\right)\right)\right) \\ &\widetilde{\subseteq} f^{-1}_{sv}\left(Cl_{\Phi}\left(Cl^{\theta}_{\Phi}(M)\right)\right) \\ &= f^{-1}_{sv}\left(Cl^{\theta}_{\Phi}(M)\right). \end{aligned}$$

(d) $\longrightarrow$ (e): Let $T \in \Phi$. Then, by (d), $pCl_{\Psi}(f^{-1}_{sv}\left(Int_{\Phi}\left(Cl^{\theta}_{\Phi}(T)\right)\right))\widetilde{\subseteq}f^{-1}_{sv}\left(Cl^{\theta}_{\Phi}(T)\right)$. Furthermore, since $T \in \Phi$, then $Cl^{\theta}_{\Phi}(T) = Cl_{\Phi}(T)$. Therefore,

$$pCl_{\Psi}(f^{-1}_{sv}(Int_{\Phi}(Cl_{\Phi}(T)))) \widetilde{\subseteq} f^{-1}_{sv}(Cl_{\Phi}(T)).$$

(e) $\longrightarrow$ (f): Let $T \in PO(\Phi)$. Then $Int_{\Phi}(Cl_{\Phi}(T)) \in \Phi$ and by (e), $pCl_{\Psi}(f^{-1}_{sv}(Int_{\Phi}(Cl_{\Phi}(Int_{\Phi}(Cl_{\Phi}(T))))))\widetilde{\subseteq}f^{-1}_{sv}(Cl_{\Phi}(Int_{\Phi}(Cl_{\Phi}(T))))$. Furthermore, since $T \in PO(\Phi)$, then by Lemma 2.21, $pCl_{\Psi}(f^{-1}_{sv}(Int_{\Phi}(Cl_{\Phi}(T))))\widetilde{\subseteq}f^{-1}_{sv}(Cl_{\Phi}(T))$.

(f) $\longrightarrow$ (g): Let $R \in RC(\Phi)$. Then $R = Cl_{\Phi}(Int_{\Phi}(R))$, and so, $Int_{\Phi}(R) = Int_{\Phi}(Cl_{\Phi}(Int_{\Phi}(R)))$. Since $Int_{\Phi}(R) \in PO(\Phi)$, then by (f),

$$\begin{aligned} pCl_{\Psi}(f^{-1}_{sv}(Int_{\Phi}(R))) &= pCl_{\Psi}(f^{-1}_{sv}(Int_{\Phi}(Cl_{\Phi}(Int_{\Phi}(R))))) \\ &\widetilde{\subseteq} f^{-1}_{sv}(Cl_{\Phi}(Int_{\Phi}(R))) \\ &= f^{-1}_{sv}(R). \end{aligned}$$

(g) $\longrightarrow$ (a): We will apply Theorem 2.20 (b). Let $T \in \Phi$. Then $Cl_{\Phi}(T) \in RC(\Phi)$ and by (g), $pCl_{\Psi}(f_{sv}^{-1}(Int_{\Phi}(Cl_{\Phi}(T))))\widetilde{\subseteq} f_{sv}^{-1}(Cl_{\Phi}(T))$. Since $T \in \Phi$, then $T\widetilde{\subseteq} Int_{\Phi}(Cl_{\Phi}(T))$, and so, $pCl_{\Psi}(f_{sv}^{-1}(T))\widetilde{\subseteq} pCl_{\Psi}(f_{sv}^{-1}(Int_{\Phi}(Cl_{\Phi}(T))))\widetilde{\subseteq} f_{sv}^{-1}(Cl_{\Phi}(T))$.

Theorem 2.23. Let $(H, \Phi, \mathcal{B})$ be a soft regular STS. The following are equivalent for a soft function $f_{sv} : (G, \Psi, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$:

(a) f_{sv} is soft pre-continuous.

(b) $f_{sv}^{-1}\left(Cl_{\Phi}^{\theta}(M)\right) \in PC(\Psi)$ for every $M \in SS(H, \mathcal{B})$.

(c) $f_{sv}^{-1}(S) \in PC(\Psi)$ for every $S \in (\Phi_{\theta})^{c}$.

(d) $f_{sv}^{-1}(T) \in PO(\Psi)$ for every $T \in \Phi_{\theta}$.

(e) f_{sv} is soft a.w.c.

Proof. (a) $\longrightarrow$ (b): Let $M \in SS(H, \mathcal{B})$. Since $Cl_{\Phi}(M) \in \Psi^{c}$, then by (a), $f_{sv}^{-1}(Cl_{\Phi}(M)) \in PC(\Psi)$. Furthermore, since $(H, \Phi, \mathcal{B})$ is soft regular, then $Cl_{\Phi}^{\theta}(M) = Cl_{\Phi}(M)$. Hence, $f_{sv}^{-1}\left(Cl_{\Phi}^{\theta}(M)\right) = f_{sv}^{-1}(Cl_{\Phi}(M)) \in PC(\Psi)$.

(b) $\longrightarrow$ (c): Let $S \in (\Phi_{\theta})^{c}$. Then $S \in \Phi^{c}$ and so $Cl_{\Phi}(S) = S$. Since $(H, \Phi, \mathcal{B})$ is soft regular, then $S = Cl_{\Phi}(S) = Cl_{\Phi}^{\theta}(S)$. Therefore, by (b), $f_{sv}^{-1}\left(Cl_{\Phi}^{\theta}(S)\right) = f_{sv}^{-1}(S) \in PC(\Psi)$.

(c) $\longrightarrow$ (d): Let $T \in \Phi_{\theta}$. Then $1_{\mathcal{B}} - T \in (\Phi_{\theta})^{c}$ and by (c), $f_{sv}^{-1}(1_{\mathcal{B}} - T) = 1_{\mathcal{A}} - f_{sv}^{-1}(T) \in PC(\Psi)$. It follows that $f_{sv}^{-1}(T) \in PO(\Psi)$.

(d) $\longrightarrow$ (e): Since $(H, \Phi, \mathcal{B})$ is soft regular, then $\Phi_{\theta} = \Phi$. Thus, f_{sv} is soft pre-continuous, and by Theorem 2.5, it is soft a.w.c.

(e) $\longrightarrow$ (a): Follows from Theorem 2.9.

Theorem 2.24. If $f_{sv} : (G, \Psi, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$ is a soft a.w.c function and $X \subseteq G$ such that $C_X \in \Psi - \{0_{\mathcal{A}}\}$, then $(f_{sv})_{|C_X} : (X, \Psi_X, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$ is soft a.w.c.

