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Research article

On nonlinear fractional Hahn integrodifference equations via nonlocal fractional Hahn integral boundary conditions

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Abstract: All authors Fractional Hahn differences and fractional Hahn integrals have various applications in fields where discrete fractional calculus plays a significant role, such as in discrete biological modeling and signal processing to handle systems with memory effects. In this study, the existence and uniqueness of solutions for a Riemann-Liouville fractional Hahn integrodifference equation with nonlocal fractional Hahn integral boundary conditions are investigated. To establish these results, we apply the Banach and Schauder fixed-point theorems. Furthermore, the Hyers-Ulam stability of solutions is studied.

Keywords: fractional Hahn integral; Riemann-Liouville fractional Hahn difference; boundary value problems; existence; Hyers-Ulam stability
Mathematics Subject Classification: 34K10, 39A10, 39A11, 39A13, 39A70

1. Introduction

Quantum calculus, which is a form of calculus without the traditional concept of limits, deals with a set of non-differentiable functions. Quantum operators are widely utilized in various mathematical fields, including hypergeometric series, complex analysis, orthogonal polynomials, combinatorics, hypergeometric functions, and the calculus of variations. Quantum calculus also has numerous applications in areas such as quantum mechanics and particle physics [1–4].

In 1949, W. Hahn introduced the Hahn difference operator [5]. This operator is a combination of two well-known operators: the forward difference operator and the Jackson q-difference operator. The Hahn difference operator is defined by

$$D_{q,\omega}f(t) = \frac{f(qt+\omega) - f(t)}{t(q-1)+\omega}, \quad t \neq \omega_0,$$

and $D_{q,\omega}f(\omega_0) = f'(\omega_0)$ where $\omega_0 := \frac{\omega}{1-q}$. We note that

$$D_{q,\omega}f(t) = \Delta_\omega f(t) \text{ whenever } q = 1, \quad D_{q,\omega}f(t) = D_q f(t) \text{ whenever } \omega = 0,$$

$$\text{and} \quad D_{q,\omega}f(t) = f'(t) \text{ whenever } q = 1, \omega \to 0.$$

The Hahn difference operator has been utilized in the study of families of orthogonal polynomials and in solving certain approximation problems (see [6–8]).

The right inverse of the Hahn difference operator was introduced by Aldwoah in 2009 [9,10]. This operator is expressed in terms of the Jackson q-integral, which contains the right inverse of D_q [11], and the Nörlund, which involves the right inverse of Δ_ω [11].

In 2010, Malinowska and Torres [12, 13] introduced the Hahn quantum variational calculus. In 2013, Malinowska and Martins [14] extended this work by presenting generalized transversality conditions for the Hahn quantum variational calculus. Subsequently, Hamza and Ahmed [15–17] developed the theory of linear Hahn difference equations and a general quantum difference calculus, studied the existence and uniqueness of solutions for initial value problems using the method of successive approximations, and proved Gronwall's and Bernoulli's inequalities in the context of the Hahn difference operator. They also investigated mean value theorems for this calculus. In 2016, Hamza and Makharesh [18] explored the Leibniz rule and Fubini's theorem associated with the Hahn difference operator. That same year, Sitthiwirattham [19] studied nonlocal boundary value problems (BVPs) for nonlinear Hahn difference equations. In 2021, Mac Quarrie et al. [20] proposed the Asymptotic Iteration Method for solving Hahn difference equations. In 2023, Hıra [21] focused on defining and proving fundamental properties of the Hahn Laplace Transform, including linearity, shifting theorems, and convolution theorems.

In 2010, Čermák and Nechvátal [22] introduced the fractional (q, h)-difference operator and the fractional (q, h)-integral for $q > 1$. In 2011, Čermák, Kisela, and Nechvátal [23] presented linear fractional difference equations with discrete Mittag-Leffler functions for $q > 1$. Rahmat [24,25] studied the (q, h)-Laplace transform and some (q, h)-analogues of integral inequalities on discrete time scales for $q > 1$. In 2016, Du, Jai, Erbe, and Peterson [26] presented the monotonicity and convexity for nabla fractional (q, h)-difference for $q > 0$, $q \neq 1$. Since fractional Hahn operators require a fractional parameter $0 < q < 1$, the operators previously mentioned are not considered fractional Hahn operators. The fractional Hahn operators have been studied by Brikshavana and Sitthiwirattham [27]. There are research papers that focus on the BVPs for Hahn difference equations, such as [28–32].

Building on the foundation of quantum calculus, the study of fractional Hahn calculus has gained increasing attention due to its ability to generalize classical difference and integral operators. The fractional Hahn difference and integral operators provide a powerful framework for capturing memory effects and non-local behaviors, which are essential in modeling complex systems. In particular, the exploration of boundary value problems within this framework allows for a deeper understanding of dynamic systems governed by fractional discrete processes.

This research focuses on investigating boundary value problems involving fractional Hahn operators, aiming to extend the applicability of fractional Hahn calculus to broader mathematical and physical contexts. Such problems not only enrich the theoretical development of fractional discrete calculus but also pave the way for applications in fields like control theory, population dynamics, and numerical simulations. Specifically, we focus on a nonlocal Riemann-Liouville fractional Hahn integrodifference BVP of the form

$$\begin{aligned} &D^{\alpha}_{q,\omega}u(t) = \lambda F\left[t, u(t), \left(\Psi^{\gamma}_{q,\omega}u\right)(t)\right] + \mu H\left[t, u(t), \left(\Upsilon^{\nu}_{q,\omega}u\right)(t)\right], \quad t \in I^{T}_{q,\omega}, \\ &\mathcal{I}^{\beta}_{q,\omega}g_1(\eta)u(\eta) = \phi_1(u), \quad \eta \in I^{T}_{q,\omega} - \{\omega_0, T\}, \\ &\mathcal{I}^{\beta}_{q,\omega}g_2(T)\,u(T) = \phi_2(u), \end{aligned} \tag{1.1}$$

where $[\omega_0, T]_{q,\omega} := \{q^k T + \omega[k]_q : k \in \mathbb{N}_0\} \cup \{\omega_0\}$; $0 < q < 1, \omega > 0$; $\alpha \in (1, 2]$; $\beta, \gamma, \nu \in (0, 1]$; $\lambda, \mu \in \mathbb{R}^+$; $F, H \in C([\omega_0, T]_{q,\omega} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $g_1, g_2 \in C([\omega_0, T]_{q,\omega}, \mathbb{R}^+)$ are given functions; $\phi_1, \phi_2 : C([\omega_0, T]_{q,\omega}, \mathbb{R}) \to \mathbb{R}$ are given functionals, and for $\varphi, \psi \in C([\omega_0, T]_{q,\omega} \times [\omega_0, T]_{q,\omega}, [0, \infty))$, we define operators

$$\begin{aligned} \left(\Psi^{\gamma}_{q,\omega}u\right)(t) &:= \left(\mathcal{I}^{\gamma}_{q,\omega}\varphi\, u\right)(t) = \frac{1}{\Gamma_q(\gamma)} \int_{\omega_0}^{t} \left(t - \sigma_{q,\omega}(s)\right)^{\underline{\gamma-1}}_{q,\omega} \varphi(t, s)\, u(s)\, d_{q,\omega}s, \\ \left(\Upsilon^{\nu}_{q,\omega}u\right)(t) &:= \left(D^{\nu}_{q,\omega}\psi\, u\right)(t) = \frac{1}{\Gamma_q(-\nu)} \int_{\omega_0}^{t} \left(t - \sigma_{q,\omega}(s)\right)^{\underline{-\nu-1}}_{q,\omega} \psi(t, s)\, u(s)\, d_{q,\omega}s. \end{aligned} \tag{1.2}$$

Section 2 lays the groundwork by presenting fundamental definitions, properties, and lemmas. In Sections 3 and 4, we delve into the existence analysis and stability analysis of problem (1.1). We employ the powerful Banach fixed-point theorem to prove the existence and uniqueness of solutions, and we use the Schauder fixed-point theorem to establish the existence of at least one solution. To concretize our findings, Section 5 offers illustrative examples.

2. Preliminaries

We establish necessary notation, definitions, and lemmas for the subsequent theorems. Let $q \in (0, 1)$, $\omega > 0$ and define

$$[n]_q := \frac{1 - q^n}{1 - q} = q^{n-1} + \dots + q + 1 \qquad \text{and} \qquad [n]_q! := \prod_{k=1}^{n} \frac{1 - q^k}{1 - q}, \qquad n \in \mathbb{R}.$$

The q-analogue of the power function $(a - b)^{\underline{n}}_{q}$ with $n \in \mathbb{N}_0 := [0, 1, 2, ...]$ is defined by

$$(a - b)^{\underline{0}}_{q} := 1, \qquad (a - b)^{\underline{n}}_{q} := \prod_{k=0}^{n-1} (a - bq^k), \qquad a, b \in \mathbb{R}.$$

The q, ω-analogue of the power function $(a - b)^{\underline{n}}_{q,\omega}$ with $n \in \mathbb{N}_0 := [0, 1, 2, ...]$ is defined by

$$(a - b)^{\underline{0}}_{q,\omega} := 1, \qquad (a - b)^{\underline{n}}_{q,\omega} := \prod_{k=0}^{n-1} \left[a - (bq^k + \omega[k]_q)\right], \qquad a, b \in \mathbb{N}.$$

In general, for $\alpha \in \mathbb{R}$, we define

$$(a-b)_q^{\underline{\alpha}} = a^\alpha \prod_{n=0}^{\infty} \frac{1-\left(\frac{b}{a}\right)q^n}{1-\left(\frac{b}{a}\right)q^{\alpha+n}}, \ a \neq 0,$$

$$(a-b)_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^\alpha \prod_{n=0}^{\infty} \frac{1-\left(\frac{b-\omega_0}{a-\omega_0}\right)q^n}{1-\left(\frac{b-\omega_0}{a-\omega_0}\right)q^{\alpha+n}} = \Big((a-\omega_0)-(b-\omega_0)\Big)_q^{\underline{\alpha}}, \ a \neq \omega_0.$$

We note that, $a_q^{\underline{\alpha}} = a^\alpha$ and $(a-\omega_0)_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^\alpha$ and use the notation $(0)_q^{\underline{\alpha}} = (\omega_0)_{q,\omega}^{\underline{\alpha}} = 0$ for $\alpha > 0$. The q-gamma and q-beta functions are defined by

$$\begin{aligned} \Gamma_q(x) &:= \frac{(1-q)_q^{\underline{x-1}}}{(1-q)^{x-1}}, \qquad x \in \mathbb{R} \setminus \{0,-1,-2,...\}, \\ B_q(x,s) &:= \int_0^1 t^{x-1}(1-qt)_q^{\underline{s-1}} d_q t = \frac{\Gamma_q(x)\Gamma_q(s)}{\Gamma_q(x+s)}. \end{aligned}$$

Definition 2.1. *For $q \in (0,1)$, $\omega > 0$ and f defined on an interval $I \subseteq \mathbb{R}$ that contains $\omega_0 := \frac{\omega}{1-q}$, the Hahn difference of f is defined by*

$$D_{q,\omega}f(t) = \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega} \quad \text{for } t \neq \omega_0,$$

and $D_{q,\omega}f(\omega_0) = f'(\omega_0)$. Providing that f is differentiable at ω_0, we call $D_{q,\omega}f$ the q,ω-derivative of f and say that f is q,ω-differentiable on I.

