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Research article

A novel fractional approach to finding the upper bounds of Simpson and Hermite-Hadamard-type inequalities in tensorial Hilbert spaces by using differentiable convex mappings

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Abstract: Function spaces are significant in the study and application of mathematical inequalities. The objective of this article is to develop several new bounds and refinements for well-known inequalities that involve Hilbert spaces within a tensorial framework. Using self-adjoint operators in tensor Hilbert spaces, we developed Simpson type inequalities by using different types of generalized convex mappings. Our next step involved developing a variety of new variations of the Hermite and Hadamard inequalities using convex mappings with some special means, specifically arithmetic and geometric means. Furthermore, we developed a number of new fractional identities, which are used in our main findings, by using Riemann-Liouville integrals. In addition, we discuss some examples involving log convex functions and their consequences.

Keywords: Hermite-Hadamard; fractional calculus; upper bounds; mathematical operators
Mathematics Subject Classification: 26A48, 26A51, 33B10, 39A12, 39B62

1. Introduction

The relationship between convexity and inequality is a rich subject of study with significant applications in applied mathematics. Convex functions have characteristics that make it easier to derive inequality and make them more useful for resolving practical issues. By utilizing the features of convex functions, one can determine bounds, optimize functions, and evaluate behaviors that are critical in mathematical, statistical, and economic settings. For instance, in economics, convexity in preferences or utility functions can lead to inequalities that describe optimal allocations of resources [1]; in numerical methods, inequalities derived from convex functions are used to estimate errors and improve algorithms [2]; in information theory, particularly in estimating entropies and divergences [3]; in statistics, help in understanding distributions and the behavior of systems under various constraints, leading to insights [4]. In [5], the authors present various applications of convex optimization issues in aerospace engineering. In [6], the authors demonstrate applications of convex optimization in signal processing and digital communication. In [7], the authors present inequality problems in mechanics and applications for convex and nonconvex energy functions. In [8], authors provide a convex analytic approach to DC programming: Theory, methods, and applications. For some further recent applications in various disciplines, we refer to [9–12].

Fractional convex integral inequalities combine the notions of convexity and fractional calculus, providing several applications in advanced mathematical analysis. These results are very useful in domains that require the analysis of non-local or memory-dependent processes, making them a strong tool in both theoretical and applied mathematics. These inequalities play a key role in numerical methods, particularly in the estimation of error bounds in numerical integration techniques such as Simpson's rule and the trapezoidal rule. Researchers have used various types of convex mappings, integral operators such as classical, fractional and stochastic various order relations such as cr-order, pseudo-order, left-right order and inclusion orders, and various other techniques to develop convex integral inequalities. For instance, in [13], authors used convex symmetric coordinated functions to create Hermite and Hadamard inequalities; in [14], authors used a fractional Riemann-Liouville integral to create Newton type inequalities for differentiable convex mappings; in [15], authors created Simpson type inequalities by using various function classes; and in [16], authors created Bullen-type inequalities using generalized fractional integrals. In [17], authors refined Young's inequality with several interesting applications, and in [18], authors developed Hölder's inequality by utilizing mean continuity to solve delay differential equations and demonstrate their uniqueness. Authors in [19] used differentiable s-convex mappings to create Ostrowski type inequalities, whereas authors in [20] employed quantum integral operators to develop midpoint and trapezoid type inequalities. Stojiljkovi'c et al. [21] provided modifications to the tensorial inequalities in Hilbert spaces. Zareen et al. [22] created several novel versions of Hermite-Hadamard and Fejér-type inequalities for the Godunova-Levin preinvex class of interval-valued functions. In [23], the authors established a novel version of the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals. In [24], the authors established new extensions of Hermite-Hadamard inequalities for generalized fractional integrals. In [25], the authors created Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities using fractional integral operators. In [26,27], the authors created fractional integral versions of the Hermite-Hadamard type inequality for generalized cr-convexity. For further detail, we refer to [28–31].

Simpson's inequality is a significant result in numerical analysis and calculus, particularly in the context of approximating definite integrals. Simpson's rule, which is the foundation of Simpson's inequality, was named after the mathematician Thomas Simpson, who popularized it in the 18th century. The rule provides a method for estimating the integral of a function by approximating it with a quadratic polynomial. Specifically, it states that for a function $\mathfrak{I}$ that is continuous on the interval $[\epsilon, \upsilon]$, the integral can be approximated as [32]:

- Simpson's $\frac{1}{3}$ rule, often known as the quadrature formula:

$$\int_{\epsilon}^{\upsilon} \mathfrak{I}(\pi_i)\mathrm{d}\pi_i \approx \frac{\upsilon - \epsilon}{6}\left(\mathfrak{I}(\epsilon) + 4\mathfrak{I}\left(\frac{\epsilon + \upsilon}{2}\right) + \mathfrak{I}(\upsilon)\right).$$

- Simpson's $\frac{3}{8}$ rule, often known as Simpson's second formula:

$$\int_{\epsilon}^{\upsilon} \mathfrak{I}(\pi_i)\mathrm{d}\pi_i \approx \frac{\upsilon - \epsilon}{8}\left[\mathfrak{I}(\epsilon) + 3\mathfrak{I}\left(\frac{2\epsilon + \upsilon}{3}\right) + 3\mathfrak{I}\left(\frac{\epsilon + 2\upsilon}{3}\right) + \mathfrak{I}(\upsilon)\right].$$

As shown below, the three-point Simpson-type inequality is the most widely used Newton-Cotes quadrature.

Theorem 1.1. *(See [32]) Let* $\mathfrak{I} : [\epsilon, \upsilon] \to \mathrm{R}$ *be a continuous mapping, and assume that* $\left\|\mathfrak{I}^{(4)}\right\|_\infty = \sup_{\pi_i \in (\epsilon, \upsilon)} \left|\mathfrak{I}^{(4)}(\pi_i)\right| < \infty$. *Then, the inequality stated below holds true:*

$$\left|\frac{1}{6}\left[\mathfrak{I}(\epsilon) + 4\mathfrak{I}\left(\frac{\epsilon + \upsilon}{2}\right) + \mathfrak{I}(\upsilon)\right] - \frac{1}{\upsilon - \epsilon}\int_{\epsilon}^{\upsilon} \mathfrak{I}(\pi_i)\mathrm{d}\pi_i\right| \leq \frac{1}{2880}\left\|\mathfrak{I}^{(4)}\right\|_\infty (\upsilon - \epsilon)^4.$$

This approximation becomes exact for polynomials of degree three or less. Researchers have used a variety of methods to investigate Simpson's inequality. For example, in [33], authors used q-class integral operators and coordinated convex type mappings to show several new bounds; in [34], authors used various fractional integral operators for differentiable mappings and found various enhanced bounds; in [35], authors used the idea of preinvex mappings in conjunction with quantum calculus to show some refinement and reversal; in [36], authors used the concept of tempered fractional integral operators; and in [37], authors used multiplicative calculus to find a variety of bounds and reversals for these kind of inequalities. For additional information on these kinds of related outcomes, readers are directed to [38–41] and the references therein.

Operator inequalities are extensions of familiar numerical inequalities to the realm of linear operators acting on Hilbert spaces. These inequalities play a crucial role in various fields, including functional analysis, matrix theory, quantum mechanics, and optimization. Many authors have recently investigated classical inequalities in the context of operators on Hilbert spaces. For instance, authors employed bounded linear operators in Hilbert spaces in [42] to create numerical radius-type inequalities, and authors produced multiple means inequalities for positive linear operators in Hilbert spaces in [43]; in [44], authors developed Hölder-type inequalities for power series with several interesting applications in Hilbert spaces; and in [45], authors studied variational problem associated with inequalities and graphs in Hilbert spaces. See [46–49] for further results on a similar kind connected to developed results.

Silvestru Sever Dragomir [50] presented several new novel modifications and refinements of Young's results in tensorial framework.

Theorem 1.2. *(See [50]) Let* $\mathtt{H}$ *be a Hilbert space. If the self-adjoint operators* ξ *and* ϕ *satisfy the conditions* $0 < \kappa_1 \leq \xi, \phi \leq \kappa_2$*, for some constants* κ_1, κ_2*, then*

$$\begin{aligned} 0 &\leq \frac{\kappa_1}{\kappa_2^2}\pi_i(1-\pi_i)\left(\frac{\xi^2\otimes 1 + 1\otimes\phi^2}{2} - \xi\otimes\phi\right) \\ &\leq (1-\pi_i)\xi\otimes 1 + \pi_i 1\otimes\phi - \xi^{1-\pi_i}\otimes\phi_i^{\pi} \\ &\leq \frac{\kappa_2}{\kappa_1^2}\pi_i(1-\pi_i)\left(\frac{\xi^2\otimes 1 + 1\otimes\phi^2}{2} - \xi\otimes\phi\right). \end{aligned}$$

Corollary 1.1. *(See [31]) Let* $\Im, \Phi$ *be continuous maps on* Δ*. If* $\xi_{\mathtt{j}}, \phi_{\mathtt{j}}$ *are self adjoint operators in Hilbert spaces and* $\mathtt{r_j}, \mathtt{s_j} \geq 0, \mathtt{j} \in \{1, \ldots, \mathtt{k}\}$*, then*

$$\begin{aligned} &\left(\sum_{\mathtt{j}=1}^{\mathtt{k}} \mathtt{r_j}\phi(\xi_{\mathtt{j}})\Im(\xi_{\mathtt{j}})\Phi(\xi_{\mathtt{j}})\right)\otimes\left(\sum_{\mathtt{i}=1}^{\mathtt{k}} \mathtt{s_i}\Im(\phi_{\mathtt{j}})\right) + \left(\sum_{\mathtt{j}=1}^{\mathtt{k}} \mathtt{r_j}\phi(\xi_{\mathtt{j}})\right)\otimes\left(\sum_{\mathtt{i}=1}^{\mathtt{k}} \mathtt{s_i}\Im(\phi_{\mathtt{j}})\Im(\phi_{\mathtt{j}})\Phi(\phi_{\mathtt{j}})\right) \\ &\geq \left(\sum_{\mathtt{j}=1}^{\mathtt{k}} \mathtt{r_j}\phi(\xi_{\mathtt{j}})\Im(\xi_{\mathtt{j}})\right)\otimes\left(\sum_{\mathtt{i}=1}^{\mathtt{k}} \mathtt{s_i}\Im(\phi_{\mathtt{j}})\Phi(\phi_{\mathtt{j}})\right) + \left(\sum_{\mathtt{j}=1}^{\mathtt{k}} \mathtt{r_j}\phi(\xi_{\mathtt{j}})\Phi(\xi_{\mathtt{j}})\right)\otimes\left(\sum_{\mathtt{i}=1}^{\mathtt{k}} \mathtt{s_i}\Im(\phi_{\mathtt{j}})\Im(\phi_{\mathtt{j}})\right). \end{aligned}$$

Vuk Stojiljkovic [51] created the Ostrowski type inequality by applying twice differentiable mappings to continuous functions on self-adjoint operators in Hilbert space.

Theorem 1.3. *(See [51]) Assume that* ξ *and* ϕ *are self-adjoint operators with associated sepctrums* $\mathcal{SP}(\xi), \mathcal{SP}(\phi) \subset \Delta$*. Let* $\Im$ *be a continous function on* Δ*, we have*

$$\begin{aligned} &\int_0^1 \Im((1-\pi_i)\xi\otimes 1 + \pi_i 1\otimes\phi)d\pi_i - \Im\left(\frac{\xi\otimes 1 + 1\otimes\phi}{2}\right) \\ =&\frac{(1\otimes\phi - \xi\otimes 1)^2}{16}\left[\int_0^1 \pi_i^2\Im''\left(\left(1-\frac{\pi_i}{2}\right)\xi\otimes 1 + \frac{\pi_i}{2}1\otimes\phi\right)d\pi_i\right. \\ &\left. + \int_0^1 (\pi_i - 1)^2\Im''\left(\left(\frac{1-\pi_i}{2}\right)\xi\otimes 1 + \left(\frac{1+\pi_i}{2}\right)1\otimes\phi\right)d\pi_i\right]. \end{aligned}$$

Shuhei employed positive semidefinite operators on a Hilbert space to derive the following double inequality.

Theorem 1.4. *(See [53]) Let* ξ *and* ϕ *be positive as well as semidefinite operators with associated sepctrums* $\mathcal{SP}(\xi), \mathcal{SP}(\phi) \subset \Delta$*. Then,*

$$\begin{aligned} (\xi\#\phi)\otimes(\xi\#\phi) &\leqslant \frac{1}{2}\left\{(\xi\sigma\phi)\otimes(\xi\sigma^{\perp}\phi) + (\xi\sigma^{\perp}\phi)\otimes(\xi\sigma\phi)\right\} \\ &\leqslant \frac{1}{2}\{(\xi\otimes\phi) + (\phi\otimes\xi)\}. \end{aligned}$$

This study is novel and significant as mathematical inequalities by using Hilbert spaces in tensor frameworks are very rarely developed so this study will open up a whole new avenue in inequality theory. Additionally, we use several new interesting fractional identities to find upper bounds for Simpson inequality using convex and differentiable mappings. We also give some interesting applications and implications of transcendental functions.

Our motivation to create a new and enhanced version of different inequalities in tensorial Hilbert spaces comes mostly from the works of [31,51,55]. The use of fresh approaches and viewpoints, which have almost ever been covered in a few papers, significantly broadens and enriches inequality theory. The work is organized into four sections, starting with the topic's preliminary introduction and relevant definitions. In Section 2, we develop many significant identities and lemmas that are employed in the main discoveries. In Section 3, we use numerous significant fractional identities to build a Simpson type inequality for differentiable convex mappings. In Section 4, we provide examples and remarks for transcendental functions. In Section 5, we discuss the main findings and some future possible work related to these results.

2. Preliminaries

In this section, we will go over some fundamental ideas related to function spaces, fractional identities, and certain arithmetic operations on tensor Hilbert spaces. Some fundamental ideas are not completely addressed here, thus we refer to [31].