Proof. Let $T \in \Phi$. Since f_{sv} is soft a.w.c, then $f_{sv}^{-1}(T)\widetilde{\subseteq} Int_{\Psi}\left(Cl_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(T))\right)\right)$. Since $\left((f_{rw})_{|C_X}\right)^{-1}(T) = C_X\widetilde{\cap} f_{sv}^{-1}(T)$, then $\left((f_{rw})_{|C_X}\right)^{-1}(T)\widetilde{\subseteq} C_X\widetilde{\cap} Int_{\Psi}\left(Cl_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(T))\right)\right)$. Since $C_X \in \Psi$, then

$$\begin{aligned} C_X\widetilde{\cap} Int_{\Psi}\left(Cl_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(T))\right)\right) &= Int_{\Psi_X}\left(C_X\widetilde{\cap} Cl_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(T))\right)\right) \\ &\widetilde{\subseteq} Int_{\Psi_X}\left(C_X\widetilde{\cap} Cl_{\Psi}\left(C_X\widetilde{\cap} f_{sv}^{-1}(Cl_{\Phi}(T))\right)\right) \\ &= Int_{\Psi_X}\left(Cl_{\Psi_X}\left(\left((f_{rw})_{|C_X}\right)^{-1}(T)\right)\right). \end{aligned}$$

Therefore, $\left((f_{rw})_{|C_X}\right)^{-1}(T)\widetilde{\subseteq} Int_{\Psi_X}\left(Cl_{\Psi_X}\left(\left((f_{rw})_{|C_X}\right)^{-1}(T)\right)\right)$. Hence, $(f_{sv})_{|C_X} : (X, \Psi_X, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$ is soft a.w.c.

In Theorem 2.24, the condition '$C_X \in \Psi - \{0_{\mathcal{A}}\}$' cannot be dropped:

Example 2.25. Let $\mathfrak{I}$ be the usual topology $\mathbb{R}$. Consider the functions $s : (\mathbb{R}, \mathfrak{I}) \longrightarrow (\mathbb{R}, \mathfrak{I})$ and $v : \{a\} \longrightarrow \{a\}$, where $s(x) = 1$ if $x \in \mathbb{Q}$, $s(x) = -1$ if $x \in \mathbb{R} - \mathbb{Q}$, and $v(a) = a$. Let $X = ([0, \infty) \cap \mathbb{Q}) \cup ((-\infty, 0) \cap (\mathbb{R} - \mathbb{Q}))$ Then $f_{sv} : (\mathbb{R}, \tau(\mathfrak{I}), \{a\}) \longrightarrow (\mathbb{R}, \tau(\mathfrak{I}), \{a\})$ is soft a.w.c while $(f_{sv})_{|C_X} : (X, \tau(\mathfrak{I}), \{a\}) \longrightarrow (\mathbb{R}, \tau(\mathfrak{I}), \{a\})$ is not soft a.w.c.

Theorem 2.26. If $f_{s_1v_1} : (G, \Psi, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$ is soft a.w.c and $f_{s_2v_2} : (H, \Phi, \mathcal{B}) \longrightarrow (M, \Pi, \mathcal{L})$ is soft continuous, then $f_{(s_2\circ s_1)(v_2\circ v_1)} : (G, \varphi, \mathcal{A}) \longrightarrow (M, \Pi, \mathcal{L})$ is soft a.w.c.

Proof. Let $f_{s_1v_1}$ be soft a.w.c and $f_{s_2v_2}$ be soft continuous. Let $T \in \Pi$. Since $f_{s_2v_2}$ is soft continuous, then $f_{s_2v_2}^{-1}(T) \in \Phi$ and $Cl_{\Phi}\left(f_{s_2v_2}^{-1}(T)\right)\widetilde{\subseteq} f_{s_2v_2}^{-1}(Cl_{\Pi}(T))$. Since $f_{s_1v_1}$ is soft a.w.c, then $f_{s_1v_1}^{-1}\left(f_{s_2v_2}^{-1}(T)\right)\widetilde{\subseteq} Int_{\Psi}\left(Cl_{\Psi}\left(f_{s_1v_1}^{-1}\left(Cl_{\Phi}\left(f_{s_2v_2}^{-1}(T)\right)\right)\right)\right)$. Thus,

$$\begin{array}{rcl} f^{-1}_{(s_2\circ s_1)(v_2\circ v_1)}(T) & = & f^{-1}_{s_1v_1}\left(f^{-1}_{s_2v_2}(T)\right) \\ & \widetilde{\subseteq} & Int_\Psi\left(Cl_\Psi\left(f^{-1}_{s_1v_1}\left(Cl_\Phi\left(f^{-1}_{s_2v_2}(T)\right)\right)\right)\right) \\ & \widetilde{\subseteq} & Int_\Psi\left(Cl_\Psi\left(f^{-1}_{s_1v_1}\left(f^{-1}_{s_2v_2}(Cl_\Pi(T))\right)\right)\right) \\ & = & Int_\Psi\left(Cl_\Psi\left(f^{-1}_{(s_2\circ s_1)(v_2\circ v_1)}(Cl_\Pi(T))\right)\right). \end{array}$$

It follows that $f_{(s_2\circ s_1)(v_2\circ v_1)}$ is soft a.w.c.

It is not necessary for the soft composition of two soft a.w.c. functions to be soft a.w.c:

Example 2.27. Let $G = \{a,b\}$, $H = M = \{a,b,c\}$, $\mathcal{A} = \{1,2\}$, $\Im = \{\emptyset, G, \{b\}\}$, $\aleph = \{\emptyset, H, \{a,c\}, \{b,c\}, \{c\}\}$, and $\wp = \{\emptyset, M, \{a\}, \{b\}, \{a,b\}\}$. Let $s_1 : (G,\Im) \longrightarrow (H,\aleph)$, $s_2 : (H,\aleph) \longrightarrow (M,\wp)$ be the inclusion functions and $v_1, v_2 : \mathcal{A} \longrightarrow \mathcal{A}$ be the identity functions. Then $f_{s_1v_1} : (G,\tau(\Im),\mathcal{A}) \longrightarrow (H,\tau(\aleph),\mathcal{A})$ and $f_{s_2v_2} : (H,\tau(\aleph),\mathcal{A}) \longrightarrow (M,\tau(\wp),\mathcal{A})$ are soft a.w.c, while $f_{(s_2\circ s_1)(v_2\circ v_1)} : (G,\tau(\Im),\mathcal{A}) \longrightarrow (M,\tau(\wp),\mathcal{A})$ is not soft a.w.c.

The composition $f_{(s_2\circ s_1)(v_2\circ v_1)}$ of a soft continuous function $f_{s_1v_1} : (G,\Psi,\mathcal{A}) \longrightarrow (H,\Phi,\mathcal{B})$ and a soft pre-continuous function $f_{s_2v_2} : (H,\Phi,\mathcal{B}) \longrightarrow (M,\Pi,\mathcal{L})$ is not necessarily soft a.w.c:

Example 2.28. Let $\Im$, $\aleph$, and $\wp$ be the usual, indiscrete, and discrete topologies on $\mathbb{R}$, respectively. Let $\mathcal{A} = \{a,b\}$. Let $s_1 : (\mathbb{R},\Im) \longrightarrow (\mathbb{R},\aleph)$, $s_2 : (\mathbb{R},\aleph) \longrightarrow (\mathbb{R},\wp)$ and $v_1, v_2 : \mathcal{A} \longrightarrow \mathcal{A}$ be the identity functions. Then $f_{s_1v_1} : (\mathbb{R},\tau(\Im),\mathcal{A}) \longrightarrow (\mathbb{R},\tau(\aleph),\mathcal{A})$ is soft continuous and $f_{s_2v_2} : (\mathbb{R},\tau(\aleph),\mathcal{A}) \longrightarrow (\mathbb{R},\tau(\wp),\mathcal{A})$ is soft pre-continuous, while $f_{(s_2\circ s_1)(v_2\circ v_1)} : (\mathbb{R},\tau(\Im),\mathcal{A}) \longrightarrow (\mathbb{R},\tau(\wp),\mathcal{A})$ is not soft a.w.c.