Remark 2.1. We give some properties for the Hahn difference as follows.

(1) $D_{q,\omega}[f(t)+g(t)] = D_{q,\omega}f(t) + D_{q,\omega}g(t)$,
(2) $D_{q,\omega}[\alpha f(t)] = \alpha D_{q,\omega}f(t)$,
(3) $D_{q,\omega}[f(t)g(t)] = f(t)D_{q,\omega}g(t) + g(qt+\omega)D_{q,\omega}f(t)$,
(4) $D_{q,\omega}\left[\frac{f(t)}{g(t)}\right] = \frac{g(t)D_{q,\omega}f(t) - f(t)D_{q,\omega}g(t)}{g(t)g(qt+\omega)}$.

Letting $a,b \in I \subseteq \mathbb{R}$ with $a < \omega_0 < b$ and $[k]_q = \frac{1-q^k}{1-q}$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we define the q,ω-interval by

$$\begin{aligned} I_{q,\omega}^{a,b} = [a,b]_{q,\omega} &:= \left\{q^k a + \omega[k]_q : k \in \mathbb{N}_0\right\} \cup \left\{q^k b + \omega[k]_q : k \in \mathbb{N}_0\right\} \cup \{\omega_0\} \\ &= [a,\omega_0]_{q,\omega} \cup [\omega_0, b]_{q,\omega} \\ &= (a,b)_{q,\omega} \cup \{a,b\} = [a,b)_{q,\omega} \cup \{b\} = (a,b]_{q,\omega} \cup \{a\}, \\ \text{and} \qquad I_{q,\omega}^T &:= I_{q,\omega}^{\omega_0,T} = [\omega_0, T]_{q,\omega}. \end{aligned}$$

Observe that for each $s \in [a,b]_{q,\omega}$, the sequence $\{\sigma_{q,\omega}^k(s)\}_{k=0}^\infty = \{q^k s + \omega[k]_q\}_{k=0}^\infty$ is uniformly convergent to ω_0.

We also define the forward jump operator as $\sigma_{q,\omega}^k(t) := q^k t + \omega[k]_q$ and the backward jump operator as $\rho_{q,\omega}^k(t) := \frac{t-\omega[k]_q}{q^k}$ for $k \in \mathbb{N}$.

Definition 2.2. *Let I be any closed interval of $\mathbb{R}$ that contains a, b and ω_0. Assuming that $f : I \to \mathbb{R}$ is a given function, we define the q,ω-integral of f from a to b by*

$$\int_a^b f(t)d_{q,\omega}t := \int_{\omega_0}^b f(t)d_{q,\omega}t - \int_{\omega_0}^a f(t)d_{q,\omega}t,$$

where

$$\int_{\omega_0}^x f(t)d_{q,\omega}t := [x(1-q)-\omega]\sum_{k=0}^{\infty} q^k f\left(xq^k + \omega[k]_q\right), \quad x \in I.$$

Providing that the series converges at $x = a$ and $x = b$, we call f is q,ω-integrable on $[a, b]$ and the sum to the right hand side of above equation will be called the Jackson-Nörlund sum.

We note that, the actual domain of function f defined on $[a, b]_{q,\omega} \subset I$.

The following lemma presents the fundamental theorem of Hahn calculus.

Lemma 2.1. *[9] Let $f : I \to \mathbb{R}$ be continuous at ω_0. Define*

$$F(x) := \int_{\omega_0}^x f(t)d_{q,\omega}t, \quad x \in I.$$

Then, F is continuous at ω_0. Furthermore, $D_{q,\omega}F(x)$ exists for every $x \in I$ and

$$D_{q,\omega}F(x) = f(x).$$

Conversely, we have

$$\int_a^b D_{q,\omega}F(t)d_{q,\omega}t = F(b) - F(a) \text{ for all } a, b \in I.$$

Lemma 2.2. *[19] Let $q \in (0, 1)$, $\omega > 0$, and $f : I \to \mathbb{R}$ be continuous at ω_0. Then,*

$$\int_{\omega_0}^t \int_{\omega_0}^r f(s)\, d_{q,\omega}s\, d_{q,\omega}r = \int_{\omega_0}^t \int_{qs+\omega}^t f(s)\, d_{q,\omega}r\, d_{q,\omega}s.$$

Lemma 2.3. *[19] Let $q \in (0, 1)$, and $\omega > 0$. Then,*

$$\int_{\omega_0}^t d_{q,\omega}s = t - \omega_0 \quad \text{and} \quad \int_{\omega_0}^t [t - \sigma_{q,\omega}(s)]\, d_{q,\omega}s = \frac{(t-\omega_0)^2}{1+q}.$$

Next, we present the definitions of the fractional Hahn integral and the Riemann-Liouville-type fractional Hahn difference.

Definition 2.3. *For $\alpha, \omega > 0$, $q \in (0, 1)$ and f defined on $[\omega_0, T]_{q,\omega}$, the fractional Hahn integral is defined by*

$$\begin{aligned} \mathcal{I}^{\alpha}_{q,\omega}f(t) &:= \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^t (t-\sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega} f(s)d_{q,\omega}s \\ &= \frac{[t(1-q)-\omega]}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n \left(t - \sigma^{n+1}_{q,\omega}(t)\right)^{\underline{\alpha-1}}_{q,\omega} f\left(\sigma^n_{q,\omega}(t)\right), \end{aligned}$$

and $(\mathcal{I}^0_{q,\omega}f)(t) = f(t)$.

Definition 2.4. *For $\alpha,\omega > 0$, $q \in (0,1)$, $N-1 < \alpha < N, N \in \mathbb{N}$, and f defined on $[\omega_0, T]_{q,\omega}$, the fractional Hahn difference of the Riemann-Liouville type of order α is defined by*

$$\begin{aligned} D^{\alpha}_{q,\omega}f(t) &:= (D^{N}_{q,\omega}\mathcal{I}^{N-\alpha}_{q,\omega}f)(t) \\ &= \frac{1}{\Gamma_q(-\alpha)}\int_{\omega_0}^{t}\left(t-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{-\alpha-1}}f(s)d_{q,\omega}s. \end{aligned}$$

The fractional Hahn difference of the Caputo type of order α is defined by

$$\begin{aligned} {}^{C}D^{\alpha}_{q,\omega}f(t) &:= (\mathcal{I}^{N-\alpha}_{q,\omega}D^{N}_{q,\omega}f)(t) \\ &= \frac{1}{\Gamma_q(N-\alpha)}\int_{\omega_0}^{t}\left(t-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{N-\alpha-1}}D^{N}_{q,\omega}f(s)d_{q,\omega}s, \end{aligned}$$

and $D^{0}_{q,\omega}f(t) = {}^{C}D^{0}_{q,\omega}f(t) = f(t)$.

Lemma 2.4. *[27] Let $\alpha > 0, q \in (0,1), \omega > 0$ and $f : I^{T}_{q,\omega} \to \mathbb{R}$. Then,*

$$\mathcal{I}^{\alpha}_{q,\omega}D^{\alpha}_{q,\omega}f(t) = f(t) + C_1(t-\omega_0)^{\alpha-1} + ... + C_N(t-\omega_0)^{\alpha-N},$$

for some $C_i \in \mathbb{R}, i = \mathbb{N}_{1,N}$ and $N-1 < \alpha \leq N, N \in \mathbb{N}$.

Lemma 2.5. *[27] Let $\alpha > 0, q \in (0,1), \omega > 0$, and $f : I^{T}_{q,\omega} \to \mathbb{R}$. Then,*

$$\mathcal{I}^{\alpha}_{q,\omega}\,{}^{C}D^{\alpha}_{q,\omega}f(t) = f(t) + C_0 + C_1(t-\omega_0) + ... + C_{N-1}(t-\omega_0)^{N-1},$$

for some $C_i \in \mathbb{R}, i = \mathbb{N}_{0,N-1}$ and $N-1 < \alpha \leq N, N \in \mathbb{N}$.

For computational efficiency, we offer these auxiliary results.

Lemma 2.6. *[27] Let $\alpha,\beta > 0$, $p,q \in (0,1)$, and $\omega > 0$. Then,*

$$\begin{aligned} \int_{\omega_0}^{t}\left(t-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}}(s-\omega_0)_{q,\omega}^{\underline{\beta}}\,d_{q,\omega}s &= (t-\omega_0)^{\alpha+\beta}B_q(\beta+1,\alpha), \\ \int_{\omega_0}^{t}\int_{\omega_0}^{x}\left(t-\sigma_{p,\omega}(x)\right)_{p,\omega}^{\underline{\alpha-1}}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\beta-1}}d_{q,\omega}s\,d_{p,\omega}x &= \frac{(t-\omega_0)^{\alpha+\beta}}{[\beta]_q}B_p(\beta+1,\alpha). \end{aligned}$$

To establish a foundation for the analysis of problem (1.1), we present a lemma addressing its linear variant and providing a corresponding solution representation.