Definition 2.1. (See [58]) An inner product on a complex linear space $\mathtt{X}$ is a map $(\cdot,\cdot) : \mathtt{X}\times\mathtt{X}\to\mathbb{C}$. A Hilbert space, generally represented as H is an inner product space that is also complete. The inner product of two elements π_{i1},π_{i2} in X is denoted by $\langle\pi_{i1},\pi_{i2}\rangle$. For all vectors $\pi_{i1},\pi_{i2},\pi_{i3}\in\mathtt{X}$ and scalars $\lambda\in\mathbb{C}$, we have

$$\begin{aligned}\langle\pi_{i1}+\pi_{i2},\pi_{i3}\rangle &= \langle\pi_{i1},\pi_{i3}\rangle+\langle\pi_{i2}+\pi_{i3}\rangle\\ \langle\lambda\pi_{i1},\pi_{i2}\rangle &= \lambda\langle\pi_{i1},\pi_{i2}\rangle\\ \langle\pi_{i1},\pi_{i2}\rangle &= \overline{\langle\pi_{i2},\pi_{i1}\rangle}\\ \langle\pi_{i1},\pi_{i1}\rangle &\geq 0,\quad \langle\pi_{i1},\pi_{i1}\rangle = 0 \Longleftrightarrow \pi_{i1}=0.\end{aligned}$$

Definition 2.2. (See [58]) A bilinear mapping $\Im : \xi\times\phi\to\mathtt{P}$ and a tensor product of ξ with ϕ provide a Hilbert space $\mathtt{P}$, such that

- the collection of all vectors $\Im(\epsilon,\upsilon)(\epsilon\in\xi,\upsilon\in\phi)$ is a total subset of $\mathtt{P}$; its closed linear span is equal to $\mathtt{P}$;
- $(\Im(\pi_{i1},\pi_{i2})\mid\Im(\pi_{i3},\pi_{i4})) = (\pi_{i1}\mid\pi_{i2})(\pi_{i3}\mid\pi_{i4})$ for $\pi_{i1},\pi_{i2}\in\xi,\pi_{i3},\pi_{i4}\in\phi$. If $(\mathtt{P},\Im)$ is a tensor product of ξ and ϕ, it is common to write $\epsilon\otimes\upsilon$ instead of $\Im(\epsilon,\upsilon)$, and $\xi\otimes\phi$ in place of $\mathtt{P}$. A tensor product of ξ with ϕ is a Hilbert space $\xi\otimes\phi$ and a mapping $(\epsilon,\upsilon)\mapsto\epsilon\otimes\upsilon$ of $\xi\times\phi$ into $\mathtt{G}\otimes\phi$ such that

$$\begin{aligned}(\pi_{i1}+\pi_{i2})\otimes\nu &= \pi_{i1}\otimes\nu+\pi_{i2}\otimes\nu\\ (\lambda\epsilon)\otimes\nu &= \lambda(\epsilon\otimes\nu)\\ \epsilon\otimes(\pi_{i3}+\pi_{i4}) &= \epsilon\otimes\pi_{i3}+\epsilon\otimes\pi_{i3}\\ \epsilon\otimes(\lambda\nu) &= \lambda(\epsilon\otimes\nu).\end{aligned}$$

Let $\Im : \Delta_1\times\ldots\times\Delta_{\mathrm{s}}\to\mathsf{R}$ be a bounded function defined in terms of the product of intervals. Assume that $\mathtt{S}=(\mathtt{S}_1,\ldots,\mathtt{S}_{\mathtt{m}})$ is an $\mathtt{m}$-tuple of self-adjoint operators associated with $\mathtt{E}_1,\ldots,\mathtt{E}_{\mathrm{s}}$ Hilbert spaces. Then,

$$\mathtt{S_i} = \int_{\Delta_\mathtt{i}} \pi_{i\mathtt{i}} \mathtt{dE_i} (\pi_{i\mathtt{i}})$$

is the spectra of possible operators for $\mathtt{i} = 1, \ldots, \mathtt{s}$; following [53], we define $\mathtt{S_i}$ as follows:

$$\Im (\mathtt{S}_1, \ldots, \mathtt{S_m}) := \int_{\Delta_1} \ldots \int_{\Delta_\mathtt{s}} \Im (\pi_{i1}, \ldots, \pi_{i1}) \, \mathtt{dE}_1 (\pi_{i1}) \otimes \ldots \otimes \mathtt{dE_z} (\pi_{i\,\mathtt{m}}).$$

If the dimensions of the Hilbert spaces are finite, integration processes can be condensed to finite summations, making functional calculus more easily applied to real-valued functions. This construction [53] extends Korányi's [54] concept for functions of two variables. It has the characteristic that

$$\Im (\mathtt{S}_1, \ldots, \mathtt{S_s}) = \Im_1 (\mathtt{S}_1) \otimes \ldots \otimes \Im_\mathtt{s} (\mathtt{S_s}),$$

whenever $\Im$ can be partitioned as a product of one varaible mappings $\Im (\mathtt{a}_1, \ldots, \mathtt{a_m}) = \Im_1 (\mathtt{a}_1) \ldots \Im_\mathtt{s} (\mathtt{a_m})$. On the interval Δ, if $\Im$ is sub(super)-multiplicative, then

$$\Im(\epsilon\upsilon) \geq (\leq) \Im(\epsilon)\Im(\upsilon) \text{ for all } \epsilon\upsilon \in [0, \infty)$$

and if $\Im$ is continuous on $[0, \infty)$, then

$$\Im(\xi \otimes \phi) \geq (\leq) \Im(\xi) \otimes \Im(\phi) \text{ for all } \xi, \phi \geq 0.$$

This leads to the conclusion that, if

$$\xi = \int_{[0,\infty)} \epsilon \mathrm{dE}(\epsilon) \text{ and } \phi = \int_{[0,\infty)} \upsilon \mathrm{dF}(\upsilon)$$

are the spectral resolutions of ξ and ϕ, then

$$\Im(\xi \otimes \phi) = \int_{[0,\infty)} \int_{[0,\infty)} \Im(\epsilon\upsilon) \mathrm{dE}(\epsilon) \otimes \mathrm{dF}(\upsilon)$$

for the $\Im$ continuous function on $[0, \infty)$.

Recall the geometric operator mean for the positive operators $\xi, \phi > 0$

$$\xi \#_\mathtt{p} \phi := \xi^{1/2} \left(\xi^{-1/2} \phi \xi^{-1/2} \right)^\mathtt{p} \xi^{1/2},$$

where $\mathtt{p} \in [0, 1]$ and

$$\xi \# \phi := \xi^{1/2} \left(\xi^{-1/2} \phi \xi^{-1/2} \right)^{1/2} \xi^{1/2}.$$

By the definitions of # and $\otimes$, we have

$$\xi \# \phi = \phi \# \xi \text{ and } (\xi \# \phi) \otimes (\phi \# \xi) = (\xi \otimes \phi) \# (\phi \otimes \xi).$$

Consider the subsequent characteristic of the tensorial product:

$$(\xi\beta) \otimes (\phi\alpha) = (\xi \otimes \phi)(\beta \otimes \alpha),$$

that holds $\forall\, \xi, \phi, \beta, \alpha \in \mathrm{B}(\upsilon)$. If we take $\beta = \xi$ and $\alpha = \phi$, then we get

$$\xi^2 \otimes \phi^2 = (\xi \otimes \phi)^2.$$

Through induction, we have

$$\xi^{\mathrm{s}} \otimes \phi^{\mathrm{s}} = (\xi \otimes \phi)^{\mathrm{s}} \text{ for natural number } \sigma \geq 0.$$

Specifically,

$$\xi^{\sigma} \otimes 1 = (\xi \otimes 1)^{\sigma} \text{ and } 1 \otimes \phi^{\sigma} = (1 \otimes \phi)^{\sigma}$$

for all $\sigma \geq 0$. Additionally, we note that the $1 \otimes \phi$ and $\xi \otimes 1$ are commutative with each other:

$$(\xi \otimes 1)(1 \otimes \phi) = (1 \otimes \phi)(\xi \otimes 1) = \xi \otimes \phi.$$

Moreover, for any two natural numbers σ_1, σ_2,

$$(\xi \otimes 1)^{\sigma_1}(1 \otimes \phi)^{\sigma_2} = (1 \otimes \phi)^{\sigma_1}(\xi \otimes 1)^{\sigma_2} = \xi^{\sigma_2} \otimes \phi^{\sigma_1}.$$

Definition 2.3. (See [52]) A mapping $\Im : \Delta \subseteq \mathrm{R} \to \mathrm{R}$ is stated to be convex (concave) on Δ, if

$$\Im(\pi_i \epsilon + (1 - \pi_i)\upsilon) \leq (\geq) \pi_i \Im(\epsilon) + (1 - \pi_i)\Im(\upsilon)$$

holds for all $\epsilon, \upsilon \in \Delta$ and $\pi_i \in [0, 1]$.

Definition 2.4. (See [52]) A mapping $\Im : \Delta \to \mathrm{R}$ is stated to be quasi-convex, if

$$\Im((1 - \pi_i)\epsilon + \pi_i \upsilon) \leq \max\{\Im(\upsilon), \Im(\epsilon)\} = \frac{1}{2}(\Im(\upsilon) + \Im(\epsilon) + |\Im(\upsilon) - \Phi(\epsilon)|)$$

for all $\epsilon, \upsilon \in \Delta$ and $\pi_i \in [0, 1]$.

Identities for Riemann-Liouville fractional integrals

In this part, we formulate fractional identities using the Riemann-Liouville fractional integral formulation and apply them to the main results.

Definition 2.5. (See [55]) Let $\Im : [\epsilon, \upsilon] \to \mathrm{R}$ be a continuous function on $[\epsilon, \upsilon]$. For $\kappa > 0$, the Riemann-Liouville integrals are represented as:

$$\mathrm{J}^{\kappa}_{\epsilon+} \Im(\wp) = \frac{1}{\Gamma(\kappa)} \int_{\epsilon}^{\wp} (\wp - \varepsilon)^{\kappa - 1} \Im(\varepsilon) \mathrm{d}\varepsilon,$$

for $\epsilon < \wp \leqslant \upsilon$ and

$$\mathrm{J}^{\kappa}_{\upsilon-} \Im(\wp) = \frac{1}{\Gamma(\kappa)} \int_{\wp}^{\upsilon} (\varepsilon - \wp)^{\kappa - 1} \Im(\varepsilon) \mathrm{d}\varepsilon,$$

for $\epsilon \leqslant \wp < \upsilon$, where Γ is the gamma function.

Lemma 2.1. *Let $\Im : [\epsilon, \upsilon] \to \mathrm{R}$ be a continuous function on $[\epsilon, \upsilon]$.*

- *For any $\wp \in (\epsilon, \upsilon)$, we have*

$$\mathsf{J}_{\epsilon+}^{\kappa}\Im(\wp)+\mathsf{J}_{\upsilon-}^{\kappa}\Im(\wp)=\frac{1}{\Gamma(\kappa+1)}\left[(\wp-\epsilon)^{\kappa}\Im(\xi)+(\upsilon-\wp)^{\kappa}\Im(\phi)\right]$$
$$+\frac{1}{\Gamma(\kappa+1)}\left[\int_{\epsilon}^{\wp}(\wp-\varepsilon)^{\kappa}\Im'(\varepsilon)\mathrm{d}\varepsilon-\int_{\wp}^{\upsilon}(\varepsilon-\wp)^{\kappa}\Im'(\varepsilon)\mathrm{d}\varepsilon\right]. \tag{2.1}$$

Proof. Since $\Im:[\epsilon,\upsilon]\to\mathsf{R}$ is a continuous function on $[\epsilon,\upsilon]$, the integrals become:

$$\int_{\epsilon}^{\wp}(\wp-\varepsilon)^{\kappa}\Im'(\varepsilon)\mathrm{d}\varepsilon \text{ and } \int_{\wp}^{\upsilon}(\varepsilon-\wp)^{\kappa}\Im'(\varepsilon)\mathrm{d}\varepsilon,$$

exist and integrating by parts, we have

$$\frac{1}{\Gamma(\kappa+1)}\int_{\epsilon}^{\wp}(\wp-\varepsilon)^{\kappa}\Im'(\varepsilon)\mathrm{d}\varepsilon=\frac{1}{\Gamma(\kappa)}\int_{\epsilon}^{\wp}(\wp-\varepsilon)^{\kappa-1}\Im(\varepsilon)\mathrm{d}\varepsilon-\frac{1}{\Gamma(\kappa+1)}(\wp-\epsilon)^{\kappa}\Im(\epsilon)$$
$$=\mathsf{J}_{\epsilon+}^{\kappa}\Im(\wp)-\frac{1}{\Gamma(\kappa+1)}(\wp-\epsilon)^{\kappa}\Im(\epsilon), \tag{2.2}$$

for $\epsilon<\wp\leqslant\upsilon$ and

$$\frac{1}{\Gamma(\kappa+1)}\int_{\wp}^{\upsilon}(\varepsilon-\wp)^{\kappa}\Im'(\varepsilon)\mathrm{d}\varepsilon=\frac{1}{\Gamma(\kappa+1)}(\upsilon-\wp)^{\kappa}\Im(\upsilon)-\frac{1}{\Gamma(\kappa)}\int_{\wp}^{\upsilon}(\varepsilon-\wp)^{\kappa-1}\Im(\varepsilon)\mathrm{d}\varepsilon$$
$$=\frac{1}{\Gamma(\kappa+1)}(\upsilon-\wp)^{\kappa}\Im(\upsilon)-\mathsf{J}_{\upsilon-}^{\kappa}\Im(\wp), \tag{2.3}$$

for $\epsilon\leqslant\wp<\upsilon$. From (2.2), we have

$$\mathsf{J}_{\epsilon+}^{\kappa}\Im(\wp)=\frac{1}{\Gamma(\kappa+1)}(\wp-\epsilon)^{\kappa}\Im(\epsilon)+\frac{1}{\Gamma(\kappa+1)}\int_{\epsilon}^{\wp}(\wp-\varepsilon)^{\kappa}\Im'(\varepsilon)\mathrm{d}\varepsilon. \tag{2.4}$$

For $\epsilon<\wp\leqslant\upsilon$ and from (2.3), we have

$$\mathsf{J}_{\upsilon-}^{\kappa}\Im(\wp)=\frac{1}{\Gamma(\kappa+1)}(\upsilon-\wp)^{\kappa}\Im(\upsilon)-\frac{1}{\Gamma(\kappa+1)}\int_{\wp}^{\upsilon}(\varepsilon-\wp)^{\kappa}\Im'(\varepsilon)\mathrm{d}\varepsilon. \tag{2.5}$$

We obtain the necessary conclusion in (2.1) by considering Eqs (2.4) and (2.5). □