Let G and H be two non-empty sets. The projection functions $h : G\times H \longrightarrow G$ and $g : G\times H \longrightarrow H$ defined by $h(x,y) = x$ and $g(x,y) = y$ for all $(x,y) \in G\times H$ will be denoted by π_G and π_H, respectively.

Theorem 2.29. Let $(G,\Psi,\mathcal{A})$, $(H,\Phi,\mathcal{B})$, and $(M,\Pi,\mathcal{L})$ be three STSs. If $f_{sv} : (G,\Psi,\mathcal{A}) \longrightarrow (H\times M, pr(\Phi\times\Pi), \mathcal{B}\times\mathcal{L})$ is soft a.w.c, then $f_{(\pi_H\circ s)(\pi_{\mathcal{B}}\circ v)} : (G,\Psi,\mathcal{A}) \longrightarrow (H,\Phi,\mathcal{B})$ and $f_{(\pi_M\circ s)(\pi_{\mathcal{L}}\circ v)} : (G,\Psi,\mathcal{A}) \longrightarrow (M,\Pi,\mathcal{L})$ are soft a.w.c.

Proof. Let f_{sv} be soft a.w.c. Since $f_{(\pi_H)(\pi_{\mathcal{B}})} : (H\times M, pr(\Phi\times\Pi), \mathcal{B}\times\mathcal{L}) \longrightarrow (H,\Phi,\mathcal{B})$ and $f_{(\pi_M)(\pi_{\mathcal{L}})} : (H\times M, pr(\Phi\times\Pi), \mathcal{B}\times\mathcal{L}) \longrightarrow (M,\Pi,\mathcal{L})$ are always soft continuous, then by Theorem 2.26, $f_{(\pi_H\circ s)(\pi_{\mathcal{B}}\circ v)} : (G,\Psi,\mathcal{A}) \longrightarrow (H,\Phi,\mathcal{B})$ and $f_{(\pi_M\circ s)(\pi_{\mathcal{L}}\circ v)} : (G,\Psi,\mathcal{A}) \longrightarrow (M,\Pi,\mathcal{L})$ are soft a.w.c.

For every function $p : G \longrightarrow H$, the function $h : G \longrightarrow G\times H$ defined by $h(x) = (x, p(x))$ is represented by $p^{\#}$.

Theorem 2.30. Let $f_{sv} : (G,\Psi,\mathcal{A}) \longrightarrow (H,\Phi,\mathcal{B})$ be a soft function. Then $f_{s^{\#}v^{\#}} : (G,\Psi,\mathcal{A}) \longrightarrow (G\times H, pr(\Psi\times\Phi), \mathcal{A}\times\mathcal{B})$ is soft a.w.c iff f_{sv} is soft a.w.c.

Proof. Necessity. Let $f_{s^{\#}v^{\#}}$ be soft a.w.c. Then, by Theorem 2.29, $f_{sv} = f_{(\pi_H\circ s^{\#})(\pi_{\mathcal{B}}\circ v^{\#})} : (G,\Psi,\mathcal{A}) \longrightarrow (H,\Phi,\mathcal{B})$ is soft a.w.c.

Sufficiency. Let f_{sv} be soft a.w.c. We will apply Theorem 2.10 (c). Let $a_x \in SP(G,\mathcal{A})$ and let $R \in pr(\Psi\times\Phi)$ such that $f_{s^{\#}v^{\#}}(a_x)\widetilde{\in}R$. Choose $U \in \Psi$ and $W \in \Phi$ such that $f_{s^{\#}v^{\#}}(a_x) = (a, v(a))_{(x,s(x))}\widetilde{\in}U\times W\widetilde{\subseteq}R$. Since f_{sv} is soft a.w.c and $f_{sv}(a_x)\widetilde{\in}W \in \Phi$, then by Theorem 2.10 (c), $Cl_\Psi\left(f^{-1}_{sv}(Cl_\Phi(W))\right)$ is a soft neighborhood of a_x and by Lemma 1.18 (a), $U\widetilde{\cap}Cl_\Psi\left(f^{-1}_{sv}(Cl_\Phi(W))\right)\widetilde{\subseteq}Cl_\Psi\left(U\widetilde{\cap}f^{-1}_{sv}(Cl_\Phi(W))\right)$. Furthermore, we have

$$\begin{array}{rcl} U\widetilde{\cap}f^{-1}_{sv}(Cl_\Phi(W)) & \widetilde{\subseteq} & f^{-1}_{s^{\#}v^{\#}}(U\times Cl_\Phi(W)) \\ & \widetilde{\subseteq} & f^{-1}_{s^{\#}v^{\#}}\left(Cl_{pr(\Psi\times\Phi)}(R)\right). \end{array}$$

Therefore, $Cl_\Psi\left(f^{-1}_{s^{\#}v^{\#}}\left(Cl_{pr(\Psi\times\Phi)}(R)\right)\right)$ is a soft neighborhood of a_x. Hence, by Theorem 2.10 (c), $f_{s^{\#}v^{\#}}$ is soft a.w.c.

Theorem 2.31. Let $f_{s_1v_1} : (G, \Psi, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$ be a soft open continuous surjection and $f_{s_2v_2} : (H, \Phi, \mathcal{B}) \longrightarrow (M, \Pi, \mathcal{L})$ be a soft function. If $f_{(s_2\circ s_1)(v_2\circ v_1)} : (G, \Psi, \mathcal{A}) \longrightarrow (M, \Pi, \mathcal{L})$ is soft a.w.c, then $f_{s_2v_2}$ is soft a.w.c.

Proof. Let $T \in \Pi$. Since $f_{(s_2\circ s_1)(v_2\circ v_1)}$ is soft a.w.c, then

$$\begin{aligned} f^{-1}_{s_1v_1}\left(f^{-1}_{s_2v_2}(T)\right) &= f^{-1}_{(s_2\circ s_1)(v_2\circ v_1)}(T) \\ &\widetilde{\subseteq} Int_\Psi\left(Cl_\Psi\left(f^{-1}_{(s_2\circ s_1)(v_2\circ v_1)}(Cl_\Pi(T))\right)\right) \\ &= Int_\Psi\left(Cl_\Psi\left(f^{-1}_{s_1v_1}\left(f^{-1}_{s_2v_2}(Cl_\Pi(T))\right)\right)\right). \end{aligned}$$

Since $f_{s_1v_1}$ is soft continuous, then $Cl_\Psi\left(f^{-1}_{s_1v_1}\left(f^{-1}_{s_2v_2}(Cl_\Pi(T))\right)\right)\widetilde{\subseteq} f^{-1}_{s_1v_1}\left(Cl_\Psi\left(\left(f^{-1}_{s_2v_2}(Cl_\Pi(T))\right)\right)\right)$. Therefore, we have

$f^{-1}_{s_1v_1}\left(f^{-1}_{s_2v_2}(T)\right)\widetilde{\subseteq} Int_\Psi\left(f^{-1}_{s_1v_1}\left(Cl_\Psi\left(\left(f^{-1}_{s_2v_2}(Cl_\Pi(T))\right)\right)\right)\right)$, and so

$f_{s_1v_1}\left(f^{-1}_{s_1v_1}\left(f^{-1}_{s_2v_2}(T)\right)\right)\widetilde{\subseteq} f_{s_1v_1}\left(Int_\Psi\left(Cl_\Psi\left(f^{-1}_{s_1v_1}\left(f^{-1}_{s_2v_2}(Cl_\Pi(T))\right)\right)\right)\right)$.

