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Research article

Optimal investment and reinsurance for the insurer and reinsurer with the joint exponential utility

Wuyuan Jiang, Zechao Miao*and Jun Liu

Department of Mathematics, Hunan Institute of Science and Technology, Yueyang 414006, China

* **Correspondence:** Email: zechao2022@163.com.

Abstract: In this paper, we consider the problem of optimal investment-reinsurance for the insurer and reinsurer under the stochastic volatility model. The surplus process of the insurer is described by a diffusion model. The insurer can purchase proportional reinsurance from the reinsurer and the premium charged by the insurer and reinsurer follows the variance principle. Both the insurer and reinsurer are allowed to invest in risk-free assets and risky assets, and the market price of risk depends on a Markovian, affine-form, and square-root stochastic factor process. Our goal is to maximize the joint exponential utility of the terminal wealth of the insurer, and reinsurer over a certain period of time. By solving the HJB equation, we obtain the optimal investment-reinsurance strategies, and present the proof of the verification theorem. Finally, we demonstrate a numerical analysis, and the economic implications of our findings are illustrated.

Keywords: reinsurance and investment; variance premium principle; stochastic volatility; joint exponential utility
Mathematics Subject Classification: 91B30, 93E20

1. Introduction

The insurance industry has developed rapidly in recent decades. Reinsurance and investment are important research issues in the field of actuarial science and have been widely studied. Reinsurance can protect insurance companies from potentially huge losses, while the investment of premiums enables insurance companies to achieve certain management goals. For example, [1] proposes two criteria of maximizing terminal wealth utility and minimizing bankruptcy probability under continuous time. The author of [2] studied the optimal investment strategy to maximize the expected exponential utility of terminal wealth under the jump-diffusion model. The author of [3] investigated optimal reinsurance and investments that take into account transaction costs. The author of [4] studied the optimal proportional reinsurance and investment strategy under the CEV model. The author of [5] studied the problem of optimal portfolio and reinsurance with two different risk assets. Moreover, a multitude of scholars have directed their focus towards diverse optimization objectives. For instance, [6–10] explored the optimal reinsurance and investment problem under the mean-variance criterion. In addition, [11–15] investigated the optimal problem for insurers and aim to minimize ruin probability.

Although there is a large literatures on optimal reinsurance and investment issues, most of the articles are conducted under the expected value premium principle. The expected value premium principle is widely used in the reinsurance premium principle because of its practicality. However, the variance of the same expected risk is not necessarily the same, so the fluctuation in claims needs to be taken into account when we set the premium. In recent years, the variance or mean-variance premium principle has received more and more attention. For example, [16,17] investigated the optimal reinsurance under the mean–variance premium principle. The author of [18] considered the optimal proportional reinsurance strategy for dependent risks and the variance premium principle under the expected utility maximization criterion. The author of [19] used the generalized variance premium principle to get the optimal investment–reinsurance strategy, which maximizes the expected utility of terminal wealth and minimizes the ruin probability. By applying the generalized variance premium principle, the author of [20] obtained the optimal reinsurance and investment strategy for insurance companies with defaulted bonds.

In addition, most of the above studies are conducted under the assumption that the prices of risky assets have constant or determined volatility, which contradicts the evidence supporting the existence of stochastic volatility, such as volatility smiles and volatility clustering. Previously, the author of [21] made a detailed study of stochastic volatility. In recent years, as an important feature of asset price models, stochastic volatility has attracted the attention of many scholars. They study the optimal reinsurance and investment of risk asset prices under a stochastic volatility model, such as the CEV model ([7, 22–24]) and the Heston model ([25–27]). We tend to consider a more general stochastic volatility model, which includes both the CEV model and the Heston model. The author of [28] studied the asset–liability management problem involving mean–variance with an affine diffusion factor process and a reinsurance option, providing us with a good idea.

Most of the existing literature considers the optimization of the insurer, but in reality there is always an interest relationship between the insurer and reinsurer, and the role of the reinsurer cannot be ignored. And both the insurer and the reinsurer want to maximize their own benefits, so it is necessary to maintain a good dynamic balance between the insurer and the reinsurer. From [29], we know that the two companies should negotiate to maximize their mutual profits and that they must reach a compromise. The author of [30] derived the expectation formula of the common survival profit of the insurer and reinsurer in a fixed time. Furthermore, the author of [31] studied the joint survival and profitable probabilities of the insurer and reinsurer. The author of [32] studied the optimal proportional reinsurance and investment to maximize the utility of the insurer and reinsurer under the CEV model. The author of [33] considered the interests of both the insurer and reinsurer. In addition, [34–37] studied the optimal reinsurance and investment problem using the weighted sum method for the wealth processes of the insurer and reinsurer.

To the best of our knowledge, there is little literature on the maximization of the common terminal wealth utility of the insurer and reinsurer. In this paper, adopting the idea of [23] and [39], we mainly study the optimal investment and reinsurance problem of the insurer and reinsurer under the joint exponential utility. The surplus process of the insurer is described by a diffusion model. The insurer can purchase proportional reinsurance from the reinsurer, and the premium charged by the insurer and reinsurer follows the variance principle. Furthermore, both the insurer and reinsurer are allowed to invest in risk-free assets and risky assets, and the market price of risk depends on a Markovian, affine-form, and square-root stochastic factor process, which is a more general stochastic volatility model including both the CEV model and the Heston one. Then, we obtain the HJB equation under the optimization criterion of maximizing the terminal joint exponential utility. By solving the HJB equation, we obtain the optimal investment–reinsurance strategies, and present the proof of the verification theorem. Finally, we demonstrate a numerical analysis, and the economic implications of our findings are illustrated.

The innovation of this paper is the use of a more general stochastic volatility model to describe the price process of risky assets, which is also is the difference of the paper from [23]. Under the criterion of maximizing the terminal joint exponential utility, we study the optimal investment-reinsurance strategies of the insurer and reinsurer in the process where the market price of risk depends on a Markovian, affine-form, and square-root stochastic factor. The model incorporates the situation in which the insurer and reinsurer can invest in different risk assets. We believe that this model will be more general than CEV model. Moreover, we present the explicit expression of the value function, and give the proof of the case $m_1 = m_2$, which [23] did not consider.

The rest of this article proceeds as follows. Section 2 introduces our mathematical model. Section 3 obtains the HJB equation under the objective of maximizing the joint exponential utility of terminal wealth and presents the optimal strategy and value function along with a verification theorem. Section 4 provides some special cases. Section 5 illustrates our results through numerical simulation. Section 6 concludes the whole paper. And the appendix contains the proof of some theorems.

2. Model setup

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ be a filtered, complete probability space satisfying the usual conditions, and let $T > 0$ be a finite time horizon representing the term of the contract. $\mathcal{F}_t$ stands for the information available until time t. We assume that all stochastic processes are adapted processes in this filtered probability space. The insurer's surplus process is described by the classical compound Poisson risk model:

$$R(t) = x_0 + ct - C(t) = x_0 + ct - \sum_{i=1}^{N(t)} Z_i,\ t \geq 0,$$

where $x_0 \geq 0$ is the initial surplus of the insurer, c represents the insurer's premium rate, and $\{Z_i, i \geq 1\}$ are independent and identically distributed positive variables representing the successive individual claim amounts with first moment $\mathrm{E}(Z_i) = \mu_Z$ and second moment $\mathrm{E}[Z_i^2] = \sigma_Z^2$, they have common distribution $F(z)$. Here $\mathrm{E}(\cdot)$ denotes the mean value under the probability measure $\mathbb{P}$, and $N(t)$ denotes the number of claims up to time t, and process $\{N(t); t \geq 0\}$ is an ordinary homogeneous renewal Poisson process with intensity λ. In addition, we assume that $N(t)$ is independent of the claim sizes $\{Z_i, i \geq 1\}$. In this paper, both the insurance and reinsurance premiums are calculated according to the variance principle. Thus, the insurance premium c can be obtained by $c = \lambda\mu_Z + \lambda\alpha_1\sigma_Z^2$, where $\alpha_1 > 0$ is a given constant, being called the safety loading of the insurer.

Assume that the insurer is permitted to purchase proportional reinsurance to disperse the underlying insurance risk. Let $q(t)$ be the reinsurance proportion at time t. i.e., for a claim Z_i occurring, the insurer pays $q(t)Z_i$, while the reinsurer needs to pay $(1-q(t))Z_i$. Then the corresponding surplus process of the insurer and reinsurer can be described by

$$R_1(t) = x_1 + c_1 t - q(t)\sum_{i=1}^{N(t)} Z_i,$$

and

$$R_2(t) = x_2 + c_2 t - (1-q(t))\sum_{i=1}^{N(t)} Z_i,$$

where

$$\begin{aligned} c_1 =& \lambda\mu_Z + \lambda\alpha_1\sigma_Z^2 - [\lambda\mu_Z(1-q(t)) + \lambda\alpha_2\sigma_Z^2(1-q(t))^2] \\ =& \lambda\mu_Z q(t) + \lambda\alpha_1\sigma_Z^2 - \lambda\alpha_2\sigma_Z^2(1-q(t))^2, \\ c_2 =& \lambda\mu_Z(1-q(t)) + \lambda\alpha_2\sigma_Z^2(1-q(t))^2, \end{aligned}$$

and α_2 denotes the safety loading of the reinsurer, x_2 is the initial surplus of the reinsurer. Suppose that $\alpha_2 > \alpha_1$, otherwise, arbitrage will exist. According to [38], the surplus processes of the insurer and reinsurer can be respectively approximated by the following diffusion processes:

$$dR_1(t) = [\lambda\alpha_1\sigma_Z^2 - \alpha_2\lambda\sigma_Z^2(1-q(t))^2]dt + q(t)\sqrt{\lambda\sigma_Z^2}dW_0(t),$$

and

$$dR_2(t) = \alpha_2\lambda\sigma_Z^2(1-q(t))^2 dt + (1-q(t))\sqrt{\lambda\sigma_Z^2}dW_0(t),$$

where $W_0(t)$ is a standard Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$.

Remark 2.1. In this paper, we require that the risk exposure $q(t)$ must meet the net profit condition, so through $\lambda\alpha_1\sigma_Z^2 - \alpha_2\lambda\sigma_Z^2(1-q(t))^2 \ge 0$ we get $0 < 1 - \sqrt{\frac{\alpha_1}{\alpha_2}} \le q(t) \le 1$. Fulfilling the net profit requirement means that the enterprise's earnings, after all expenses, costs, and taxes have been subtracted, are not in the red.

In addition to reinsurance, both the insurer and the reinsurer can invest the company's surplus in a financial market consisting of one risk-free asset and two risky assets. The price process of the risk-free asset satisfies the ordinary differential equation

$$dS_0(t) = r_0 S_0(t)dt, \quad S_0(0) = s_0, \tag{2.1}$$

where $r_0 > 0$ represents the risk-free interest rate. The risk assets that the insurer and reinsurer can invest in are represented by $S_1(t)$ and $S_2(t)$, respectively. The price process of the risk asset $S_i(t)$ is described by

$$dS_i(t) = S_i(t)[\mu_i(t)dt + \sigma_i(t)dW_i(t)], S_i(0) = s_{0i} > 0, \tag{2.2}$$

where $\mu_i(t), \sigma_i(t) > 0$ are the appreciation rate and volatility rate of risk assets at time t, respectively. $W_i(t)(i = 1, 2)$ is a standard Brownian motion and independent of $\{W_j(t)\}(j = 0, 1, 2, j \ne i)$, $\{N(t)\}_{t\in[0,T]}$, $\{Z_i, i \ge 1\}$.We assume that $\{\mu_i(t)\}_{t\in[0,T]}$ and $\{\sigma_i(t)\}_{t\in[0,T]}$ are $\mathcal{F}_t$-predictable processes and that they are continuously bounded deterministic functions or stochastic processes. The market price of risk $\{\omega_i(t)\}_{t\in[0,T]}$ is

$$\omega_i(t) := \frac{\mu_i(t) - r_0}{\sigma_i(t)}, \forall t \in [0, T]. \tag{2.3}$$

$\{\omega_i(t)\}_{t\in[0,T]}$ is related to a stochastic factor process $\{\vartheta_i(t)\}_{t\in[0,T]}$ as

$$\omega_i(t) = \omega_i \sqrt{\vartheta_i(t)}, \forall t \in [0, T], \omega_i \in \mathbb{R}_0 := \mathbb{R}\backslash\{0\}, \tag{2.4}$$

where $\{\vartheta_i(t)\}_{t\in[0,T]}$ satisfies the following Markovian, affine-form square-root model

$$\mathrm{d}\vartheta_i(t) = \kappa_i[\phi_i - \vartheta_i(t)]\mathrm{d}t + \sqrt{\vartheta_i(t)}[\rho_{i1}\mathrm{d}W_i(t) + \rho_{i2}\mathrm{d}\overline{W}_i(t)], \vartheta_i(0) = \vartheta_{0i} \geq 0, \tag{2.5}$$

and $\kappa_i, \phi_i, \rho_{i1}, \rho_{i2}$ are positive constants. $\{\overline{W}_i(t)\}(i = 1, 2)$ is another standard Brownian motion that is independent of $\{\overline{W}_i(t)\}(j = 1, 2, j \neq i)$, $\{W_i(t)\}(i = 0, 1, 2)$ and $\{N(t)\}_{t\in[0,t]}, \{Z_i, i \geq 1\}$. In addition, we assume that the solution to the square-root model (2.5) is non negative for all $t \in [0, T]$.

Remark 2.2. The model that the insurer and reinsurer are allowed to invest in two different types of risky assets, respectively, is more common. In reality, the insurer and reinsurer are two individuals/companies who may choose different risk assets to invest in. If all parameters of both risk assets are the same, then both the insurer and the reinsurer invest in the same risk asset, which is the special case of our model.

Remark 2.3. According to [9], let $\mu_i(t) = \mu_i, \sigma_i(t) = \sigma_i(S_i(t))^{\nu_i}$, where $\mu_i, r_0, \sigma_i \geq 1$ and $\nu_i \in \mathbb{R}$ such that $\mu_i \neq r_0$, then the risk asset price is given by CEV model

$$\mathrm{d}S_i(t) = S_i(t)[\mu_i\mathrm{d}t + \sigma_i(S_i(t))^{\nu_i}\mathrm{d}W_i(t)], S_i(0) = s_{0i} > 0, \tag{2.6}$$

where ν_i is the elasticity parameter of the risky asset. Set

$$\vartheta_i(t) = (S_i(t))^{-2\nu_i}, \kappa_i = 2\nu_i\mu_i, \phi_i = (\nu_i + \frac{1}{2})\frac{\sigma_i^2}{\mu_i}, \rho_{i1} = -2\nu_i\sigma_i, \rho_{i2} = 0 \textit{ and } \omega_i = \frac{\mu_i - r_0}{\sigma_i},$$

then applying Itô's formula to $S_i^{-2\nu_i}(t)$, we obtain

$$\mathrm{d}(S_i(t))^{-2\nu_i} = 2\nu_i\mu_i\left[\left(\nu_i + \frac{1}{2}\right)\frac{\sigma_i^2}{\mu_i} - (S_i(t))^{-2\nu_i}\right]\mathrm{d}t - 2\nu_i\sigma_i(S_i(t))^{-\nu_i}\mathrm{d}W_i(t). \tag{2.7}$$

It is a special case of the CEV model. If $\nu_i = 0$ in equation (2.6), the CEV model reduces to the GBM model.