Lemma 2.7. *Let $\Omega \neq 0$, $\alpha \in (1,2]$, $\beta \in (0,1]$, $\omega > 0$, $p,q \in (0,1)$, $p = q^m$, $m \in \mathbb{N}$, $\theta = \omega\left(\frac{1-p}{1-q}\right)$, and $h \in C([\omega_0,T]_{q,\omega},\mathbb{R})$ be given function. Then the problem*

$$\begin{aligned} &D^{\alpha}_{q,\omega}u(t) = h(t), \\ &\mathcal{I}^{\beta}_{q,\omega}g_1(\eta)u(\eta) = \phi_1(u), \quad \eta \in [\omega_0,T]_{q,\omega} - \{\omega_0, T\}, \\ &\mathcal{I}^{\beta}_{q,\omega}g_2(T)\,u(T) = \phi_2(u), \end{aligned} \tag{2.1}$$

has the unique solution

$$u(t) = \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{t}\left[t-\sigma_{q,\omega}(s)\right]_{q,\omega}^{\underline{\alpha-1}}h(s)d_{q,\omega}s + \frac{(t-\omega_0)^{\alpha-1}}{\Lambda}\left[\mathcal{B}_{\eta}O_T - \mathcal{B}_T O_{\eta}\right] \tag{2.2}$$

$$+\frac{(t-\omega_0)^{\alpha-2}}{\Lambda}\Big[\mathcal{A}_T O_\eta - \mathcal{A}_\eta O_T\Big],$$

where the functionals and the constants are defined by

$$\Lambda := \mathcal{A}_T\mathcal{B}_\eta - \mathcal{A}_\eta\mathcal{B}_T, \tag{2.3}$$

$$\mathcal{A}_\eta := \frac{1}{\Gamma_q(\beta)}\int_{\omega_0}^{\eta} g_1(s)(\eta-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\beta-1}}(s-\omega_0)^{\alpha-1}d_{q,\omega}s, \tag{2.4}$$

$$\mathcal{B}_\eta := \frac{1}{\Gamma_q(\beta)}\int_{\omega_0}^{\eta} g_1(s)(\eta-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\beta-1}}(s-\omega_0)^{\alpha-2}d_{q,\omega}s, \tag{2.5}$$

$$\mathcal{A}_T := \frac{1}{\Gamma_q(\beta)}\int_{\omega_0}^{T} g_2(s)(T-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\beta-1}}(s-\omega_0)^{\alpha-1}d_{q,\omega}s, \tag{2.6}$$

$$\mathcal{B}_T := \frac{1}{\Gamma_q(\beta)}\int_{\omega_0}^{T} g_2(s)(T-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\beta-1}}(s-\omega_0)^{\alpha-2}d_{q,\omega}s, \tag{2.7}$$

$$O_\eta[\phi_1,h] := \phi_1(u(\eta)) - \frac{1}{\Gamma_q(\beta)}\frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{\eta}\int_{\omega_0}^{x} g_1(x)(\eta-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}}[x-\sigma_{q,\omega}(s)]_{q,\omega}^{\underline{\alpha-1}}\times h(s)d_{q,\omega}s\,d_{q,\omega}x, \tag{2.8}$$

$$O_T[\phi_2,h] := \phi_2(u(T)) - \frac{1}{\Gamma_q(\beta)}\frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{T}\int_{\omega_0}^{x} g_2(x)(T-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}}[x-\sigma_{q,\omega}(s)]_{q,\omega}^{\underline{\alpha-1}}\times h(s)d_{q,\omega}s\,d_{q,\omega}x. \tag{2.9}$$

Proof. Taking the fractional Hahn integral of order α for (2.1), we obtain

$$\begin{aligned} u(t) &= \mathcal{I}_{q,\omega}^{\alpha}h(t) + C_1(t-\omega_0)^{\alpha-1} + C_2(t-\omega_0)^{\alpha-2} \qquad (2.10)\\ &= \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{t}\Big(t-\sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\alpha-1}} h(s)\,d_{q,\omega}s + C_1(t-\omega_0)^{\alpha-1} + C_2(t-\omega_0)^{\alpha-2}. \end{aligned}$$

Multiplying (2.10) by $g_1(t)$ and taking the fractional Hahn difference of order β, we obtain

$$\begin{aligned} \mathcal{I}_{q,\omega}^{\beta}g_1(t)u(t) &= \frac{1}{\Gamma_q(\beta)\Gamma_q(\alpha)}\int_{\omega_0}^{t}\int_{\omega_0}^{x} g_1(x)(t-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}}[x-\sigma_{q,\omega}(s)]_{q,\omega}^{\underline{\alpha-1}}h(s)d_{q,\omega}s\,d_{q,\omega}x\\ &\quad+\frac{1}{\Gamma_q(\beta)}\int_{\omega_0}^{t} g_1(s)(t-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\beta-1}}[C_1(s-\omega_0)^{\alpha-1}+C_2(s-\omega_0)^{\alpha-2}]d_{q,\omega}s. \qquad (2.11) \end{aligned}$$

Multiplying (2.10) by $g_2(t)$ and taking fractional Hahn difference of order β, we obtain

$$\begin{aligned} \mathcal{I}_{q,\omega}^{\beta}g_2(t)u(t) &= \frac{1}{\Gamma_q(\beta)\Gamma_q(\alpha)}\int_{\omega_0}^{t}\int_{\omega_0}^{x} g_2(x)(t-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}}[x-\sigma_{q,\omega}(s)]_{q,\omega}^{\underline{\alpha-1}}h(s)d_{q,\omega}s\,d_{q,\omega}x\\ &\quad+\frac{1}{\Gamma_q(\beta)}\int_{\omega_0}^{t} g_2(s)(t-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\beta-1}}[C_1(s-\omega_0)^{\alpha-1}+C_2(s-\omega_0)^{\alpha-2}]d_{q,\omega}s. \qquad (2.12) \end{aligned}$$

Substituting $t=\eta$ into (2.11) and using the first condition of (2.1), we have

$$\Big[\frac{1}{\Gamma_q(\beta)}\int_{\omega_0}^{\eta} g_1(s)(\eta-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\beta-1}}(s-\omega_0)^{\alpha-1}d_{q,\omega}s\Big]C_1$$

$$
\begin{aligned}
&+\Big[\frac{1}{\Gamma_q(\beta)}\int_{\omega_0}^{\eta} g_1(s)(\eta-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\beta-1}}(s-\omega_0)^{\alpha-2}d_{q,\omega}s\Big]C_2 \qquad (2.13)\\
=\;& \mathcal{A}_\eta C_1+\mathcal{B}_\eta C_2 \;=\; O_\eta[\phi_1,h].
\end{aligned}
$$

Substituting $t = T$ into (2.12) and using the second condition of (2.1), we have

$$
\begin{aligned}
&\Big[\frac{1}{\Gamma_q(\beta)}\int_{\omega_0}^{T} g_2(s)(T-\sigma_{q,\omega}(s))_{q\omega}^{\underline{\beta-1}}(s-\omega_0)^{\alpha-1}d_{q,\omega}s\Big]C_1\\
&+\Big[\frac{1}{\Gamma_q(\beta)}\int_{\omega_0}^{T} g_2(s)(T-\sigma_{q,\omega}(s))_{q\omega}^{\underline{\beta-1}}(s-\omega_0)^{\alpha-2}d_{q,\omega}s\Big]C_2\\
=\;& \mathcal{A}_T C_1+\mathcal{B}_T C_2 \;=\; O_T[\phi_2,h].
\end{aligned}
$$

To find C_1 and C_2, we solve the system of Eqs (2.10) and (2.12). Then, we obtain

$$
\begin{aligned}
C_1 &= \frac{\mathcal{B}_\eta O_T-\mathcal{B}_T O_\eta}{\Lambda},\\
C_2 &= \frac{\mathcal{A}_T O_\eta-\mathcal{A}_\eta O_T}{\Lambda},
\end{aligned}
$$

where $\Lambda, \mathcal{A}_\eta, \mathcal{A}_T, \mathcal{B}_\eta, \mathcal{B}_T, O_\eta, O_T$ are defined as (2.3) – (2.9), respectively.

Substituting the constants C_1, C_2 into (2.10), we obtain the solution for 2.1, as shown in Eq (2.2). □

To prove the existence of a solution to Eq (1.1), we will employ the well-known Schauder's fixed point theorem.

Lemma 2.8. *[33] (Arzelá-Ascoli theorem) A set of functions in $C[a, b]$ with the sup norm, is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a, b]$.*

Lemma 2.9. *[33] If a set is closed and relatively compact, then it is compact.*

Lemma 2.10. *[34] (Schauder's fixed point theorem) Let (D, d) be a complete metric space, U be a closed convex subset of D, and $T : D \to D$ be the map such that the set $Tu : u \in U$ is relatively compact in D. Then the operator T has at least one fixed point $u^* \in U$: $Tu^* = u^*$.*

3. Existence and uniqueness results

In this section, we prove the existence results for problem (1.1). Let $C = C\left(I_{q,\omega}^T, \mathbb{R}\right)$ be a Banach space of all functions u with the norm defined by

$$
\|u\|_C = \|u\| + \|\mathcal{D}_{q,\omega}^{\nu} u\|,
$$

where $\|u\| = \max_{t\in I_{q,\omega}^T}\{|u(t)|\}$ and $\|\mathcal{D}_{q,\omega}^{\nu}u\| = \max_{t\in I_{q,\omega}^T}\{|\mathcal{D}_{q,\omega}^{\nu}u(t)|\}$.

Lemma 3.1. *$(C, \|\cdot\|_C)$ is Banach space.*

Proof. Let $\{u_n\}_{n=1}^{\infty}$ be any Cauchy sequence in the space $(C, \|\cdot\|_C)$. Then $\forall\varepsilon > 0$, there exists $N > 0$ such that

$$
\|u_n - u_m\|_C = \|u_n - u_m\| + \|\mathcal{D}_{q,\omega}^{\nu}u_n - \mathcal{D}_{q,\omega}^{\nu}u_m\| < \varepsilon,
$$

for $n, m > N$. Therefore, for any fixed $t_0 \in I^T_{q,\omega}$, the sequence $\{u_n(t_0)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. In this way, the unique $u(t)$ can be associated for each $t \in I^T_{q,\omega}$. This defines (pointwise) a function u on $I^T_{q,\omega}$. And can be shown $u \in C$ and $u_m \to u$ with $\|u_n - u_m\| + \|\mathcal{D}^{\nu}_{q,\omega}u_n - \mathcal{D}^{\nu}_{q,\omega}u_m\| < \infty$. Letting $n \to \infty$, for every $t \in I^T_{q,\omega}$, the following inequality holds.

$$|u(t) - u_m(t)| \leq \varepsilon \text{ for all } m > N.$$

This means that $u_m(t)$ converges to $u(t)$ uniformly on $I^T_{q,\omega}$. Since the u_m are continuous on $I^T_{q,\omega}$ and the convergence is uniform, the limit function u is continuous on $I^T_{q,\omega}$. Hence $u \in C$ and $u_m \to u$. Next, that $\|u_n - u_m\| + \|\mathcal{D}^{\nu}_{q,\omega}u_n - \mathcal{D}^{\nu}_{q,\omega}u_m\| < \varepsilon$ will be proven. Consider for $t \in I^T_{q,\omega}$

$$\begin{aligned} |u| + |\mathcal{D}^{\nu}_{q,\omega}u| &= |u(t) - u_m(t) + u_m(t)| + |\mathcal{D}^{\nu}_{q,\omega}u - \mathcal{D}^{\nu}_{q,\omega}u_m + \mathcal{D}^{\nu}_{q,\omega}u_m| \\ &\leq |u(t) - u_m(t)| + |u_m(t)| + |\mathcal{D}^{\nu}_{q,\omega}u - \mathcal{D}^{\nu}_{q,\omega}u_m| + |\mathcal{D}^{\nu}_{q,\omega}u_m| \\ &< \varepsilon + \varepsilon. \end{aligned}$$