Since $f_{s_1v_1}$ is surjective, then $f_{s_1v_1}\left(f^{-1}_{s_1v_1}\left(f^{-1}_{s_2v_2}(T)\right)\right) = f^{-1}_{s_2v_2}(T)$.

Since $f_{s_1v_1}$ is soft open, then

$$f_{s_1v_1}\left(Int_\Psi\left(Cl_\Psi\left(f^{-1}_{s_1v_1}\left(f^{-1}_{s_2v_2}(Cl_\Pi(T))\right)\right)\right)\right) \widetilde{\subseteq} Int_\Phi\left(f_{s_1v_1}\left(Cl_\Psi\left(f^{-1}_{s_1v_1}\left(f^{-1}_{s_2v_2}(Cl_\Pi(T))\right)\right)\right)\right).$$

Since $f_{s_1v_1}$ is soft continuous, then

$$\begin{aligned} f_{s_1v_1}\left(Cl_\Psi\left(f^{-1}_{s_1v_1}\left(f^{-1}_{s_2v_2}(Cl_\Pi(T))\right)\right)\right) &\widetilde{\subseteq} Cl_\Phi\left(f_{s_1v_1}\left(f^{-1}_{s_1v_1}\left(f^{-1}_{s_2v_2}(Cl_\Pi(T))\right)\right)\right) \\ &\widetilde{\subseteq} Cl_\Phi\left(f^{-1}_{s_2v_2}(Cl_\Pi(T))\right). \end{aligned}$$

Therefore, we have $f^{-1}_{s_2v_2}(T)\widetilde{\subseteq} Int_\Phi\left(Cl_\Phi\left(f^{-1}_{s_2v_2}(Cl_\Pi(T))\right)\right)$ and hence, $f_{s_2v_2}$ is soft a.w.c.

Theorem 2.32. Let $f_{s_1v_1} : (G, \Psi, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$ and $f_{s_2v_2} : (M, \Gamma, \mathcal{L}) \longrightarrow (N, \Pi, \mathcal{F})$ be two soft functions. Let $s^* : G \times M \longrightarrow H \times N$ and $v^* : \mathcal{A} \times \mathcal{L} \longrightarrow \mathcal{B} \times \mathcal{F}$ be the functions defined by $s^*(x, z) = (s_1(x), s_2(z))$ and $v^*(a, l) = (v_1(a), v_2(l))$. Then $f_{s^*v^*} : (G \times M, pr(\Psi\times\Gamma), \mathcal{A} \times \mathcal{L}) \longrightarrow (H \times N, pr(\Phi\times\Pi), \mathcal{B} \times \mathcal{F})$ is soft a.w.c iff $f_{s_1v_1}$ and $f_{s_2v_2}$ are soft a.w.c.

Proof. Necessity. Let $T \in \Phi - \{0_\mathcal{B}\}$ and $R \in \mathcal{F} - \{0_\mathcal{F}\}$. Then $T \times R \in pr(\Phi\times\Pi)$ and so,

$$\begin{aligned} f^{-1}_{s_1v_1}(T) \times f^{-1}_{s_2v_2}(R) &= f^{-1}_{s^*v^*}(T \times R) \\ &\widetilde{\subseteq} Int_{pr(\Psi\times\Gamma)}\left(Cl_{pr(\Psi\times\Gamma)}\left(f^{-1}_{s^*v^*}\left(Cl_{pr(\Phi\times\Pi)}(T \times R)\right)\right)\right) \\ &= Int_{pr(\Psi\times\Gamma)}\left(Cl_{pr(\Psi\times\Gamma)}\left(f^{-1}_{s^*v^*}(Cl_\Phi(T) \times Cl_\Pi(R))\right)\right) \\ &= Int_{pr(\Psi\times\Gamma)}Cl_{pr(\Psi\times\Gamma)}\left(f^{-1}_{s_1v_1}(Cl_\Phi(T)) \times f^{-1}_{s_2v_2}(Cl_\Pi(R))\right) \\ &= Int_{pr(\Psi\times\Gamma)}\left(Cl_\Psi\left(f^{-1}_{s_1v_1}(Cl_\Phi(T))\right) \times Cl_\Gamma\left(f^{-1}_{s_2v_2}(Cl_\Pi(R))\right)\right) \\ &= Int_\Psi\left(Cl_\Psi\left(f^{-1}_{s_1v_1}(Cl_\Phi(T))\right)\right) \times Int_\Psi\left(Cl_\Gamma\left(f^{-1}_{s_2v_2}(Cl_\Pi(R))\right)\right). \end{aligned}$$

This implies that $f^{-1}_{s_1v_1}(T)\widetilde{\subseteq} Int_\Psi\left(Cl_\Psi\left(f^{-1}_{s_1v_1}(Cl_\Phi(T))\right)\right)$ and

$f^{-1}_{s_2v_2}(R)\widetilde{\subseteq} Int_\Gamma\left(Cl_\Gamma\left(f^{-1}_{s_2v_2}(Cl_\Pi(R))\right)\right)$. It follows that $f_{s_1v_1}$ and $f_{s_2v_2}$ are soft a.w.c.

Sufficiency. Let $f_{s_1v_1}$ and $f_{s_2v_2}$ be soft a.w.c. We will apply Theorem 2.10 (c). Let $(a, b)_{(x,y)} \in SP(G \times M, \mathcal{A} \times \mathcal{L})$ and let $Z \in pr(\Phi\times\Pi)$ such that $f_{s^*v^*}\left((a, b)_{(x,y)}\right)\widetilde{\in} Z$. Choose $U \in \Psi$ and $W \in \Phi$ such that $f_{s^*v^*}\left((a, b)_{(x,y)}\right) = (v_1(a), v_2(b))_{(s_1(x),s_2(y))}\widetilde{\in} U \times W\widetilde{\subseteq} Z$. Then $(v_1(a))_{s_1(x)}\widetilde{\in} U$ and $(v_2(b))_{s_2(y)}\widetilde{\in} W$. Since $f_{s_1v_1}$ and $f_{s_2v_2}$ are soft a.w.c, then $Cl_\Psi\left(f^{-1}_{s_1v_1}(Cl_\Phi(U))\right)$ is a soft neighborhood of $(v_1(a))_{s_1(x)}$ and $Cl_\Gamma\left(f^{-1}_{s_2v_2}(Cl_\Pi(W))\right)$ is a soft neighborhood of $(v_2(b))_{s_2(y)}$. Then we have $Cl_\Psi\left(f^{-1}_{s_1v_1}(Cl_\Phi(U))\right) \times Cl_\Gamma\left(f^{-1}_{s_2v_2}(Cl_\Pi(W))\right)$ is a soft neighborhood of $(v_1(a), v_2(b))_{(s_1(x),s_2(y))}$. But