And if $\mu_i(t) = r_0 + \omega_i\vartheta_i(t), \sigma_i(t) = \sqrt{\vartheta_i(t)}, \rho_{i1} = \sigma_{0i}\rho_i, \rho_{i2} = \sigma_{0i}\sqrt{1-\rho_i^2}$, where $\mathrm{r}_0, \sigma_i > 0, \omega_i \in \mathbb{R}_0, \rho_i \in (-1, 1)$, then the risky asset's price is reduced to the Heston model

$$\mathrm{d}S_i(t) = S_i(t)\Big[(r_0 + \omega_i\vartheta_i(t))\mathrm{d}t + \sqrt{\vartheta_i(t)}\mathrm{d}W_i(t)\Big], S_i(0) = s_{0i} > 0, \tag{2.8}$$

and

$$\begin{aligned} &\mathrm{d}\vartheta_i(t) = \kappa_i[\phi_i - \vartheta_i(t)]\mathrm{d}t + \sqrt{\vartheta_i(t)}\Big[\sigma_{0i}\rho_i\mathrm{d}W_i(t) + \sigma_{0i}\sqrt{1-\rho_i^2}\mathrm{d}\overline{W}_i(t)\Big], \\ &\vartheta_i(0) = \vartheta_{0i} \geq 0, \end{aligned} \tag{2.9}$$

where $\{\vartheta_i(t)\}_{t\in[0,T]}$ is the variance process, κ_i is the variance rate, ϕ_i is the long-run level, σ_{0i} is the volatility of risky asset and ρ_i is the correlation coefficient between the risky asset's price and the variance. In the Heston model, the market price of risk is $\omega_i(t) = \omega_i\sqrt{\vartheta_i(t)}$. It is required that the Feller condition is satisfied, i.e., $2\kappa_i\phi_i \geq \sigma_{0i}^2$ for all $t \in [0, T]$.

Denote $\pi_1(t)$ and $\pi_2(t)$ as the money amounts invested in the first risky asset $S_1(t)$ and the second risky asset $S_2(t)$ by the insurer and reninsurer at the time t, respectively. Then $X_1(t) - \pi_1(t)$ and $X_2(t) - \pi_2(t)$ are the money amounts invested in the risk-free asset by the insurer and reinsurer, respectively. An investment-reinsurance strategy is described by $u := \{(\pi_1(t), \pi_2(t), q(t))\}_{t\in[0,T]}$. Then the insurer's wealth process $X_1^u(t)$ and the reinsurer's wealth process $X_2^u(t)$ follow the following dynamic:

$$\begin{cases} \mathrm{d}X_1^u(t) = [r_0X_1^u(t) + (\mu_1(t) - r_0)\pi_1(t) + \lambda\alpha_1\sigma_Z^2 - \lambda\alpha_2\sigma_Z^2(1 - q(t))^2]dt \\ \qquad\qquad + \dfrac{\pi_1(t)(\mu_1(t) - r_0)}{\omega_1\sqrt{\vartheta_1(t)}}\mathrm{d}W_1(t) + q(t)\sqrt{\lambda\sigma_Z^2}\mathrm{d}W_0(t), \\ X_1(0) = x_{01}, \end{cases} \tag{2.10}$$

and

$$\begin{cases} \mathrm{d}X_2^u(t) = [r_0X_2^u(t) + (\mu_2(t) - r_0)\pi_2(t) + \lambda\alpha_2\sigma_Z^2(1 - q(t))^2]dt \\ \qquad\qquad + \dfrac{\pi_2(t)(\mu_2(t) - r_0)}{\omega_2\sqrt{\vartheta_2(t)}}\mathrm{d}W_2(t) + (1 - q(t))\sqrt{\lambda\sigma_Z^2}\mathrm{d}W_0(t), \\ X_2(0) = x_{02}. \end{cases} \tag{2.11}$$

3. The optimal strategy for the insurer and reinsurer

In this paper, we consider the expected utility maximization of the terminal wealth for the insurer and reinsurer. Inspired by [39], we suppose that the insurer and reinsurer have the joint exponential utility function

$$U(x, y) = -\frac{1}{m_1m_2}e^{-m_1x-m_2y}, m_1 \neq m_2,$$

where $m_1, m_2 > 0$ are the risk aversion coefficients of the insurer and reinsurer, respectively.

Definition 3.1. (Admissible strategy). An investment–reinsurance strategy $u = \{(\pi_1(t), \pi_2(t), q(t)\}_{t\in\{0,T\}}$ is said to be admissible if
(1) $\forall t \in [0, T]$, $q(t) \in [1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, 1]$.
(2) $\mathrm{E}\{\int_0^T [(\pi_1(t)\sigma_1(t))^2 + (\pi_2(t)\sigma_2(t))^2 + q(t)^2]dt\} < \infty$ and u is $\mathcal{F}_t$-progres-sively measurable.
(3) $\forall(t, x_1, x_2, v_1, v_2) \in [0, T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+$, Eqs (2.10) and (2.11) have unique solution $\{X_1^u(t)\}_{t\in[0,T]}$ and $\{X_2^u(t)\}_{t\in[0,T]}$ with $X_1^u(t) = x_1, X_2^u(t) = x_2, \vartheta_1(t) = v_1$ and $\vartheta_2 = v_2$, respectively.
(4) $E^u\{U[X(T), Y(T)]|X_1(t) = x_1, X_2(t) = x_2, \vartheta_1(t) = v_1, \vartheta_2(t) = v_2\} < \infty$, where $u \in \mathcal{U}$, $t \in [0, T]$ is the proportional reinsurance and investment strategy, and $\mathcal{U}$ is the set of all admissible strategies u.

Suppose that we are interested in maximizing the joint exponential utility of terminal wealth at a fixed time T. In order to apply the classical tools of stochastic optimal control, we now introduce the relevant value function.

$$\begin{aligned} V(t, x_1, x_2, v_1, v_2) = &\sup_{u\in\mathcal{U}} E\{U[X(T), Y(T)]|X_1(t) = x_1, X_2(t) = x_2, \\ &\vartheta_1(t) = v_1, \vartheta_2(t) = v_2\}, t \in [0, T], \end{aligned} \tag{3.1}$$

with boundary condition $V(T, x_1, x_2, v_1, v_2) = U(x_1, x_2)$.

To resolve the problem outlined above, we adopt the dynamic programming method. Let $C^{1,2,2,2,2}([0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)$ is the space of $V(t, x_1, x_2, v_1, v_2)$, which are first-order continuously differentiable in $t \in [0,T]$, second-order continuously differentiable in $x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, v_1 \in \mathbb{R}^+, v_2 \in \mathbb{R}^+$. Denote $V_t, V_{x_1}, V_{x_2}, V_{v_1}, V_{v_2}, V_{x_1x_1}, V_{x_2x_2}, V_{v_1v_1}, V_{v_2v_2}, V_{x_1v_1}, V_{x_2v_2}$ and $V_{x_1x_2}$ as the first and second partial derivatives of V, which are continuous on $C^{1,2,2,2,2}([0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)$. Then we define a variational operator $\mathcal{A}^u$: for $\forall(t, x_1, x_2, v_1, v_2) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+, \forall V(t, x_1, x_2, v_1, v_2) \in C^{1,2,2,2,2}([0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)$, denote

$$\begin{aligned}
&\mathcal{A}^u V(t, x_1, x_2, v_1, v_2)\\
&=V_t + [r_0x_1 + (\mu_1(t) - r_0)\pi_1(t) + \lambda\alpha_1\sigma_Z^2 - \lambda\alpha_2\sigma_Z^2(1-q)^2]V_{x_1}\\
&\quad + [r_0x_2 + (\mu_2(t) - r_0)\pi_2(t) + \lambda\alpha_2\sigma_Z^2(1-q)^2]V_{x_2} + \kappa_1[\phi_1 - v_1]V_{v_1}\\
&\quad + \kappa_2[\phi_2 - v_2]V_{v_2} + \frac{v_1(\rho_{11}^2 + \rho_{12}^2)}{2}V_{v_1v_1} + \frac{v_2(\rho_{21}^2 + \rho_{22}^2)}{2}V_{v_2v_2}\\
&\quad + [\frac{\pi_1^2(\mu_1(t) - r_0)^2}{2\omega_1^2 v_1} + \frac{1}{2}\lambda\sigma_Z^2 q^2]V_{x_1x_1} + [\frac{\pi_2^2(\mu_2(t) - r_0)^2}{2\omega_2^2 v_2} + \frac{1}{2}\lambda\sigma_Z^2(1-q)^2]V_{x_2x_2}\\
&\quad + \frac{\pi_1(\mu_1(t) - r_0)\rho_{11}}{\omega_1}V_{x_1v_1} + \frac{\pi_2(\mu_2(t) - r_0)\rho_{21}}{\omega_2}V_{x_2v_2} + \lambda\sigma_Z^2 q(1-q)V_{x_1x_2}.
\end{aligned} \tag{3.2}$$

Then V satisfies the following Hamilton–Jacobi–Bellman (HJB) equation:

$$\sup_{u\in\mathcal{U}} \mathcal{A}^u V(t, x_1, x_2, v_1, v_2) = 0. \tag{3.3}$$

Lemma 3.1. If $\psi(t, x_1, x_2, v_1, v_2)$ is the solution of HJB equation (3.3) with the boundary $\psi(T, x_1, x_2, v_1, v_2) = U(x, y)$, then we have

$$E[\psi(t, X_1^{u^*}(t), X_2^{u^*}(t), \vartheta_1(t), \vartheta_2(t))]^2 < \infty.$$

Proof. See the Appendix. □

Theorem 3.2. (Verification theorem). Let $\psi(t, x_1, x_2, v_1, v_2) \in C^{1,2,2,2,2}$, and ψ satisfies HJB equation (3.3) with boundary conditions $\psi(T, x_1, x_2, v_1, v_2) = U(x, y)$. Let $u^*(t) = (\pi_1^*(t), \pi_2^*(t), q^*(t)) \in U$ such that $\mathcal{A}^{u^*} V(t, x_1, x_2, v_1, v_2) = 0$, then the value function $V(t, x_1, x_2, v_1, v_2) = \psi(t, x_1, x_2, v_1, v_2)$ and u^* is the optimal strategy.

Proof. See the Appendix. □

Remark 3.1. Due to differences in the model, the proof of Lemma 3.1 is significantly different from that of Lemma 3.2 in [23], and the equation in the proof of Theorem 3.2 also differs from the one presented in [23].

Theorem 3.3. (The optimal strategy and value function). Denote $t_1 = T - \frac{\ln\hat{\Delta}_1}{r_0}, t_2 = T - \frac{\ln\hat{\Delta}_2}{r_0}$, where $\hat{\Delta}_1 = \frac{2\alpha_2}{(m_2-m_1)}, \hat{\Delta}_2 = \frac{2\alpha_2}{[m_2+m_1(\sqrt{\frac{\alpha_2}{\alpha_1}}-1)]}$. Therefore, we can deduce that $\hat{\Delta}_1 > \hat{\Delta}_2 > 0$ when $m_1 < m_2$, and

$\hat{\Delta}_2 > 0 > \hat{\Delta}_1$ when $m_1 > m_2$, i.e., $0 \le t_1 < t_2 \le T$ when $m_1 < m_2$, and $0 \le t_2 \le T$ when $m_1 > m_2$. For problem (3.1), the optimal investment strategies are given by

$$\pi_1^*(t) = \begin{cases} \frac{\omega_1^2 v_1(c_1 - c_2 e^{-\frac{\rho_{12}^2}{2}(c_1-c_2)(T-t)}) + \rho_{11}\omega_1 v_1 c_1 c_2 (1 - e^{-\frac{\rho_{12}^2}{2}(c_1-c_2)(T-t)})}{(\mu_1(t)-r_0)m_1(c_1 - c_2 e^{-\frac{\rho_{12}^2}{2}(c_1-c_2)(T-t)})} e^{-r_0(t-t)}, & \rho_{12} \neq 0, \\ \frac{2\omega_1^2 v_1(\kappa_1+\omega_1\rho_{11}) + \omega_1^3 \rho_{11} v_1 (e^{-(\kappa_1+\omega_1\rho_{11})(T-t)} - 1)}{2(\mu_1(t)-r_0)m_1(\kappa_1+\omega_1\rho_{11})} e^{-r_0(T-t)}, & \rho_{12} = 0, \end{cases} \tag{3.4}$$

and

$$\pi_2^*(t) = \begin{cases} \frac{\omega_2^2 v_2(d_1 - d_2 e^{-\frac{\rho_{22}^2}{2}(d_1-d_2)(T-t)}) + \rho_{21}\omega_2 v_2 d_1 d_2 (1 - e^{-\frac{\rho_{22}^2}{2}(d_1-d_2)(T-t)})}{(\mu_2(t)-r_0)m_1(d_1 - d_2 e^{-\frac{\rho_{22}^2}{2}(d_1-d_2)(T-t)})} e^{-r_0(T-t)}, & \rho_{22} \neq 0, \\ \frac{2\omega_2^2 v_2(\kappa_2+\omega_2\rho_{21}) + \omega_2^3 \rho_{21} v_2 (e^{-(\kappa_2+\omega_2\rho_{21})(T-t)} - 1)}{2(\mu_2(t)-r_0)m_2(\kappa_2+\omega_2\rho_{21})} e^{-r_0(T-t)}, & \rho_{22} = 0. \end{cases} \tag{3.5}$$

The optimal reinsurance strategies are given by

Case (I), If $m_1 > m_2$ and $\hat{\Delta}_2 \ge 1$, then

$$q^* = \begin{cases} 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, & 0 \le t \le t_2, \\ \hat{q}(t), & t_2 \le t \le T. \end{cases}$$

Case (II), If $m_1 > m_2$ and $\hat{\Delta}_2 < 1$, then

$$q^* = 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, 0 \le t \le T.$$

Case (III), If $m_1 < m_2$ and $\hat{\Delta}_1 > \hat{\Delta}_2 \ge 1$, when $L\Big(1 - \sqrt{\frac{\alpha_1}{\alpha_2}}\Big) \ge L(1)$, then

$$q^* = \begin{cases} 1, & 0 \le t \le t_1, \\ 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, & t_1 \le t \le t_2, \\ 1, & t_2 \le t \le T, \end{cases}$$

when $L\Big(1 - \sqrt{\frac{\alpha_1}{\alpha_2}}\Big) < L(1)$, then

$$q^* = \begin{cases} 1, & 0 \le t \le t_1, \\ 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, & t_1 \le t \le T. \end{cases}$$

Case (IV), If $m_1 < m_2$ and $\hat{\Delta}_2 < 1 \le \hat{\Delta}_1$, then

$$q^* = \begin{cases} 1, & 0 \le t \le t_1, \\ 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, & t_1 \le t \le T. \end{cases}$$