This implies

$$\|u\| + \|\mathcal{D}^{\nu}_{q,\omega}u\| < \infty.$$

Hence, $(C, \|\cdot\|_C)$ is a Banach space. □

By Lemma 2.7, replacing $h(t)$ by $\lambda F\left[t, u(t), \left(\Psi^{\gamma}_{q,\omega}u\right)(t)\right] + \mu H\left[t, u(t), \left(\Upsilon^{\nu}_{q,\omega}u\right)(t)\right]$, we define an operator $\mathcal{A} : C \to C$ by

$$\begin{aligned} (\mathcal{A}u)(t) := &\frac{1}{\Gamma_q(\alpha)} \int_{\omega_0}^{t} \left[t - \sigma_{q,\omega}(s)\right]_{q\omega}^{\underline{\alpha-1}} \Big\{\lambda F\left[s, u(s), \left(\Psi^{\gamma}_{q,\omega}u\right)(s)\right] + \mu H\left[s, u(s), \left(\Upsilon^{\nu}_{q,\omega}u\right)(s)\right]\Big\} d_{q,\omega}s \\ &- \frac{(t-\omega_0)^{\alpha-1}}{\Lambda}\Big\{\mathcal{B}_T O^*_{\eta}(\phi_1, F_u + H_u) - \mathcal{B}_{\eta} O^*_T(\phi_2, F_u + H_u)\Big\} \\ &+ \frac{(t-\omega_0)^{\alpha-2}}{\Lambda}\Big\{\mathcal{A}_T O^*_{\eta}(\phi_1, F_u + H_u) - \mathcal{A}_{\eta} O^*_T(\phi_2, F_u + H_u)\Big\}, \end{aligned} \tag{3.1}$$

where $\Lambda, \mathcal{A}_{\eta}, \mathcal{B}_{\eta}, \mathcal{A}_T$, and $\mathcal{B}_T$ are defined in (2.3)–(2.7), respectively, and the functionals $O^*_{\eta}[\phi_1, F_u + H_u]$, $O^*_T[\phi_2, F_u + H_u]$ are defined by

$$\begin{aligned} O^*_{\eta}[\phi_1, F_u + H_u] := &\ \phi_1(u(\eta)) - \frac{1}{\Gamma_q(\beta)\Gamma_q(\alpha)} \int_{\omega_0}^{\eta}\int_{\omega_0}^{x} g_1(x)(\eta - \sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}}[x - \sigma_{q,\omega}(s)]_{q,\omega}^{\underline{\alpha-1}} \times \\ &\Big\{\lambda F\left[s, u(s), \left(\Psi^{\gamma}_{q,\omega}u\right)(s)\right] + \mu H\left[s, u(s), \left(\Upsilon^{\nu}_{q,\omega}u\right)(s)\right]\Big\} d_{q,\omega}s\, d_{q,\omega}x, \end{aligned} \tag{3.2}$$

$$\begin{aligned} O^*_T[\phi_2, F_u + H_u] := &\ \phi_2(u(T)) - \frac{1}{\Gamma_q(\beta)\Gamma_q(\alpha)} \int_{\omega_0}^{T}\int_{\omega_0}^{x} g_2(x)(T - \sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}}[x - \sigma_{q,\omega}(s)]_{q,\omega}^{\underline{\alpha-1}} \times \\ &\Big\{\lambda F\left[s, u(s), \left(\Psi^{\gamma}_{q,\omega}u\right)(s)\right] + \mu H\left[s, u(s), \left(\Upsilon^{\nu}_{q,\omega}u\right)(s)\right]\Big\} d_{q,\omega}s\, d_{q,\omega}x. \end{aligned} \tag{3.3}$$

Obviously, problem (1.1) has solutions if and only if the operator $\mathcal{A}$ has fixed points.

Theorem 3.1. *Assume that $F, H : I^T_{q,\omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $\varphi, \psi : I^T_{q,\omega} \times I^T_{q,\omega} \to [0, \infty)$ is continuous with $\varphi_0 = \max\left\{\varphi(t,s) : (t,s) \in I^T_{q,\omega} \times I^T_{q,\omega}\right\}$ and $\psi_0 = \max\left\{\psi(t,s) : (t,s) \in I^T_{q,\omega} \times I^T_{q,\omega}\right\}$. In addition, suppose that the following conditions hold:*

(H_1) *There exist constants* $M_i > 0$ *such that for each* $t \in I^T_{q,\omega}$ *and* $u_i, v_i \in \mathbb{R},\ i = 1, 2,$

$$\left|F\left[t, u_1, u_2\right] - F\left[t, v_1, v_2\right]\right| \leq M_1\left|u_1 - v_1\right| + M_2\left|u_2 - v_2\right|.$$

(H_2) *There exist constants* $N_i > 0$ *such that for each* $t \in I^T_{q,\omega}$ *and* $u_i, v_i \in \mathbb{R},\ i = 1, 2,$

$$\left|H\left[t, u_1, u_2\right] - H\left[t, v_1, v_2\right]\right| \leq N_1\left|u_1 - v_1\right| + N_2\left|u_2 - v_2\right|.$$

(H_3) *There exist constants* $\omega_1, \omega_2 > 0$ *such that for each* $u, v \in C$,

$$\left|\phi_1(u) - \phi_1(v)\right| \leq \omega_1\left\|u - v\right\|_C \ \text{and}\ \left|\phi_2(u) - \phi_2(v)\right| \leq \omega_2\left\|u - v\right\|_C.$$

(H_4) *For each* $t \in I^T_{q,\omega}$, $\hat{g}_1 \leq g_1(t) \leq G_1$ *and* $\hat{g}_2 \leq g_2(t) \leq G_2$.

(H_5) $\mathcal{X} = \mathcal{L}\Big[\Phi + \frac{G_1(\eta-\omega_0)^{\alpha+\beta}\Theta^*_T + G_2(T-\omega_0)^{\alpha+\beta}\Theta^*_\eta}{\Gamma_q(\alpha+\beta+1)}\Big] + \Theta^*_T\omega_1 + \Theta^*_\eta\omega_2 \leq 1,$

where

$$\mathcal{L} := \lambda\Big[M_1 + M_2\frac{\varphi_0(T-\omega_0)^\gamma}{\Gamma_q(\gamma+1)}\Big] + \mu\Big[N_1 + N_2\frac{\psi_0(T-\omega_0)^{-\gamma}}{\Gamma_q(-\gamma+1)}\Big], \tag{3.4}$$

$$\Phi := \frac{(T-\omega_0)^\alpha}{\Gamma_q(\alpha+1)} + \frac{(T-\omega_0)^{\alpha-\nu}}{\Gamma_q(\alpha-\nu+1)}, \tag{3.5}$$

$$\Theta^*_\eta := \Theta_\eta + \bar{\Theta}_\eta, \tag{3.6}$$

$$\Theta^*_T := \Theta_T + \bar{\Theta}_T, \tag{3.7}$$

$$\Theta_\eta := \frac{1}{\min|\Lambda|}\Big[\max|B_\eta|(T-\omega_0)^{\alpha-1} + \max|A_\eta|(T-\omega_0)^{\alpha-2}\Big], \tag{3.8}$$

$$\Theta_T := \frac{1}{\min|\Lambda|}\Big[\max|B_T|(T-\omega_0)^{\alpha-1} + \max|A_T|(T-\omega_0)^{\alpha-2}\Big], \tag{3.9}$$

$$\bar{\Theta}_\eta := \frac{1}{\min|\Lambda|}\Big[\max|B_\eta|(T-\omega_0)^{-\nu+\alpha-1}\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\nu)} + \max|A_\eta|(T-\omega_0)^{-\nu+\alpha-2}\frac{\Gamma_q(\alpha-1)}{\Gamma_q(\alpha-1-\nu)}\Big], \tag{3.10}$$

$$\bar{\Theta}_T := \frac{1}{\min|\Lambda|}\Big[\max|B_T|(T-\omega_0)^{-\nu+\alpha-1}\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\nu)} + \max|A_T|(T-\omega_0)^{-\nu+\alpha-2}\frac{\Gamma_q(\alpha-1)}{\Gamma_q(\alpha-1-\nu)}\Big]. \tag{3.11}$$

Then, problem (1.1) *has a unique solution.*

Proof. For each $t \in I^T_{q,\omega}$ and $u, v \in C$, we find that

$$\begin{aligned}\left|(\Psi^\gamma_{q,\omega}u)(t) - (\Psi^\gamma_{q,\omega}v)(t)\right| &\leq \frac{\varphi_0}{\Gamma_q(\gamma)}\int_{\omega_0}^T (T-\sigma_{q,\omega}(s))^{\underline{\gamma-1}}_{q,\omega}\left|u(s) - v(s)\right|d_{q,\omega}s \\ &\leq \frac{\varphi_0\|u-v\|}{\Gamma_q(\gamma)}\int_{\omega_0}^T (T-\sigma_{q,\omega}(s))^{\underline{\gamma-1}}_{q,\omega}d_{q,\omega}s \\ &= \frac{\varphi_0\|u-v\|(T-\omega_0)^\gamma}{\Gamma_q(\gamma+1)}.\end{aligned}$$

Similarly, we have $\left|(\Upsilon^\nu_{q,\omega}u)(t) - (\Upsilon^\nu_{q,\omega}v)(t)\right| \leq \frac{\psi_0(T-\omega_0)^{-\nu}\|u-v\|}{\Gamma_q(-\nu+1)}$.