$$\begin{aligned} Cl_{\Psi}\left(f^{-1}_{s_1v_1}\left(Cl_{\Phi}\left(U\right)\right)\right)\times Cl_{\Gamma}\left(f^{-1}_{s_2v_2}\left(Cl_{\Pi}\left(W\right)\right)\right) &= Cl_{pr(\Psi\times\Gamma)}\left(f^{-1}_{s_1v_1}\left(Cl_{\Phi}\left(U\right)\right)\times f^{-1}_{s_2v_2}\left(Cl_{\Pi}\left(W\right)\right)\right) \\ &= Cl_{pr(\Psi\times\Gamma)}\left(f^{-1}_{s^*v^*}\left(Cl_{\Phi}\left(U\right)\times Cl_{\Pi}\left(W\right)\right)\right) \\ &= Cl_{pr(\Psi\times\Gamma)}\left(f^{-1}_{s^*v^*}\left(Cl_{pr(\Phi\times\Pi)}\left(U\times W\right)\right)\right) \\ &\widetilde{\subseteq} Cl_{pr(\Psi\times\Gamma)}\left(f^{-1}_{s^*v^*}\left(Cl_{pr(\Phi\times\Pi)}\left(Z\right)\right)\right). \end{aligned}$$

This shows that $Cl_{pr(\Psi\times\Gamma)}\left(f^{-1}_{s^*v^*}\left(Cl_{pr(\Phi\times\Pi)}\left(Z\right)\right)\right)$ is a soft neighborhood of $\left(v_1\left(a\right), v_2\left(b\right)\right)_{(s_1(x),s_2(y))}$. Hence, $f_{s^*v^*}$ is soft a.w.c.

3. Soft Hausdorff spaces and soft almost weakly continuous functions

Theorem 3.1. If $f_{sv}:(G,\Psi,\mathcal{A})\longrightarrow(H,\Phi,\mathcal{B})$ is a soft a.w.c function and $(H,\Phi,\mathcal{B})$ is soft Hausdorff, then $Graph\left(f_{sv}\right)\in PC\left(pr\left(\Psi\times\Phi\right)\right)$.

Proof. We will show that $1_{\mathcal{A}\times\mathcal{B}}-Graph\left(f_{sv}\right)\widetilde{\subseteq}1_{\mathcal{A}\times\mathcal{B}}-pCl_{\Psi\times\Phi}\left(Graph\left(f_{sv}\right)\right)$. Let $(d,e)_{(x,y)}\widetilde{\in}1_{\mathcal{A}\times\mathcal{B}}-Graph\left(f_{sv}\right)$. Then $f_{sv}\left(d_x\right)\neq e_y$. Since $(H,\Phi,\mathcal{B})$ is soft Hausdorff, then there are $T,R\in\Phi$ such that $f_{sv}\left(d_x\right)\widetilde{\in}T$, $e_y\widetilde{\in}R$, and $T\widetilde{\cap}R=0_{\mathcal{B}}$; hence $Cl_{\Phi}\left(T\right)\widetilde{\cap}R=0_{\mathcal{B}}$. Since f_{sv} is soft a.w.c, by Theorem 2.20 (d), there exists $K\in PO\left(\Psi\right)$ such that $a_x\widetilde{\in}K$ and $f_{sv}(K)\widetilde{\subseteq}Cl_{\Phi}\left(T\right)$. Thus, $f_{sv}(K)\widetilde{\cap}R=0_{\mathcal{B}}$. Therefore, we have $(d,e)_{(x,y)}\widetilde{\in}K\times R\in PO\left(pr\left(\Psi\times\Phi\right)\right)$ and $\left(K\times R\right)\widetilde{\cap}Graph\left(f_{sv}\right)=0_{\mathcal{A}\times\mathcal{B}}$. This implies that $(d,e)_{(x,y)}\widetilde{\in}1_{\mathcal{A}\times\mathcal{B}}-pCl_{\Psi\times\Phi}\left(Graph\left(f_{sv}\right)\right)$.

Corollary 3.2. If $f_{sv}:(G,\Psi,\mathcal{A})\longrightarrow(H,\Phi,\mathcal{B})$ is a soft weakly continuous function and $(H,\Phi,\mathcal{B})$ is soft Hausdorff, then $Graph\left(f_{sv}\right)\in PC\left(pr\left(\Psi\times\Phi\right)\right)$.

Proof. Theorems 2.4 and 3.1 provide the proof.

Corollary 3.3. If $f_{sv}:(G,\Psi,\mathcal{A})\longrightarrow(H,\Phi,\mathcal{B})$ is a soft pre-continuous function and $(H,\Phi,\mathcal{B})$ is soft Hausdorff, then $Graph\left(f_{sv}\right)\in PC\left(pr\left(\Psi\times\Phi\right)\right)$.

Proof. Theorems 2.5 and 3.1 provide the proof.

Lemma 3.4. Let $(G,\Psi,\mathcal{A})$ be a STS. If $X\in\alpha\left(\Psi\right)$ and $Y\in PO\left(\Psi\right)$, then $X\widetilde{\cap}Y\in PO\left(\Psi\right)$.

Proof. Let $X\in\alpha\left(\Psi\right)$ and $Y\in PO\left(\Psi\right)$. Then $X\widetilde{\subseteq}Int_{\Psi}\left(Cl_{\Psi}\left(Int_{\Psi}\left(X\right)\right)\right)$ and $Y\widetilde{\subseteq}Int_{\Psi}\left(Cl_{\Psi}\left(Y\right)\right)$. By Lemma 2.18 (a),

$$\begin{aligned} X\widetilde{\cap}Y &\widetilde{\subseteq} Int_{\Psi}\left(Cl_{\Psi}\left(Int_{\Psi}\left(X\right)\right)\right)\widetilde{\cap}Int_{\Psi}\left(Cl_{\Psi}\left(Y\right)\right) \\ &= Int_{\Psi}\left(Cl_{\Psi}\left(Int_{\Psi}\left(X\right)\right)\widetilde{\cap}Int_{\Psi}\left(Cl_{\Psi}\left(Y\right)\right)\right) \\ &\widetilde{\subseteq} Int_{\Psi}\left(Cl_{\Psi}\left(Int_{\Psi}\left(X\right)\widetilde{\cap}Int_{\Psi}\left(Cl_{\Psi}\left(Y\right)\right)\right)\right) \\ &\widetilde{\subseteq} Int_{\Psi}\left(Cl_{\Psi}\left(Int_{\Psi}\left(X\right)\widetilde{\cap}\left(Cl_{\Psi}\left(Y\right)\right)\right)\right) \\ &\widetilde{\subseteq} Int_{\Psi}\left(Cl_{\Psi}\left(Cl_{\Psi}\left(Int_{\Psi}\left(X\right)\widetilde{\cap}Y\right)\right)\right) \\ &= Int_{\Psi}\left(Cl_{\Psi}\left(Int_{\Psi}\left(X\right)\widetilde{\cap}Y\right)\right) \\ &\widetilde{\subseteq} Int_{\Psi}\left(Cl_{\Psi}\left(X\widetilde{\cap}Y\right)\right). \end{aligned}$$

Therefore, $X\widetilde{\cap}Y\in PO\left(\Psi\right)$.