Case (V), If $m_1 < m_2$ and $\hat{\Delta}_2 < \hat{\Delta}_1 < 1$, then

$$q^* = 1, 0 \le t \le T.$$

When q^* takes different values, the explicit expression of the value function is as follows:

$$V(t, x_1, x_2, v_1, v_2) = -\frac{1}{m_1 m_2} e^{[-m_1 x_1 - m_2 x_2 - d(t)]e^{r_0(T-t)} + g(t, v_1, v_2)}, \tag{3.6}$$

where

$$g(t, v_1, v_2) = I(t) + J_1(t)v_1 + J_2(t)v_2.$$

(1) When $q^* = 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$,

$$\begin{aligned} d(t) = &-\frac{m_2 \lambda \alpha_1 \mu_2}{r_0}[e^{-r_0(T-t)} - 1] + \frac{\sigma_0^2}{2r_0}\Big[\Big(\frac{1}{2} - \sqrt{\frac{\alpha_1}{\alpha_2}}\Big)m_1^2 + \frac{\alpha_1}{2\alpha_2}(m_1 - m_2)^2 \\ &+ \sqrt{\frac{\alpha_1}{\alpha_2}} m_1 m_2\Big] \times [e^{-r_0(T-t)} - e^{r_0(T-t)}], \end{aligned} \tag{3.7}$$

$$I(t) = \begin{cases} \kappa_1\phi_1 c_1(T-t) - \frac{2\kappa_1\phi_1}{\rho_{12}^2} \ln \frac{c_1 e^{\frac{\rho_{12}^2}{2}(c_1 - c_2)(T-t)} - c_2}{c_1 - c_2} \\ \quad + \kappa_2\phi_2 d_1(T-t) - \frac{2\kappa_2\phi_2}{\rho_{22}^2} \ln \frac{d_1 e^{\frac{\rho_{22}^2}{2}(d_1 - d_2)(T-t)} - d_2}{d_1 - d_2}, & \rho_{i2} \neq 0, \\ \frac{\kappa_1\phi_1\omega_1^2}{2(\kappa_1 + \omega_1\rho_{11})}\Big[\frac{1 - e^{-(\kappa_1 + \omega_1\rho_{11})(T-t)}}{\kappa_1 + \omega_1\rho_{11}} - (T-t)\Big] \\ \quad + \frac{\kappa_2\phi_2\omega_2^2}{2(\kappa_2 + \omega_2\rho_{21})}\Big[\frac{1 - e^{-(\kappa_2 + \omega_2\rho_{21})(T-t)}}{\kappa_2 + \omega_2\rho_{21}} - (T-t)\Big], & \rho_{i2} = 0, \end{cases} \tag{3.8}$$

$$J_1(t) = \begin{cases} \frac{c_1 c_2 (1 - e^{-\frac{\rho_{12}^2}{2}(c_1 - c_2)(T-t)})}{c_1 - c_2 e^{-\frac{\rho_{12}^2}{2}(c_1 - c_2)(T-t)}}, & \rho_{i2} \neq 0, \\ \frac{\omega_1^2}{2(\kappa_1 + \omega_1\rho_{11})}(e^{-(\kappa_1 + \omega_1\rho_{11})(T-t)} - 1), & \rho_{i2} = 0, \end{cases} \tag{3.9}$$

and

$$J_2(t) = \begin{cases} \frac{d_1 d_2 (1 - e^{-\frac{\rho_{22}^2}{2}(d_1 - d_2)(T-t)})}{d_1 - d_2 e^{-\frac{\rho_{22}^2}{2}(d_1 - d_2)(T-t)}}, & \rho_{i2} \neq 0, \\ \frac{\omega_2^2}{2(\kappa_2 + \omega_2\rho_{21})}(e^{-(\kappa_2 + \omega_2\rho_{21})(T-t)} - 1), & \rho_{i2} = 0. \end{cases} \tag{3.10}$$

(2) When $q^* = \hat{q}(t)$,

$$d(t) = -\frac{\lambda \sigma_Z^2 m_1 \alpha_1}{r_0}[e^{-r_0(T-t)} - 1], \tag{3.11}$$

$$\begin{aligned} I(t) = &\kappa_1\phi_1 c_1(T-t) - \frac{2\kappa_1\phi_1}{\rho_{12}^2} \ln \frac{c_1 e^{\frac{\rho_{12}^2}{2}(c_1 - c_2)(T-t)} - c_2}{c_1 - c_2} \\ &+ \kappa_2\phi_2 d_1(T-t) - \frac{2\kappa_2\phi_2}{\rho_{22}^2} \ln \frac{d_1 e^{\frac{\rho_{22}^2}{2}(d_1 - d_2)(T-t)} - d_2}{d_1 - d_2} \\ &+ \frac{2\lambda\alpha_2^2\sigma_Z^2 m_1^2}{r_0(m_1 - m_2)^2} \ln\left|\frac{2\alpha_2 + (m_1 - m_2)}{2\alpha_2 + (m_1 - m_2)e^{r_0(T-t)}}\right| \\ &+ \frac{\lambda\alpha_2\sigma_Z^2 m_1^2}{r_0(m_1 - m_2)}[e^{r_0(T-t)} - 1], \quad \rho_{i2} \neq 0, \end{aligned} \tag{3.12}$$

and

$$
\begin{aligned}
I(t) =&\frac{\kappa_1\phi_1\omega_1^2}{2(\kappa_1+\omega_1\rho_{11})}[\frac{1-e^{-(\kappa_1+\omega_1\rho_{11})(T-t)}}{\kappa_1+\omega_1\rho_{11}}-(T-t)]\\
&+\frac{\kappa_2\phi_2\omega_2^2}{2(\kappa_2+\omega_2\rho_{21})}[\frac{1-e^{-(\kappa_2+\omega_2\rho_{21})(T-t)}}{\kappa_2+\omega_2\rho_{21}}-(T-t)]\\
&+\frac{2\lambda\alpha_2^2\sigma_Z^2m_1^2}{r_0(m_1-m_2)^2}\ln\left|\frac{2\alpha_2+(m_1-m_2)}{2\alpha_2+(m_1-m_2)e^{r_0(T-t)}}\right|\\
&+\frac{\lambda\alpha_2\sigma_Z^2m_1^2}{r_0(m_1-m_2)}[e^{r_0(T-t)}-1],\quad \rho_{i2}=0.
\end{aligned}
\tag{3.13}
$$

$J_1(t)$ and $J_2(t)$ are given by Eqs (3.9) and (3.10), respectively.
(3) When $q^*=1$,

$$
d(t)=\frac{m_1^2\sigma_0^2}{4r_0}[e^{-r_0(T-t)}-e^{r_0(T-t)}]+\frac{m_1\lambda\alpha_1\mu_2}{r_0}[1-e^{-r_0(T-t)}].
\tag{3.14}
$$

$I(t)$, $J_1(t)$, and $J_2(t)$ are given by Eqs (3.8)–(3.10), respectively, where

$$
\begin{aligned}
&\hat{q}(t)=\frac{2\alpha_2-m_2e^{r_0(T-t)}}{2\alpha_2+(m_1-m_2)e^{r_0(T-t)}}=1-\frac{m_1e^{r_0(T-t)}}{2\alpha_2+(m_1-m_2)e^{r_0(T-t)}},\\
&c_1=\frac{\kappa_1+\omega_1\rho_{11}+\sqrt{\Delta_1}}{\rho_{12}^2},\quad c_2=\frac{\kappa_1+\omega_1\rho_{11}-\sqrt{\Delta_1}}{\rho_{12}^2},\\
&d_1=\frac{\kappa_2+\omega_2\rho_{21}+\sqrt{\Delta_2}}{\rho_{22}^2},\quad d_2=\frac{\kappa_2+\omega_2\rho_{21}-\sqrt{\Delta_2}}{\rho_{22}^2},\\
&\Delta_i=(\kappa_i+\omega_i\rho_{i1})^2+\omega_i^2\rho_{i2}^2>0, i=1,2,\\
&L(1-\sqrt{\frac{\alpha_1}{\alpha_2}})=(m_1-m_2)e^{r_0(T-t)}\lambda\alpha_1\sigma_Z^2+[\frac{\alpha_1}{2\alpha_2}(m_1-m_2)^2\\
&\qquad\qquad+(\frac{1}{2}-\sqrt{\frac{\alpha_1}{\alpha_2}})m_1{}^2+\sqrt{\frac{\alpha_1}{\alpha_2}}m_1m_2]\lambda\sigma_Z^2e^{2r_0(T-t)},\\
&L(1)=\frac{1}{2}\lambda\sigma_Z^2e^{2r_0(T-t)}m_1^2.
\end{aligned}
$$

Proof. See the Appendix. □

Remark 3.2. Since we employ a more general stochastic volatility model to describe the price dynamics of risky assets, the analytical solution of the entire model becomes more complex, and the research findings have broader applications in the financial market.

Theorem 3.4. For the optimal problem with $m_1=m_2$, π_1^* and π_2^* given in Eqs (3.4) and (3.5), they are also the optimal investment strategies for the insurer and reinsurer, and any measurable function $q^*(t):[0,T]\rightarrow[1-\sqrt{\frac{\alpha_1}{\alpha_2}},1]$ is an optimal reinsurance strategy. Furthermore, the optimal value function is

$$
V(t,x_1,x_2,v_1,v_2)=-\frac{1}{m_1m_2}e^{[-m_1x_1-m_2x_2-d(t)]e^{r_0(T-t)}+g(t,v_1,v_2)},
$$

where $d(t)$ is given by equation (3.12), and

$$
g(t,v_1,v_2)=I(t)+J_1(t)v_1+J_2(t)v_2,
$$

where $I(t)$, $J_1(t)$, and $J_2(t)$ are given by Eqs (3.8)–(3.10), respectively.

Proof. If $m_1 = m_2$, then equation (6.12) can be rewritten as

$$\begin{aligned}&[r_0 d(t) - d_t - m_1\lambda\alpha_1\sigma_Z^2 + \frac{1}{2}m_1^2\lambda\sigma_Z^2 e^{r_0(T-t)}]e^{r_0(T-t)} + g_t + \kappa_1[\phi_1 - v_1]g_{v_1}\\&+ \kappa_2[\phi_2 - v_2]g_{v_2} + \frac{v_1(\rho_{11}^2 + \rho_{12}^2)}{2}(g_{v_1v_1} + g_{v_1}^2) + \frac{v_2(\rho_{21}^2 + \rho_{22}^2)}{2}(g_{v_2v_2} + g_{v_2}^2)\\&- \frac{v_1(\omega_1 + \rho_{11}g_{v_1})^2}{2} - \frac{v_2(\omega_2 + \rho_{21}g_{v_2})^2}{2}) = 0.\end{aligned} \tag{3.15}$$

Equation (3.15) is independent of $q^*(t)$ and can be divided into the following two equations:

$$r_0 d(t) - d_t - m_1\lambda\alpha_1\sigma_Z^2 + \frac{1}{2}{m_1}^2\lambda\sigma_Z^2 e^{r_0(T-t)} = 0, \tag{3.16}$$

and Eq (6.20). Thus, we obtain the expressions of $g(t, v_1, v_2), I(t), J_1(t)$, and $J_2(t)$ by Eqs (6.22) and (3.8)–(3.10).

Note that Eq (3.16) is a linear ordinary differential equation with the boundary condition $d(T) = 0$; it is not difficult to derive that

$$d(t) = -\frac{m_1\lambda\alpha_1\mu_2}{r_0}[e^{-r_0(T-t)} - 1] + \frac{\lambda\sigma_Z^2}{4r_0}m_1^2[e^{-r_0(T-t)} - e^{r_0(T-t)}], \tag{3.17}$$

then we can get the explicit expression of the value function $V(t, x_1, x_2, v_1, v_2)$. Similar to Theorem 3.3, we can easily derive the optimal investment strategies for the insurer and reinsurer. The procedure is similar to that of $m_1 \neq m_2$, so we omit it here. □

4. Special cases

This section is devoted to seeking optimal reinsurance and investment strategies for some of the relevant models and corresponding value functions.

4.1. Optimal strategy for the insurer and reinsurer under the CEV model

In this case, we discuss the optimization problem under the CEV model in Remark 2.3. Then the wealth process (2.10) and (2.11) are rewritten as

$$\begin{cases} dX_1^u(t) = [r_0 X_1^u(t) + (\mu_1(t) - r_0)\pi_1(t) + \lambda\alpha_1\sigma_Z^2 - \lambda\alpha_2\sigma_Z^2(1 - q(t))^2]dt \\ \qquad + \pi_1(t)\sigma_1(S_1(t))^{\nu_1}dW_1(t) + q(t)\sqrt{\lambda\sigma_Z^2}dW_0(t), \\ X_1(0) = x_{01}, \end{cases} \tag{4.1}$$

and

$$\begin{cases} dX_2^u(t) = [r_0(t)X_2^u(t) + (\mu_2 - r_0)\pi_2(t) + \lambda\alpha_2\sigma_Z^2(1 - q(t))^2]dt \\ \qquad + \pi_2(t)\sigma_2(S_2(t))^{\nu_2}dW_2(t) + (1 - q(t))\sqrt{\lambda\sigma_Z^2}dW_0(t), \\ X_2(0) = x_{02}. \end{cases} \tag{4.2}$$

Proposition 4.1. For optimization problem (3.1), if the price process of risky asset $S_i(t)(i = 1, 2)$ is governed by the CEV model, the optimal investment strategies are given by

$$\pi_1^*(t) = \frac{2(\mu_1 - r_0) - (\mu_1 - r_0)^2(e^{-2r_0\nu_1(T-t)} - 1)}{2r_0 m_1 \sigma_1^2 (s_1)^{2\nu_1}} e^{-r_0(T-t)}, \tag{4.3}$$

and

$$\pi_2^*(t) = \frac{2(\mu_2 - r_0) - (\mu_2 - r_0)^2(e^{-2r_0\nu_2(T-t)} - 1)}{2r_0 m_2 \sigma_2^2 (s_2)^{2\nu_2}} e^{-r_0(T-t)}. \tag{4.4}$$

The optimal reinsurance strategies are given by

Case (I), If $m_1 > m_2$ and $\hat{\Delta}_2 \geq 1$, then

$$q^* = \begin{cases} 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, & 0 \leq t \leq t_2, \\ \hat{q}(t), & t_2 \leq t \leq T. \end{cases}$$

Case (II), If $m_1 > m_2$ and $\hat{\Delta}_2 < 1$, then

$$q^* = 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, 0 \leq t \leq T.$$

Case (III), If $m_1 < m_2$ and $\hat{\Delta}_1 > \hat{\Delta}_2 \geq 1$, when $L\Big(1 - \sqrt{\frac{\alpha_1}{\alpha_2}}\Big) \geq L(1)$, then

$$q^* = \begin{cases} 1, & 0 \leq t \leq t_1, \\ 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, & t_1 \leq t \leq t_2, \\ 1, & t_2 \leq t \leq T, \end{cases}$$

when $L\Big(1 - \sqrt{\frac{\alpha_1}{\alpha_2}}\Big) < L(1)$, then

$$q^* = \begin{cases} 1, & 0 \leq t \leq t_1, \\ 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, & t_1 \leq t \leq T. \end{cases}$$

Case (IV), If $m_1 < m_2$ and $\hat{\Delta}_2 < 1 \leq \hat{\Delta}_1$, then

$$q^* = \begin{cases} 1, & 0 \leq t \leq t_1, \\ 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, & t_1 \leq t \leq T. \end{cases}$$

Case (V), If $m_1 < m_2$ and $\hat{\Delta}_2 < \hat{\Delta}_1 < 1$, then

$$q^* = 1, 0 \leq t \leq T.$$

Case (VI), If $m_1 = m_2$, then any measurable function $q^*(t) : [0, T] \to [1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, 1]$ is an optimal reinsurance strategy.

When q^* takes different values, the explicit expression of the value function is as follows:

$$V(t,x_1,x_2,v_1,v_2) = -\frac{1}{m_1 m_2} e^{[-m_1x_1-m_2x_2-d(t)]e^{r_0(T-t)}+g(t,v_1,v_2)},$$

where

$$g(t,v_1,v_2) = I(t) + J_1(t)v_1 + J_2(t)v_2.$$

(1) When $q^* = 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$,

$$\begin{aligned} d(t) =& -\frac{m_2\lambda\alpha_1\mu_2}{r_0}[e^{-r_0(T-t)} - 1] + \frac{\sigma_0^2}{2r_0}\Big[\Big(\frac{1}{2} - \sqrt{\frac{\alpha_1}{\alpha_2}}\Big)m_1^2 + \frac{\alpha_1}{2\alpha_2}(m_1-m_2)^2 \\ &+ \sqrt{\frac{\alpha_1}{\alpha_2}}m_1m_2\Big] \times [e^{-r_0(T-t)} - e^{r_0(T-t)}], \end{aligned} \tag{4.5}$$

$$\begin{aligned} I(t) =& \frac{(2v_1+1)(\mu_1-r_0)^2}{4r_0}[\frac{1-e^{-2v_1r_0(T-t)}}{2v_1r_0} - (T-t)] \\ &+ \frac{(2v_2+1)(\mu_2-r_0)^2}{4r_0}[\frac{1-e^{-2v_2r_0(T-t)}}{2v_2r_0} - (T-t)], \end{aligned} \tag{4.6}$$

$$J_1(t) = \frac{(\mu_1-r_0)^2}{4r_0v_1\sigma_1^2}(e^{-2r_0v_1(T-t)} - 1), \tag{4.7}$$

and

$$J_2(t) = \frac{(\mu_2-r_0)^2}{4r_0v_2\sigma_2^2}(e^{-2r_0v_2(T-t)} - 1). \tag{4.8}$$

(2) When $q^* = \hat{q}(t)$,

$$d(t) = -\frac{\lambda\sigma_Z^2m_1\alpha_1}{r_0}[e^{-r_0(T-t)} - 1], \tag{4.9}$$

$$\begin{aligned} I(t) =& \frac{(2v_1+1)(\mu_1-r_0)^2}{4r_0}[\frac{1-e^{-2v_1r_0(T-t)}}{2v_1r_0} - (T-t)] \\ &+ \frac{(2v_2+1)(\mu_2-r_0)^2}{4r_0}[\frac{1-e^{-2v_2r_0(T-t)}}{2v_2r_0} - (T-t)] \\ &+ \frac{2\lambda\alpha_2^2\sigma_Z^2m_1^2}{r_0(m_1-m_2)^2}\ln\left|\frac{2\alpha_2+(m_1-m_2)}{2\alpha_2+(m_1-m_2)e^{r_0(T-t)}}\right| \\ &+ \frac{\lambda\alpha_2\sigma_Z^2m_1^2}{r_0(m_1-m_2)}[e^{r_0(T-t)} - 1]. \end{aligned} \tag{4.10}$$

$J_1(t)$ and $J_2(t)$ are given by Eqs (4.7) and (4.8), respectively.
(3) When $q^* = 1$,

$$d(t) = \frac{m_1^2\sigma_0^2}{4r_0}[e^{-r_0(T-t)} - e^{r_0(T-t)}] + \frac{m_1\lambda\alpha_1\mu_2}{r_0}[1 - e^{-r_0(T-t)}]. \tag{4.11}$$

$I(t)$, $J_1(t)$ and $J_2(t)$ are given by Eqs (4.6)–(4.8), respectively.