We set

$$\mathcal{F}|u-v|(t) := \Big|F[t,u(t),(\Psi^{\gamma}_{q,\omega}u)(t)] - F[t,v(t),(\Psi^{\gamma}_{q,\omega}v)(t)]\Big|$$
$$\mathcal{H}|u-v|(t) := \Big|H[t,u(t),(\Upsilon^{\nu}_{q,\omega}u)(t)] - H[t,v(t),(\Upsilon^{\nu}_{q,\omega}v)(t)]\Big|.$$

Then, we obtain

$$\begin{aligned}
&\Big|O^*_\eta[\phi_1, F_u + H_u,] - O^*_\eta[\phi_1, F_v + H_v]\Big| \\
\le\ & \Big|\phi_1(u(\eta)) - \phi_1(v(\eta))\Big| + \frac{1}{\Gamma_q(\beta)\Gamma_q(\alpha)}\int_{\omega_0}^{\eta}\int_{\omega_0}^{x} g_1(x)(\eta-\sigma_{q,\omega}(x))^{\underline{\beta-1}}_{q,\omega}[x-\sigma_{q,\omega}(s)]^{\underline{\alpha-1}}_{q,\omega} \\
& \times\{\lambda\mathcal{F}|u-v|(s) + \mu\mathcal{H}|u-v|(s)\}d_{q,\omega}s d_{q,\omega}x \\
\le\ & \omega_1\|u-v\|_C + \frac{G_1(\eta-\omega_0)^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)}\Big(\lambda[M_1 + M_2\frac{\varphi_0(T-\omega_0)^{\gamma}}{\Gamma_q(\gamma+1)}] + \mu[N_1 + N_2\frac{\psi_0(T-\omega_0)^{-\nu}}{\Gamma_q(-\nu+1)}]\Big)\|u-c\|_C \\
=\ & \omega_1\|u-v\|_C + \frac{\mathcal{L}G_1(\eta-\omega_0)^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)}\|u-c\|_C \\
=\ & \Big[\omega_1 + \frac{\mathcal{L}G_1(\eta-\omega_0)^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)}\Big]\|u-v\|_C.
\end{aligned}$$

Similarly, we obtain

$$\Big|O^*_T[\phi_2, F_u + H_u] - O^*_T[\phi_2, F_v + H_v]\Big| \le \Big(\omega_2 + \frac{\mathcal{L}G_2(T-\omega_0)^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)}\Big)\|u-v\|_C.$$

Next, we find that

$$\begin{aligned}
&\Big|\mathcal{A}(u)(t) - \mathcal{A}(v)(t)\Big| \\
\le\ & \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{T}(T-\sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}\{\lambda\mathcal{F}|u-v|(s) + \mu\mathcal{H}|u-v|(s)\}d_{q,\omega}s \\
& + \frac{(T-\omega_0)^{\alpha-1}}{|\Lambda|}\Big[\big|\mathcal{B}_T\big|\big|O^*_\eta[\phi_1, F_u + H_u] - O^*_\eta[\phi_1, F_v + H_v]\big| + \big|\mathcal{B}_\eta\big|\big|O^*_T[\phi_2, F_u + H_u] - O^*_T[\phi_2, F_v + H_v]\big|\Big] \\
& + \frac{(T-\omega_0)^{\alpha-2}}{|\Lambda|}\Big[\big|\mathcal{A}_T\big|\big|O^*_\eta[\phi_1, F_u + H_u] - O^*_\eta[\phi_1, F_v + H_v]\big| + \big|\mathcal{A}_\eta\big|\big|O^*_T[\phi_2, F_u + H_u - O^*_T[\phi_2, F_v + H_v]\big|\Big] \\
\le\ & \Big[\mathcal{L}\Big(\frac{(T-\omega_0)^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{G_1(\eta-\omega_0)^{\alpha+\beta}\Theta_T + G_2(T-\omega_0)^{\alpha+\beta}\Theta_\eta}{\Gamma_q(\alpha+\beta+1)}\Big) + \Theta_T\omega_1 + \Theta_\eta\omega_2\Big]\|u-v\|_C. \qquad (3.12)
\end{aligned}$$

Considering $(\mathcal{D}^{\nu}_q\mathcal{A}u)$, we have

$$\begin{aligned}
(\mathcal{D}^{\nu}_{q,\omega}\mathcal{A}u) =\ & \frac{1}{\Gamma_q(-\nu)\Gamma_q(\alpha)}\int_{\omega_0}^{t}\int_{\omega_0}^{x}(t-\sigma_{q,\omega}(x))^{-\nu-1}(x-\sigma_{q,\omega}(s))^{\underline{\alpha-1}}_{q,\omega}\{\lambda F(s,u(s),(\Psi^{\gamma}_{q,\omega}u)(s)) \\
& + \mu H(s,u(s),(\Upsilon^{\nu}_{q,\omega}u)(s))\}d_{q,\omega}s d_{q,\omega}x - \{\mathcal{B}_T O^*_\eta[\phi_1, F_u + H_u] - \mathcal{B}_\eta O^*_T[\phi_2, F_u + H_u]\} \\
& \times\frac{1}{\Gamma_q(-\nu)}\int_{\omega_0}^{t}(t-\sigma_{q,\omega}(s))^{\underline{-\nu-1}}_{q,\omega}\frac{(s-\omega_0)^{\alpha-1}}{\Lambda}d_{q,\omega}s + \{\mathcal{A}_T O^*_\eta[\phi_1, F_u + H_u]
\end{aligned}$$

$$-\mathcal{B}_\eta O_T^*[\phi_2, F_u + H_u]\}\frac{1}{\Gamma_q(-\nu)}\int_{\omega_0}^{t}(t-\sigma_{q,\omega}(s))_{q,\omega}^{-\nu-1}\frac{(s-\omega_0)^{\alpha-2}}{\Lambda}d_{q,\omega}s. \tag{3.13}$$

Hence,

$$\begin{aligned}\left|(\mathcal{D}_{q,\omega}^{\nu}\mathcal{A}u)(t)-(\mathcal{D}_{q,\omega}^{\nu}\mathcal{A}v)(t)\right| \le\ & \Big[\mathcal{L}\Big(\frac{(T-\omega_0)^{\alpha-\nu}}{\Gamma_q(\alpha-\nu+1)}+\frac{G_1(\eta-\omega_0)^{\alpha+\beta}\bar{\Theta}_T+G_2(T-\omega_0)^{\alpha+\beta}\bar{\Theta}_\eta}{\Gamma_q(\alpha+\beta+1)}\Big)\\ &+\omega_1\bar{\Theta}_T+\omega_2\bar{\Theta}_\eta\Big]\|u-v\|_C. \end{aligned}\tag{3.14}$$

From (3.12) and (3.14), we find that

$$\|\mathcal{A}(u)-\mathcal{A}(v)\|_C \le \ \chi\|u-v\|_C.$$

Thus, the operator $\mathcal{A}$ is a contraction. Then, by the Banach contraction mapping principle, $\mathcal{A}$ has a fixed point, which is the unique solution for (1.1).

We next show the existence of a solution to (1.1) by the following Schauder's fixed point theorem.

Theorem 3.2. *Let us assume that $F, H : I_{q,\omega}^T \times \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions and $\phi_1, \phi_2 : C(I_{q,\omega}^T, \mathbb{R}) \to \mathbb{R}$ are given functionals. Let us suppose that the following conditions hold:*

(H_6) There exist positive constants $\hat{\mathcal{F}}$, $\hat{\mathcal{H}}$ such that for each $t \in I_{q,\omega}^T$ and $u_i \in \mathbb{R}, i = 1, 2$,

$$\left|F[t,u_1,u_2]\right| \le \hat{\mathcal{F}} \text{ and } \left|\hat{\mathcal{H}}[t,u_1,u_2]\right| \le \hat{\mathcal{H}}.$$

(H_7) There exist positive constants O_1, O_2 such that for each $u \in C$,

$$\left|\phi_1(u)\right| \le O_1 \text{ and } \left|\phi_2(u)\right| \le O_2.$$

Then, problem (1.1) has at least one solution on $I_{q,\omega}^T$.

Proof. We organize the proof into three steps.

(i) For each $t \in I_{q,\omega}^T$ and $u \in B_R$, we obtain

$$\begin{aligned}\left|O_\eta^*[\phi_1, F_u + H_u]\right| \le\ & O_1 + \frac{(\lambda\hat{\mathcal{F}}+\mu\hat{\mathcal{H}})G_1}{\Gamma_q(\beta)\Gamma_q(\alpha)}\int_{\omega_0}^{\eta}\int_{\omega_0}^{x}(\eta-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}}[x-\sigma_{q,\omega}(s)]_{q,\omega}^{\underline{\alpha-1}}d_{q,\omega}s\,d_{q,\omega}x\\ \le\ & O_1 + \frac{G_1(\eta-\omega_0)^{\alpha+\beta}(\lambda\hat{\mathcal{F}}+\mu\hat{\mathcal{H}})}{\Gamma_q(\alpha+\beta+1)}.\end{aligned}\tag{3.15}$$

Similarly, we have

$$\left|O_T^*[\phi_2, F_u + H_u]\right| \le\ O_2 + \frac{G_2(\lambda\hat{\mathcal{F}}+\mu\hat{\mathcal{H}})(T-\omega_0)^{\alpha+\beta}}{\Gamma_q(\alpha+\beta+1)}. \tag{3.16}$$

From (3.15) and (3.16) and for each $t \in I_{q,\omega}^T$, we find that

$$\left|(\mathcal{A}u)(t)\right| \le \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{T}(T-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}}\left|\lambda F[s,u(s),(\psi_{q,\omega}^{\gamma}u)(s)]+\mu H[s,u(s),(\Upsilon_{q,\omega}^{\nu}u)(s)]\right|d_{q,\omega}s$$

$$+\frac{(T-\omega_0)^{\alpha-1}}{|\Lambda|}\Big[|\mathcal{B}_T||\mathcal{O}_\eta^*[\phi_1, F_u+H_u]|+|\mathcal{B}_\eta||\mathcal{O}_T^*[\phi_2, F_u+H_u]|\Big]$$
$$+\frac{(T-\omega_0)^{\alpha-2}}{|\Lambda|}\Big[|\mathcal{A}_T||\mathcal{O}_\eta^*[\phi_1, F_u+H_u]|+|\mathcal{A}_\eta||\mathcal{O}_T^*[\phi_2, F_u+H_u]|\Big]$$
$$\leq (\lambda\hat{\mathcal{F}}+\mu\hat{\mathcal{H}})\Big[\frac{(T-\omega_0)^{\alpha}}{\Gamma_q(\alpha+1)}+\frac{G_1(\eta-\omega_0)^{\alpha+\beta}\Theta_T+G_2(T-\omega_0)^{\alpha+\beta}\Theta_\eta}{\Gamma_q(\alpha+\beta+1)}\Big]+\mathcal{O}_1\Theta_T+\mathcal{O}_2\Theta_\eta. \tag{3.17}$$

In addition, we obtain

$$\Big|(D_{q,\omega}^{\nu}\mathcal{A}u)(t)\Big| \leq (\lambda\hat{\mathcal{F}}+\mu\hat{\mathcal{H}})\Big[\frac{(T-\omega_0)^{\alpha-\nu}}{\Gamma_q(\alpha-\nu+1)}+\frac{G_1(\eta-\omega_0)^{\alpha+\beta}\bar{\Theta}_T+G_2(T-\omega_0)^{\alpha+\beta}\bar{\Theta}_\eta}{\Gamma_q(\alpha+\beta+1)}\Big]$$
$$+\mathcal{O}_1\bar{\Theta}_T+\mathcal{O}_2\bar{\Theta}_\eta. \tag{3.18}$$

From (3.17) and (3.18) we obtain

$$\Big\|(\mathcal{A}u)\Big\|_C \leq (\lambda\hat{\mathcal{F}}+\mu\hat{\mathcal{H}})\Big[\Phi+\frac{G_1(\eta-\omega_0)^{\alpha+\beta}\Theta_T^*+G_2(T-\omega_0)^{\alpha+\beta}\Theta_\eta^*}{\Gamma_q(\alpha+\beta+1)}\Big]+\mathcal{O}_1\Theta_T^*+\mathcal{O}_2\Theta_\eta^*$$
$$< \infty,$$

which implies that $\mathcal{A}(B_R)$ is uniformly bounded.