Theorem 3.5. Let $f_{sv},f_{pu}:(G,\Psi,\mathcal{A})\longrightarrow(H,\Phi,\mathcal{B})$ be soft functions, and $(H,\Phi,\mathcal{B})$ be soft Hausdorff. If f_{sv} is soft almost α-continuous, f_{pu} is soft a.w.c, and $Z=\widetilde{\cup}\left\{a_x\in SP\left(G,\mathcal{A}\right):f_{sv}\left(a_x\right)=f_{pu}\left(a_x\right)\right\}$, then $Z\in PC\left(\Psi\right)$.

Proof. We will show that $1_{\mathcal{A}}-Z\widetilde{\subseteq}1_{\mathcal{A}}-pCl_{\Psi}\left(Z\right)$. Let $a_x\widetilde{\in}1_{\mathcal{A}}-Z$. Then $f_{sv}\left(a_x\right)\neq f_{pu}\left(a_x\right)$. Since $(H,\Phi,\mathcal{B})$ is soft Hausdorff, then there are $U,V\in\Phi$ such that $f_{sv}\left(a_x\right)\widetilde{\in}U$, $f_{pu}\left(a_x\right)\widetilde{\in}V$, and $U\widetilde{\cap}V=0_{\mathcal{B}}$; hence $Int_{\Phi}\left(Cl_{\Phi}\left(U\right)\right)\widetilde{\cap}Cl_{\Phi}\left(V\right)=0_{\mathcal{B}}$. Since f_{sv} is soft almost α-continuous, and $Int_{\Phi}\left(Cl_{\Phi}\left(U\right)\right)\in RO\left(\Phi\right)$, then there exists $Y\in\alpha\left(\Psi\right)$ such that $a_x\widetilde{\in}Y$ and $f_{sv}\left(Y\right)\widetilde{\subseteq}Int_{\Phi}\left(Cl_{\Phi}\left(U\right)\right)$. Since f_{pu} is soft a.w.c, then by Theorem 2.20 (d), there exists $K \in PO(\Psi)$ such that $a_x \widetilde{\in} K$ and $f_{pu}(K) \widetilde{\subseteq} Cl_{\Phi}(V)$. We have $f_{sv}(Y) \widetilde{\cap} f_{pu}(K) = 0_{\mathcal{B}}$ and thus, $\left(K \widetilde{\cap} Y\right) \widetilde{\cap} Z = 0_{\mathcal{A}}$. Furthermore, by Lemma 3.4, $K \widetilde{\cap} Y \in PO(\Psi)$. This shows that $a_x \widetilde{\in} 1_{\mathcal{A}} - pCl_{\Psi}(Z)$.

Corollary 3.6. Let $f_{sv}, f_{pu} : (G, \Psi, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$ be soft functions, and $(H, \Phi, \mathcal{B})$ is soft Hausdorff. If f_{sv} is soft almost α-continuous, f_{pu} is soft weakly continuous, and $Z = \widetilde{\cup}\left\{a_x \in SP(G, \mathcal{A}) : f_{sv}(a_x) = f_{pu}(a_x)\right\}$, then $Z \in PC(\Psi)$.

Proof. Theorems 2.4 and 3.5 provide the proof.

Corollary 3.7. Let $f_{sv}, f_{pu} : (G, \Psi, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$ be soft functions, and $(H, \Phi, \mathcal{B})$ is soft Hausdorff. If f_{sv} is soft almost α-continuous, f_{pu} is soft pre-continuous, and $Z = \widetilde{\cup}\left\{a_x \in SP(G, \mathcal{A}) : f_{sv}(a_x) = f_{pu}(a_x)\right\}$, then $Z \in PC(\Psi)$.

Proof. Theorems 2.5 and 3.5 provide the proof.

Theorem 3.8. Let $(G, \Psi, \mathcal{A})$ be soft Hausdorff, and let Y be a non-empty subset of G. If there is a soft a.w.c function $f_{sv} : (G, \Psi, \mathcal{A}) \longrightarrow (Y, \Psi_Y, \mathcal{A})$ such that $f_{sv}(a_x) = a_x$ for all $a_x \in SP(G, \mathcal{A})$, then $C_Y \in PC(\Psi)$.

Proof. Suppose, on the contrary, there exists $a_x \widetilde{\in} pCl_{\Psi}(C_Y) - C_Y$. Then $f_{sv}(a_x) \neq a_x$. Since $(G, \Psi, \mathcal{A})$ is soft Hausdorff, there exist $U, V \in \Psi$ such that $a_x \widetilde{\in} U$, $f_{sv}(a_x) \widetilde{\in} V$, and $U \widetilde{\cap} V = 0_{\mathcal{A}}$; hence $U \widetilde{\cap} Cl_{\Psi}(V) = 0_{\mathcal{A}}$. Since $V \widetilde{\cap} C_Y \in \Psi_Y$ and f_{sv} is soft a.w.c, then by Theorem 2.20 (d), there exists $K \in PO(\Psi)$ such that $a_x \widetilde{\in} K$ and $f_{sv}(K) \widetilde{\subseteq} Cl_{\Psi_Y}\left(V \widetilde{\cap} C_Y\right) \widetilde{\subseteq} Cl_{\Psi}(V)$. Since $a_x \widetilde{\in} U \widetilde{\cap} K \in PO(\Psi)$ and $a_x \widetilde{\in} pCl_{\Psi}(C_Y)$, then $\left(U \widetilde{\cap} K\right) \widetilde{\cap} C_Y \neq 0_{\mathcal{A}}$. Choose $b_y \widetilde{\in} \left(U \widetilde{\cap} K\right) \widetilde{\cap} C_Y$. Since $b_y \widetilde{\in} C_Y$, then $f_{sv}\left(b_y\right) = b_y$. Since $b_y \widetilde{\in} K$, then $f_{sv}\left(b_y\right) = b_y \widetilde{\in} f_{sv}(K) \widetilde{\subseteq} Cl_{\Psi}(V)$. Since $U \widetilde{\cap} Cl_{\Psi}(V) = 0_{\mathcal{A}}$, then $b_y \widetilde{\notin} U$. This is a contradiction.

Corollary 3.9. Let $(G, \Psi, \mathcal{A})$ be soft Hausdorff, and let Y be a non-empty subset of G. If there is a soft weakly continuous function $f_{sv} : (G, \Psi, \mathcal{A}) \longrightarrow (Y, \Psi_Y, \mathcal{A})$ such that $f_{sv}(a_x) = a_x$ for all $a_x \in SP(G, \mathcal{A})$, then $C_Y \in PC(\Psi)$.

Proof. Theorems 2.4 and 3.8 provide the proof.

Corollary 3.10. Let $(G, \Psi, \mathcal{A})$ be soft Hausdorff, and let Y be a non-empty subset of G. If there is a soft pre-continuous function $f_{sv} : (G, \Psi, \mathcal{A}) \longrightarrow (Y, \Psi_Y, \mathcal{A})$ such that $f_{sv}(a_x) = a_x$ for all $a_x \in SP(G, \mathcal{A})$, then $C_Y \in PC(\Psi)$.

Proof. Theorems 2.5 and 3.8 provide the proof.

Theorem 3.11. If $f_{sv} : (G, \Psi, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$ is a soft a.w.c injection such that $Graph(f_{sv})$ is soft strongly closed with respect to $(G \times H, pr(\Psi \times \Phi), \mathcal{A} \times \mathcal{B})$, then $(G, \Psi, \mathcal{A})$ is soft Hausdorff.