Specifically, when $m_1 = m_2$, $d(t)$, $J_1(t)$, $J_2(t)$, and $I(t)$ are given by Eqs (3.18) and (4.6)–(4.8), respectively.

Remark 4.1. The CEV model is a mathematical model used to describe the volatility of financial asset returns. By introducing an elasticity parameter, it provides a more flexible and realistic framework to describe and analyze the volatility of financial asset prices, enabling investors and risk managers to make more precise decisions in derivatives pricing, risk management, and the formulation of quantitative investment strategies.

4.2. Optimal strategy for the insurer and reinsurer under the Heston model

In this case, we discuss the optimization problem under the Heston model in Remark 2.4. Then the wealth process (2.10) and (2.11) are rewritten as

$$\begin{cases} \mathrm{d}X_1^u(t) =[r_0(t)X_1^u(t) + \omega_1\vartheta_1(t)\pi_1(t) + \lambda\alpha_1\sigma_Z^2 - \lambda\alpha_2\sigma_Z^2(1-q(t))^2]dt \\ \qquad\qquad + \pi_1(t)\sqrt{\vartheta_1(t)}\mathrm{d}W_1(t) + q(t)\sqrt{\lambda\sigma_Z^2}\mathrm{d}W_0(t), \\ X_1(0) =x_{01}, \end{cases} \tag{4.12}$$

and

$$\begin{cases} \mathrm{d}X_2^u(t) =[r_0(t)X_2^u(t) + \omega_2\vartheta_2(t)\pi_2(t) + \lambda\alpha_2\sigma_Z^2(1-q(t))^2]dt \\ \qquad\qquad + \pi_2(t)\sqrt{\vartheta_2(t)}\mathrm{d}W_2(t) + (1-q(t))\sqrt{\lambda\sigma_Z^2}\mathrm{d}W_0(t), \\ X_2(0) =x_{02}. \end{cases} \tag{4.13}$$

Proposition 4.2. For optimization problem (3.1), if the price process of risky asset $S_i(t)(i = 1, 2)$ is governed by the Heston model, the optimal investment strategies are given by

$$\begin{aligned} \pi_1^*(t) =&\frac{\omega_1(c_1 - c_2e^{-\frac{\sigma_{01}^2(1-\rho_1^2)}{2}(c_1-c_2)(T-t)})}{\sigma_{01}\rho_1c_1c_2(1-e^{-\frac{\sigma_{01}^2(1-\rho_1^2)}{2}(c_1-c_2)(T-t)})}e^{-r_0(T-t)} \\ &+ \frac{\sigma_{01}\rho_1c_1c_2(1-e^{-\frac{\sigma_{01}^2(1-\rho_1^2)}{2}(c_1-c_2)(T-t)})}{\sigma_{01}\rho_1c_1c_2(1-e^{-\frac{\sigma_{01}^2(1-\rho_1^2)}{2}(c_1-c_2)(T-t)})}e^{-r_0(T-t)}, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \pi_2^*(t) =&\frac{\omega_2(d_1 - d_2e^{-\frac{\sigma_{02}^2(1-\rho_2^2)}{2}(d_1-d_2)(T-t)})}{m_2(d_1 - d_2e^{-\frac{\sigma_{02}^2(1-\rho_2^2)}{2}(d_1-d_2)(T-t)})}e^{-r_0(T-t)} \\ &+ \frac{\sigma_{02}\rho_2d_1d_2(1-e^{-\frac{\sigma_{02}^2(1-\rho_2^2)}{2}(d_1-d_2)(T-t)})}{m_2(d_1 - d_2e^{-\frac{\sigma_{02}^2(1-\rho_2^2)}{2}(d_1-d_2)(T-t)})}e^{-r_0(T-t)}. \end{aligned} \tag{4.15}$$

The optimal reinsurance strategies are given by:
Case (I), If $m_1 > m_2$ and $\hat{\Delta}_2 \geq 1$, then

$$q^* = \begin{cases} 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, & 0 \leq t \leq t_2, \\ \hat{q}(t), & t_2 \leq t \leq T. \end{cases}$$

Case (II), If $m_1 > m_2$ and $\hat{\Delta}_2 < 1$, then

$$q^* = 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, 0 \leq t \leq T.$$

Case (III), If $m_1 < m_2$ and $\hat{\Delta}_1 > \hat{\Delta}_2 \geq 1$, when $L\Big(1 - \sqrt{\frac{\alpha_1}{\alpha_2}}\Big) \geq L(1)$, then

$$q^* = \begin{cases} 1, & 0 \leq t \leq t_1, \\ 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, & t_1 \leq t \leq t_2, \\ 1, & t_2 \leq t \leq T, \end{cases}$$

when $L\Big(1 - \sqrt{\frac{\alpha_1}{\alpha_2}}\Big) < L(1)$, then

$$q^* = \begin{cases} 1, & 0 \leq t \leq t_1, \\ 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, & t_1 \leq t \leq T. \end{cases}$$

Case (IV), If $m_1 < m_2$ and $\hat{\Delta}_2 < 1 \leq \hat{\Delta}_1$, then

$$q^* = \begin{cases} 1, & 0 \leq t \leq t_1, \\ 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, & t_1 \leq t \leq T. \end{cases}$$

Case (V), If $m_1 < m_2$ and $\hat{\Delta}_2 < \hat{\Delta}_1 < 1$, then

$$q^* = 1, 0 \leq t \leq T.$$

Case (VI), If $m_1 = m_2$, then any measurable function $q^*(t) : [0, T] \to [1 - \sqrt{\frac{\alpha_1}{\alpha_2}}, 1]$ is an optimal reinsurance strategy.

When q^* takes different values, the explicit expression of the value function is as follows:

$$V(t, x_1, x_2, v_1, v_2) = -\frac{1}{m_1 m_2} e^{[-m_1 x_1 - m_2 x_2 - d(t)]e^{r_0(T-t)} + g(t, v_1, v_2)},$$

where

$$g(t, v_1, v_2) = I(t) + J_1(t)v_1 + J_2(t)v_2.$$

(1) When $q^* = 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$,

$$\begin{aligned} d(t) = &- \frac{m_2 \lambda \alpha_1 \mu_2}{r_0}[e^{-r_0(T-t)} - 1] + \frac{\sigma_0^2}{2r_0}\Big[\Big(\frac{1}{2} - \sqrt{\frac{\alpha_1}{\alpha_2}}\Big)m_1^2 + \frac{\alpha_1}{2\alpha_2}(m_1 - m_2)^2 \\ &+ \sqrt{\frac{\alpha_1}{\alpha_2}} m_1 m_2\Big] \times [e^{-r_0(T-t)} - e^{r_0(T-t)}], \end{aligned} \tag{4.16}$$

$$\begin{aligned} I(t) =& \kappa_1 \phi_1 c_1 (T-t) - \frac{2\kappa_1 \phi_1}{\sigma_{01}(1-\rho_1^2)} \ln \frac{c_1 e^{\frac{\sigma_{01}(1-\rho_1^2)}{2}(c_1 - c_2)(T-t)} - c_2}{c_1 - c_2} \\ &+ \kappa_2 \phi_2 d_1 (T-t) - \frac{2\kappa_2 \phi_2}{\sigma_{02}(1-\rho_2^2)} \ln \frac{d_1 e^{\frac{\sigma_{02}(1-\rho_2^2)}{2}(d_1 - d_2)(T-t)} - d_2}{d_1 - d_2}, \end{aligned} \tag{4.17}$$

$$J_1(t) = \frac{c_1c_2(1 - e^{-\frac{\sigma_{01}(1-\rho_1^2)}{2}(c_1-c_2)(T-t)})}{c_1 - c_2e^{-\frac{\sigma_{01}(1-\rho_1^2)}{2}(c_1-c_2)(T-t)}}, \tag{4.18}$$

and

$$J_2(t) = \frac{d_1d_2(1 - e^{-\frac{\sigma_{02}(1-\rho_2^2)}{2}(d_1-d_2)(T-t)})}{d_1 - d_2e^{-\frac{\sigma_{02}(1-\rho_2^2)}{2}(d_1-d_2)(T-t)}}. \tag{4.19}$$

(2) When $q^* = \hat{q}(t)$,

$$d(t) = -\frac{\lambda\sigma_Z^2 m_1\alpha_1}{r_0}[e^{-r_0(T-t)} - 1], \tag{4.20}$$

and

$$\begin{aligned} I(t) =&\kappa_1\phi_1c_1(T-t) - \frac{2\kappa_1\phi_1}{\sigma_{01}(1-\rho_1^2)}\ln\frac{c_1e^{\frac{\sigma_{01}(1-\rho_1^2)}{2}(c_1-c_2)(T-t)} - c_2}{c_1 - c_2} \\ &+ \kappa_2\phi_2d_1(T-t) - \frac{2\kappa_2\phi_2}{\sigma_{02}(1-\rho_2^2)}\ln\frac{d_1e^{\frac{\sigma_{02}(1-\rho_2^2)}{2}(d_1-d_2)(T-t)} - d_2}{d_1 - d_2} \\ &+ \frac{2\lambda\alpha_2^2\sigma_Z^2m_1^2}{r_0(m_1-m_2)^2}\ln\left|\frac{2\alpha_2 + (m_1-m_2)}{2\alpha_2 + (m_1-m_2)e^{r_0(T-t)}}\right| \\ &+ \frac{\lambda\alpha_2\sigma_Z^2m_1^2}{r_0(m_1-m_2)}[e^{r_0(T-t)} - 1]. \end{aligned} \tag{4.21}$$

$J_1(t)$ and $J_2(t)$ are given by Eqs (4.18) and (4.19), respectively.

(3) When $q^* = 1$, where

$$d(t) = \frac{m_1^2\sigma_0^2}{4r_0}[e^{-r_0(T-t)} - e^{r_0(T-t)}] + \frac{m_1\lambda\alpha_1\mu_2}{r_0}[1 - e^{-r_0(T-t)}]. \tag{4.22}$$

$I(t)$, $J_1(t)$ and $J_2(t)$ are given by Eqs (4.17)–(4.19), respectively, where

$$\begin{aligned} &c_1 = \frac{\kappa_1 + \omega_1\sigma_{01}\rho_1 + \sqrt{\Delta_1}}{\sigma_{01}(1-\rho_1^2)}, \quad c_2 = \frac{\kappa_1 + \omega_1\sigma_{01}\rho_1 - \sqrt{\Delta_1}}{\sigma_{01}(1-\rho_1^2)}, \\ &d_1 = \frac{\kappa_2 + \omega_2\sigma_{02}\rho_2 + \sqrt{\Delta_2}}{\sigma_{02}(1-\rho_2^2)}, \quad d_2 = \frac{\kappa_2 + \omega_2\sigma_{02}\rho_2 - \sqrt{\Delta_2}}{\sigma_{02}(1-\rho_2^2)}, \\ &\Delta_i = (\kappa_i + \omega_i\sigma_{0i}\rho_i)^2 + \omega_i^2\sigma_{0i}(1-\rho_i^2) > 0, i = 1, 2. \end{aligned}$$

Specially, when $m_1 = m_2$, $d(t), J_1(t), J_2(t)$ and $I(t)$ and are given by Eqs (3.18) and (4.17)–(4.19), respectively.

Remark 4.2. The Heston model is a stochastic volatility model used for pricing financial derivatives. By introducing stochastic volatility and the mean-reverting characteristic of volatility, it provides a framework for derivative pricing that is closer to the actual behavior of financial markets. This model allows investors and risk managers to make more precise decisions in derivative pricing, risk management, and the formulation of quantitative investment strategies.

Remark 4.3. The CEV model and the Heston model are two distinct models within the field of financial mathematics, each playing a unique role and offering advantages in the areas of option pricing and financial derivatives analysis. Depending on the specific risk market environment, different models are chosen, and there is no inclusion relationship between these two models.

5. Numerical experiment and analysis

In this section, we provide some numerical examples to show the effects of some model parameters on the optimal reinsurance and investment strategy. We assume that the claim size Z_i follows an exponential distribution with parameter λ_Z, i.e., the density function of Z_i is given by $f(z) = \lambda_Z e^{-\lambda_Z z}, z \geq 0$. Throughout this section, unless otherwise stated, the basic parameters are given by Tables 1–3. Specifically, we have set the risk-free interest rate $r_0 = 0.1$.

5.1. Effects of model parameters on the optimal reinsurance strategy

Table 1. Model general parameters.

Time parameters		Insurer parameters	
T	t	α_1	m_1
5	0	0.8	1.8
Reinsurer parameters		Insurance claim parameters	
α_2	m_2	λ_Z	λ
1.2	1.3	1	1

In Figure 1, we let $\Delta_2 > 1$ with $\alpha_2 = 1.1, 1.2, m_1 = 1.8, m_2 = 1.3$, and $\Delta_2 < 1$ with $\alpha_2 = 1.1, 1.2, m_1 = 2, m_2 = 1.9$. From Theorem 3.2, the optimal reinsurance strategy is a fixed constant $1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$ when $\Delta_2 < 1$. When $\Delta_2 > 1$, we find that the initial retention level q increases with the increase of α_2, and the larger α_2 is, the earlier the optimal strategy changes. This result can be explained by the fact that the larger α_2, the higher the reinsurance price and the less reinsurance the insurer buys.

Let $m_2 = 1.2$, then we get $\Delta_2 > 1$ with $m_1 = 1.9 \sim 2.1$, and we obtain $q^* = \hat{q}$ when $t \in (t_2, T)$. Figure 2 shows that the optimal reinsurance strategy q^* is a decreasing function of the insurer's risk aversion coefficient m_1. We find that when the risk aversion coefficient of the reinsurer is constant, the insurer with a higher risk aversion coefficient is willing to buy more reinsurance.

Figure 3 displays that the optimal reinsurance strategy q^* is a decreasing function of the reinsurer's risk aversion coefficient m_2. Let $m_1 = 2$, then we can calculate $\Delta_2 > 1$ with $m_2 = 1.2 \sim 1.4$, and we obtain $q^* = \hat{q}$ when $t \in (t_2, T)$. We find that when the risk aversion coefficient of the insurer is constant, the reinsurer with higher risk aversion coefficient is willing to accept more claim risk. One possible reason for this is that the reinsurer with a higher risk aversion invest less in risky assets and have more cash to hedge against claims.

Figure 4 shows that q^* increases with time t and the security load of reinsurer α_2. It can be explained that the greater the safety load of the reinsurer, the more premium the insurer will pay, and then the insurer will appropriately reduce the reinsurance ratio and increase the retention level.