Corollary 2.1. *If* $\Im:[\epsilon,\upsilon]\to\mathsf{R}$ *is a continuous function on* $[\epsilon,\upsilon]$*, we get the following double equality for the midpoint of intervals:*

$$\mathsf{J}_{\epsilon+}^{\kappa}\Im\left(\frac{\epsilon+\upsilon}{2}\right)+\mathsf{J}_{\upsilon-}^{\kappa}\Im\left(\frac{\epsilon+\upsilon}{2}\right)$$
$$=\frac{1}{2^{\kappa-1}\Gamma(\kappa+1)}\frac{\Im(\epsilon)+\Im(\upsilon)}{2}$$
$$+\frac{1}{\Gamma(\kappa+1)}\left[\int_{\epsilon}^{\frac{\epsilon+\upsilon}{2}}\left(\frac{\epsilon+\upsilon}{2}-\varepsilon\right)^{\kappa}\Im'(\varepsilon)\mathrm{d}\varepsilon-\int_{\frac{\epsilon+\upsilon}{2}}^{\upsilon}\left(\varepsilon-\frac{\epsilon+\upsilon}{2}\right)^{\kappa}\Im'(\varepsilon)\mathrm{d}\varepsilon\right]$$

and

$$\begin{aligned}
&\mathsf{J}^{\kappa}_{\frac{\epsilon+\upsilon}{2}-}\Im(\epsilon)+\mathsf{J}^{\kappa}_{\frac{\epsilon+\upsilon}{2}+}\Im(\upsilon)\\
&=\frac{1}{2^{\kappa-1}\Gamma(\kappa+1)}\Im\left(\frac{\epsilon+\upsilon}{2}\right)(\upsilon-\epsilon)^{\kappa}\\
&+\frac{1}{\Gamma(\kappa+1)}\left[\int_{\frac{\epsilon+\upsilon}{2}}^{\upsilon}(\varepsilon-\upsilon)^{\kappa}\Im'(\varepsilon)\mathrm{d}\varepsilon-\int_{\epsilon}^{\frac{\epsilon+\upsilon}{2}}(\varepsilon-\epsilon)^{\kappa}\Im'(\varepsilon)\mathrm{d}\varepsilon\right].
\end{aligned}\tag{2.6}$$

Proof. For $\epsilon \leqslant \frac{\epsilon+\upsilon}{2} < \upsilon$ and from (2.6), we have

$$\begin{aligned}
\mathsf{J}^{\kappa}_{\frac{\epsilon+\upsilon}{2}-}\Im(\epsilon)&=\frac{1}{2^{\kappa-1}\Gamma(\kappa+1)}\Im\left(\frac{\epsilon+\upsilon}{2}\right)(\upsilon-\epsilon)^{\kappa}-\frac{1}{\Gamma(\kappa+1)}\left[\int_{\epsilon}^{\frac{\epsilon+\upsilon}{2}}(\varepsilon-\epsilon)^{\kappa}\Im'(\varepsilon)\mathrm{d}\varepsilon\right]\\
&=\frac{1}{2^{\kappa-1}\Gamma(\kappa+1)}\Im\left(\frac{\epsilon+\upsilon}{2}\right)(\upsilon-\epsilon)^{\kappa}-\frac{\pi_i^{\kappa}(\upsilon-\epsilon)^{\kappa+1}}{2^{\kappa+1}\Gamma(\kappa+1)}\left[\int_0^1\Im'\left((1-\pi_i)\epsilon+\left(\frac{\epsilon+\upsilon}{2}\right)\pi_i\right)\mathrm{d}\pi_i\right].
\end{aligned}\tag{2.7}$$

For $\epsilon < \frac{\epsilon+\upsilon}{2} \leqslant \upsilon$ and from (2.6), we have

$$\begin{aligned}
\mathsf{J}^{\kappa}_{\frac{\epsilon+\upsilon}{2}+}\Im(\upsilon)&=\frac{1}{2^{\kappa-1}\Gamma(\kappa+1)}\Im\left(\frac{\epsilon+\upsilon}{2}\right)(\upsilon-\epsilon)^{\kappa}+\frac{1}{\Gamma(\kappa+1)}\left[\int_{\frac{\epsilon+\upsilon}{2}}^{\upsilon}(\upsilon-\varepsilon)^{\kappa}\Im'(\varepsilon)\mathrm{d}\varepsilon\right]\\
&=\frac{1}{2^{\kappa-1}\Gamma(\kappa+1)}\Im\left(\frac{\epsilon+\upsilon}{2}\right)(\upsilon-\epsilon)^{\kappa}-\frac{(1-\pi_i)^{\kappa}(\upsilon-\epsilon)^{\kappa+1}}{2^{\kappa+1}\Gamma(\kappa+1)}\left[\int_0^1\Im'\left((1-\pi_i)\left(\frac{\epsilon+\upsilon}{2}\right)+\upsilon\pi_i\right)\mathrm{d}\pi_i\right].
\end{aligned}\tag{2.8}$$

□

3. The main results

In this part, we use new fractional identities to find upper bounds for Simpson type inequalities involving differentiable convex mappings and various generalized convex mappings.

Lemma 3.1. *Let ξ and ϕ be self-adjoint operators with $\mathcal{SP}(\xi) \subset \Delta_1$ and $\mathcal{SP}(\phi) \subset \Delta_2$. Suppose that $\Im, \vartheta$ are continuous on Δ_1, $\Phi, \Im$ are continuous on Δ_2, and φ is convex on Δ. Then sum of intervals $\vartheta(\Delta_1) + \Im(\Delta_2)$ has the following equality:*

$$\begin{aligned}
&(\Im(\xi)\otimes 1+1\otimes\Phi(\phi))\varphi(\vartheta(\xi)\otimes 1+1\otimes\Im(\phi))\\
&=\int_{\Delta_1}\int_{\Delta_2}(\Im(\upsilon)+\Phi(\epsilon))\varphi(\vartheta(\upsilon)+\Im(\epsilon))\mathtt{dE}_{\Delta_1}\otimes\mathtt{dF}_{\Delta_2},
\end{aligned}\tag{3.1}$$

where ξ and ϕ have the spectral resolutions

$$\xi=\int_{\Delta_1}\upsilon\mathtt{dE}(\upsilon)\ \textit{and}\ \phi=\int_{\Delta_2}\epsilon\mathtt{dF}(\epsilon).$$

Proof. According to Stone-Weierstrass, any continous function can be represented in terms of a polynomial sequence, hence simply checking its equivalence is adequate. Consider $\varphi(\mu) = e^{\mu^{\sigma_1}}$. If σ_1 is a natural number, then we have

$$\begin{aligned}
\Im &:= \int_{\Delta_1}\int_{\Delta_2}(\Im(\upsilon)+\Phi(\epsilon))e^{(\vartheta(\upsilon)+\Im(\epsilon))^{\sigma_1}}\mathtt{dE}_{\Delta_1}\otimes\mathtt{dF}_{\Delta_2}\\
&= \int_{\Delta_1}\int_{\Delta_2}(\Im(\upsilon)+\Phi(\epsilon))\sum_{\sigma_1=0}^{\sigma_2}C_{\sigma_1}^{\sigma_2}e^{[\vartheta(\upsilon)]^{\sigma_2}}e^{[\Im(\epsilon)]^{\sigma_2-\sigma_1}}\mathtt{dE}_{\Delta_1}\otimes\mathtt{dF}_{\Delta_2}\\
&= \sum_{\sigma_1=0}^{\sigma_2}C_{\sigma_1}^{\sigma_2}\int_{\Delta_1}\int_{\Delta_2}(\Im(\upsilon)+\Phi(\epsilon))e^{[\vartheta(\upsilon)]^{\sigma_2}}e^{[\Im(\epsilon)]^{\sigma_2-\sigma_1}}\mathtt{dE}_{\Delta_1}\otimes\mathtt{dF}_{\Delta_2}\\
&= \sum_{\sigma_1=0}^{\sigma_2}C_{\sigma_1}^{\sigma_2}\Bigg[\int_{\Delta_1}\int_{\Delta_2}\Im(\upsilon)e^{[\vartheta(\upsilon)]^{\sigma_2}}e^{[\Im(\epsilon)]^{\sigma_2-\sigma_1}}\mathtt{dE}_{\Delta_1}\otimes\mathtt{dF}_{\Delta_2}\\
&+ \int_{\Delta_1}\int_{\Delta_2}e^{[\vartheta(\upsilon)]^{\sigma_2}}\Phi(\epsilon)e^{[\Im(\epsilon)]^{\sigma_2-\sigma_1}}\mathtt{dE}_{\Delta_1}\otimes\mathtt{dF}_{\Delta_2}\Bigg].
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_{\Delta_1}\int_{\Delta_2}\Im(\upsilon)e^{[\vartheta(\upsilon)]^{\sigma_2}}e^{[\Im(\epsilon)]^{\sigma_2-\sigma_1}}\mathtt{dE}_{\Delta_1}\otimes\mathtt{dF}_{\Delta_2}\\
&= \Im(\xi)e^{[\vartheta(\xi)]^{\sigma_2}}\otimes e^{[\Im(\phi)]^{\sigma_2-\sigma_1}} = (\Im(\xi)\otimes 1)e^{\left([\vartheta(\xi)]^{\sigma_2}\otimes[\Im(\phi)]^{\sigma_2-\sigma_1}\right)}\\
&= (\Im(\xi)\otimes 1)e^{([\vartheta(\xi)]^{\sigma_2}\otimes 1)}e^{\left(1\otimes[\Im(\phi)]^{\sigma_2-\sigma_1}\right)}\\
&= (\Im(\xi)\otimes 1)e^{(\vartheta(\xi)\otimes 1)^{\sigma_2}}e^{(1\otimes\Im(\phi))^{\sigma_2-\sigma_1}}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Delta_1}\int_{\Delta_2}e^{[\vartheta(\upsilon)]^{\sigma_2}}\Phi(\epsilon)e^{[\Im(\epsilon)]^{\sigma_2-\sigma_1}}\mathtt{dE}_{\Delta_1}\otimes\mathtt{dF}_{\Delta_2}\\
&= e^{[\vartheta(\xi)]^{\sigma_2}}\otimes\left(\Phi(\phi)e^{[\Im(\phi)]^{\sigma_2-\sigma_1}}\right) = (1\otimes\Phi(\phi))e^{\left([\vartheta(\xi)]^{\sigma_2}\otimes[\Im(\phi)]^{\sigma_2-\sigma_1}\right)}\\
&= (1\otimes\Phi(\phi))e^{([\vartheta(\xi)]^{\sigma_2}\otimes 1)}e^{\left(1\otimes[\Im(\phi)]^{\sigma_2-\sigma_1}\right)}\\
&= (1\otimes\Phi(\phi))e^{(\vartheta(\xi)\otimes 1)^{\sigma_2}}e^{(1\otimes\Im(\phi))^{\sigma_2-\sigma_1}}
\end{aligned}$$

where $e^{(\vartheta(\xi)\otimes 1)}$ and $e^{(1\otimes\Im(\phi))}$ commute with each other. Therefore,

$$\begin{aligned}
\Im &= (\Im(\xi)\otimes 1 + 1\otimes\Phi(\phi))\sum_{\sigma_1=0}^{\sigma_2}C_{\sigma_1}^{\sigma_2}e^{(\vartheta(\xi)\otimes 1)^{\sigma_2}}e^{(1\otimes\Im(\phi))^{\sigma_2-\sigma_1}}\\
&= (\Im(\xi)\otimes 1 + 1\otimes\Phi(\phi))e^{(\vartheta(\xi)\otimes 1+1\otimes\Im(\phi))^{\sigma_1}}.
\end{aligned}$$

□

Lemma 3.2. *Let ξ and ϕ be self-adjoint operators with $\mathcal{SP}(\xi)\subset\Delta_1$ and $\mathcal{SP}(\phi)\subset\Delta_2$. Suppose that $\Im,\vartheta$ are continuous on Δ_1, $\Phi,\Im$ are continuous on Δ_2, and φ is convex on Δ. Then product of intervals $\vartheta(\Delta_1)+\Im(\Delta_2)$ has the following equality:*

$$\varphi(\Im(\xi)\otimes\Phi(\phi))\chi(\vartheta(\xi)\otimes\Im(\phi)) = \int_{\Delta_1}\int_{\Delta_2}\varphi(\Im(\upsilon)\Phi(\epsilon))\chi(\vartheta(\upsilon)\Im(\epsilon))\mathtt{dE}_\epsilon\otimes\mathtt{dF}_\upsilon \tag{3.2}$$

where ξ and ϕ have the spectral resolutions

$$\xi = \int_{\Delta_1} \upsilon \mathtt{dE}(\upsilon) \text{ and } \phi = \int_{\Delta_2} \epsilon \mathtt{dF}(\epsilon).$$

Proof. According to Stone-Weierstrass, any continous function can be represented in terms of a polynomial sequence, hence simply checking its equivalence is adequate. Let two non-negative mappings $\varphi(\mu) = e^{\mu^{\sigma_2}}, \chi(\mu) = e^{\mu^{\sigma_1}}$, where σ_1 and σ_2 are each natural numbers. Then, one has

$$\begin{aligned}
&\int_{\Delta_1}\int_{\Delta_2} (e^{\epsilon}e^{\upsilon})^{\sigma_2}(e^{\epsilon}e^{\upsilon})^{\sigma_1}\mathtt{dE}_{\epsilon}\otimes \mathtt{dF}_{\upsilon} = \int_{\Delta_1}\int_{\Delta_2} [e^{\epsilon}]^{\sigma_2}[e^{\upsilon}]^{\sigma_2}[e^{\epsilon}]^{\sigma_1}[e^{\upsilon}]^{\sigma_1}\mathtt{dE}_{\epsilon}\otimes \mathtt{dF}_{\upsilon}\\
&= \int_{\Delta_1}\int_{\Delta_2} [e^{\epsilon}]^{\sigma_2}[e^{\epsilon}]^{\sigma_1}[e^{\upsilon}]^{\sigma_2}[e^{\upsilon}]^{\sigma_1}\mathtt{dE}_{\epsilon}\otimes \mathtt{dF}_{\upsilon} = \left([e^{\xi}]^{\sigma_2}[e^{\xi}]^{\sigma_1}\right)\otimes\left([e^{\phi}]^{\sigma_2}[e^{\phi}]^{\sigma_1}\right)\\
&= \left([e^{\xi}]^{\sigma_2}\otimes[e^{\phi}]^{\sigma_2}\right)\left([e^{\xi}]^{\sigma_1}\otimes[e^{\phi}]^{\sigma_1}\right) = (e^{\xi}\otimes e^{\phi})^{\sigma_2}(e^{\xi}\otimes e^{\phi})^{\sigma_1}
\end{aligned}$$

and the equality (3.2) is proven.