Proof. Let $a_x, d_z \in SP(G, \mathcal{A})$ such that $a_x \neq d_z$. Since f_{sv} is injective, then $f_{sv}(a_x) \neq f_{sv}(d_z)$. Thus, we have $(a, v(d))_{(x, s(z))} \widetilde{\in} 1_{\mathcal{A} \times \mathcal{B}} - Graph(f_{sv})$. Since $Graph(f_{sv})$ is soft strongly closed with respect to $(G \times H, pr(\Psi \times \Phi), \mathcal{A} \times \mathcal{B})$, then there exist $U \in \Psi$ and $V \in \Phi$ such that $a_x \widetilde{\in} U$, $f_{sv}(d_z) \widetilde{\in} V$, and $(U \times Cl_{\Phi}(V)) \widetilde{\cap} Graph(f_{sv}) = 0_{\mathcal{A} \times \mathcal{B}}$. Thus, we have $U \widetilde{\cap} f_{sv}^{-1}(Cl_{\Phi}(V)) = 0_{\mathcal{A}}$ and hence, $U \widetilde{\cap} Int_{\Psi}\left(Cl_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(V))\right)\right) = 0_{\mathcal{A}}$. Since f_{sv} is soft a.w.c, then $d_z \widetilde{\in} f_{sv}^{-1}(V) \widetilde{\subseteq} Int_{\Psi}\left(Cl_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(V))\right)\right)$. It follows that $(G, \Psi, \mathcal{A})$ is soft Hausdorff.

Theorem 3.12. If $f_{sv} : (G, \Psi, \mathcal{A}) \longrightarrow (H, \Phi, \mathcal{B})$ is a soft a.w.c injection and $(H, \Phi, \mathcal{B})$ is soft Urysohn, then $(G, \Psi, \mathcal{A})$ is soft pre-T_2.

Proof. $a_x, d_z \in SP(G, \mathcal{A})$ such that $a_x \neq d_z$. Since f_{sv} is injective, then $f_{sv}(a_x) \neq f_{sv}(d_z)$. Since $(H, \Phi, \mathcal{B})$ is soft Urysohn, then there exist $T, R \in \Phi$ such that $f_{sv}(a_x) \widetilde{\in} T$, $f_{sv}(d_z) \widetilde{\in} R$, and $Cl_{\Phi}(T) \widetilde{\cap} Cl_{\Phi}(R) = 0_{\mathcal{B}}$. Since f_{sv} is soft a.w.c, then by Theorem 2.20 (d), there are $K, M \in PO(\Psi)$ such that $a_x \widetilde{\in} K$, $d_z \widetilde{\in} M$, $f_{sv}(K) \widetilde{\subseteq} Cl_{\Phi}(T)$, and $f_{sv}(M) \widetilde{\subseteq} Cl_{\Phi}(R)$. Thus,

$f_{sv}(K)\widetilde{\cap}f_{sv}(M)\widetilde{\subseteq}Cl_{\Phi}(T)\widetilde{\cap}Cl_{\Phi}(R) = 0_{\mathcal{B}}$ and hence, $f_{sv}(K)\widetilde{\cap}f_{sv}(M) = 0_{\mathcal{B}}$. Since f_{sv} is injective, then $f_{sv}(K)\widetilde{\cap}f_{sv}(M) = f_{sv}\left(K\widetilde{\cap}M\right)$. Hence, $K\widetilde{\cap}M = 0_{\mathcal{A}}$. It follows that $(G,\Psi,\mathcal{A})$ is soft pre-T_2.

Theorem 3.13. Let $f_{sv}, f_{pu} : (G,\Psi,\mathcal{A}) \longrightarrow (H,\Phi,\mathcal{B})$ be soft a.w.c, $(G,\Psi,\mathcal{A})$ is soft submaximal, and $(H,\Phi,\mathcal{B})$ is soft Urysohn. Then

$$\widetilde{\cup}\left\{a_x \in SP(G,\mathcal{A}) : f_{sv}(a_x) = f_{pu}(a_x)\right\} \in \Psi^c.$$

Proof. Let $N = \widetilde{\cup}\left\{a_x \in SP(G,\mathcal{A}) : f_{sv}(a_x) = f_{pu}(a_x)\right\}$. We will show that $1_{\mathcal{A}} - N\widetilde{\subseteq}1_{\mathcal{A}} - Cl_{\Psi}(N)$. Let $a_x\widetilde{\in}1_{\mathcal{A}} - N$. Then $f_{sv}(a_x) \neq f_{pu}(a_x)$. Since $(H,\Phi,\mathcal{B})$ is soft Urysohn, then there exist $T, R \in \Phi$ such that $f_{sv}(a_x)\widetilde{\in}T$, $f_{pu}(a_x)\widetilde{\in}R$, and $Cl_{\Phi}(T)\widetilde{\cap}Cl_{\Phi}(R) = 0_{\mathcal{B}}$. Since f_{sv}, f_{pu} are soft a.w.c, by Theorem 2.20 (d), we find $K, M \in PO(\Psi)$ such that $a_x\widetilde{\in}K\widetilde{\cap}M$, $f_{sv}(K)\widetilde{\subseteq}Cl_{\Phi}(T)$, and $f_{pv}(M)\widetilde{\subseteq}Cl_{\Phi}(R)$. Thus, $f_{sv}\left(K\widetilde{\cap}M\right)\widetilde{\cap}f_{pu}\left(K\widetilde{\cap}M\right)\widetilde{\subseteq}Cl_{\Phi}(T)\widetilde{\cap}Cl_{\Phi}(R) = 0_{\mathcal{B}}$. Since $(G,\Psi,\mathcal{A})$ is soft submaximal, then by Theorem 4.2 of [60], we have $K, M \in \Psi$ and so, $K\widetilde{\cap}M \in \Psi$. Since $f_{sv}\left(K\widetilde{\cap}M\right)\widetilde{\cap}f_{pu}\left(K\widetilde{\cap}M\right) = 0_{\mathcal{B}}$, then $\left(K\widetilde{\cap}M\right)\widetilde{\cap}N = 0_{\mathcal{A}}$. Since $a_x\widetilde{\in}K\widetilde{\cap}M \in \Psi$ and $\left(K\widetilde{\cap}M\right)\widetilde{\cap}N = 0_{\mathcal{A}}$, then $a_x\widetilde{\in}1_{\mathcal{A}} - Cl_{\Psi}(N)$.