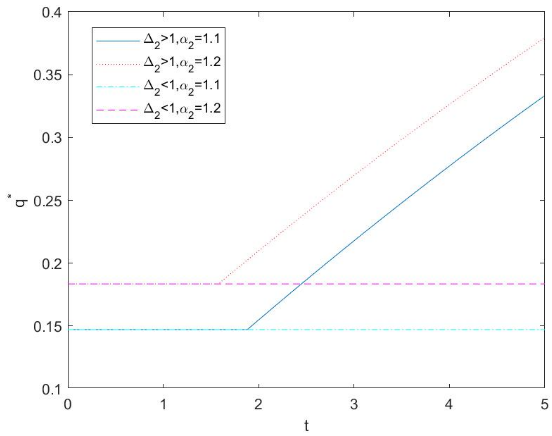

Figure 1. The optimal reinsurance retention level q^* varies over time when $m_1 > m_2$.

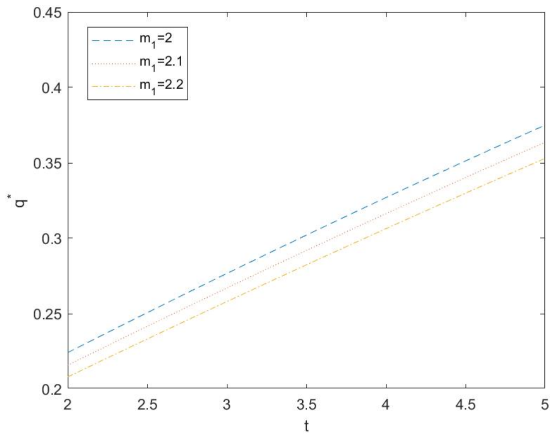

Figure 2. The effect of m_1 on the optimal reinsurance retention level q^*.

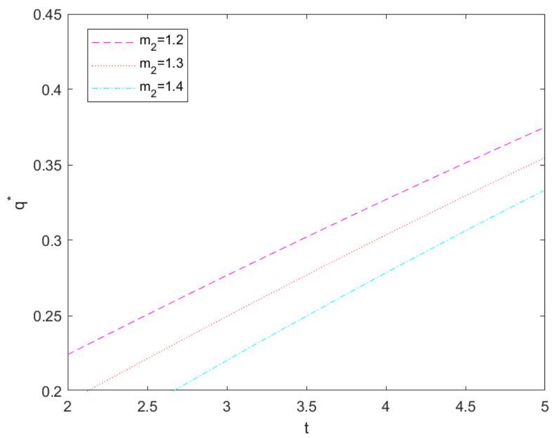

Figure 3. The effect of m_2 on the optimal reinsurance retention level q^*.

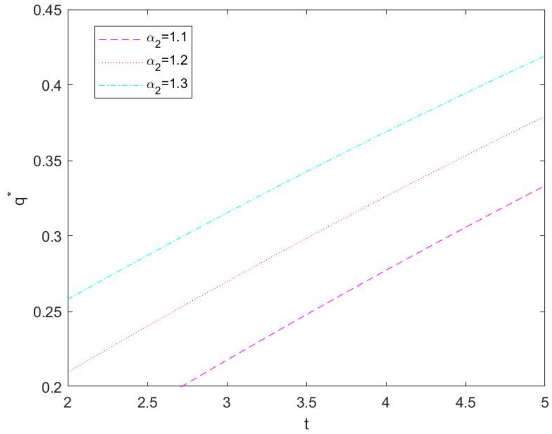

Figure 4. The effect of α_2 on the optimal reinsurance retention level q^*.

5.2. Effects of model parameters on the optimal investment strategy under the CEV model

Table 2. CEV model parameters.

Financial market parameters under the CEV model							
s_1	s_2	μ_1	μ_2	ν_1	ν_2	σ_1	σ_2
1	2	0.2	0.3	-0.8	-0.7	1	2

Figure 5 shows that the optimal investment strategy decreases with the increase of the risk aversion coefficient. The reason is that when the risk aversion coefficient becomes larger, the insurer will increase the reinsurance proportion and reduce the investment amount of risky assets.

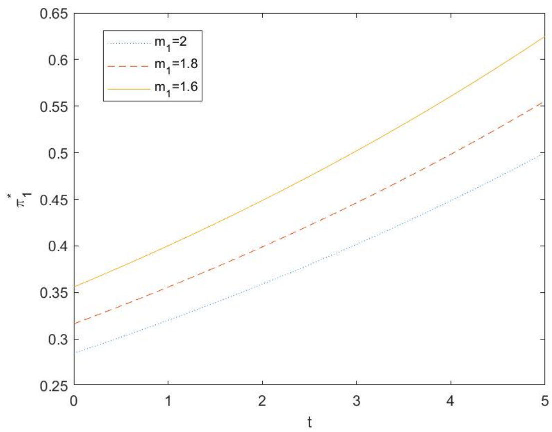

Figure 5. The effect of m_1 on the optimal investment strategy π_1^* under the CEV model.

From Figure 6, we find that near the initial time, the greater the risk aversion coefficient of the reinsurer, the greater the amount of investment in risky assets. This is because the greater the risk aversion coefficient, the more reinsurance premiums reinsurance companies charge, and they can invest more money in risky assets. In addition, we also find that the amount of investment in risky assets by reinsurers increases more gently with the increase of the risk aversion coefficient.

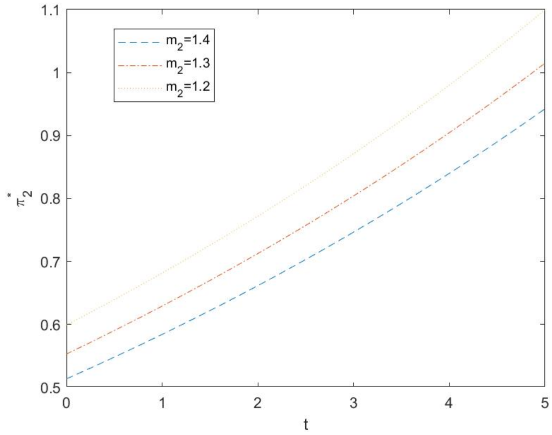

Figure 6. The effect of m_2 on the optimal investment strategy π_2^* under the CEV model.

Figures 7 and 8 present that when the risk-free intersets rate is fixed and the instantaneous rate of return of risky assets increases, both the insurer and the reinsurer will increase their investment in risky assets. This is consistent with our intuition.

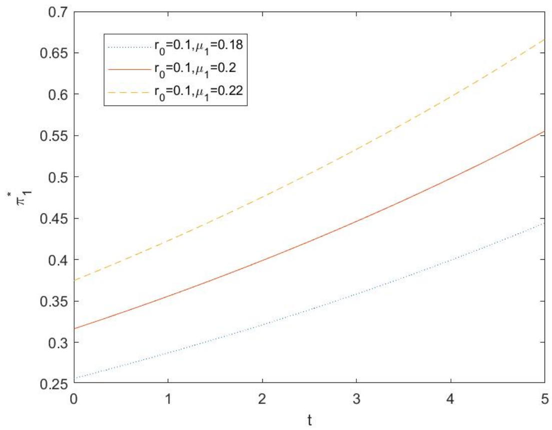

Figure 7. The effect of μ_1 on the optimal investment strategy π_1^* under the CEV model.

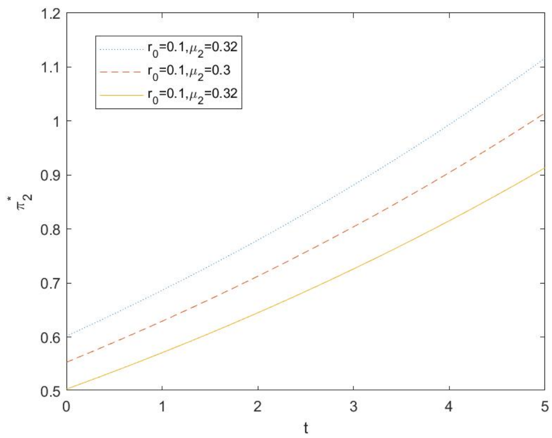

Figure 8. The effect of μ_2 on the optimal investment strategy π_2^* under the CEV model.

5.3. *Effects of model parameters on the optimal investment strategy under the Heston model*

In Figure 9, we know that the insurer's investment strategy π_1^* decreases with m_1, which means that when m_1 becomes larger, the insurer will reduce its investment in risky assets. Figure 10 also displays the negative correlation between the reinsurer's optimal investment strategy π_2^* and its risk aversion coefficient m_2.

Table 3. Heston model parameters.

Financial market parameters under the Heston model							
ω_1	ω_2	κ_1	κ_2	σ_{01}	σ_{02}	ρ_1	ρ_2
2	1.2	3	1	1	1	0.3	0.3

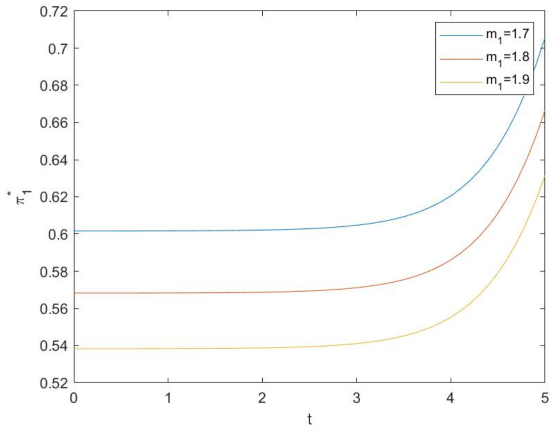

Figure 9. The effect of m_1 on the optimal investment strategy π_1^* under the Heston model.

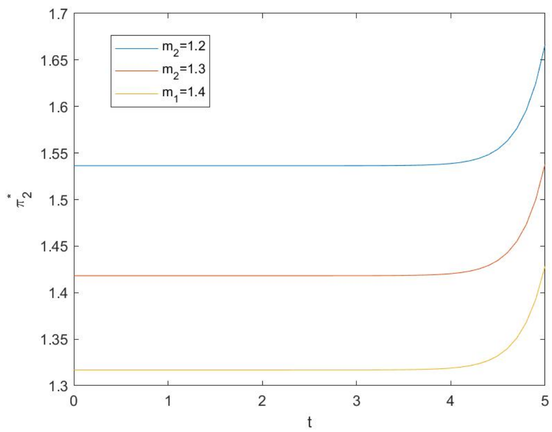

Figure 10. The effect of m_2 on the optimal investment strategy π_2^* under the Heston model.

Figure 11 demonstrates that the optimal investment strategy π_1^* increases with respect to ω_1. A larger ω_1 leads to a higher appreciation rate of the risky asset. Thus, the insurer will invest more in the risky asset when ω_1 becomes larger. Figure 12 also exhibits the positive correlation between the reinsurer's optimal investment strategy π_2^* and ω_2.

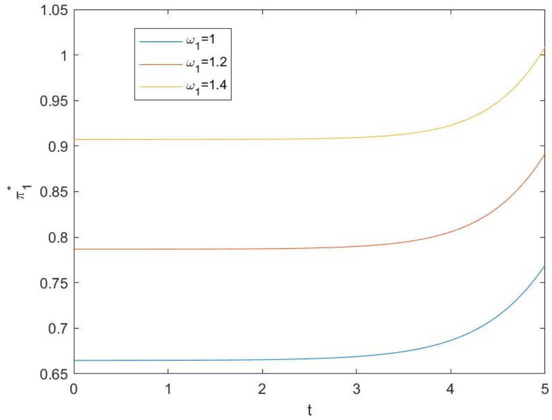

Figure 11. The effect of ω_1 on the optimal investment strategy π_1^* under the Heston model.

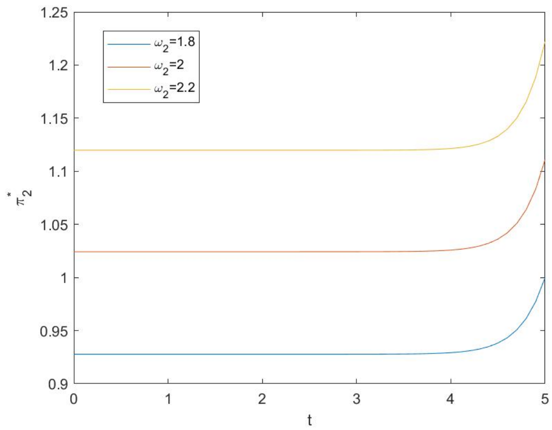

Figure 12. The effect of ω_2 on the optimal investment strategy π_2^* under the Heston model.

6. Conclusions

In this paper, the problem of optimal investment and proportional reinsurance with a joint exponential effect between the insurer and reinsurer is studied under the stochastic volatility model. Our aim is to maximize the expectation of the joint exponential utility of the terminal wealth of the insurer and reinsurer over a certain period of time. The surplus process of the insurer is described by a diffusion model. The insurer can purchase proportional reinsurance from the reinsurer, and the premium charged by the insurer and reinsurer follows the variance principle. Both the insurer and reinsurer are allowed to invest in risk-free assets and risky assets. The price process of risky assets is described by a Markov, affine-form, square-root stochastic factor process, which is a general stochastic volatility model, including the CEV model and Heston model. By solving the extended HJB equation, the optimal proportional reinsurance and investment strategy and its corresponding value function are explicitly derived. It is found that the optimal reinsurance strategy can be divided into several cases, which are related to the risk aversion coefficient of the insurer and reinsurer, and are not related to the price of risk assets. There are still some issues to be discussed in the future. For example, other reinsurance may be considered, such as overage or stop-loss reinsurance. Dependent risk model, such as common-shock dependence or thinning dependence, can also be taken into account. Or consider a financial market consisting of one risk-free asset and n risky assets, where the risk premium is dependent on the affine diffusion factor process.