(ii) We show that $\mathcal{A}(B_R)$ is equicontinuous. for any $t_1, t_2 \in I_{q,\omega}^T$ with $t_1 < t_2$, we have

$$\Big|(\mathcal{A}u)(t_2)-(\mathcal{A}u)(t_1)\Big|$$
$$\leq \frac{(\lambda\hat{\mathcal{F}}+\lambda\hat{\mathcal{H}})}{\Gamma_q(\alpha+1)}\Big|(t_2-\omega_0)^{\alpha}-(t_1-\omega_0)^{\alpha}\Big|$$
$$+\frac{\Big|(t_2-\omega_0)^{\alpha-1}-(t_1-\omega_0)^{\alpha-1}\Big|}{|\Lambda|}\{|\mathcal{B}_T||\mathcal{O}_\eta^*[\phi_1, F_u+H_u]|+|\mathcal{B}_\eta||\mathcal{O}_T^*[\phi_2, F_u+H_u]|\}$$
$$+\frac{\Big|(t_2-\omega_0)^{\alpha-2}-(t_1-\omega_0)^{\alpha-2}\Big|}{|\Lambda|}\{|\mathcal{A}_T||\mathcal{O}_\eta^*[\phi_1, F_u+H_u]|+|\mathcal{A}_\eta||\mathcal{O}_T^*[\phi_2, F_u+H_u]|\} \tag{3.19}$$

and

$$\Big|(D_{q,\omega}^{\nu}\mathcal{A}u)(t_2)-(D_{q,\omega}^{\nu}\mathcal{A}u)(t_1)\Big|$$
$$\leq \frac{(\lambda\hat{\mathcal{F}}+\lambda\hat{\mathcal{H}})}{\Gamma_q(\alpha-\nu+1)}\Big|(t_2-\omega_0)^{\alpha-\nu}-(t_1-\omega_0)^{\alpha-\nu}\Big|$$
$$+\frac{\Gamma_q(\alpha)\Big|(t_2-\omega_0)^{\alpha-\nu-1}-(t_1-\omega_0)^{\alpha-\nu-1}\Big|}{|\Lambda|\Gamma_q(\alpha-\nu)}\{|\mathcal{B}_T||\mathcal{O}_\eta^*[\phi_1, F_u+H_u]|+|\mathcal{B}_\eta||\mathcal{O}_T^*[\phi_2, F_u+H_u]|\}$$
$$+\frac{\Gamma_q(\alpha-1)\Big|(t_2-\omega_0)^{\alpha-\nu-2}-(t_1-\omega_0)^{\alpha-\nu-2}\Big|}{|\Lambda|\Gamma_q(\alpha-\nu-1)}\{|\mathcal{A}_T||\mathcal{O}_\eta^*[\phi_1, F_u+H_u]|+|\mathcal{A}_\eta||\mathcal{O}_T^*[\phi_2, F_u+H_u]|\}. \tag{3.20}$$

The right-hand side of (3.19) and (3.20) tends to zero as $t_1 \to t_2$, independently of u, which implies that $\mathcal{A}(B_R)$ is an equicontinuous set. By using the Arzela–Ascoli theorem, the set $\mathcal{A}(B_R)$ is compact.

(iii) Finally, we show that $\mathcal{W} = \{u \in C : u = \zeta \mathcal{A}u, 0 < \zeta < 1\}$ is a bounded set. Let $u \in \mathcal{W}$. Since $|u| = \zeta \|\mathcal{A}u\| \le \zeta \|\mathcal{A}u\|_C$, and from (i) $\|\mathcal{A}u\|_C$ is bounded, hence $\mathcal{W}$ is bounded. Then, as in (i), we have

$$\begin{aligned} \left|u(t)\right| &\le \zeta \|\mathcal{A}u\|_C \\ &\le (\lambda\hat{\mathcal{F}} + \mu\hat{\mathcal{H}})\Big[\Phi + \frac{G_1(\eta-\omega_0)^{\alpha+\beta}\Theta_T^* + G_2(T-\omega_0)^{\alpha+\beta}\Theta_\eta^*}{\Gamma_q(\alpha+\beta+1)}\Big] + O_1\Theta_T^* + O_2\Theta_\eta^*. \end{aligned}$$

Therefore, $\mathcal{W}$ is bounded.

4. Hyers-Ulam stability analysis result

In this section, we study the Hyers-Ulam stability of system (1.1). Let $\varepsilon > 0$ and $\delta : I_{q,\omega}^T \to \mathbb{R}$ be a continuous function. Consider

$$\begin{aligned} &\left|D_{q,\omega}^{\alpha}u(t) - \lambda F\left[t, u(t), \left(\Psi_{q,\omega}^{\gamma}u\right)(t)\right] - \mu H\left[t, u(t), \left(\Upsilon_{q,\omega}^{\nu}u\right)(t)\right]\right| \le \varepsilon\delta(t), \quad t \in I_{q,\omega}^T, \\ &\mathcal{I}_{q,\omega}^{\beta} g_1(\eta)u(\eta) = \phi_1(u), \quad \eta \in I_{q,\omega}^T - \{\omega_0, T\}, \\ &\mathcal{I}_{q,\omega}^{\beta} g_2(T)\, u(T) = \phi_2(u). \end{aligned} \tag{4.1}$$

Now, we give out the definition of Hyers-Ulam stability of system (1.1).

Definition 4.1. *System (1.1) is Hyers-Ulam stable with respect to system (4.1), if there exists $A_{F,H} > 0$ such that*

$$|\bar{u} - \tilde{u}| \le \varepsilon A_{F,H}$$

for all $t \in I_{q,\omega}^T$, where $\bar{u}$ is the solution of (4.1) and $\tilde{u}$ is the solution for system (1.1).

Theorem 4.1. *Assume that $(H_1) - (H_5)$ hold, and $\max_{t \in I_{q,\omega}^T} \delta(t) \le 1$. Then the system (1.1) is Hyers–Ulam stable with respect to system (4.1).*

Proof. Let $D_{q,\omega}^{\alpha}\bar{u}(t) = \lambda F\left[t, \bar{u}(t), \left(\Psi_{q,\omega}^{\gamma}\bar{u}\right)(t)\right] + \mu H\left[t, \bar{u}(t), \left(\Upsilon_{q,\omega}^{\nu}\bar{u}\right)(t)\right] + k(t)$. Consider

$$\begin{aligned} &D_{q,\omega}^{\alpha}\bar{u}(t) = \lambda F\left[t, \bar{u}(t), \left(\Psi_{q,\omega}^{\gamma}\bar{u}\right)(t)\right] + \mu H\left[t, \bar{u}(t), \left(\Upsilon_{q,\omega}^{\nu}\bar{u}\right)(t)\right] + k(t), \quad t \in I_{q,\omega}^T, \\ &\mathcal{I}_{q,\omega}^{\beta} g_1(\eta)u(\eta) = \phi_1(u), \quad \eta \in I_{q,\omega}^T - \{\omega_0, T\}, \\ &\mathcal{I}_{q,\omega}^{\beta} g_2(T)\, u(T) = \phi_2(u). \end{aligned} \tag{4.2}$$

Similarly to the system in Theorem 3.1, system (4.2) is equivalent to the following equation in Lemma 2.7.

$$\begin{aligned} \bar{u}(t) := &\frac{1}{\Gamma_q(\alpha)} \int_{\omega_0}^{t} \left[t - \sigma_{q,\omega}(s)\right]_{q\omega}^{\underline{\alpha-1}} \Big\{\lambda F\left[s, \bar{u}(s), \left(\Psi_{q,\omega}^{\gamma}\bar{u}\right)(s)\right] + \mu H\left[s, \bar{u}(s), \left(\Upsilon_{q,\omega}^{\nu}\bar{u}\right)(s)\right] + k(s)\Big\} d_{q,\omega}s \\ &- \frac{(t-\omega_0)^{\alpha-1}}{\Lambda}\Big\{\mathcal{B}_T\bar{O}_\eta^*(\phi_1, F_{\bar{u}} + H_{\bar{u}} + k) - \mathcal{B}_\eta\bar{O}_T^*(\phi_2, F_{\bar{u}} + H_{\bar{u}} + k)\Big\} \end{aligned}$$

$$+\frac{(t-\omega_0)^{\alpha-2}}{\Lambda}\Big\{\mathcal{A}_T\bar{O}_\eta^*(\phi_1,F_{\bar{u}}+H_{\bar{u}}+k)-\mathcal{A}_\eta\bar{O}_T^*(\phi_2,F_{\bar{u}}+H_{\bar{u}}+k)\Big\}, \tag{4.3}$$

where $\Lambda, \mathcal{A}_\eta, \mathcal{B}_\eta, \mathcal{A}_T$, and $\mathcal{B}_T$ are defined in (2.3)–(2.7), respectively, and the functionals $\bar{O}_\eta^*[\phi_1, F_{\bar{u}} + H_{\bar{u}} + k]$, $\bar{O}_T^*[\phi_2, F_{\bar{u}} + H_{\bar{u}} + k]$ are defined by