□

3.1. Simpson type inequalities utilizing self-adjoint operators on Hilbert spaces

Lemma 3.3. *Assume ξ and ϕ are two self-adjoint operators with $\mathcal{SP}(\xi) \subset \Delta$ and $\mathcal{SP}(\phi) \subset \Delta$. Let $\Im$ be a convex mapping on Δ. Then, the equality stated below holds true:*

$$\begin{aligned}
&\left[\frac{1}{8}(\Im(\xi)\otimes 1) + \frac{3}{8}\Im\left(\frac{2\xi\otimes 1 + 1\otimes\phi}{2}\right) + \frac{3}{8}\Im\left(\frac{\xi\otimes 1 + 2\otimes\phi}{2}\right) + \frac{1}{8}(1\otimes\Im(\phi))\right]\\
&-\left[\Im\left(\frac{\xi\otimes 1 + 1\otimes\phi}{2}\right) - \frac{\pi_i^k(\nu-\epsilon)}{4}\left[\int_0^1 \Im'\left(\left(1-\frac{\pi_i}{2}\right)\xi\otimes 1 + \left(\frac{\pi_i 1\otimes\phi}{2}\right)\right)\mathrm{d}\pi_i\right]\right.\\
&+\Im\left(\frac{\xi\otimes 1 + 1\otimes\phi}{2}\right) - \frac{(1-\pi_i)^{\kappa}(\nu-\epsilon)}{4}\left[\int_0^1 \Im'\left(\left(\frac{1-\pi_i}{2}\right)\xi\otimes 1 + \left(\frac{1+\pi_i}{2}\right)1\otimes\phi\right)\mathrm{d}\pi_i\right]\\
&= \frac{1\otimes\phi - \xi\otimes 1}{4}\left[\int_0^{\frac{2}{3}}(\pi_i^K - \frac{1}{4})\left[\Im'\left(1\otimes\xi\frac{\pi_i}{2} + 1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right) - \Im'\left(1\otimes\phi\frac{\pi_i}{2} + 1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\right.\right.\\
&+\int_{\frac{2}{3}}^1 \left(\pi_i{}^K - 1\right)\left[\Im'\left(1\otimes\xi\frac{\pi_i}{2} + 1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right) - \Im'\left(1\otimes\phi\frac{\pi_i}{2} + 1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\right]\Bigg]\mathrm{d}\pi_i. \qquad (3.3)
\end{aligned}$$

Proof. Take into account the following result from 2024 [55], which refines Simpson type inequalities in the fractional framework via differentiable convex mappings.

Let $\Im : [\epsilon, \upsilon] \to \mathsf{R}$ be a differentiable mapping (ϵ, υ) such that $\Im' \in \mathtt{L}_1([\epsilon, \upsilon])$. Then, the following double equality holds true:

$$\begin{aligned}
&\frac{1}{8}\left[\Im(\epsilon) + 3\Im\left(\frac{\epsilon+2\upsilon}{3}\right) + 3\Im\left(\frac{2\epsilon+\upsilon}{3}\right) + \Im(\upsilon)\right] - \frac{2^{\kappa-1}\Gamma(\kappa+1)}{(\upsilon-\epsilon)^{\kappa}}\left[\mathsf{J}^{\kappa}_{\frac{\epsilon+\upsilon}{2}-}\Im(\epsilon) + \mathsf{J}^{\kappa}_{\frac{\epsilon+\upsilon}{2}+}\Im(\upsilon)\right]\\
&= \frac{\upsilon-\epsilon}{4}\left[\int_0^{\frac{2}{3}}\left(\pi_i{}^{\kappa} - \frac{1}{4}\right)\left[\Im'\left(\epsilon\frac{\pi_i}{2} + \epsilon\left(1-\frac{\pi_i}{2}\right)\right) - \Im'\left(\upsilon\frac{\pi_i}{2} + \upsilon\left(1-\frac{\pi_i}{2}\right)\right)\right]\mathrm{d}\pi_i\right.
\end{aligned}$$

$$+\int_{\frac{2}{3}}^{1}(\pi_i{}^{\kappa}-1)\left[\Im'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(\upsilon\frac{\pi_i}{2}+\upsilon\left(1-\frac{\pi_i}{2}\right)\right)\right]\mathrm{d}\pi_i\Bigg]. \tag{3.4}$$

By using substitution from Eqs (2.7) and (2.8), we have

$$\begin{aligned}
&\frac{1}{8}\left[\Im(\epsilon)+3\Im\left(\frac{\epsilon+2\upsilon}{3}\right)+3\Im\left(\frac{2\epsilon+\upsilon}{3}\right)+\Im(\upsilon)\right]-\frac{2^{\kappa-1}\Gamma(\kappa+1)}{(\upsilon-\epsilon)^{\kappa}}\left[\frac{1}{2^{\kappa-1}\Gamma(\kappa+1)}\Im\left(\frac{\epsilon+\upsilon}{2}\right)(\upsilon-\epsilon)^{\kappa}\right.\\
&-\frac{\pi_i^{\kappa}(\upsilon-\epsilon)^{\kappa+1}}{2^{\kappa+1}\Gamma(\kappa+1)}\left[\int_0^1\Im'\left((1-\pi_i)\epsilon+\left(\frac{\epsilon+\upsilon}{2}\right)\pi_i\right)\mathrm{d}\pi_i\right]+\frac{1}{2^{\kappa-1}\Gamma(\kappa+1)}\Im\left(\frac{\epsilon+\upsilon}{2}\right)(\upsilon-\epsilon)^{\kappa}\\
&\left.-\frac{(1-\pi_i)^{\kappa}(\upsilon-\epsilon)^{\kappa+1}}{2^{\kappa+1}\Gamma(\kappa+1)}\left[\int_0^1\Im'\left((1-\pi_i)\left(\frac{\epsilon+\upsilon}{2}\right)+\upsilon\pi_i\right)\mathrm{d}\pi_i\right]\right]\\
=&\frac{\upsilon-\epsilon}{4}\Bigg[\int_0^{\frac{2}{3}}\left(\pi_i{}^{\kappa}-\frac{1}{4}\right)\left[\Im'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(\upsilon\frac{\pi_i}{2}+\upsilon\left(1-\frac{\pi_i}{2}\right)\right)\right]\mathrm{d}\pi_i\\
&+\int_{\frac{2}{3}}^{1}(\pi_i{}^{\kappa}-1)\left[\Im'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(\upsilon\frac{\pi_i}{2}+\upsilon\left(1-\frac{\pi_i}{2}\right)\right)\right]\mathrm{d}\pi_i\Bigg].
\end{aligned} \tag{3.5}$$

By making several simplifications, we may have

$$\begin{aligned}
&\left[\frac{1}{8}\Im(\epsilon)+\frac{3}{8}\Im\left(\frac{\epsilon+2v}{3}\right)+\frac{3}{8}\Im\left(\frac{2\epsilon+v}{3}\right)+\frac{1}{8}\Im(v)\right]\\
&-\left[\Im\left(\frac{\epsilon+v}{2}\right)-\frac{\pi_i^{\kappa}(v-\epsilon)}{4}\left[\int_0^1\Im'\left(\left(1-\frac{\pi_i}{2}\right)\epsilon+\left(\frac{v}{2}\right)\pi_i\right)\mathrm{d}\pi_i\right]\right.\\
&\left.+\Im\left(\frac{\epsilon+v}{2}\right)-\frac{(1-\pi_i)^{\kappa}(v-\epsilon)}{4}\left[\int_0^1\Im'\left(\left(\frac{1-\pi_i}{2}\right)\epsilon+\left(\left(\frac{1+\pi_i}{2}\right)v\right)\mathrm{d}\pi_i\right]\right]\\
=&\frac{v-\epsilon}{4}\Bigg[\int_0^{\frac{2}{3}}\left(\pi_i{}^{K}-\frac{1}{4}\right)\left[\Im'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(v\frac{\pi_i}{2}+v\left(1-\frac{\pi_i}{2}\right)\right)\right]\mathrm{d}\pi_i\\
&+\int_{\frac{2}{3}}^{1}\left(\pi_i{}^{K}-1\right)\left[\Im'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(v\frac{\pi_i}{2}+v\left(1-\frac{\pi_i}{2}\right)\right)\right]\mathrm{d}\pi_i\Bigg].
\end{aligned} \tag{3.6}$$

Assume that the spectral resolutions of ξ and ϕ are

$$\xi=\int_{\Delta}\upsilon\mathrm{dE}(\upsilon)\text{ and }\phi=\int_{\Delta}\epsilon\mathrm{dF}(\epsilon).$$

Taking, $\int_{\Delta}\int_{\Delta}$ over $\mathrm{dE}_{\epsilon}\otimes\mathrm{dF}_{\upsilon}$ in (3.5), we get

$$\begin{aligned}
&\int_{\Delta}\int_{\Delta}\left(\frac{1}{8}\Im(\epsilon)+\frac{3}{8}\Im\left(\frac{\epsilon+2\upsilon}{3}\right)+\frac{3}{8}\Im\left(\frac{2\epsilon+\upsilon}{3}\right)+\frac{1}{8}\Im(\upsilon)\right)\mathrm{dE}_{\epsilon}\otimes\mathrm{dF}_{\upsilon}\\
&-\left[\int_{\Delta}\int_{\Delta}\left(\Im\left(\frac{\epsilon+\upsilon}{2}\right)-\frac{\pi_i^{\kappa}(\upsilon-\epsilon)}{4}\left[\int_0^1\Im'\left(\left(1-\frac{\pi_i}{2}\right)\epsilon+\left(\frac{\upsilon}{2}\right)\pi_i\right)\mathrm{d}\pi_i\right)\right]\mathrm{dE}_{\epsilon}\otimes\mathrm{dF}_{\upsilon}\\
&+\int_{\Delta}\int_{\Delta}\left(\Im\left(\frac{\epsilon+\upsilon}{2}\right)-\frac{(1-\pi_i)^{\kappa}(\upsilon-\epsilon)}{4}\left[\int_0^1\Im'\left(\left(\frac{1-\pi_i}{2}\right)\epsilon+\left(\left(\frac{1+\pi_i}{2}\right)\upsilon\right)\mathrm{d}\pi_i\right]\right)\mathrm{dE}_{\epsilon}\otimes\mathrm{dF}_{\upsilon}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Delta \int_\Delta \frac{\upsilon-\epsilon}{4}\Bigg[\int_0^{\frac{2}{3}} \left(\pi_i{}^{\kappa} - \frac{1}{4}\right)\left[\Im'\left(\epsilon\frac{\pi_i}{2} + \epsilon\left(1-\frac{\pi_i}{2}\right)\right) - \Im'\left(\upsilon\frac{\pi_i}{2} + \upsilon\left(1-\frac{\pi_i}{2}\right)\right)\right] \mathrm{d}\pi_i \\
&\quad + \int_{\frac{2}{3}}^1 (\pi_i{}^{\kappa} - 1)\left[\Im'\left(\epsilon\frac{\pi_i}{2} + \epsilon\left(1-\frac{\pi_i}{2}\right)\right) - \Im'\left(\upsilon\frac{\pi_i}{2} + \upsilon\left(1-\frac{\pi_i}{2}\right)\right)\right]\Bigg] \mathrm{d}\pi_i\Bigg] \mathtt{dE}_\epsilon \otimes \mathtt{dF}_\upsilon. \qquad (3.7)
\end{aligned}
$$

Considering Lemma 3.1 and Fubini's theorem, we have

$$\int_{\Delta}\int_{\Delta}\Im(\upsilon)\mathrm{dE}_{\epsilon}\otimes\mathrm{dF}_{\upsilon}=(\Im(\xi)\otimes 1),$$
$$\int_{\Delta}\int_{\Delta}\Im\left(\frac{\epsilon+\upsilon}{2}\right)\mathrm{dE}_{\epsilon}\otimes\mathrm{dF}_{\upsilon}=\Im\left(\frac{\xi\otimes 1+1\otimes\phi}{2}\right),$$
$$\int_{\Delta}\int_{\Delta}\Im(\epsilon)\mathrm{dE}_{\epsilon}\otimes\mathrm{dF}_{\upsilon}=(1\otimes\Im(\phi)),$$
$$\int_{\Delta}\int_{\Delta}\int_0^1\Im'\left(\left(1-\frac{\pi_i}{2}\right)\epsilon+\left(\frac{\upsilon}{2}\right)\pi_i\right)\mathrm{d}\pi_i\mathrm{dE}_{\epsilon}\otimes\mathrm{dF}_{\upsilon}$$
$$=\int_0^1\int_{\Delta}\int_{\Delta}\Im'\left(\left(1-\frac{\pi_i}{2}\right)\epsilon+\left(\frac{\upsilon}{2}\right)\pi_i\right)\mathrm{dE}_{\epsilon}\otimes\mathrm{dF}_{\upsilon}\mathrm{d}\pi_i$$
$$=\int_0^1\Im'\left(\left(1-\frac{\pi_i}{2}\right)\xi\otimes 1+\left(\frac{\pi_i 1\otimes\phi}{2}\right)\right)\mathrm{d}\pi_i,$$
$$\int_{\Delta}\int_{\Delta}\int_0^1\Im'\left(\left(\frac{1-\pi_i}{2}\right)\epsilon+\left(\frac{1+\pi_i}{2}\right)\upsilon\right)\mathrm{d}\pi_i\mathrm{dE}_{\epsilon}\otimes\mathrm{dF}_{\upsilon}$$
$$=\int_0^1\int_{\Delta}\int_{\Delta}\Im'\left(\left(\frac{1-\pi_i}{2}\right)\epsilon+\left(\frac{1+\pi_i}{2}\right)\upsilon\right)\mathrm{dE}_{\epsilon}\otimes\mathrm{dF}_{\upsilon}\mathrm{d}\pi_i$$
$$=\int_0^1\int_{\Delta}\int_{\Delta}\Im'\left(\left(\frac{1-\pi_i}{2}\right)\xi\otimes 1+\left(\frac{1+\pi_i}{2}\right)1\otimes\phi\right)\mathrm{d}\pi_i,$$
$$\left(\Im'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(\upsilon\frac{\pi_i}{2}+\upsilon\left(1-\frac{\pi_i}{2}\right)\right)\right)\mathrm{d}\pi_i\mathrm{dE}_{\epsilon}\otimes\mathrm{dF}_{\upsilon}$$
$$=\left(\Im'\left(\xi\otimes 1\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\right)\mathrm{d}\pi_i. \tag{3.8}$$