Author contributions

Wuyuan Jiang: Conceptualization, Investigation, Analysis, Writing-review; Zechao Miao: Conceptualization, Investigation, Analysis, Writing-original draft preparation, Writing-review and editing; Jun Liu: Contributed to the literature review, linking the study to prior research and assisting in the interpretation of findings. All authors participated in drafting the manuscript, revising it critically for important intellectual content. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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Appendix

Proof of Lemma 3.1

Plugging u^* into Eqs (2.10) and (2.11), respectively, we have

$$\begin{aligned}X_1^{u^*}(t) =& x_{01}e^{r_0 t} + \int_0^t e^{r_0(t-s)}A_1(s)ds + \int_0^t e^{r_0(t-s)}\frac{\pi_1^*(s)(\mu_1(s)-r_0)}{\omega_1\sqrt{\vartheta_1(s)}}dW_1(s)\\ &+\int_0^t e^{r_0(t-s)}q^*(s)\sqrt{\lambda\sigma_Z^2}dW_0(s),\end{aligned}$$

and

$$\begin{aligned}X_2^{u^*}(t) =& x_{02}e^{r_0 t} + \int_0^t e^{r_0(t-s)}A_2(s)ds + \int_0^t e^{r_0(t-s)}\frac{\pi_2^*(s)(\mu_2(s)-r_0)}{\omega_2\sqrt{\vartheta_2(s)}}dW_2(s)\\ &+\int_0^t e^{r_0(t-s)}q^*(s)\sqrt{\lambda\sigma_Z^2}dW_0(s),\end{aligned}$$

where $A_1(s) = (\mu_1(s)-r_0)\pi_1^*(s) + \lambda\alpha_1\sigma_Z^2 - \lambda\alpha_2\sigma_Z^2(1-q(s)^*)^2$ and $A_2(s) = (\mu_2(s)-r_0)\pi_2^*(s) + \lambda\alpha_2\sigma_Z^2(1-q(s)^*)^2$. Then

$$\psi(t, X_1^{u^*}(t), X_2^{u^*}(t), \vartheta_1(t), \vartheta_2(t))^2 = \frac{1}{m_1^2 m_2^2}e^{[-2m_1X_1^{u^*}-2m_2X_2^{u^*}-2d(t)]e^{r_0(T-t)}+2g(t,\vartheta_1(t),\vartheta_2(t))}.$$

Furthermore, due to $d(t), g(t,\vartheta_1(t),\vartheta_2(t)), x_{01}e^{r_0t}, x_{02}e^{r_0t}, \int_0^t e^{r_0(t-s)}A_1(s)ds$ and $\int_0^t e^{r_0(t-s)}A_2(s)ds$ are deterministic and bounded, so we can get the following estimate with a apprppriate positive constant M

$$\psi(t, X_1^{u^*}(t), X_2^{u^*}(t), \vartheta_1(t), \vartheta_2(t))^2 \le MD_1(t)D_2(t)D_3(t)D_4(t),$$

where

$$\begin{aligned}D_1(t) &= e^{-2m_1e^{r_0(T-t)}\int_0^t e^{r_0(t-s)}\frac{\pi_1^*(s)(\mu_1(s)-r_0)}{\omega_1\sqrt{\vartheta_1(s)}}dW_1(s)},\\ D_2(t) &= e^{-2m_1e^{r_0(T-t)}\int_0^t e^{r_0(t-s)}q^*(s)\sqrt{\lambda\sigma_Z^2}dW_0(s)},\\ D_3(t) &= e^{-2m_2e^{r_0(T-t)}\int_0^t e^{r_0(t-s)}\frac{\pi_2^*(s)(\mu_2(s)-r_0)}{\omega_2\sqrt{\vartheta_2(s)}}dW_2(s)},\\ D_4(t) &= e^{-2m_2e^{r_0(T-t)}\int_0^t e^{r_0(t-s)}q^*(s)\sqrt{\lambda\sigma_Z^2}dW_0(s)}.\end{aligned}$$

It is evident that $D_1(t), D_2(t), D_3(t)$ and $D_4(t)$ are all martingales. Hence

$$E[\psi(t, X_1^{u^*}(t), X_2^{u^*}(t), \vartheta_1(t), \vartheta_2(t))]^2 < \infty.$$

Proof of Theorem 3.2

Since ψ is a function in $C^{1,2,2,2,2}([0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+)$, for all $t \in [0,T], u \in \mathcal{U}$ and any stopping time $\tau \in [0,\infty)$, applying Itô's formula to ψ between t and $T \wedge \tau$, we obtain that

$$\begin{aligned}
&\psi(T \wedge \tau, X_1^u(T \wedge \tau), X_2^u(T \wedge \tau), \vartheta_1(T \wedge \tau), \vartheta_2(T \wedge \tau)) \\
&= \psi(t, x_1, x_2, v_1, v_2) + \int_t^{T\wedge\tau} \mathcal{A}^u\psi(s, , X_1^u(s), X_2^u(s), \vartheta_1(s), \vartheta_2(s))ds \\
&\quad + \int_t^{T\wedge\tau} [\psi_{x_1} q(s) + \psi_{x_2}(1 - q(s))] \sqrt{\lambda\sigma_Z^2} dW_0(s) \\
&\quad + \int_t^{T\wedge\tau} [\psi_{x_1} \frac{\pi_1(s)(\mu_1(s) - r_0)}{\omega_1 \sqrt{\vartheta_1(s)}} + \psi_{v_1}\rho_{11} \sqrt{\vartheta_1(s)}]dW_1(s) \\
&\quad + \int_t^{T\wedge\tau} [\psi_{x_2} \frac{\pi_2(s)(\mu_2(s) - r_0)}{\omega_2 \sqrt{\vartheta_2(s)}} + \psi_{v_2}\rho_{21} \sqrt{\vartheta_2(s)}]dW_2(s) \\
&\quad + \int_t^{T\wedge\tau} \psi_{v_1}\rho_{12} \sqrt{\vartheta_1(s)}d\overline{W}_1(s) + \int_t^{T\wedge\tau} \psi_{v_2}\rho_{22} \sqrt{\vartheta_2(s)}d\overline{W}_2(s).
\end{aligned}$$

Since the last five terms are square-integrable martingales with zero expectation, taking conditional expectation given (t, x_1, x_2, v_1, v_2) on both sides of the above formula and taking Eq (3.3) into account result that

$$\begin{aligned}
&E^{t,x_1,x_2,v_1,v_2}[\psi(T \wedge \tau, X_1^u(T \wedge \tau), X_2^u(T \wedge \tau), \vartheta_1(T \wedge \tau), \vartheta_2(T \wedge \tau))] \\
&= \psi(t, x_1, x_2, v_1, v_2) + E^{t,x_1,x_2,v_1,v_2}[\int_t^{T\wedge\tau} \mathcal{A}^u\psi(s, , X_1^u(s), X_2^u(s), \vartheta_1(s), \vartheta_2(s))ds] \\
&\leq \psi(t, x_1, x_2, v_1, v_2).
\end{aligned}$$

By virtue of Lemma 3.1, $\psi(\tau_i \wedge T, X_1^u(\tau_i \wedge T), X_2^u(\tau_i \wedge T), \vartheta_1(\tau_i \wedge T), \vartheta_1(\tau_i \wedge T)), i = 1, 2, ...$are uniformly integrable. Thus we have

$$\begin{aligned}
&V(t, x_1, x_2, v_1, v_2) = \sup_{u\in\mathcal{U}} E^{t,x_1,x_2,v_1,v_2}[U[X^u(T), Y^u(T)] \\
&= \lim_{i\to\infty} E^{t,x_1,x_2,v_1,v_2}[\psi(\tau_i \wedge T, X_1^u(\tau_i \wedge T), X_2^u(\tau_i \wedge T), \vartheta_1(\tau_i \wedge T), \vartheta_1(\tau_i \wedge T))] \\
&\leq \psi(t, x_1, x_2, v_1, v_2).
\end{aligned}$$

Assuming that u^* is a measurable function valued in the set $\mathcal{U}$, such that

$$-\frac{\partial\psi}{\partial t}(t, x_1, x_2, v_1, v_2) - \sup_{u\in\mathcal{U}} \mathcal{L}^u\psi(t, x_1, x_2, v_1, v_2) = -\frac{\partial\psi}{\partial t}(t, x_1, x_2, v_1, v_2) - \mathcal{L}^{u^*}\psi(t, x_1, x_2, v_1, v_2) = 0.$$

Thus, it's easy for the aforementioned inequality to become an equality when $u = \mathcal{U}$. Theorem 3.2 is proved.

Proof of Theorem 3.3

Substituting Eq (3.2) into (3.3), we have the following HJB equation

$$\begin{aligned}
&\sup_{u\in U}\Big\{V_t + [r_0x_1 + (\mu_1(t) - r_0)\pi_1(t) + \lambda\alpha_1\sigma_Z^2 - \lambda\alpha_2\sigma_Z^2(1-q)^2]V_{x_1}\\
&+ [r_0x_2 + (\mu_2(t) - r_0)\pi_2(t) + \lambda\alpha_2\sigma_Z^2(1-q)^2]V_{x_2} + \kappa_1[\phi_1 - v_1]V_{v_1}\\
&+ \kappa_2[\phi_2 - v_2]V_{v_2} + \frac{v_1(\rho_{11}^2+\rho_{12}^2)}{2}V_{v_1v_1} + \frac{v_2(\rho_{21}^2+\rho_{22}^2)}{2}V_{v_2v_2}\\
&+ [\frac{\pi_1^2(\mu_1(t)-r_0)^2}{2\omega_1^2v_1} + \frac{1}{2}\lambda\sigma_Z^2q^2]V_{x_1x_1} + [\frac{\pi_2^2(\mu_2(t)-r_0)^2}{2\omega_2^2v_2} + \frac{1}{2}\lambda\sigma_Z^2(1-q)^2]V_{x_2x_2}\\
&+ \frac{\pi_1(\mu_1(t)-r_0)\rho_{11}}{\omega_1}V_{x_1v_1} + \frac{\pi_2(\mu_2(t)-r_0)\rho_{21}}{\omega_2}V_{x_2v_2} + \lambda\sigma_Z^2q(1-q)V_{x_1x_2}\Big\} = 0.
\end{aligned} \tag{6.1}$$

Inspried by [23], we try a solution to equation (6.1) by

$$V(t, x_1, x_2, v_1, v_2) = -\frac{1}{m_1m_2}e^{[-m_1x_1-m_2x_2-d(t)]e^{r_0(T-t)}+g(t,v_1,v_2)}, \tag{6.2}$$

with the boundary condition $g(T, v_1, v_2) = 0$ and $d(T) = 0$. Let $g_t, g_{v_1}, g_{v_2}, g_{v_1v_1}, g_{v_2v_2}$ be the first and second partial derivatives of g with respect to t, v_1, v_2, which are given by

$$\begin{aligned}
&V_t = \{-r_0e^{r_0(T-t)}[-m_1x - m_2y - d(t)] - d_te^{r_0(T-t)} + g_t\}V,\\
&V_{x_1} = -m_1e^{r_0(T-t)}V, V_{x_2} = -m_2e^{r_0(T-t)}V,\\
&V_{v_1} = g_{v_1}V, V_{v_2} = g_{v_2}V, V_{x_1x_1} = m_1^2e^{2r_0(T-t)}V, V_{x_2x_2} = m_2^2e^{2r_0(T-t)}V,\\
&V_{v_1v_1} = (g_{v_1v_1} + g_{v_1}^2)V, V_{v_2v_2} = (g_{v_2v_2} + g_{v_2}^2)V,\\
&V_{x_1v_1} = -m_1e^{r_0(T-t)}g_{v_1}V, V_{x_2v_2} = -m_2e^{r_0(T-t)}g_{v_2}V, V_{x_1x_2} = m_1m_2e^{2r_0(T-t)}V.
\end{aligned} \tag{6.3}$$

Substituting Eq (6.3) into (6.1), we have

$$\begin{aligned}
&\inf_{u\in U}\Big\{-r_0e^{r_0(T-t)}[-m_1x_1 - m_2x_2 - d(t)] - d_te^{r_0(T-t)} + g_t\\
&- m_1e^{r_0(T-t)}[r_0x_1 + (\mu_1(t) - r_0)\pi_1 + \lambda\alpha_1\sigma_Z^2 - \lambda\alpha_2\sigma_Z^2(1-q)^2]\\
&- m_2e^{r_0(T-t)}[r_0x_2 + (\mu_2(t) - r_0)\pi_2 + \lambda\alpha_2\sigma_Z^2(1-q)^2] + \kappa_1[\phi_1 - v_1]g_{v_1}\\
&+ \kappa_2[\phi_2 - v_2]g_{v_2} + \frac{v_1(\rho_{11}^2+\rho_{12}^2)}{2}(g_{v_1v_1} + g_{v_1}^2) + \frac{v_2(\rho_{21}^2+\rho_{22}^2)}{2}(g_{v_2v_2} + g_{v_2}^2)\\
&+ [\frac{\pi_1^2(\mu_1(t)-r_0)^2}{2\omega_1^2v_1} + \frac{1}{2}\lambda\sigma_Z^2q^2]m_1^2e^{2r_0(T-t)}\\
&+ [\frac{\pi_2^2(\mu_2(t)-r_0)^2}{2\omega_2^2v_2} + \frac{1}{2}\lambda\sigma_Z^2(1-q)^2]m_2^2e^{2r_0(T-t)} - \frac{\pi_1(\mu_1(t)-r_0)\rho_{11}}{\omega_1}m_1e^{2r_0(T-t)}g_{v_1}\\
&- \frac{\pi_2(\mu_2(t)-r_0)\rho_{21}}{\omega_2}m_2e^{2r_0(T-t)}g_{v_2} + \lambda\sigma_Z^2m_1m_2q(1-q)e^{2r_0(T-t)}\} = 0.
\end{aligned} \tag{6.4}$$

Differentiating Eq (6.4) with respect to π_1 and π_2, we obtain the following first-order optimality conditions

$$\pi_1^*(t) = \frac{\omega_1^2v_1 + \rho_{11}\omega_1v_1g_{v_1}}{(\mu_1(t) - r_0)m_1}e^{-r_0(T-t)}, \tag{6.5}$$

$$\pi_2^*(t) = \frac{\omega_2^2 v_2 + \rho_{21}\omega_2 v_2 g_{v_2}}{(\mu_2(t) - r_0)m_2} e^{-r_0(T-t)}. \tag{6.6}$$

Let

$$\begin{aligned} L(q,t) =& m_1 e^{r_0(T-t)}\lambda\alpha_2\sigma_Z^2(1-q)^2 - m_2 e^{r_0(T-t)}\lambda\alpha_2\sigma_Z^2(1-q)^2 \\ &+ \lambda\sigma_Z^2 m_1 m_2 q(1-q)e^{2r_0(T-t)} + \frac{1}{2}\lambda\sigma_Z^2 e^{2r_0(T-t)}[m_1^2 q^2 + m_2^2(1-q)^2]. \end{aligned} \tag{6.7}$$

In order to find the value of $q^*(t)$ that minimizes $L(q,t)$, we need to take the first and the second derivatives of $L(q,t)$ w.r.t q. Then $\frac{\partial L(q,t)}{\partial q}$ and $\frac{\partial^2 L(q,t)}{\partial q^2}$ are given by

$$\frac{\partial L(q,t)}{\partial q} = (m_1 - m_2)\lambda\sigma_Z^2 e^{r_0(T-t)}[2\alpha_2(q-1) + q(m_1 - m_2)e^{r_0(T-t)} + m_2 e^{r_0(T-t)}], \tag{6.8}$$

and

$$\frac{\partial^2 L(q,t)}{\partial q^2} = (m_1 - m_2)\lambda\sigma_Z^2 e^{r_0(T-t)}[2\alpha_2 + (m_1 - m_2)e^{r_0(T-t)}]. \tag{6.9}$$

Let $\frac{\partial L(q,t)}{\partial q} = 0$, we have

$$\hat{q}(t) = \frac{2\alpha_2 - m_2 e^{r_0(T-t)}}{2\alpha_2 + (m_1 - m_2)e^{r_0(T-t)}} = 1 - \frac{m_1 e^{r_0(T-t)}}{2\alpha_2 + (m_1 - m_2)e^{r_0(T-t)}}. \tag{6.10}$$

We first classify the optimal reinsurance strategy when q takes three different values and give the corresponding optimal investment π_1^* and π_2^* values, and finally we get the explicit expression of the corresponding value function. Plugging Eqs (6.5), (6.6) and the optimal reinsurance strategy q^* into (6.4), we have

$$\begin{aligned} &- r_0 e^{r_0(T-t)}[-m_1 x_1 - m_2 x_2 - d(t)] - d_t e^{r_0(T-t)} + g_t - m_1 e^{r_0(T-t)}[r_0 x_1 \\ &+ \lambda\alpha_1\sigma_z^2] - m_2 e^{r_0(T-t)} r_0 x_2 + \kappa_1[\phi_1 - v_1]g_{v_1} + \kappa_2[\phi_2 - v_2]g_{v_2} \\ &+ \frac{v_1(\rho_{11}^2 + \rho_{12}^2)}{2}(g_{v_1 v_1} + g_{v_1}^2) + \frac{v_2(\rho_{21}^2 + \rho_{22}^2)}{2}(g_{v_2 v_2} + g_{v_2}^2) \\ &- m_1 e^{r_0(T-t)}(\mu_1(t) - r_0)\pi_1^* + \frac{\pi_1^{*2}(\mu_1(t) - r_0)^2}{2\omega_1^2 v_1} m_1^2 e^{2r_0(T-t)} \\ &- \frac{\pi_1^*(\mu_1(t) - r_0)\rho_{11}}{\omega_1} m_1 e^{r_0(T-t)} g_{v_1} - m_2 e^{r_0(T-t)}(\mu_2(t) - r_0)\pi_2^* \\ &+ \frac{\pi_2^{*2}(\mu_2(t) - r_0)^2}{2\omega_2^2 v_2} m_2^2 e^{2r_0(T-t)} - \frac{\pi_2^*(\mu_2(t) - r_0)\rho_{21}}{\omega_2} m_2 e^{r_0(T-t)} g_{v_2} + L(q^*, t) = 0. \end{aligned} \tag{6.11}$$

Simplify Eq (6.11), we get

$$\begin{aligned} &[r_0 d(t) - d_t - m_1\lambda\alpha_1\sigma_Z^2]e^{r_0(T-t)} + g_t + \kappa_1[\phi_1 - v_1]g_{v_1} + \kappa_2[\phi_2 - v_2]g_{v_2} \\ &+ \frac{v_1(\rho_{11}^2 + \rho_{12}^2)}{2}(g_{v_1 v_1} + g_{v_1}^2) + \frac{v_2(\rho_{21}^2 + \rho_{22}^2)}{2}(g_{v_2 v_2} + g_{v_2}^2) \\ &- \frac{v_1(\omega_1 + \rho_{11} g_{v_1})^2}{2} - \frac{v_2(\omega_2 + \rho_{21} g_{v_2})^2}{2} + L(q^*, t) = 0. \end{aligned} \tag{6.12}$$

In order to find the optimal value of q for the minimizes $L(q,t)$ given by Eq (6.7), we need to discuss the concavity of $L(q,t)$ and the relationship between the sizes of $\hat{q}(t), 1-\frac{\alpha_1}{\alpha_2}$ and 1. We can easily observe that $\hat{\Delta}_1 > \hat{\Delta}_2 > 0$ when $m_1 < m_2$, and $\hat{\Delta}_2 > 0 > \hat{\Delta}_1$ when $m_1 > m_2$.