Theorem 3.14. If $f_{sv} : (G,\Psi,\mathcal{A}) \longrightarrow (H,\Phi,\mathcal{B})$ is a soft a.w.c and $(H,\Phi,\mathcal{B})$ is soft Urysohn, then the soft set

$$E = \widetilde{\cup}\left\{(a,d)_{(x,z)} \in SP(G\times G,\mathcal{A}\times\mathcal{A}) : f_{sv}(a_x) = f_{sv}(d_z)\right\} \in PC(\Psi\times\Psi).$$

Proof. We will show that $1_{\mathcal{A}\times\mathcal{A}} - E\widetilde{\subseteq}1_{\mathcal{A}\times\mathcal{A}} - pCl_{\Psi\times\Psi}(E)$. Let $(a,d)_{(x,z)}\widetilde{\in}1_{\mathcal{A}\times\mathcal{A}} - E$. Then $f_{sv}(a_x) \neq f_{sv}(d_z)$. Since $(H,\Phi,\mathcal{B})$ is soft Urysohn, then there exist $T, R \in \Phi$ such that $f_{sv}(a_x)\widetilde{\in}T$, $f_{sv}(d_z)\widetilde{\in}R$, and $Cl_{\Phi}(T)\widetilde{\cap}Cl_{\Phi}(R) = 0_{\mathcal{B}}$. Since f_{sv} is soft a.w.c, then by Theorem 2.20 (d), there are $K, M \in PO(\Psi)$ such that $a_x\widetilde{\in}K$, $d_z\widetilde{\in}M$, $f_{sv}(K)\widetilde{\subseteq}Cl_{\Phi}(T)$, and $f_{sv}(M)\widetilde{\subseteq}Cl_{\Phi}(R)$. Thus, $f_{sv}(K)\widetilde{\cap}f_{sv}(M)\widetilde{\subseteq}Cl_{\Phi}(T)\widetilde{\cap}Cl_{\Phi}(R) = 0_{\mathcal{B}}$ and hence, $f_{sv}(K)\widetilde{\cap}f_{sv}(M) = 0_{\mathcal{B}}$. Thus, we have $(a,d)_{(x,z)}\widetilde{\in}K\times M \in PO(\Psi\times\Psi)$ and $(K\times M)\widetilde{\cap}E = 0_{\mathcal{A}\times\mathcal{A}}$. This implies that $(a,d)_{(x,z)}\widetilde{\in}1_{\mathcal{A}\times\mathcal{A}} - pCl_{\Psi\times\Psi}(E)$.

Definition 3.15. Let $f_{sv} : (G,\Psi,\mathcal{A}) \longrightarrow (H,\Phi,\mathcal{B})$ be a soft function. Then $Graph(f_{sv})$ is called soft strongly p-closed with respect to $(G\times H, pr(\Psi\times\Phi), \mathcal{A}\times\mathcal{B})$ if for each $(a,d)_{(x,z)}\widetilde{\in}1_{\mathcal{A}\times\mathcal{B}} - Graph(f_{sv})$, there exist $U \in PO(\Psi)$ and $V \in \Phi$ such that $a_x\widetilde{\in}U$, $d_z\widetilde{\in}V$, and $(U\times Cl_{\Phi}(V))\widetilde{\cap}Graph(f_{sv}) = 0_{\mathcal{A}\times\mathcal{B}}$.

Theorem 3.16. If $f_{sv} : (G,\Psi,\mathcal{A}) \longrightarrow (H,\Phi,\mathcal{B})$ is a soft a.w.c and $(H,\Phi,\mathcal{B})$ is soft Urysohn, then $Graph(f_{sv})$ is soft strongly p-closed with respect to $(G\times H, pr(\Psi\times\Phi), \mathcal{A}\times\mathcal{B})$.

Proof. Let $(a,d)_{(x,z)}\widetilde{\in}1_{\mathcal{A}\times\mathcal{B}} - Graph(f_{sv})$. Then $f_{sv}(a_x) \neq d_z$. Since $(H,\Phi,\mathcal{B})$ is soft Urysohn, then there exist $T, R \in \Phi$ such that $f_{sv}(a_x)\widetilde{\in}T$, $d_z\widetilde{\in}R$, and $Cl_{\Phi}(T)\widetilde{\cap}Cl_{\Phi}(R) = 0_{\mathcal{B}}$. Since f_{sv} is soft a.w.c, then by Theorem 2.20 (d), there is $K \in PO(\Psi)$ such that $a_x\widetilde{\in}K$ and $f_{sv}(K)\widetilde{\subseteq}Cl_{\Phi}(T)$. This implies that $f_{sv}(K)\widetilde{\cap}Cl_{\Phi}(R) = 0_{\mathcal{B}}$; hence $(K\times Cl_{\Phi}(R))\widetilde{\cap}Graph(f_{sv}) = 0_{\mathcal{A}\times\mathcal{B}}$. This shows that $Graph(f_{sv})$ is soft strongly p-closed with respect to $(G\times H, pr(\Psi\times\Phi), \mathcal{A}\times\mathcal{B})$.

Theorem 3.17. If $f_{sv} : (G,\Psi,\mathcal{A}) \longrightarrow (H,\Phi,\mathcal{B})$ is a soft a.w.c injection with $Graph(f_{sv})$ soft strongly p-closed with respect to $(G\times H, pr(\Psi\times\Phi), \mathcal{A}\times\mathcal{B})$, then $(G,\Psi,\mathcal{A})$ is soft pre-T_2.

Proof. Let $a_x, b_y \in SP(G,\mathcal{A})$ such that $a_x \neq b_y$. Since f_{sv} is injective, then $f_{sv}(a_x) \neq f_{sv}\left(b_y\right)$, and so $(a, v(b))_{(x,s(y))}\widetilde{\in}1_{\mathcal{A}\times\mathcal{B}} - Graph(f_{sv})$. Since $Graph(f_{sv})$ is soft strongly p-closed with respect to $(G\times H, pr(\Psi\times\Phi), \mathcal{A}\times\mathcal{B})$, then there exist $U \in PO(\Psi)$ and $V \in \Phi$ such that $a_x\widetilde{\in}U$, $f_{sv}\left(b_y\right)\widetilde{\in}V$, and $(U\times Cl_{\Phi}(V))\widetilde{\cap}Graph(f_{sv}) = 0_{\mathcal{A}\times\mathcal{B}}$. Thus, we have $f_{sv}(U)\widetilde{\cap}Cl_{\Phi}(V) = 0_{\mathcal{B}}$ and hence, $U\widetilde{\cap}f_{sv}^{-1}(Cl_{\Phi}(V)) = 0_{\mathcal{A}}$. It follows that $U\widetilde{\cap}pInt_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(V))\right) = 0_{\mathcal{A}}$. Since f_{sv} is a soft a.w.c, then by Theorem 2.20 (c), $b_y\widetilde{\in}f_{sv}^{-1}(V)\widetilde{\subseteq}pInt_{\Psi}\left(f_{sv}^{-1}(Cl_{\Phi}(V))\right) \in PO(\Psi)$. This shows that $(G,\Psi,\mathcal{A})$ is soft pre-T_2.

4. Conclusions

This paper has successfully introduced and explored the novel concept of soft almost weakly continuous functions, a generalized family of soft continuous functions encompassing soft pre-continuous and weakly continuous functions. Through a comprehensive analysis, we have established various characterizations of soft almost weakly continuous functions and investigated their connections to their counterparts in general topology. We have also found the conditions that are needed for a soft almost weakly continuous function to change into a soft weakly continuous or soft pre-continuous function. This helps us understand how these function classes are related. The results on soft composition, limitation, preservation, product, and soft graph theorems in the setting of soft almost weakly continuous functions lay a solid foundation for future study in this field. As the subject of soft topology advances, this study sets the door for future investigation of soft continuous functions and their applications in a variety of disciplines.

Author contributions

Samer Al-Ghour and Jawaher Al-Mufarrij: Conceptualization, methodology, formal analysis, writing-original draft, writing-review and editing, and funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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