Taking the same technique into consideration, we have

$$\int_{\Delta}\int_{\Delta}\frac{\upsilon-\epsilon}{4}\Bigg[\int_0^{\frac{2}{3}}\left(\pi_i^{\kappa}-\frac{1}{4}\right)\left[\Im'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(\upsilon\frac{\pi_i}{2}+\upsilon\left(1-\frac{\pi_i}{2}\right)\right)\right]\mathrm{d}\pi_i$$
$$+\int_{\frac{2}{3}}^1(\pi_i^{\kappa}-1)\left[\Im'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(\upsilon\frac{\pi_i}{2}+\upsilon\left(1-\frac{\pi_i}{2}\right)\right)\right]\mathrm{d}\pi_i\Bigg]\mathrm{dE}_{\epsilon}\otimes\mathrm{dF}_{\upsilon}$$
$$=\frac{1\otimes\phi-\xi\otimes 1}{4}\Bigg[\int_0^{\frac{2}{3}}\left(\pi_i^{\kappa}-\frac{1}{4}\right)\left[\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\right]$$
$$+\int_{\frac{2}{3}}^1(\pi_i^{\kappa}-1)\left[\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\right]\Bigg]\mathrm{d}\pi_i. \tag{3.9}$$

Using Eqs (3.20) and (3.21) in (3.7), we get the needed result. □

Theorem 3.1. *Assume ξ and ϕ are two self-adjoint operators with $\mathcal{SP}(\xi)\subset\Delta$ and $\mathcal{SP}(\phi)\subset\Delta$. Let $\Im$ be differentiable on Δ with $\left\|\Im'\right\|_{\Delta,\infty}:=\sup_{\kappa\in\Delta}\left|\Im'(\kappa)\right|<\infty$. Then, we have*

$$\left\|\left(\frac{1}{8}(\Im(\xi)\otimes 1)+\frac{3}{8}\Im\left(\frac{2\xi\otimes 1+1\otimes\phi}{2}\right)+\frac{3}{8}\Im\left(\frac{\xi\otimes 1+2\otimes\phi}{2}\right)+\frac{1}{8}(1\otimes\Im(\phi))\right)\right.$$

$$. - \left[\Im\left(\frac{\xi\otimes 1+1\otimes\phi}{2}\right) - \frac{\pi_i^{\kappa}(\upsilon-\epsilon)}{4}\left[\int_0^1 \Im'\left(\left(1-\frac{\pi_i}{2}\right)\xi\otimes 1+\left(\frac{\pi_i 1\otimes\phi}{2}\right)\right)\mathrm{d}\pi_i\right]\right.$$
$$\left. + \Im\left(\frac{\xi\otimes 1+1\otimes\phi}{2}\right) - \frac{(1-\pi_i)^{\kappa}(\upsilon-\epsilon)}{4}\left[\int_0^1 \Im'\left(\left(\frac{1-\pi_i}{2}\right)\xi\otimes 1+\left(\frac{1+\pi_i}{2}\right)1\otimes\phi\right)\mathrm{d}\pi_i\right]\right\|$$
$$\leq \frac{\|1\otimes\phi-\xi\otimes 1\|}{4}\left(\frac{(\kappa+1)\cdot 3^{\kappa}+2^{\kappa+2}}{(6\kappa+6)\cdot 3^{\kappa}}\left(\left\|\Im'\right\|_{\Delta,+\infty}+\left\|\Im'\right\|_{\Delta,+\infty}\right)+\frac{2^{\kappa+1}+(\kappa+2)\cdot 3^{\kappa}}{(3\kappa+3)\cdot 3^{\kappa}}\left(\left\|\Im'\right\|_{\Delta,+\infty}+\left\|\Im'\right\|_{\Delta,+\infty}\right)\right).$$

Proof. Considering Lemma 3.3 and applying the triangle inequality, we arrive at

$$\left\|\left(\frac{1}{8}(\Im(\xi)\otimes 1)+\frac{3}{8}\Im\left(\frac{2\xi\otimes 1+1\otimes\phi}{2}\right)+\frac{3}{8}\Im\left(\frac{\xi\otimes 1+2\otimes\phi}{2}\right)+\frac{1}{8}(1\otimes\Im(\phi))\right)\right.$$
$$. - \left[\Im\left(\frac{\xi\otimes 1+1\otimes\phi}{2}\right) - \frac{\pi_i^{\kappa}(\nu-\epsilon)}{4}\left[\int_0^1 \Im'\left(\left(1-\frac{\pi_i}{2}\right)\xi\otimes 1+\left(\frac{\pi_i 1\otimes\phi}{2}\right)\right)\mathrm{d}\pi_i\right]\right.$$
$$\left. + \Im\left(\frac{\xi\otimes 1+1\otimes\phi}{2}\right) - \frac{(1-\pi_i)^{\kappa}(\nu-\epsilon)}{4}\left[\int_0^1 \Im'\left(\left(\frac{1-\pi_i}{2}\right)\xi\otimes 1+\left(\frac{1+\pi_i}{2}\right)1\otimes\phi\right)\mathrm{d}\pi_i\right]\right\|$$
$$\leq \frac{\|1\otimes\phi-\xi\otimes 1\|}{4}\left\|\left[\int_0^{\frac{2}{3}}(\pi_i^{\kappa}-\frac{1}{4})\left[\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi^{2-\pi_i}\frac{2}{2}\right)-\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\right.\right.\right.$$
$$\left.\left.\left.+\int_{\frac{2}{3}}^1(\pi_i^{\kappa}-1)\left[\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\right]\right]\mathrm{d}\pi_i\right\|$$
$$\leq \frac{\|1\otimes\phi-\xi\otimes 1\|}{4}\|\int_0^{\frac{2}{3}}\left(\pi_i^{\kappa}-\frac{1}{4}\right)\left[\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\mathrm{d}\pi_i\|\right.$$
$$+\left\|\int_{\frac{2}{3}}^1(\pi_i^{\kappa}-1)\left[\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi^{2-\pi_i}\frac{2}{2}\right)-\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\right]\mathrm{d}\pi_i\right\|. \tag{3.10}$$

Observe that, by Lemma 3.1

$$\left|\left(\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\right)\right|$$
$$=\int_\Delta\int_\Delta\left|\left(\Im'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(\upsilon\frac{\pi_i}{2}+\upsilon\left(1-\frac{\pi_i}{2}\right)\right)\right)\right|\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon.$$

As by convexity, we have

$$\left|\Im'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)\right|\leqslant\left\|\Im'\right\|_{\Delta,+\infty}$$

for all $\tau\in[0,1]$ and $\epsilon,\upsilon\in\Delta$.

Taking $\int_\Delta\int_\Delta$ over $\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon$, we get

$$\left|\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)\right|=\int_\Delta\int_\Delta\left|\Im'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)\right|\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon$$
$$\leqslant\left\|\Im'\right\|_{\Delta,+\infty}\int_\Delta\int_\Delta\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon=\left\|\Im'\right\|_{\Delta,+\infty}. \tag{3.11}$$

Similarly, we get

$$\left|\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\right|=\int_\Delta\int_\Delta\left|\Im'\left(\upsilon\frac{\pi_i}{2}+\upsilon\left(1-\frac{\pi_i}{2}\right)\right)\right|\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon$$

$$\leqslant \left\|\mathfrak{I}'\right\|_{\Delta,+\infty} \int_{\Delta}\int_{\Delta} \mathtt{dE}_{\epsilon} \otimes \mathtt{dF}_{\upsilon} = \left\|\mathfrak{I}'\right\|_{\Delta,+\infty}. \tag{3.12}$$

Considering Eq (3.10), it now follows that

$$\begin{aligned}
&\frac{\|1\otimes\phi-\xi\otimes 1\|}{4}\Bigg(\Bigg\|\int_0^{\frac{2}{3}}\left(\pi_i{}^{\kappa}-\frac{1}{4}\right)\left[\mathfrak{I}'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)-\mathfrak{I}'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\mathrm{d}\pi_i\right\|\\
&\quad+\left\|\int_{\frac{2}{3}}^{1}(\pi_i{}^{\kappa}-1)\left[\mathfrak{I}'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)-\mathfrak{I}'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\right]\mathrm{d}\pi_i\right\|\Bigg)\\
&\le\frac{\|1\otimes\phi-\xi\otimes 1\|}{4}\Bigg(\left\|\int_0^{\frac{2}{3}}\left(\pi_i{}^{\kappa}-\frac{1}{4}\right)\right\|\Bigg(\left\|\left[\mathfrak{I}'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)\right\|+\left\|\mathfrak{I}'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\mathrm{d}\pi_i\right\|\Bigg)\\
&\quad+\left(\left\|\int_{\frac{2}{3}}^{1}(\pi_i{}^{\kappa}-1)\right\|\left(\left\|\left[\mathfrak{I}'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)\mathrm{d}\pi_i\right\|+\left\|\mathfrak{I}'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\mathrm{d}\pi_i\right\|\right)\right)\\
&\le\frac{\|1\otimes\phi-\xi\otimes 1\|}{4}\left(\frac{(\kappa+1)\cdot 3^{\kappa}+2^{\kappa+2}}{(6\kappa+6)\cdot 3^{\kappa}}\left(\left\|\mathfrak{I}'\right\|_{\Delta,+\infty}+\left\|\mathfrak{I}'\right\|_{\Delta,+\infty}\right)+\frac{2^{\kappa+1}+(\kappa+2)\cdot 3^{\kappa}}{(3\kappa+3)\cdot 3^{\kappa}}\left(\left\|\mathfrak{I}'\right\|_{\Delta,+\infty}+\left\|\mathfrak{I}'\right\|_{\Delta,+\infty}\right)\right).
\end{aligned} \tag{3.13}$$

Using Eq (3.13) in (3.10), we get needed output. □

Theorem 3.2. *Assume ξ and ϕ are two self-adjoint operators with $\mathcal{SP}(\xi)\subset\Delta$ and $\mathcal{SP}(\phi)\subset\Delta$. Let $\mathfrak{I}$ be differentiable as well as convex $\left|\mathfrak{I}'\right|$ on Δ. Then, the following inequality holds true:*

$$\begin{aligned}
&\Bigg\|\left(\frac{1}{8}(\mathfrak{I}(\xi)\otimes 1)+\frac{3}{8}\mathfrak{I}\left(\frac{2\xi\otimes 1+1\otimes\phi}{2}\right)+\frac{3}{8}\mathfrak{I}\left(\frac{\xi\otimes 1+2\otimes\phi}{2}\right)+\frac{1}{8}(1\otimes\mathfrak{I}(\phi))\right)\\
&\quad.-\left[\mathfrak{I}\left(\frac{\xi\otimes 1+1\otimes\phi}{2}\right)-\frac{\pi_i^{\kappa}(\upsilon-\epsilon)}{4}\left[\int_0^1\mathfrak{I}'\left(\left(1-\frac{\pi_i}{2}\right)\xi\otimes 1+\left(\frac{\pi_i 1\otimes\phi}{2}\right)\right)\mathrm{d}\pi_i\right]\right.\\
&\quad\left.+\mathfrak{I}\left(\frac{\xi\otimes 1+1\otimes\phi}{2}\right)-\frac{(1-\pi_i)^{\kappa}(\upsilon-\epsilon)}{4}\left[\int_0^1\mathfrak{I}'\left(\left(\frac{1-\pi_i}{2}\right)\xi\otimes 1+\left(\frac{1+\pi_i}{2}\right)1\otimes\phi\right)\mathrm{d}\pi_i\right]\right\|\\
&\le\frac{\|1\otimes\phi-\xi\otimes 1\|}{4}\left(\frac{3^{1+\kappa}\kappa^2+2^{\kappa+3}\kappa+2^{\kappa+3}+8\kappa\cdot 3^{\kappa}+5\cdot 3^{\kappa}}{3^{\kappa}(\kappa+1)(6\kappa+6)}\left(\left\|\mathfrak{I}'(\xi)\right\|+\left\|\mathfrak{I}'(\phi)\right\|\right)\right).
\end{aligned}$$

Proof. By assuming that $\left|\mathfrak{I}'\right|$ is convex on Δ, we have

$$\left|\mathfrak{I}'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)\right|\le\frac{\pi_i}{2}\left|\mathfrak{I}'(\epsilon)\right|+\left(1-\frac{\pi_i}{2}\right)\left|\mathfrak{I}'(\epsilon)\right|.$$

Similarly, we get

$$\left|\mathfrak{I}'\left(\upsilon\frac{\pi_i}{2}+\upsilon\left(1-\frac{\pi_i}{2}\right)\right)\right|\le\frac{\pi_i}{2}\left|\mathfrak{I}'(\upsilon)\right|+\left(\left(1-\frac{\pi_i}{2}\right)\right)\left|\mathfrak{I}'(\upsilon)\right|$$

for all for $\tau\in[0,1]$ and $\epsilon,\upsilon\in\Delta$.