On the hand, $\frac{\partial^2 L(q,t)}{\partial q^2} > 0$ if and only if one of the following conditions holds

$$\begin{aligned} &(1) \quad m_1 > m_2, \\ &(2) \quad m_1 < m_2, \hat{\Delta}_1 \geq 1, 0 \leq t \leq t_1, \\ &(3) \quad m_1 < m_2, \hat{\Delta}_1 < 1, 0 \leq t \leq T, \end{aligned} \tag{6.13}$$

and $\frac{\partial^2 L(q,t)}{\partial q^2} < 0$ if only and if

$$m_1 < m_2, \hat{\Delta}_1 \geq 1, t_1 \leq t \leq T. \tag{6.14}$$

On the other hand, note that $\hat{q}(t) \leq 1-\sqrt{\frac{\alpha_1}{\alpha_2}}$ if and only if one of the following conditions holds

$$\begin{aligned} &(1) \quad m_1 > m_2, \hat{\Delta}_2 \geq 1, 0 \leq t \leq t_2, \\ &(2) \quad m_1 > m_2, \hat{\Delta}_2 < 1, 0 \leq t \leq T \\ &(3) \quad m_1 < m_2, \hat{\Delta}_1 > \hat{\Delta}_2 \geq 1, t_1 \leq t \leq t_2, \\ &(4) \quad m_1 < m_2, \hat{\Delta}_1 > 1 > \hat{\Delta}_2, t_1 \leq t \leq T, \end{aligned} \tag{6.15}$$

$1-\sqrt{\frac{\alpha_1}{\alpha_2}} < \hat{q}(t) < 1$ if and only if one of the following conditions holds

$$\begin{aligned} &(1) \quad m_1 > m_2, \hat{\Delta}_2 \geq 1, t_2 < t < T, \\ &(2) \quad m_1 < m_2, \hat{\Delta}_1 > \hat{\Delta}_2 \geq 1, t_2 < t < T, \end{aligned} \tag{6.16}$$

and $\hat{q}(t) \geq 1$ if and only if one of the following conditions holds

$$\begin{aligned} &(1) \quad m_1 < m_2, \hat{\Delta}_1 \geq 1, 0 < t < t_1, \\ &(2) \quad m_1 < m_2, \hat{\Delta}_1 < 1, 0 < t < T. \end{aligned} \tag{6.17}$$

Based on the above analysis, we draw the following conclusions.

(1) Combining Eqs (6.13) and (6.15), we get that when $m_1 > m_2, \hat{\Delta}_2 \geq 1, 0 \leq t \leq t_2$ or $m_1 > m_2, \hat{\Delta}_2 \geq 1, 0 \leq t \leq T$ is satisfied, there are $\frac{\partial^2 L(q,t)}{\partial q^2} > 0$ and $\hat{q}(t) \leq 1-\sqrt{\frac{\alpha_1}{\alpha_2}}$, then $q^* = 1$.

(2) Combining Eqs (6.13) and (6.16), we find that when $m_1 > m_2, \hat{\Delta}_2 \geq 1, t_2 < t < T$ is satisfied, there are $\frac{\partial^2 L(q,t)}{\partial q^2} > 0$ and $1-\sqrt{\frac{\alpha_1}{\alpha_2}} < \hat{q}(t) < 1$, then $q^* = \hat{q}(t)$.

(3) Combining Eqs (6.13) and (6.17), we obtain that when $m_1 < m_2, \hat{\Delta}_1 \geq 1, 0 < t < t_1$ or $m_1 < m_2, \hat{\Delta}_1 < 1, 0 < t < T$ is satisfied, there are $\frac{\partial^2 L(q,t)}{\partial q^2} > 0$ and $\hat{q}(t) \geq 1$, then $q^*(t) = 1$.

(4) Combining Eqs (6.14) and (6.15), we get that when $m_1 < m_2, \hat{\Delta}_1 > \hat{\Delta}_2 \geq 1, t_1 \leq t \leq t_2$ or $m_1 < m_2, \hat{\Delta}_1 > 1 > \hat{\Delta}_2, t_1 \leq t \leq T$ is satisfied, there are $\frac{\partial^2 L(q,t)}{\partial q^2} < 0$ and $\hat{q}(t) \leq 1-\sqrt{\frac{\alpha_1}{\alpha_2}}$, then $q^* = 1-\sqrt{\frac{\alpha_1}{\alpha_2}}$.

(5) Combining Eqs (6.14) and (6.16), we find that when $m_1 < m_2, \hat{\Delta}_1 > \hat{\Delta}_2 \geq 1, t_2 < t < T$ is satisfied, there are $\frac{\partial^2 L(q,t)}{\partial q^2} < 0$ and $1 - \sqrt{\frac{\alpha_1}{\alpha_2}} < \hat{q}(t) < 1$, then $q^* = \hat{q}(t)$.

(6) Combining Eqs (6.14) and (6.17), we find that the intersection of the two is empty. Thus, If $\frac{\partial^2 L(q,t)}{\partial q^2} < 0$ and $\hat{q}(t) \geq 1$, then $q^*(t) = 1$ does not exist.

Combining above (1)–(6), we get the optimal reinsurance strategy. Next we prove the optimal investment strategy and the value function when $q^*(t)$ is different.

(1) When $q^*(t) = 1 - \sqrt{\frac{\alpha_1}{\alpha_2}}$, substituting it into Eq (6.12) yields

$$\begin{aligned}&[r_0 d(t) - d_t - m_1\lambda\alpha_1\sigma_Z^2]e^{r_0(T-t)} + [(\frac{1}{2} - \sqrt{\frac{\alpha_1}{\alpha_2}})m_1{}^2 + \frac{\alpha_1}{2\alpha_2}(m_1 - m_2)^2 \\ &+ \sqrt{\frac{\alpha_1}{\alpha_2}}m_1 m_2]\lambda\sigma_Z^2 e^{2r_0(T-t)} + g_t + \kappa_1[\phi_1 - v_1]g_{v_1} + \kappa_2[\phi_2 - v_2]g_{v_2} \\ &+ \frac{v_1(\rho_{11}^2 + \rho_{12}^2)}{2}(g_{v_1v_1} + g_{v_1}^2) + \frac{v_2(\rho_{21}^2 + \rho_{22}^2)}{2}(g_{v_2v_2} + g_{v_2}^2) \\ &- \frac{v_1(\omega_1 + \rho_{11}g_{v_1})^2}{2} - \frac{v_2(\omega_2 + \rho_{21}g_{v_2})^2}{2} = 0,\end{aligned} \tag{6.18}$$

which can be split into following two equations

$$\begin{aligned}&[r_0 d(t) - d_t - m_2\lambda\alpha_1\sigma_Z^2]e^{r_0(T-t)} + [(\frac{1}{2} - \sqrt{\frac{\alpha_1}{\alpha_2}})m_1{}^2 + \frac{\alpha_1}{2\alpha_2}(m_1 - m_2)^2 \\ &+ \sqrt{\frac{\alpha_1}{\alpha_2}}m_1 m_2]\lambda\sigma_Z^2 e^{2r_0(T-t)} = 0,\end{aligned} \tag{6.19}$$

and

$$\begin{aligned}&g_t + \kappa_1[\phi_1 - v_1]g_{v_1} + \kappa_2[\phi_2 - v_2]g_{v_2} + \frac{v_1(\rho_{11}^2 + \rho_{12}^2)}{2}(g_{v_1v_1} + g_{v_1}^2) \\ &+ \frac{v_2(\rho_{21}^2 + \rho_{22}^2)}{2}(g_{v_2v_2} + g_{v_2}^2) - \frac{v_1(\omega_1 + \rho_{11}g_{v_1})^2}{2} - \frac{v_2(\omega_2 + \rho_{21}g_{v_2})^2}{2} = 0.\end{aligned} \tag{6.20}$$

Note that Eq (6.19) is a linear ordinary differential equation with the boundary condition d(T) = 0, it is not difficult to derive that

$$\begin{aligned}d(t) =& -\frac{m_2\lambda\alpha_1\mu_2}{r_0}[e^{-r_0(T-t)} - 1] + \frac{\lambda\sigma_Z^2}{2r_0}\Big[\Big(\frac{1}{2} - \sqrt{\frac{\alpha_1}{\alpha_2}}\Big)m_1^2 + \frac{\alpha_1}{2\alpha_2}(m_1 - m_2)^2 \\ &+ \sqrt{\frac{\alpha_1}{\alpha_2}}m_1 m_2\Big] \times [e^{-r_0(T-t)} - e^{r_0(T-t)}].\end{aligned} \tag{6.21}$$

Trying to solve Eq (6.20), we put

$$g(t, v_1, v_2) = I(t) + J_1(t)v_1 + J_2(t)v_2, \tag{6.22}$$

with the boundary condition given by $I(T) = J_1(T) = J_2(T) = 0$. Then, we obtain the partial derivatives of g as

$$g_t = I_t + J_{1_t}v_1 + J_{2_t}v_2, g_{v_1} = J_1(t), g_{v_2} = J_2(t), g_{v_1v_1} = 0, g_{v_2v_2} = 0. \tag{6.23}$$

Substituting Eq (6.23) into Eq (6.20), we have

$$\begin{aligned} & I_t + J_{1_t} v_1 + J_{2_t} v_2 + \kappa_1[\phi_1 - v_1]J_1(t) + \kappa_2[\phi_2 - v_2]J_2(t) + \frac{v_1\rho_{12}^2}{2}J_1^2(t) \\ & + \frac{v_1\rho_{22}^2}{2}J_2^2(t) - v_1\omega_1\rho_{11}J_1(t) - v_2\omega_2\rho_{21}J_2(t) - \frac{v_1\omega_1^2}{2} - \frac{v_2\omega_2^2}{2} = 0. \end{aligned} \tag{6.24}$$

We can split Eq (6.24) into three equations:

$$J_{1_t} - (\kappa_1 + \omega_1\rho_{11})J_1(t) + \frac{\rho_{12}^2}{2}J_1^2(t) - \frac{\omega_1^2}{2} = 0, \tag{6.25}$$

$$J_{2_t} - (\kappa_2 + \omega_2\rho_{21})J_2(t) + \frac{\rho_{22}^2}{2}J_2^2(t) - \frac{\omega_2^2}{2} = 0, \tag{6.26}$$

and

$$I_t + \kappa_1\phi_1 J_1(t) + \kappa_2\phi_2 J_2(t) = 0. \tag{6.27}$$

Since Eqs (6.25) and (6.26) is linear ordinary differential equations with the boundary condition $J_1(T) = J_2(T) = 0$.

Thus, when $\rho_{i2} \neq 0$, due to

$$\Delta_i = (\kappa_i + \omega_i\rho_{i1})^2 + \omega_i^2\rho_{i2}^2 > 0, i = 1, 2.$$

Thus Eqs (6.25) and (6.26) have two different roots,respectively

$$\begin{aligned} c_1 &= \frac{\kappa_1 + \omega_1\rho_{11} + \sqrt{\Delta_1}}{\rho_{12}^2}, \quad c_2 = \frac{\kappa_1 + \omega_1\rho_{11} - \sqrt{\Delta_1}}{\rho_{12}^2}, \\ d_1 &= \frac{\kappa_2 + \omega_2\rho_{21} + \sqrt{\Delta_2}}{\rho_{22}^2}, \quad d_2 = \frac{\kappa_2 + \omega_2\rho_{21} - \sqrt{\Delta_2}}{\rho_{22}^2}. \end{aligned} \tag{6.28}$$

Substituting Eq (6.28) into (6.25) and (6.26), we obtain

$$\begin{aligned} & J_{1_t} = -\frac{\rho_{12}^2}{2}(J_1(t) - c_1)(J_1(t) - c_2) \\ & \Rightarrow \frac{1}{c_1 - c_2}\left(\frac{1}{J_1(t) - c_1} - \frac{1}{J_1(t) - c_2}\right)J_{1_t} = -\frac{\rho_{12}^2}{2} \\ & \Rightarrow \int_t^T \left(\frac{1}{J_1(t) - c_1} - \frac{1}{J_1(t) - c_2}\right)dJ_1(t) = -\frac{\rho_{12}^2}{2}(c_1 - c_2)(T - t), \end{aligned} \tag{6.29}$$

and

$$\begin{aligned} & J_{2_t} = -\frac{\rho_{22}^2}{2}(J_2(t) - d_1)(J_2(t) - d_2) \\ & \Rightarrow \frac{1}{d_1 - d_2}\left(\frac{1}{J_2(t) - d_1} - \frac{1}{J_2(t) - d_2}\right)J_{2_t} = -\frac{\rho_{22}^2}{2}, \\ & \Rightarrow \int_t^T \left(\frac{1}{J_2(t) - d_1} - \frac{1}{J_2(t) - d_2}\right)dJ_2(t) = -\frac{\rho_{22}^2}{2}(d_1 - d_2)(T - t). \end{aligned} \tag{6.30}$$

Slove Eqs (6.29) and (6.30), we get

$$J_1(t) = \frac{c_1c_2(1 - e^{-\frac{\rho_{12}^2}{2}(c_1-c_2)(T-t)})}{c_1 - c_2e^{-\frac{\rho_{12}^2}{2}(c_1-c_2)(T-t)}}, \tag{6.31}$$

and

$$J_2(t) = \frac{d_1d_2(1 - e^{-\frac{\rho_{22}^2}{2}(d_1-d_2)(T-t)})}{d_1 - d_2e^{-\frac{\rho_{22}^2}{2}(d_1-d_2)(T-t)}}. \tag{6.32}$$

Combining Eqs (6.27), (6.31) and (6.32), we have

$$\begin{aligned} I(t) =& \int_t^T [\kappa_1\phi_1J_1(s) + \kappa_2\phi_2J_2(s)]ds \\ =&\kappa_1\phi_1c_1(T-t) - \frac{2\kappa_1\phi_1}{\rho_{12}^2}\ln\frac{c_1e^{\frac{\rho_{12}^2}{2}(c_1-c_2)(T-t)} - c_2}{c_1 - c_2} \\ &+ \kappa_2\phi_2d_1(T-t) - \frac{2\kappa_2\phi_2}{\rho_{22}^2}\ln\frac{d_1e^{\frac{\rho_{22}^2}{2}(d_1-d_2)(T-t)} - d_2}{d_1 - d_2}. \end{aligned} \tag{6.33}$$