$$\begin{aligned}\bar{O}_\eta^*[\phi_1,F_{\bar{u}}+H_{\bar{u}}+k] &:= \phi_1(u(\eta))-\frac{1}{\Gamma_q(\beta)\Gamma_q(\alpha)}\int_{\omega_0}^{\eta}\int_{\omega_0}^{x}g_1(x)(\eta-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}}[x-\sigma_{q,\omega}(s)]_{q,\omega}^{\underline{\alpha-1}}\\ &\quad\times\Big\{\lambda F\Big[s,\bar{u}(s),\Big(\Psi_{q,\omega}^{\gamma}\bar{u}\Big)(s)\Big]+\mu H\Big[s,\bar{u}(s),\Big(\Upsilon_{q,\omega}^{\nu}\bar{u}\Big)(s)\Big]+k(s)\Big\}d_{q,\omega}s\,d_{q,\omega}x,\end{aligned} \tag{4.4}$$

$$\begin{aligned}\bar{O}_T^*[\phi_2,F_{\bar{u}}+H_{\bar{u}}+k] &:= \phi_2(u(T))-\frac{1}{\Gamma_q(\beta)\Gamma_q(\alpha)}\int_{\omega_0}^{T}\int_{\omega_0}^{x}g_2(x)(T-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}}[x-\sigma_{q,\omega}(s)]_{q,\omega}^{\underline{\alpha-1}}\\ &\quad\times\Big\{\lambda F\Big[s,\bar{u}(s),\Big(\Psi_{q,\omega}^{\gamma}\bar{u}\Big)(s)\Big]+\mu H\Big[s,\bar{u}(s),\Big(\Upsilon_{q,\omega}^{\nu}\bar{u}\Big)(s)\Big]+k(s)\Big\}d_{q,\omega}s\,d_{q,\omega}x.\end{aligned} \tag{4.5}$$

Now, we define the operator as

$$(\tilde{\mathcal{A}}u)(t)=(\mathcal{A}u)(t)+\mathcal{K}(t), \tag{4.6}$$

where

$$\begin{aligned}\mathcal{K}(t) &= \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{t}\Big[t-\sigma_{q,\omega}(s)\Big]_{q,\omega}^{\underline{\alpha-1}}k(s)d_{q,\omega}s+\frac{(t-\omega_0)^{\alpha-1}}{\Lambda}\Big[\mathcal{B}_\eta O_T[k]-\mathcal{B}_T O_\eta[k]\Big]\\ &\quad+\frac{(t-\omega_0)^{\alpha-2}}{\Lambda}\Big[\mathcal{A}_T O_\eta[k]-\mathcal{A}_\eta O_T[k]\Big],\end{aligned} \tag{4.7}$$

where the functionals are defined by

$$O_\eta[k] := k(\eta)-\frac{1}{\Gamma_q(\beta)}\frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{\eta}\int_{\omega_0}^{x}g_1(x)(\eta-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}}[x-\sigma_{q,\omega}(s)]_{q,\omega}^{\underline{\alpha-1}}k(s)d_{q,\omega}s\,d_{q,\omega}x, \tag{4.8}$$

$$O_T[k] := k(T)-\frac{1}{\Gamma_q(\beta)}\frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{T}\int_{\omega_0}^{x}g_2(x)(T-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}}[x-\sigma_{q,\omega}(s)]_{q,\omega}^{\underline{\alpha-1}}k(s)d_{q,\omega}s\,d_{q,\omega}. \tag{4.9}$$

Note that

$$\|\tilde{\mathcal{A}}u-\tilde{\mathcal{A}}v\|=\|\mathcal{A}u-\mathcal{A}v\|. \tag{4.10}$$

Then the existence of a solution of (1.1) implies the existence of a solution to (4.2). It follows from Theorem 3.1 that $\tilde{\mathcal{A}}$ is a contraction. Thus there is a unique fixed point $\bar{u}$ of $\tilde{\mathcal{A}}$, and $\tilde{u}$ of $\mathcal{A}$.

Since $t \in I_{q,\omega}^T$ and $\max_{t\in I_{q,\omega}^T}\delta(t) \le 1$, we obtain

$$\|\mathcal{K}\|=\max_{t\in I_{q,\omega}^T}|\mathcal{K}(t)|\le\varepsilon\hat{\chi}, \tag{4.11}$$

where

$$\hat{\chi}=\Big[\frac{\varphi_0(T-\omega_0)^{\gamma}}{\Gamma_q(\gamma+1)}+\frac{\psi_0(T-\omega_0)^{-\gamma}}{\Gamma_q(-\gamma+1)}\Big]\Big[\Phi+\frac{G_1(\eta-\omega_0)^{\alpha+\beta}\Theta_T^*+G_2(T-\omega_0)^{\alpha+\beta}\Theta_\eta^*}{\Gamma_q(\alpha+\beta+1)}\Big]+\Theta_T^*+\Theta_\eta^*, \tag{4.12}$$

Φ, Θ_T^* and Θ_η^* are defined by (3.5)–(3.7), respectively.

Hence, we obtain

$$\|\bar{u} - \tilde{u}\| = \|\tilde{\mathcal{A}}\bar{u} - \mathcal{A}\tilde{u}\| = \|\mathcal{A}\bar{u} - \mathcal{A}\tilde{u} + \mathcal{K}(t)\| = \|\mathcal{A}\bar{u} - \mathcal{A}\tilde{u}\| + \|\mathcal{K}(t)\| = \chi\|\bar{u} - \tilde{u}\| + \varepsilon\hat{\chi}. \tag{4.13}$$

By condition (H_5), we obtain

$$\|\bar{u} - \tilde{u}\| \leq \frac{\varepsilon\hat{\chi}}{1-\chi}. \tag{4.14}$$

Let $A_{F,H} = \frac{\hat{\chi}}{1-\chi}$, then

$$\|\bar{u} - \tilde{u}\| \leq \varepsilon A_{F,H}. \tag{4.15}$$

This completes the proof. □

5. Some examples

To elucidate our results, we present several illustrative examples.

Example 5.1. Consider the following fractional Hahn BVP

$$\begin{aligned} D^{\frac{3}{2}}_{\frac{1}{2},\frac{1}{3}} u(t) = &\Big[\frac{e^{-[\sin^2(2\pi t)+\pi]}}{100 + e^{\cos^2(2\pi t)}}\Big]\frac{|u(t)| + e^{-(5t+\pi)}\Big|\Psi^{\frac{1}{4}}_{\frac{1}{2},\frac{1}{3}} u(t)\Big|}{1 + |u(t)|} \\ &+ \Big[\frac{e^{-[\cos^2(2\pi t)+5]}}{(t+10)^2}\Big]\frac{|u(t)| + e^{-(20t+\frac{\pi}{3})}\Big|\Upsilon^{\frac{2}{3}}_{\frac{1}{2},\frac{1}{3}} u(t)\Big|}{1 + |u(t)|}, \quad t \in [\frac{2}{3}, 10]_{\frac{1}{2},\frac{1}{3}} \end{aligned} \tag{5.1}$$

subject to fractional Hahn integral boundary condition

$$\begin{aligned} I^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{3}}\Big(2e + \sin(\frac{23}{24})\Big)^2 u(\frac{23}{24}) &= \sum_{i=0}^{\infty} \frac{C_i|u(t_i)|}{1 + |u(t_i)|}, \quad t_i \in \sigma^i_{\frac{1}{2},\frac{1}{3}}(\frac{23}{24}), \\ I^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{3}}(2\pi + \cos(10))^2 u(10) &= \sum_{i=0}^{\infty} \frac{D_i|u(t_i)|}{1 + |u(t_i)|}, \quad t_i \in \sigma^i_{\frac{1}{2},\frac{1}{3}}(10), \end{aligned} \tag{5.2}$$

where $\varphi(t,s) = \frac{e^{-|t-s|}}{(t+2\pi)^5}$, $\psi(t,s) = \frac{e^{-|t-s|}}{(t+2e)^4}$ and C_i, D_i are given constants, with $\frac{1}{2000} \leq \sum_{i=0}^{\infty} C_i \leq \frac{\pi}{2000}$ and $\frac{1}{1000} \leq \sum_{i=0}^{\infty} D_i \leq \frac{e}{1000}$.

Here $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{4}$, $q = \frac{1}{2}$, $\omega = \frac{1}{3}$, $\nu = \frac{2}{3}$, $\omega_0 = \frac{\omega}{1-q} = \frac{2}{3}$, $T = 10$, $\eta = \sigma^5_{\frac{1}{2},\frac{1}{3}}(10) = \frac{23}{24}$, $\lambda = e^{-\pi}$, $\mu = e^{-5}$, $\phi_1 = \sum_{i=0}^{\infty} \frac{C_i|u(t_i)|}{1 + |u(t_i)|}$, $\phi_2 = \sum_{i=0}^{\infty} \frac{D_i|u(t_i)|}{1 + |u(t_i)|}$, $g_1(t) = (2e + \sin t)^2$ and $g_2(t) = (2\pi + \cos t)^2$.

$$F[t, u(t), (\Psi^{\gamma}_{q,\omega})(t)] = \Big[\frac{e^{-[\sin^2(2\pi t)]}}{100 + e^{\cos^2(2\pi t)}}\Big]\frac{|u(t)| + e^{-(5t+\pi)}\Big|\Psi^{\frac{1}{4}}_{\frac{1}{2},\frac{1}{3}} u(t)\Big|}{1 + |u(t)|},$$

and

$$H[t,u(t),(\Upsilon^{\nu}_{q,\omega})(t)] = \Big[\frac{e^{-[\cos^2(2\pi t)]}}{(t+10)^2}\Big]\frac{|u(t)| + e^{-(20t+\frac{\pi}{3})}\Big|\Upsilon^{\frac{2}{3}}_{\frac{1}{2},\frac{1}{3}}u(t)\Big|}{1+|u(t)|}.$$

For all $t \in I^{10}_{\frac{1}{2},\frac{1}{3}}$ and $u, v \in \mathbb{R}$, we have

$$\Big|F[t,u,\Psi^{\gamma}_{q,\omega}u] - F[t,v,\Psi^{\gamma}_{q,\omega}v]\Big| \le \frac{1}{101}|u-v| + \frac{1}{101e^{\pi}}\Big|\Psi^{\gamma}_{u} - \Psi^{\gamma}_{v}\Big|,$$
$$\Big|H[t,u,\Upsilon^{\nu}_{q,\omega}u] - H[t,v,\Upsilon^{\nu}_{q,\omega}v]\Big| \le \frac{1}{100}|u-v| + \frac{1}{100e^{\frac{\pi}{3}}}\Big|\Upsilon^{\gamma}_{u} - \Upsilon^{\gamma}_{v}\Big|.$$

Thus (H_1) and (H_2) hold with $M_1 = 0.0099, M_2 = 0.000428, N_1 = 0.01$ and $N_2 = 0.00351$.