Taking $\int_\Delta \int_\Delta$ over $\mathtt{dE}_\epsilon \otimes \mathtt{dF}_\upsilon$, then we get

$$\begin{aligned}
&\left|\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)\right| = \int_\Delta\int_\Delta \left|\Im'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)\right| \mathtt{dE}_\epsilon \otimes \mathtt{dF}_\upsilon \\
&\leq \int_\Delta\int_\Delta \frac{\pi_i}{2}\left|\Im'(\epsilon)\right| + \left(1-\frac{\pi_i}{2}\right)\left|\Im'(\epsilon)\right| \mathtt{dE}_\epsilon \otimes \mathtt{dF}_\upsilon \\
&\leq \frac{\pi_i}{2} 1\otimes\left|\Im'(\xi)\right| + \left(1-\frac{\pi_i}{2}\right)\left|\Im'(\xi)\right|\otimes 1.
\end{aligned} \tag{3.14}$$

If we apply the norm in (3.29), then we have

$$\begin{aligned}
&\left\|\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)\right\| \\
&\leq \left\|\frac{\pi_i}{2}1\otimes\left|\Im'(\xi)\right| + \left(1-\frac{\pi_i}{2}\right)\left|\Im'(\xi)\right|\otimes 1\right\| \leq \frac{\pi_i}{2}\left\|\Im'(\xi)\right\| + \left(1-\frac{\pi_i}{2}\right)\left\|\Im'(\xi)\right\|.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\left\|\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\right\| \\
&\leq \left\|\frac{\pi_i}{2}1\otimes\left|\Im'(\phi)\right| + \left(\left(1-\frac{\pi_i}{2}\right)\right)\left|\Im'(\phi)\right|\otimes 1\right\| \leq \frac{\pi_i}{2}\left\|\Im'(\phi)\right\| + \left(\left(1-\frac{\pi_i}{2}\right)\right)\left\|\Im'(\phi)\right\|.
\end{aligned}$$

Using the norm in (3.5) and considering the triangle inequality, we have

$$\begin{aligned}
&\left\|\left(\frac{1}{8}(\Im(\xi)\otimes 1)+\frac{3}{8}\Im\left(\frac{2\xi\otimes 1+1\otimes\phi}{2}\right)+\frac{3}{8}\Im\left(\frac{\xi\otimes 1+2\otimes\phi}{2}\right)+\frac{1}{8}(1\otimes\Im(\phi))\right)\right. \\
&\quad -\left[\Im\left(\frac{\xi\otimes 1+1\otimes\phi}{2}\right)-\frac{\pi_i^\kappa(\upsilon-\epsilon)}{4}\left[\int_0^1\Im'\left(\left(1-\frac{\pi_i}{2}\right)\xi\otimes 1+\left(\frac{\pi_i 1\otimes\phi}{2}\right)\right)\mathrm{d}\pi_i\right]\right. \\
&\quad \left.\left.+\Im\left(\frac{\xi\otimes 1+1\otimes\phi}{2}\right)-\frac{(1-\pi_i)^\kappa(\upsilon-\epsilon)}{4}\left[\int_0^1\Im'\left(\left(\frac{1-\pi_i}{2}\right)\xi\otimes 1+\left(\frac{1+\pi_i}{2}\right)1\otimes\phi\right)\mathrm{d}\pi_i\right]\right]\right\| \\
&\leq \frac{\|1\otimes\phi-\xi\otimes 1\|}{4}\left(\left\|\int_0^{\frac{2}{3}}\left(\pi_i{}^\kappa-\frac{1}{4}\right)\left[\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\mathrm{d}\pi_i\right\|\right. \\
&\quad \left.+\left\|\int_{\frac{2}{3}}^1(\pi_i{}^\kappa-1)\left[\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\right]\mathrm{d}\pi_i\right\|\right) \\
&\leq \frac{\|1\otimes\phi-\xi\otimes 1\|}{4}\left(\left\|\int_0^{\frac{2}{3}}\left(\pi_i{}^\kappa-\frac{1}{4}\right)\right\|\left(\left\|\left[\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)\right\|+\left\|\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\mathrm{d}\pi_i\right\|\right)\right. \\
&\quad +\left(\left\|\int_{\frac{2}{3}}^1(\pi_i{}^\kappa-1)\right\|\left(\left\|\left[\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)\mathrm{d}\pi_i\right\|+\left\|\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\mathrm{d}\pi_i\right\|\right)\right) \\
&\leq \frac{\|1\otimes\phi-\xi\otimes 1\|}{4}\left(\left\|\int_0^{\frac{2}{3}}\left(\pi_i{}^\kappa-\frac{1}{4}\right)\right\|\left(\left\|\frac{\pi_i}{2}1\otimes\left|\Im'(\xi)\right|+\left(1-\frac{\pi_i}{2}\right)\left|\Im'(\xi)\right|\otimes 1\right\|\right.\right. \\
&\quad +\left\|\frac{\pi_i}{2}1\otimes\left|\Im'(\phi)\right|+\left(1-\frac{\pi_i}{2}\right)\left|\Im'(\phi)\right|\otimes 1\right\|+\left(\left\|\int_{\frac{2}{3}}^1(\pi_i{}^\kappa-1)\right\|\left(\left\|\frac{\pi_i}{2}1\otimes\left|\Im'(\xi)\right|+\left(1-\frac{\pi_i}{2}\right)\left|\Im'(\xi)\right|\otimes 1\right\|\right.\right.
\end{aligned}$$

$$+\left\|\frac{\pi_i}{2}1\otimes\left|\Im'(\phi)\right|+\left(1-\frac{\pi_i}{2}\right)\left|\Im'(\phi)\right|\otimes 1\right\|\Bigg)$$

$$\leq\frac{\|1\otimes\phi-\xi\otimes 1\|}{4}\left(\frac{3^{1+\kappa}\kappa^2+2^{\kappa+3}\kappa+2^{\kappa+3}+8\kappa\cdot 3^{\kappa}+5\cdot 3^{\kappa}}{3^{\kappa}(\kappa+1)(6\kappa+6)}\left(\left\|\Im'(\xi)\right\|+\left\|\Im'(\phi)\right\|\right)\right).\tag{3.15}$$

□

Remark 3.1. *In Theorem 3.2, if we set $\kappa=1$ and tensorial arithmetic operations are degenerated, we get Remark 3 from [57].*

Theorem 3.3. *Assume ξ and ϕ are two self-adjoint operators with $\mathcal{SP}(\xi)\subset\Delta$ and $\mathcal{SP}(\phi)\subset\Delta$. Let $\Im$ be differentiable as well as quasi convex $\left|\Im'\right|$ on Δ. Then, the following inequality holds true:*

$$\Bigg\|\left(\frac{1}{8}(\Im(\xi)\otimes 1)+\frac{3}{8}\Im\left(\frac{2\xi\otimes 1+1\otimes\phi}{2}\right)+\frac{3}{8}\Im\left(\frac{\xi\otimes 1+2\otimes\phi}{2}\right)+\frac{1}{8}(1\otimes\Im(\phi))\right)$$
$$.-\left[\Im\left(\frac{\xi\otimes 1+1\otimes\phi}{2}\right)-\frac{\pi_i^{\kappa}(\upsilon-\epsilon)}{4}\left[\int_0^1\Im'\left(\left(1-\frac{\pi_i}{2}\right)\xi\otimes 1+\left(\frac{\pi_i 1\otimes\phi}{2}\right)\right)d\pi_i\right]\right.$$
$$\left.+\Im\left(\frac{\xi\otimes 1+1\otimes\phi}{2}\right)-\frac{(1-\pi_i)^{\kappa}(\upsilon-\epsilon)}{4}\left[\int_0^1\Im'\left(\left(\frac{1-\pi_i}{2}\right)\xi\otimes 1+\left(\frac{1+\pi_i}{2}\right)1\otimes\phi\right)d\pi_i\right]\right\|$$
$$\leq\frac{\|1\otimes\phi-\xi\otimes 1\|}{4}\left(\frac{3\cdot 3^{\kappa}\,\kappa^2+8\cdot 2^{\kappa}\,\kappa+8.2^{\kappa}+8\,\kappa\cdot\ 3^{\kappa}+5\ \cdot\ 3^{\kappa}}{3^{\kappa}\,(\kappa+1)\,(48\,\kappa+48)}\right)$$
$$\times\left(\left\|\left|\Im'(\xi)\right|\otimes 1+1\otimes\left|\Im'(\phi)\right|\right\|+\left\|\left|\Im'(\xi)\right|\otimes 1-1\otimes\left|\Im'(\phi)\right|\right\|\right).$$

Proof. By assuming that $\left|\Im'\right|$ is quasi convex on Δ, we have

$$\left|\left(\Im'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(\upsilon\frac{\pi_i}{2}+\upsilon\left(1-\frac{\pi_i}{2}\right)\right)\right)\right|$$
$$\leq\left|\left(\Im'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)+\Im'\left(\upsilon\frac{\pi_i}{2}+\upsilon\left(1-\frac{\pi_i}{2}\right)\right)\right)\right|\leq\frac{1}{2}\left(\left|\Im'(\upsilon)\right|+\left|\Im'(\epsilon)\right|+\|\Im'(\upsilon)|-|\Im'(\epsilon)\|\right),$$

$\forall\,\tau\in[0,1]$ and $\epsilon,\upsilon\in\Delta$.
Taking $\int_\Delta\int_\Delta$ over $dE_\epsilon\otimes dF_\upsilon$ yields:

$$\left|\left(\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\right)\right|$$
$$=\int_\Delta\int_\Delta\left|\left(\Im'\left(\epsilon\frac{\pi_i}{2}+\epsilon\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(\upsilon\frac{\pi_i}{2}+\upsilon\left(1-\frac{\pi_i}{2}\right)\right)\right)\right|dE_\epsilon\otimes dF_\upsilon$$
$$\leq\frac{1}{2}\int_\Delta\int_\Delta\left(\left|\Im'(\upsilon)\right|+\left|\Im'(\epsilon)\right|+\|\Im'(\upsilon)|-|\Im'(\epsilon)\|\right)dE_\epsilon\otimes dF_\upsilon$$
$$=\frac{1}{2}\left(\left|\Im'(\xi)\right|\otimes 1+1\otimes\left|\Im'(\phi)\right|+\||\Im'(\xi)|\otimes 1-1\otimes|\Im'(\phi)|\|\right).$$

Applying the norm in the above inequality result gives

$$\left\|\left(\Im'\left(1\otimes\xi\frac{\pi_i}{2}+1\otimes\xi\left(1-\frac{\pi_i}{2}\right)\right)-\Im'\left(1\otimes\phi\frac{\pi_i}{2}+1\otimes\phi\left(1-\frac{\pi_i}{2}\right)\right)\right)\right\|$$

$$\leq \left\| \frac{1}{2}\left(\left|\Im'(\xi)\right| \otimes 1 + 1 \otimes \left|\Im'(\phi)\right| + \|\Im'(\xi)| \otimes 1 - 1 \otimes |\Im'(\phi)\| \right) \right\|$$
$$\leqslant \frac{1}{2}\left(\left\|\left|\Im'(\xi)\right| \otimes 1 + 1 \otimes \left|\Im'(\phi)\right|\right\| + \left\|\left|\Im'(\xi)\right| \otimes 1 - 1 \otimes \left|\Im'(\phi)\right|\right\| \right).$$

Using the norm in (3.5) and considering triangular inequality, we have

$$\begin{aligned}
&\Bigg\| \left(\frac{1}{8}(\Im(\xi) \otimes 1) + \frac{3}{8}\Im\left(\frac{2\xi \otimes 1 + 1 \otimes \phi}{2}\right) + \frac{3}{8}\Im\left(\frac{\xi \otimes 1 + 2 \otimes \phi}{2}\right) + \frac{1}{8}(1 \otimes \Im(\phi)) \right) \\
&\quad . - \left[\Im\left(\frac{\xi \otimes 1 + 1 \otimes \phi}{2}\right) - \frac{\pi_i^{\kappa}(\upsilon - \epsilon)}{4}\left[\int_0^1 \Im'\left(\left(1 - \frac{\pi_i}{2}\right)\xi \otimes 1 + \left(\frac{\pi_i 1 \otimes \phi}{2}\right)\right) \mathrm{d}\pi_i \right] \right. \\
&\quad + \Im\left(\frac{\xi \otimes 1 + 1 \otimes \phi}{2}\right) - \frac{(1 - \pi_i)^{\kappa}(\upsilon - \epsilon)}{4}\left[\int_0^1 \Im'\left(\left(\frac{1 - \pi_i}{2}\right)\xi \otimes 1 + \left(\frac{1 + \pi_i}{2}\right) 1 \otimes \phi\right) \mathrm{d}\pi_i \right] \Bigg\| \\
&\leq \frac{\|1 \otimes \phi - \xi \otimes 1\|}{4}\Bigg(\Bigg\| \int_0^{\frac{2}{3}} \left(\pi_i{}^{\kappa} - \frac{1}{4}\right)\left[\Im'\left(1 \otimes \xi\frac{\pi_i}{2} + 1 \otimes \xi\left(1 - \frac{\pi_i}{2}\right)\right) - \Im'\left(1 \otimes \phi\frac{\pi_i}{2} + 1 \otimes \phi\left(1 - \frac{\pi_i}{2}\right)\right) \mathrm{d}\pi_i \right\| \\
&\quad + \left\| \int_{\frac{2}{3}}^1 (\pi_i{}^{\kappa} - 1)\left[\Im'\left(1 \otimes \xi\frac{\pi_i}{2} + 1 \otimes \xi\left(1 - \frac{\pi_i}{2}\right)\right) - \Im'\left(1 \otimes \phi\frac{\pi_i}{2} + 1 \otimes \phi\left(1 - \frac{\pi_i}{2}\right)\right) \right] \mathrm{d}\pi_i \right\| \Bigg) \\
&\leq \frac{\|1 \otimes \phi - \xi \otimes 1\|}{4}\Bigg(\left\| \int_0^{\frac{2}{3}} \left(\pi_i{}^{\kappa} - \frac{1}{4}\right) \right\| \left(\left\| \left[\Im'\left(1 \otimes \xi\frac{\pi_i}{2} + 1 \otimes \xi\left(1 - \frac{\pi_i}{2}\right)\right) \right\| + \left\| \Im'\left(1 \otimes \phi\frac{\pi_i}{2} + 1 \otimes \phi\left(1 - \frac{\pi_i}{2}\right)\right) \mathrm{d}\pi_i \right\| \right) \\
&\quad + \left(\left\| \int_{\frac{2}{3}}^1 (\pi_i{}^{\kappa} - 1) \right\| \left(\left\| \left[\Im'\left(1 \otimes \xi\frac{\pi_i}{2} + 1 \otimes \xi\left(1 - \frac{\pi_i}{2}\right)\right) \mathrm{d}\pi_i \right\| + \left\| \Im'\left(1 \otimes \phi\frac{\pi_i}{2} + 1 \otimes \phi\left(1 - \frac{\pi_i}{2}\right)\right) \mathrm{d}\pi_i \right\| \right) \right) \\
&\leq \frac{\|1 \otimes \phi - \xi \otimes 1\|}{4}\Bigg(\left\| \int_0^{\frac{2}{3}} \left(\pi_i{}^{\kappa} - \frac{1}{4}\right) \right\| \left(\frac{1}{2}\left(\left\|\left|\Im'(\xi)\right| \otimes 1 + 1 \otimes \left|\Im'(\phi)\right|\right\| + \left\|\left|\Im'(\xi)\right| \otimes 1 - 1 \otimes \left|\Im'(\phi)\right|\right\| \right) \right. \\
&\quad + \frac{1}{2}\left(\left\|\left|\Im'(\xi)\right| \otimes 1 + 1 \otimes \left|\Im'(\phi)\right|\right\| + \left\|\left|\Im'(\xi)\right| \otimes 1 - 1 \otimes \left|\Im'(\phi)\right|\right\| \right)) \\
&\quad + \left(\left\| \int_{\frac{2}{3}}^1 (\pi_i{}^{\kappa} - 1) \right\| \left(\frac{1}{2}\left(\left\|\left|\Im'(\xi)\right| \otimes 1 + 1 \otimes \left|\Im'(\phi)\right|\right\| + \left\|\left|\Im'(\xi)\right| \otimes 1 - 1 \otimes \left|\Im'(\phi)\right|\right\| \right) \right. \right. \\
&\quad + \frac{1}{2}\left(\left\|\left|\Im'(\xi)\right| \otimes 1 + 1 \otimes \left|\Im'(\phi)\right|\right\| + \left\|\left|\Im'(\xi)\right| \otimes 1 - 1 \otimes \left|\Im'(\phi)\right|\right\| \right)) \\
&\leq \frac{\|1 \otimes \phi - \xi \otimes 1\|}{4}\left(\frac{3 \cdot 3^{\kappa}\, \kappa^2 + 8 \cdot 2^{\kappa}\, \kappa + 8.2^{\kappa} + 8\, \kappa \cdot\ 3^{\kappa} + 5\ \cdot\ 3^{\kappa}}{3^{\kappa}\, (\kappa + 1)\, (48\, \kappa + 48)} \right) \\
&\times \left(\left\|\left|\Im'(\xi)\right| \otimes 1 + 1 \otimes \left|\Im'(\phi)\right|\right\| + \left\|\left|\Im'(\xi)\right| \otimes 1 - 1 \otimes \left|\Im'(\phi)\right|\right\| \right). \qquad (3.16)
\end{aligned}$$