Using Eqs (6.5), (6.6), (6.23), (6.31) and (6.32), we obtain

$$\begin{aligned} \pi_1^*(t) =& \frac{\omega_1(c_1 - c_2e^{-\frac{\sigma_{01}^2(1-\rho_1^2)}{2}(c_1-c_2)(T-t)})}{\sigma_{01}\rho_1c_1c_2(1 - e^{-\frac{\sigma_{01}^2(1-\rho_1^2)}{2}(c_1-c_2)(T-t)})}e^{-r_0(T-t)} \\ &+ \frac{\sigma_{01}\rho_1c_1c_2(1 - e^{-\frac{\sigma_{01}^2(1-\rho_1^2)}{2}(c_1-c_2)(T-t)})}{\sigma_{01}\rho_1c_1c_2(1 - e^{-\frac{\sigma_{01}^2(1-\rho_1^2)}{2}(c_1-c_2)(T-t)})}e^{-r_0(T-t)}, \end{aligned} \tag{6.34}$$

and

$$\begin{aligned} \pi_2^*(t) =& \frac{\omega_2(d_1 - d_2e^{-\frac{\sigma_{02}^2(1-\rho_2^2)}{2}(d_1-d_2)(T-t)})}{m_2(d_1 - d_2e^{-\frac{\sigma_{02}^2(1-\rho_2^2)}{2}(d_1-d_2)(T-t)})}e^{-r_0(T-t)} \\ &+ \frac{\sigma_{02}\rho_2d_1d_2(1 - e^{-\frac{\sigma_{02}^2(1-\rho_2^2)}{2}(d_1-d_2)(T-t)})}{m_2(d_1 - d_2e^{-\frac{\sigma_{02}^2(1-\rho_2^2)}{2}(d_1-d_2)(T-t)})}e^{-r_0(T-t)}, \end{aligned} \tag{6.35}$$

when $\rho_{i2} = 0$, Eqs (6.25) and (6.26) can be rewritten as

$$J_{1_t} - (\kappa_1 + \omega_1\rho_{11})J_1(t) - \frac{\omega_1^2}{2} = 0, \tag{6.36}$$

and

$$J_{2_t} - (\kappa_2 + \omega_2\rho_{21})J_2(t) - \frac{\omega_2^2}{2} = 0. \tag{6.37}$$

Since Eqs (6.36) and (6.37) are linear ordinary differential equations with the boundary condition $J_1(T) = J_2(T) = 0$, we derive that

$$J_1(t) = \frac{\omega_1^2}{2(\kappa_1 + \omega_1\rho_{11})}(e^{-(\kappa_1+\omega_1\rho_{11})(T-t)} - 1), \tag{6.38}$$

and

$$J_2(t) = \frac{\omega_2^2}{2(\kappa_2 + \omega_2\rho_{21})}(e^{-(\kappa_2+\omega_2\rho_{21})(T-t)} - 1). \tag{6.39}$$

Combining Eqs (6.27), (6.38) and (6.39), we have

$$\begin{aligned} I(t) =& \int_t^T [\kappa_1\phi_1 J_1(s) + \kappa_2\phi_2 J_2(s)]ds \\ =& \frac{\kappa_1\phi_1\omega_1^2}{2(\kappa_1 + \omega_1\rho_{11})}[\frac{1 - e^{-(\kappa_1+\omega_1\rho_{11})(T-t)}}{\kappa_1 + \omega_1\rho_1 1} - (T-t)] \\ &+ \frac{\kappa_2\phi_2\omega_2^2}{2(\kappa_2 + \omega_2\rho_{21})}[\frac{1 - e^{-(\kappa_2+\omega_2\rho_{21})(T-t)}}{\kappa_2 + \omega_2\rho_2 1} - (T-t)]. \end{aligned} \tag{6.40}$$

Using Eqs (6.5), (6.6), (6.23), (6.38) and (6.39), we get

$$\pi_1^*(t) = \frac{2\omega_1^2 v_1(\kappa_1 + \omega_1\rho_{11}) + \omega_1^3\rho_{11}v_1(e^{-(\kappa_1+\omega_1\rho_{11})(T-t)} - 1)}{2(\mu_1(t) - r_0)m_1(\kappa_1 + \omega_1\rho_{11})}e^{-r_0(T-t)}, \tag{6.41}$$

and

$$\pi_2^*(t) = \frac{2\omega_2^2 v_2(\kappa_2 + \omega_2\rho_{21}) + \omega_2^3\rho_{21}v_2(e^{-(\kappa_2+\omega_2\rho_{21})(T-t)} - 1)}{2(\mu_2(t) - r_0)m_2(\kappa_2 + \omega_2\rho_{21})}e^{-r_0(T-t)}. \tag{6.42}$$

Above all, we obtion the expression of $d(t), g(t, v_1, v_2), I(t), J_1(t)$, and $J_2(t)$ by Eqs (6.21), (6.22), (6.31)–(6.33) and (6.38)–(6.40), then we can get the explicit expression of the value function $V(t, x_1, x_2, v_1, v_2)$.

(2) When $q^*(t) = \hat{q}(t)$, substituting it into Eq (6.12) yields

$$\begin{aligned} &[r_0 d(t) - d_t - m_1\lambda\alpha_1\sigma_Z^2]e^{r_0(T-t)} + g_t + \kappa_1[\phi_1 - v_1]g_{v_1} + \kappa_2[\phi_2 - v_2]g_{v_2} \\ &+ \frac{v_1(\rho_{11}^2 + \rho_{12}^2)}{2}(g_{v_1v_1} + g_{v_1}^2) + \frac{v_2(\rho_{21}^2 + \rho_{22}^2)}{2}(g_{v_2v_2} + g_{v_2}^2) \\ &- \frac{v_1(\omega_1 + \rho_{11}g_{v_1})^2}{2} - \frac{v_2(\omega_2 + \rho_{21}g_{v_2})^2}{2} + L(\hat{q}, t) = 0, \end{aligned} \tag{6.43}$$

which can be split into following two equations

$$[r_0 d(t) - d_t - m_1\lambda\alpha_1\sigma_Z^2]e^{r_0(T-t)} = 0, \tag{6.44}$$

and

$$\begin{aligned} &g_t + \kappa_1[\phi_1 - v_1]g_{v_1} + \kappa_2[\phi_2 - v_2]g_{v_2} + \frac{v_1(\rho_{11}^2 + \rho_{12}^2)}{2}(g_{v_1v_1} + g_{v_1}^2) \\ &+ \frac{v_2(\rho_{21}^2 + \rho_{22}^2)}{2}(g_{v_2v_2} + g_{v_2}^2) - \frac{v_1(\omega_1 + \rho_{11}g_{v_1})^2}{2} - \frac{v_2(\omega_2 + \rho_{21}g_{v_2})^2}{2} \\ &+ L(\hat{q}, t) = 0. \end{aligned} \tag{6.45}$$

Note that Eq (6.44) is a linear ordinary differential equation with the boundary condition $d(T) = 0$, it is not difficult to derive that

$$d(t) = -\frac{\lambda\sigma_Z^2 m_1\alpha_1}{r_0}[e^{-r_0(T-t)} - 1]. \tag{6.46}$$

Since Eq (6.45) is similar with (6.20),we can get the expression of $I(t)$ which is similar with Eqs (6.33) and (6.40)

$$\begin{aligned} I(t) &= \int_t^T [\kappa_1\phi_1 J_1(s) + \kappa_2\phi_2 J_2(s) + L(\hat{q}, s)]ds \\ &= \kappa_1\phi_1 c_1(T-t) - \frac{2\kappa_1\phi_1}{\rho_{12}^2} \ln \frac{c_1 e^{\frac{\rho_{12}^2}{2}(c_1-c_2)(T-t)} - c_2}{c_1 - c_1} \\ &\quad + \kappa_2\phi_2 d_1(T-t) - \frac{2\kappa_2\phi_2}{\rho_{22}^2} \ln \frac{d_1 e^{\frac{\rho_{22}^2}{2}(d_1-d_2)(T-t)} - d_2}{d_1 - d_1} \\ &\quad + \int_t^T L(\hat{q}, s)ds, \end{aligned} \tag{6.47}$$

and

$$\begin{aligned} I(t) &= \int_t^T [\kappa_1\phi_1 J_1(s) + \kappa_2\phi_2 J_2(s) + L(\hat{q}, s)]ds \\ &= \frac{\kappa_1\phi_1\omega_1^2}{2(\kappa_1+\omega_1\rho_{11})}\Big[\frac{1-e^{-(\kappa_1+\omega_1\rho_{11})(T-t)}}{\kappa_1+\omega_1\rho_{11}} - (T-t)\Big] \\ &\quad + \frac{\kappa_2\phi_2\omega_2^2}{2(\kappa_2+\omega_2\rho_{21})}\Big[\frac{1-e^{-(\kappa_2+\omega_2\rho_{21})(T-t)}}{\kappa_2+\omega_2\rho_{21}} - (T-t)\Big] \\ &\quad + \int_t^T L(\hat{q}, s)ds, \end{aligned} \tag{6.48}$$

where

$$\begin{aligned} L(\hat{q}, t) &= m_1 e^{r_0(T-t)}\lambda\alpha_2\sigma_Z^2(1-\hat{q})^2 - m_2 e^{r_0(T-t)}\lambda\alpha_2\sigma_Z^2(1-\hat{q})^2 \\ &\quad + \lambda\sigma_Z^2 m_1 m_2 \hat{q}(1-\hat{q})e^{2r_0(T-t)} + \frac{1}{2}\lambda\sigma_Z^2 e^{2r_0(T-t)}[m_1^2\hat{q}^2 + m_2^2(1-\hat{q})^2] \\ &= \frac{\lambda\alpha_2\sigma_Z^2 m_1^2(m_1-m_2)e^{3r_0(T-t)} + 2\lambda\alpha_2^2\sigma_Z^2 m_1^2 e^{2r_0(T-t)}}{[2\alpha_2 + (m_1-m_2)e^{r_0(T-t)}]^2} \\ &= \frac{\lambda\alpha_2\sigma_Z^2 m_1^2(m_1-m_2)e^{3r_0(T-t)} + (4-2)\lambda\alpha_2^2\sigma_Z^2 m_1^2 e^{2r_0(T-t)}}{[2\alpha_2 + (m_1-m_2)e^{r_0(T-t)}]^2} \\ &= -\frac{1}{r_0}\Big[\frac{\lambda\alpha_2\sigma_Z^2 m_1^2 e^{2r_0(T-t)}}{2\alpha_2 + (m_1-m_2)e^{r_0(T-t)}}\Big]' + \frac{e^{r_0(T-t)}}{r_0}\Big[\frac{\lambda\alpha_2\sigma_Z^2 m_1^2 e^{r_0(T-t)}}{2\alpha_2 + (m_1-m_2)e^{r_0(T-t)}}\Big]'. \end{aligned} \tag{6.49}$$

Then

$$\begin{aligned} \int_t^T L(\hat{q}, s)ds &= \int_t^T -\frac{1}{r_0}\Big[\frac{\lambda\alpha_2\sigma_Z^2 m_1^2 e^{2r_0(T-s)}}{2\alpha_2 + (m_1-m_2)e^{r_0(T-s)}}\Big]' \\ &\quad + \frac{e^{r_0(T-s)}}{r_0}\Big[\frac{\lambda\alpha_2\sigma_Z^2 m_1^2 e^{r_0(T-s)}}{2\alpha_2 + (m_1-m_2)e^{r_0(T-s)}}\Big]' ds \\ &= \frac{2\lambda\alpha_2^2\sigma_Z^2 m_1^2}{r_0(m_1-m_2)^2} \ln\Big|\frac{2\alpha_2 + (m_1-m_2)}{2\alpha_2 + (m_1-m_2)e^{r_0(T-t)}}\Big| \\ &\quad + \frac{\lambda\alpha_2\sigma_Z^2 m_1^2}{r_0(m_1-m_2)}[e^{r_0(T-t)} - 1]. \end{aligned} \tag{6.50}$$

As a result, Eqs (6.47) and (6.48) is converted to

$$\begin{aligned}I(t) =& \int_t^T [\kappa_1\phi_1 J_1(s) + \kappa_2\phi_2 J_2(s) + L(\hat{q}, s)]ds\\=&\kappa_1\phi_1 c_1(T-t) - \frac{2\kappa_1\phi_1}{\rho_{12}^2}\ln\frac{c_1 e^{\frac{\rho_{12}^2}{2}(c_1-c_2)(T-t)} - c_2}{c_1 - c_1}\\&+ \kappa_2\phi_2 d_1(T-t) - \frac{2\kappa_2\phi_2}{\rho_{22}^2}\ln\frac{d_1 e^{\frac{\rho_{22}^2}{2}(d_1-d_2)(T-t)} - d_2}{d_1 - d_1}\\&+ \frac{2\lambda\alpha_2^2\sigma_Z^2 m_1^2}{r_0(m_1-m_2)^2}\ln\left|\frac{2\alpha_2 + (m_1-m_2)}{2\alpha_2 + (m_1-m_2)e^{r_0(T-t)}}\right|\\&+ \frac{\lambda\alpha_2\sigma_Z^2 m_1^2}{r_0(m_1-m_2)}[e^{r_0(T-t)} - 1],\end{aligned} \tag{6.51}$$

and

$$\begin{aligned}I(t) =& \int_t^T [\kappa_1\phi_1 J_1(s) + \kappa_2\phi_2 J_2(s) + L(\hat{q}, s)]ds\\=&\frac{\kappa_1\phi_1\omega_1^2}{2(\kappa_1+\omega_1\rho_{11})}[\frac{1 - e^{-(\kappa_1+\omega_1\rho_{11})(T-t)}}{\kappa_1 + \omega_1\rho_1 1} - (T-t)]\\&+ \frac{\kappa_2\phi_2\omega_2^2}{2(\kappa_2+\omega_2\rho_{21})}[\frac{1 - e^{-(\kappa_2+\omega_2\rho_{21})(T-t)}}{\kappa_2 + \omega_2\rho_2 1} - (T-t)]\\&+ \frac{2\lambda\alpha_2^2\sigma_Z^2 m_1^2}{r_0(m_1-m_2)^2}\ln\left|\frac{2\alpha_2 + (m_1-m_2)}{2\alpha_2 + (m_1-m_2)e^{r_0(T-t)}}\right|\\&+ \frac{\lambda\alpha_2\sigma_Z^2 m_1^2}{r_0(m_1-m_2)}[e^{r_0(T-t)} - 1].\end{aligned} \tag{6.52}$$

Above all, we get the display expression of the value function $V(t, x_1, x_2, v_1, v_2)$.
(3) When $q^*(t) = 1$, substituting it into Eq (6.12) yields

$$\begin{aligned}&[r_0 d(t) - d_t - m_1\lambda\alpha_1\sigma_Z^2 + \frac{1}{2}\lambda\sigma_Z^2 m_1^2 e^{r_0(T-t)}]e^{r_0(T-t)}\\&+ g_t + \kappa_1[\phi_1 - v_1]g_{v_1} + \kappa_2[\phi_2 - v_2]g_{v_2}\\&+ \frac{v_1(\rho_{11}^2 + \rho_{12}^2)}{2}(g_{v_1v_1} + g_{v_1}^2) + \frac{v_2(\rho_{21}^2 + \rho_{22}^2)}{2}(g_{v_2v_2} + g_{v_2}^2)\\&- \frac{v_1(\omega_1 + \rho_{11}g_{v_1})^2}{2} - \frac{v_2(\omega_2 + \rho_{21}g_{v_2})^2}{2}\\&= 0.\end{aligned} \tag{6.53}$$

Also Eq (6.53) can be split into (6.20) and

$$r_0 d(t) - d_t - m_1\lambda\alpha_1\mu_2 + \frac{1}{2}m_1^2\sigma_0^2 e^{r_0(T-t)} = 0. \tag{6.54}$$

Note that Eq (6.54) is a linear ordinary differential equation with the boundary condition $d(T) = 0$, it is not difficult to derive that

$$d(t) = \frac{m_1^2\sigma_0^2}{4r_0}[e^{-r_0(T-t)} - e^{r_0(T-t)}] + \frac{m_1\lambda\alpha_1\mu_2}{r_0}[1 - e^{-r_0(T-t)}]. \tag{6.55}$$

Thus, we get the expression $V(t, x_1, x_2, v_1, v_2)$ for the value function when $q^* = 1$.

The proof of Theorem 3.3 is completed.