For all $u, v \in C$,

$$\Big|\phi_1(u) - \phi_1(v)\Big| \le \frac{\pi}{2000}\|u-v\|_C,$$
$$\Big|\phi_2(u) - \phi_2(v)\Big| \le \frac{e}{1000}\|u-v\|_C.$$

So (H_3) hold with $\omega_1 = \frac{\pi}{2000} = 0.00157$ and $\omega_2 = \frac{e}{1000} = 0.00272$.

Moreover (H_4) hold with $\hat{g}_1 = 19.683, G_1 = 41.429, \hat{g}_2 = 27.912$ and $G_2 = 53.048$.

After calculating, we find that

$$\mathcal{A}_{\eta} = 11.1274, \quad \mathcal{A}_T = 455.939, \quad \mathcal{B}_{\eta} = 65.1277, \quad \mathcal{B}_T = 83.3932,$$

$$|\Lambda| = 28766.3089, \quad \varphi_0 = 0.0000617, \quad \psi_0 = 0.00072.$$

We can show that

$$\mathcal{L} = 0.0004952, \quad \Theta_T = 0.0140446, \quad \Theta_{\eta} = 0.00704333, \quad \bar{\theta}_T = 0.0012905,$$

$$\bar{\Theta}_{\eta} = 0.00130252, \quad \Theta^*_T = 0.01531365, \quad \Theta^*_{\eta} = 0.0083485.$$

So, (H_5) holds with

$$\mathcal{X} \approx 0.027724 < 1.$$

Hence, by Theorem 3.1, the BVP (5.1)–(5.2) has a unique solution on $I^{10}_{\frac{1}{2},\frac{1}{3}}$.

In view of Theorem 4.1, we have $\hat{\chi} = 0.049859$ and

$$A_{F,H} \approx 0.051282.$$

Therefore, the BVP (5.1)–(5.2) is Hyers-Ulam stable.

For the specific case where $\phi_1 = 10, \phi_2 = 20$, and $g_1 = g_2 = 30$, we examine the numerical solution to the BVP (5.1)–(5.2) when we let $h(t) = \lambda F\left[t, u(t), \left(\Psi^{\gamma}_{q,\omega}u\right)(t)\right] + \mu H\left[t, u(t), \left(\Upsilon^{\nu}_{q,\omega}u\right)(t)\right] = (t-\omega_0)^{\theta}$. From Figure 1, we obtain the numerical solution graph for $\theta = 0, 0.25, 0.5, 0.75, 1, 1.5, 2$. The graph shows that the solution of the equation converges to zero as $t \to \omega_0$.

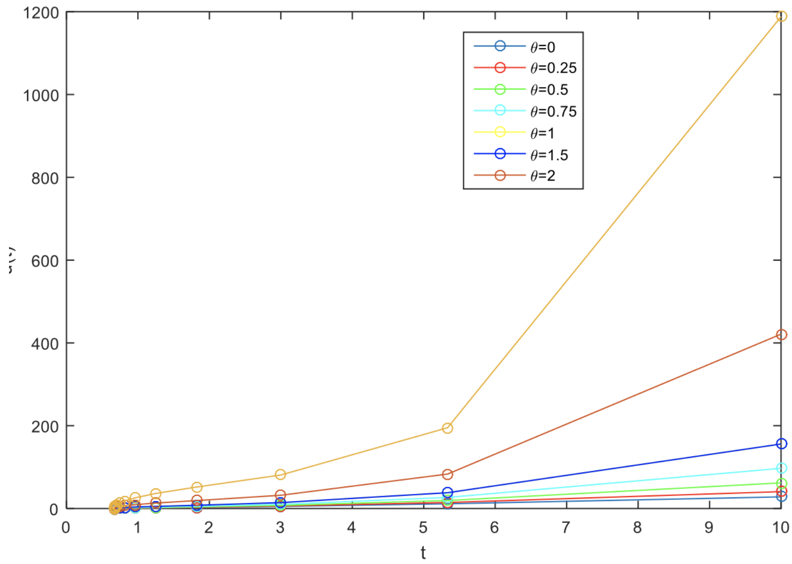

Figure 1. The numerical solution when $\theta = 0, 0.25, 0.5, 0.75, 1, 1.5, 2$.

□

Example 5.2. Consider the following fractional Hahn BVP

$$D^{\frac{5}{4}}_{\frac{1}{4},\frac{3}{2}}u(t) = \frac{1}{15}(t+\frac{2}{5})e^{-(t+5)\left[u(t)+\Psi^{\frac{1}{2}}_{\frac{1}{4},\frac{3}{2}}u(t)\right]} + \frac{1}{5}(t+\frac{1}{3})e^{-(t+\pi)\left[u(t)+\Upsilon^{\frac{1}{2}}_{\frac{1}{4},\frac{3}{2}}u(t)\right]}, \quad t \in I^{15}_{\frac{1}{4},\frac{3}{2}} \tag{5.3}$$

with fractional Hahn integral boundary condition

$$\begin{aligned} I^{\frac{1}{2}}_{\frac{1}{4},\frac{3}{2}}[\pi + \sin(\frac{525}{256})]u(\frac{525}{256}) &= \sum_{i=0}^{\infty} C_i e^{-|u(t_i)|}, \\ I^{\frac{1}{2}}_{\frac{1}{4},\frac{3}{2}}[e + \cos(15)]u(15) &= \sum_{i=0}^{\infty} D_i e^{-|u(t_i)|}, \quad t_i \in \sigma^{i}_{\frac{1}{4},\frac{3}{2}}(15), \end{aligned} \tag{5.4}$$

where C_i and D_i are given constants with $\dfrac{1}{500} \leq \displaystyle\sum_{i=0}^{\infty} C_i \leq \dfrac{e}{500}$ and $\dfrac{1}{1000} \leq \displaystyle\sum_{i=0}^{\infty} D_i \leq \dfrac{\pi}{1000}$

Here $\alpha = \frac{5}{4}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{2}$, $q = \frac{1}{4}$, $\nu = \frac{1}{3}$, $\omega = \frac{3}{2}$, $\omega_0 = \frac{\omega}{1-q} = 2$, $T = 15$, $\eta = \sigma^4_{\frac{1}{4},\frac{3}{2}} = \frac{525}{256}$, $\lambda = e^{-5}$, $\mu = e^{-\pi}$.

It is clear that $\left|F[t, u, \Psi^{\gamma}_{q,\omega}u]\right| \leq \frac{1}{30} = \hat{F}$, $\left|H[t, u, \Upsilon^{\nu}_{q,\omega}u]\right| \leq \frac{1}{15} = \hat{H}$ for $t \in I^{15}_{\frac{1}{4},\frac{3}{2}}$ and $\left|\phi_1(u)\right| \leq \frac{e}{500} = O_1$, $\left|\phi_2(u)\right| \leq \frac{\pi}{1000} = O_2$

Hence, $(H6)$ and $(H7)$ hold. Therefore, the BVP (5.3)–(5.4) has at least one solution on $I^{15}_{\frac{1}{4},\frac{3}{2}}$ by theorem 3.2.

For the specific case where $\phi_1 = \frac{e}{500}, \phi_2 = \frac{\pi}{1000}$ and $g_1 = g_2 = 1$, we examine the numerical solution to the BVP (5.3)–(5.4) when we let $h(t) = \lambda F\left[t, u(t), \left(\Psi^{\gamma}_{q,\omega}u\right)(t)\right] + \mu H\left[t, u(t), \left(\Upsilon^{\nu}_{q,\omega}u\right)(t)\right] = (t-\omega_0)^{\theta}$. From Figure 2, we obtain the numerical solution graph for $\theta = 0, 0.25, 0.5, 0.75, 1, 1.5, 2$. The graph shows that the solution of the equation converges to zero as $t \to \omega_0$.

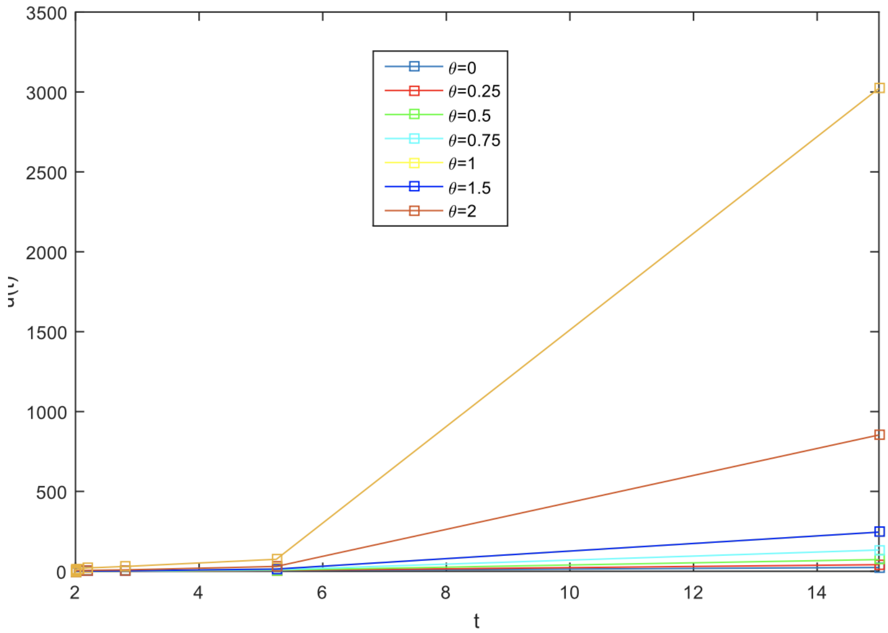

Figure 2. The numerical solution when $\theta = 0, 0.25, 0.5, 0.75, 1, 1.5, 2$.

□

6. Conclusions

We have successfully demonstrated the uniqueness and stability of solutions for the nonlocal Riemann-Liouville fractional Hahn integrodifference BVP through the application of the Banach fixed point theorem. Furthermore, we have established the existence of at least one solution using Schauder's fixed point theorem. Our innovative approach features the integration of two fractional Hahn difference operators and three fractional Hahn integrals.

Author contributions

Nichaphat Patanarapeelert, Jiraporn Reunsumrit, and Thanin Sitthiwirattham: Study conception and design, material preparation, data collection and analysis, and writing the first draft of the manuscript and commenting on previous versions of the manuscript. All the authors read and approved the final manuscript.

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Conflict of interest

The authors declare no conflicts of interest.

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