□

3.2. Hermite-Hadamard inequality involving arithmetic-geometric mean type convexity

Lemma 3.4. *Assume ξ and ϕ are two self-adjoint operators with $\mathcal{SP}(\xi) \subset \Delta$ and $\mathcal{SP}(\phi) \subset \Delta$. Let $\Im$ be convex on Δ. Then, the equality stated below holds true:*

$$\frac{\xi \otimes 1 + 1 \otimes \phi}{2} - \frac{2(\phi \otimes 1 - \xi \otimes 1)}{(\sqrt{\upsilon} - \sqrt{\epsilon})^2}\int_0^1 \Im((1 - \pi_i)\xi \otimes 1 + \pi_i \phi \otimes 1))\mathrm{d}\pi_i$$

$$=\frac{(\sqrt{1\otimes\phi}-\sqrt{\xi\otimes 1})^2}{4}\left[\int_0^1((1-2\pi_i))\,\Im'\left((1-\pi_i)\xi\otimes 1+\pi_i 1\otimes\phi\right)\right]\mathrm{d}\pi_i. \tag{3.17}$$

Proof. Considering [56, Lemma 4.1] based on a differentiable convex mapping, then one has

$$\begin{aligned}&\frac{\Im(\epsilon)+\Im(\upsilon)}{2}-\frac{2}{(\sqrt{\upsilon}-\sqrt{\epsilon})^2}\int_\epsilon^\upsilon\Im(\sigma)\mathrm{d}\sigma\\&=\frac{\Im(\epsilon)+\Im(\upsilon)}{2}-\frac{2(\upsilon-\epsilon)}{(\sqrt{\upsilon}-\sqrt{\epsilon})^2}\int_0^1\Im((1-\pi_i)\epsilon+\pi_i\upsilon))\mathrm{d}\pi_i\\&=\frac{(\sqrt{\upsilon}-\sqrt{\epsilon})^2}{4}\left[\int_0^1(1-2\pi_i)\Im'\left((1-\pi_i)\epsilon+\pi_i\upsilon\right)\right]\mathrm{d}\pi_i\,.\end{aligned} \tag{3.18}$$

Assume that the spectral resolutions of ξ and ϕ are

$$\xi=\int_\Delta\upsilon\mathrm{dE}(\upsilon)\text{ and }\phi=\int_\Delta\epsilon\mathrm{dF}(\epsilon).$$

Taking $\int_\Delta\int_\Delta$ over $\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon$ in (3.18), we have

$$\begin{aligned}&\int_\Delta\int_\Delta\left[\frac{\Im(\epsilon)+\Im(\upsilon)}{2}\right]\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon-\frac{2(\upsilon-\epsilon)}{(\sqrt{\upsilon}-\sqrt{\epsilon})^2}\int_\Delta\int_\Delta\int_0^1\Im((1-\pi_i)\epsilon+\pi_i\upsilon))\mathrm{d}\pi_i\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon\\&=\frac{(\sqrt{\upsilon}-\sqrt{\epsilon})^2}{4}\int_\Delta\int_\Delta\left[\int_0^1(1-2\pi_i)\Im'\left((1-\pi_i)\epsilon+\pi_i\upsilon\right)\right]\mathrm{d}\pi_i\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon.\end{aligned} \tag{3.19}$$

Considering Lemma 3.1 and Fubini's theorem, we have

$$\begin{aligned}&\int_\Delta\int_\Delta\Im(\upsilon)\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon=(\Im(\phi)\otimes 1),\\&\int_\Delta\int_\Delta\int_0^1\Im((1-\pi_i)\epsilon+\pi_i\upsilon))\mathrm{d}\pi_i\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon=\int_0^1\Im((1-\pi_i)\phi\otimes 1+\pi_i\otimes 1\xi))\mathrm{d}\pi_i,\\&\int_\Delta\int_\Delta\Im(\epsilon)\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon=(1\otimes\Im(\xi)).\end{aligned} \tag{3.20}$$

Taking the same technique into consideration, we have

$$\begin{aligned}&\frac{(\sqrt{\nu}-\sqrt{\epsilon})^2}{4}\int_\Delta\int_\Delta\left[\int_0^1(1-2\pi_i)\,\mathfrak{Y}'\left((1-\pi_i)\,\epsilon+\pi_i\nu\right)\right]\mathrm{d}_i\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\nu\\&=\frac{(\sqrt{1\otimes\phi}-\sqrt{\xi\otimes 1})^2}{4}\left[\int_0^1((1-2\pi_i))\,\Im'\left((1-\pi_i)\,\xi\otimes 1+\pi_i 1\otimes\phi\right)\right]\mathrm{d}\pi_i.\end{aligned} \tag{3.21}$$

Using Eqs (3.20) and (3.21) in (3.19), we get needed output. □

Theorem 3.4. *Assume ξ and ϕ are two self-adjoint operators with $\mathcal{SP}(\xi) \subset \Delta$ and $\mathcal{SP}(\phi) \subset \Delta$. Let $\Im$ be differentiable on Δ with $\left\|\Im'\right\|_{\Delta,\infty} := \sup_{\kappa\in\Delta}\left|\Im'(\kappa)\right| < \infty$. Then, we have*

$$\left\|\frac{\xi\otimes 1+1\otimes\phi}{2}-\frac{2(\phi\otimes 1-\xi\otimes 1)}{(\sqrt{\upsilon}-\sqrt{\epsilon})^2}\int_0^1\Im((1-\pi_i)\xi\otimes 1+\pi_i\phi\otimes 1))d\pi_i\right\| \\ \leq \frac{(\sqrt{1\otimes\phi}-\sqrt{\xi\otimes 1})^2}{4}\frac{\left(\left\|\Im'(\phi)\right\|+\left\|\Im'(\xi)\right\|\right)}{2}. \tag{3.22}$$

Proof. Using the triangle inequality and the operator norm of the previously derived Lemma 3.4, we may obtain

$$\left\|\frac{\xi\otimes 1+1\otimes\phi}{2}-\frac{2(\phi\otimes 1-\xi\otimes 1)}{(\sqrt{\upsilon}-\sqrt{\epsilon})^2}\int_0^1\Im((1-\pi_i)\xi\otimes 1+\pi_i\phi\otimes 1))d\pi_i\right\| \\ \leq \frac{(\sqrt{1\otimes\phi}-\sqrt{\xi\otimes 1})^2}{4}\int_0^1\|((1-2\pi_i))\|\left\|\Im'\left((1-\pi_i)\xi\otimes 1+\pi_i 1\otimes\phi\right\|d\pi_i\right. . \tag{3.23}$$

Considering Lemma 3.1, we obtain

$$\left|\left(\Im'\left(\pi_i\xi\otimes 1+(1-\pi_i)\,1\otimes\phi\right)\right|=\int_\Delta\int_\Delta\left|\left(\Im'\pi_i\epsilon+(1-\pi_i)\,\nu\right)\right|\,dE_\epsilon\otimes dF_\nu.$$

As by convexity, we have

$$\left|\left(\Im'(\pi_i\epsilon+(1-\pi_i)\,\upsilon)\right|\leqslant\left\|\Im'\right\|_{\Delta,+\infty}$$

for all $\pi_i \in [0,1]$ and $\epsilon, \upsilon \in \Delta$.

Taking $\int_\Delta\int_\Delta$ over $dE_\epsilon \otimes dF_\upsilon$, then we have

$$\left|\left(\Im'\left(\pi_i\xi\otimes 1+(1-\pi_i)\,1\otimes\phi\right)\right|=\int_\Delta\int_\Delta\left|\left(\Im'\left(\pi_i\epsilon\right)+(1-\pi_i)\,\nu\right)\right|\,dE_\epsilon\otimes dF_\nu \\ \leqslant\|\Im'\|_{\Delta,+\infty}\int_\Delta\int_\Delta dE_\epsilon\otimes dF_\nu=\left\|\Im'\right\|_{\Delta,+\infty}, \tag{3.24}$$

from which we further get

$$\int_0^1\|((1-2\pi_i))\|\,\|\Im'\left((1-\pi_i)\,\xi\otimes 1+\pi_i 1\otimes\phi\|d\pi_i\right. \\ \leq\left\|\Im'\right\|_{\Delta,+\infty}\int_0^1\left\|\left((1-\pi_i)^\kappa-\pi_i^K\right)\right\|d\pi_i=\frac{1}{1+\kappa}\left(2-2^{1-\kappa}\right)\left\|\Im'\right\|_{\Delta,+\infty}. \tag{3.25}$$

Using Eqs (3.24) and (3.25) in (3.32), we get the needed result. □

Theorem 3.5. *Assume ξ and ϕ are two self-adjoint operators with $\mathcal{SP}(\xi) \subset \Delta$ and $\mathcal{SP}(\phi) \subset \Delta$. Let $\Im$ be differentiable as well as convex $\left|\Im'\right|$ on Δ. Then, the inequality stated below is true:*

$$\left\|\frac{\xi\otimes 1+1\otimes\phi}{2}-\frac{2(\phi\otimes 1-\xi\otimes 1)}{(\sqrt{\upsilon}-\sqrt{\epsilon})^2}\int_0^1\Im((1-\pi_i)\xi\otimes 1+\pi_i\phi\otimes 1))d\pi_i\right\| \\ \leq \frac{(\sqrt{1\otimes\phi}-\sqrt{\xi\otimes 1})^2}{4}\frac{\left(\left\|\Im'(\phi)\right\|+\left\|\Im'(\xi)\right\|\right)}{4}. \tag{3.26}$$

Proof. By assuming that $\left|\Im'\right|$ is convex on Δ, we have

$$\left|\Im'(\pi_i\epsilon)+(1-\pi_i)\upsilon)\right|\leq\pi_i\left|\Im'(\upsilon)\right|+(1-\pi_i)\left|\Im'(\epsilon)\right|$$

for all for $\pi_i\in[0,1]$ and $\epsilon,\upsilon\in\Delta$.

Taking $\int_\Delta\int_\Delta$ over $\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon$, we get

$$\begin{aligned}&\left|(\Im'\left(\pi_i\left(\xi\otimes 1\right)+\left(1-\pi_i\right)1\otimes\phi\right)\right|=\int_\Delta\int_\Delta\left|(\Im'\left(\pi_i\left(\epsilon\right)+\left(1-\pi_i\right)\upsilon\right)\right|\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon\\&\leq\left(1-\pi_i\right)\int_\Delta\int_\Delta\left|\Im'(\epsilon)\right|\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon+\pi_i\int_\Delta\int_\Delta\left|\Im'(\upsilon)\right|\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon,\end{aligned}\tag{3.27}$$

namely

$$\left|\Im'((1-\pi_i)\xi\otimes 1+\pi_i 1\otimes\phi)\right|\leq\left(1-\pi_i\right)\left|\Im'(\xi)\right|\otimes 1+\pi_i\left|\Im'(\phi)\right|\otimes 1\tag{3.28}$$

for all $\pi_i\in[0,1]$. If we take the norm in (3.28), then we get

$$\begin{aligned}\left\|\Im'((1-\pi_i)\xi\otimes 1+\pi_i 1\otimes\phi)\right\|&\leq\left\|(1-\pi_i)\left|\Im'(\xi)\right|\otimes 1+\pi_i\left|\Im'(\phi)\right|\otimes 1\right\|\\&\leq\left(1-\pi_i\right)\left\|\Im'(\xi)\right\|+\pi_i\left\|\Im'(\phi)\right\|.\end{aligned}\tag{3.29}$$

Again, using the triangle inequality and the operator norm of the previously derived Lemma 3.4, we obtain

$$\begin{aligned}&\left\|\frac{\xi\otimes 1+1\otimes\phi}{2}-\frac{2(\phi\otimes 1-\xi\otimes 1)}{(\sqrt{\upsilon}-\sqrt{\epsilon})^2}\int_0^1\Im((1-\pi_i)\xi\otimes 1+\pi_i\phi\otimes 1))\mathrm{d}\pi_i\right\|\\&\leq\frac{(\sqrt{1\otimes\phi}-\sqrt{\xi\otimes 1})^2}{4}\int_0^1\|((1-2\pi_i))\|\,\|\Im'\left((1-\pi_i)\,\xi\otimes 1+\pi_i 1\otimes\phi\right\|\mathrm{d}\pi_i\\&\leq\frac{(\sqrt{1\otimes\phi}-\sqrt{\xi\otimes 1})^2}{4}\int_0^1\|(1-2\pi_i)\|\left(\pi_i\left\|\Im'(\phi)\right\|+(1-\pi_i)\left\|\Im'(\xi)\right\|\right)\mathrm{d}\pi_i\\&=\frac{(\sqrt{1\otimes\phi}-\sqrt{\xi\otimes 1})^2}{4}\frac{\left(\left\|\Im'(\phi)\right\|+\left\|\Im'(\xi)\right\|\right)}{4}.\end{aligned}\tag{3.30}$$

□

Remark 3.2. *In Theorem 3.5, if tensorial arithmetic operations are degenerated, we obtain the Theorem 3.2 provided in [56].*

Theorem 3.6. *Assume ξ and ϕ are two self-adjoint operators with $\mathcal{SP}(\xi)\subset\Delta$ and $\mathcal{SP}(\phi)\subset\Delta$. Let $\Im$ be differentiable as well as quasi convex $\left|\Im'\right|$ on Δ. Then the inequality stated below is true:*

$$\begin{aligned}&\left\|\frac{\xi\otimes 1+1\otimes\phi}{2}-\frac{2(\phi\otimes 1-\xi\otimes 1)}{(\sqrt{\upsilon}-\sqrt{\epsilon})^2}\int_0^1\Im((1-\pi_i)\xi\otimes 1+\pi_i\phi\otimes 1))\mathrm{d}\pi_i\right\|\\&\leq\frac{(\sqrt{1\otimes\phi}-\sqrt{\xi\otimes 1})^2}{4}\frac{1}{1+\kappa}\left(2-2^{1-\kappa}\right)\frac{1}{2}\left(\left\|\left|\Im'(\xi)\right|\otimes 1+1\otimes\left|\Im'(\phi)\right|\right\|+\left\|\left|\Im'(\xi)\right|\otimes 1-1\otimes\left|\Im'(\phi)\right|\right\|\right).\end{aligned}\tag{3.31}$$

Proof. By assuming $\left|\Im'\right|$ is quasiconvex on Δ, then one has

$$\left|\Im'\left(\pi_i(\epsilon)+(1-\pi_i)\,\upsilon\right)\right|\leq\frac{1}{2}\left(\left|\Im'(\upsilon)\right|+\left|\Im'(\epsilon)\right|+\left|\left|\Im'(\upsilon)\right|-\left|\Im'(\epsilon)\right|\right|\right)$$

for all for $\pi_i\in[0,1]$ and $\epsilon,\upsilon\in\Delta$.

Taking $\int_\Delta\int_\Delta$ over $\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon$ yields

$$\begin{aligned}
&\left|(\Im'\left(\pi_i(\xi\otimes 1)+(1-\pi_i)\,1\otimes\phi\right)\right|\\
&=\int_\Delta\int_\Delta\left|(\Im'\left(\pi_i(\upsilon)+(1-\pi_i)\,\upsilon\right)\right|\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon\\
&\leq\frac{1}{2}\int_\Delta\int_\Delta\left(\left|\Im'(\upsilon)\right|+\left|\Im'(\epsilon)\right|+\left|\left|\Im'(\upsilon)\right|-\left|\Im'(\epsilon)\right|\right|\right)\mathrm{dE}_\epsilon\otimes\mathrm{dF}_\upsilon\\
&=\frac{1}{2}\left(\left|\Im'(\xi)\right|\otimes 1+1\otimes\left|\Im'(\phi)\right|+\left|\left|\Im'(\xi)\right|\otimes 1-1\otimes\left|\Im'(\phi)\right|\right|\right)
\end{aligned}$$

for all $\pi_i\in[0,1]$.

By using the norm of the inequality above, we get the following:

$$\begin{aligned}
&\left\|(\Im'\left(\pi_i(\xi\otimes 1)+(1-\pi_i)\,1\otimes\phi\right)\right\|\\
&\leq\left\|\frac{1}{2}\left(\left|\Im'(\xi)\right|\otimes 1+1\otimes\left|\Im'(\phi)\right|+\left|\left|\Im'(\xi)\right|\otimes 1-1\otimes\left|\Im'(\phi)\right|\right|\right)\right\|\\
&\leqslant\frac{1}{2}\left(\left\|\left|\Im'(\xi)\right|\otimes 1+1\otimes\left|\Im'(\phi)\right|\right\|+\left\|\left|\Im'(\xi)\right|\otimes 1-1\otimes\left|\Im'(\phi)\right|\right\|\right).
\end{aligned}$$

Using the triangle inequality and applying the norm of the previously derived Lemma 3.4, we may obtain

$$\begin{aligned}
&\left\|\frac{\xi\otimes 1+1\otimes\phi}{2}-\frac{2(\phi\otimes 1-\xi\otimes 1)}{(\sqrt{\upsilon}-\sqrt{\epsilon})^2}\int_0^1\Im((1-\pi_i)\xi\otimes 1+\pi_i\phi\otimes 1))\mathrm{d}\pi_i\right\|\\
&\leq\frac{(\sqrt{1\otimes\phi}-\sqrt{\xi\otimes 1})^2}{4}\int_0^1\|((1-2\pi_i))\|\,\|\Im'\left((1-\pi_i)\,\xi\otimes 1+\pi_i 1\otimes\phi\right\|\mathrm{d}\pi_i\\
&\leq\frac{(\sqrt{1\otimes\phi}-\sqrt{\xi\otimes 1})^2}{4}\int_0^1\|(1-2\pi_i)\|\left(\pi_i\left\|\Im'(\phi)\right\|+(1-\pi_i)\left\|\Im'(\xi)\right\|\right)\mathrm{d}\pi_i\\
&\leq\frac{(\sqrt{1\otimes\phi}-\sqrt{\xi\otimes 1})^2}{4}\int_0^1\|(1-2\pi_i)\|\left\|\frac{1}{2}\left(\left|\Im'(\xi)\right|\otimes 1+1\otimes\left|\Im'(\phi)\right|+\left\|\Im'(\xi)\right|\otimes 1-1\otimes\left|\Im'(\phi)\right|\right|\right)\right\|\\
&=\frac{(\sqrt{1\otimes\phi}-\sqrt{\xi\otimes 1})^2}{4}\frac{1}{1+\kappa}\left(2-2^{1-\kappa}\right)\frac{1}{2}\left(\left\|\left|\Im'(\xi)\right|\otimes 1+1\otimes\left|\Im'(\phi)\right|\right\|+\left\|\,\Im'(\xi)\right|\otimes 1-1\otimes\left|\Im'(\phi)\right\|\,|\right).
\end{aligned}\tag{3.32}$$

□

4. Examples and consequences

For an exponential function, if self-adjoint operators ξ and ϕ commute, we obtain

$$e^{\xi}e^{\phi}=e^{\phi}e^{\xi}=e^{(\xi+\phi)}.$$

Further, if ξ is invertible and $\epsilon,\upsilon\in\mathsf{R}$ with $\epsilon<\upsilon$, then

$$\int_{\epsilon}^{\upsilon}e^{\pi_i\xi}\mathrm{d}\pi_i=\frac{[e^{\upsilon\xi}-e^{\epsilon\xi}]}{\xi}.$$

Further, if $\phi-\xi$ is invertible, then we have

$$\begin{aligned}\int_0^1 e^{((1-\kappa)\xi+\kappa\phi)}\mathrm{d}\kappa&=\int_0^1 e^{(\kappa(\phi-\xi))}e^{\xi}\mathrm{d}\kappa=\left(\int_0^1 e^{(\kappa(\phi-\xi))}\mathrm{d}\kappa\right)e^{\xi}\\&=\frac{[e^{(\phi-\xi)}-\mathtt{I}]e^{\xi}}{\phi-\xi}=\frac{[e^{\phi}-e^{\xi}]}{\phi-\xi}.\end{aligned}$$

Corollary 4.1. *Assume ξ and ϕ are two self-adjoint operators with $\mathcal{SP}(\xi)\subset\Delta$ and $\mathcal{SP}(\phi)\subset\Delta$. Let $\mathfrak{I}$ be a differentiable and convex mapping on Δ, with $\kappa=\frac{1}{3}$. Then, by Theorem 3.1 we have*

$$\begin{aligned}&\Bigg\|\left(\frac{1}{8}(\mathfrak{I}(\xi)\otimes 1)+\frac{3}{8}\mathfrak{I}\left(\frac{2\xi\otimes 1+1\otimes\phi}{2}\right)+\frac{3}{8}\mathfrak{I}\left(\frac{\xi\otimes 1+2\otimes\phi}{2}\right)+\frac{1}{8}(1\otimes\mathfrak{I}(\phi))\right)\\&.-\left[\mathfrak{I}\left(\frac{\xi\otimes 1+1\otimes\phi}{2}\right)-\frac{\pi_i^{\frac{1}{3}}(\upsilon-\epsilon)}{4}\left[\int_0^1\mathfrak{I}'\left(\left(1-\frac{\pi_i}{2}\right)\xi\otimes 1+\left(\frac{\pi_i 1\otimes\phi}{2}\right)\right)\mathrm{d}\pi_i\right]\right.\\&\left.+\mathfrak{I}\left(\frac{\xi\otimes 1+1\otimes\phi}{2}\right)-\frac{(1-\pi_i)^{\frac{1}{3}}(\upsilon-\epsilon)}{4}\left[\int_0^1\mathfrak{I}'\left(\left(\frac{1-\pi_i}{2}\right)\xi\otimes 1+\left(\frac{1+\pi_i}{2}\right)1\otimes\phi\right)\mathrm{d}\pi_i\right]\right]\Bigg\|\\&\le\frac{\|1\otimes\phi-\xi\otimes 1\|}{4}\left(\frac{\left(\frac{4}{3}\right)\cdot 3^{\frac{1}{3}}+2^{\frac{7}{2}}}{(4)\cdot 3^{\frac{1}{3}}}\left(\left\|\mathfrak{I}'\right\|_{\Delta,+\infty}\right)+\frac{2^{\frac{1}{3}}+\left(\frac{7}{2}\right)\cdot 3^{\frac{1}{2}}}{(4)\cdot 3^{\frac{1}{3}}}\left(\left\|\mathfrak{I}'\right\|_{\Delta,+\infty}\right)\right).\end{aligned}$$

Corollary 4.2. *Assume ξ and ϕ are two self-adjoint operators with $\mathcal{SP}(\xi)\subset\Delta$ and $\mathcal{SP}(\phi)\subset\Delta$ with $\left\|\mathfrak{I}'\right\|_{\Delta,\infty}:=\sup_{\kappa\in\Delta}\left|\mathfrak{I}'(\kappa)\right|<\infty$. Then, by Theorem 3.4 we have*

$$\begin{aligned}&\left\|\frac{\xi\otimes 1+1\otimes\phi}{2}-\frac{2(\phi\otimes 1-\xi\otimes 1)}{(\sqrt{\upsilon}-\sqrt{\epsilon})^2}\int_0^1\exp((1-\pi_i)\xi\otimes 1+\pi_i\phi\otimes 1))\mathrm{d}\pi_i\right\|\\&\le\frac{(\sqrt{1\otimes\phi}-\sqrt{\xi\otimes 1})^2}{2}\frac{1}{3}\left(2-2^{\frac{1}{2}}\right)\left\|\exp'\right\|_{\Delta,\infty}.\end{aligned}\tag{4.1}$$

Corollary 4.3. *Assume ξ and ϕ are two self-adjoint operators with $\mathcal{SP}(\xi)\subset\Delta$ and $\mathcal{SP}(\phi)\subset\Delta$. Let $\mathfrak{I}$ be differentiable quasiconvex on Δ with $\kappa=\frac{1}{3}$. Then, by Theorem 3.3 we have*

$$\left\|\left(\frac{1}{8}(\ln(\xi)\otimes 1)+\frac{3}{8}\ln\left(\frac{2\xi\otimes 1+1\otimes\phi}{2}\right)+\frac{3}{8}\ln\left(\frac{\xi\otimes 1+2\otimes\phi}{2}\right)+\frac{1}{8}(1\otimes\ln(\phi))\right)\right.$$

$$
\begin{aligned}
&. - \left[\ln\left(\frac{\xi\otimes 1 + 1\otimes\phi}{2}\right) - \frac{\pi_i^{\kappa}(\upsilon-\epsilon)}{4}\left[\int_0^1 \ln'\left(\left(1-\frac{\pi_i}{2}\right)\xi\otimes 1 + \left(\frac{\pi_i 1\otimes\phi}{2}\right)\right) d\pi_i\right]\right. \\
&\left. + \ln\left(\frac{\xi\otimes 1 + 1\otimes\phi}{2}\right) - \frac{(1-\pi_i)^{\kappa}(\upsilon-\epsilon)}{4}\left[\int_0^1 \ln'\left(\left(\frac{1-\pi_i}{2}\right)\xi\otimes 1 + \left(\frac{1+\pi_i}{2}\right)1\otimes\phi\right) d\pi_i\right]\right\| \\
&\leq \frac{\|1\otimes\phi - \xi\otimes 1\|}{4}\left(\frac{3\cdot 3^{\frac{1}{3}}\ \frac{1}{9} + 8\cdot 2^{\frac{1}{3}}\ \frac{1}{3} + 8.2^{\frac{1}{3}} + 8\ \frac{1}{3}\cdot\ 3^{\frac{1}{3}} + 5\ \cdot\ 3^{\frac{1}{3}}}{3^{\frac{1}{3}}\left(\frac{4}{3}\right)\left(48\ \frac{1}{3} + 48\right)}\right) \\
&\times\left(\left\|\left|\ln'(\xi)\right|\otimes 1 + 1\otimes\left|\ln'(\phi)\right|\right\| + \left\|\left|\ln'(\xi)\right|\otimes 1 - 1\otimes\left|\ln'(\phi)\right|\right\|\right).
\end{aligned}
$$

5. Conclusions and future remarks

The tensor Hilbert space and its inequalities are important topics in mathematical and physics fields such as functional analysis and quantum mechanics. The first step in this note was to develop two important lemmas by using the Stone-Weierstrass theorem, which can be used to support our main findings. Our next step was to build different variations of the Simpson and Hermite-Hadamard inequalities using two different kinds of convex mappings. These results were obtained using spectral resolution of Hilbert spaces containing self-adjoint operators. Furthermore, we established upper bounds for these inequalities and provided further examples and consequences for transcendental functions using various types of extended convex mappings. This paper contributes to mathematical inequality theory by exploring inequalities supporting tensor Hilbert spaces, which is a rare topic in the literature. In the future, we will advise readers, motivated by these results, to try to develop results by using quantum fractional integral inequalities as well as fuzzy-valued mappings and fuzzy normed spaces instead of the standard norm.

Author contributions

W. Afzal: Conceptualization, data curation, writing-original draft, investigation, visualization; M. Abbas: Conceptualization, formal analysis, writing-original draft, supervision, validation, writing-review & editing; J. Ro: Investigation, project administration, visualization; Z. A. Khan: Data curation, formal analysis, methodology; N. A. Aloraini: Methodology, project administration, software. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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