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Research article

Explicit solutions of nonlocal reverse-time Hirota-Maxwell-Bloch system

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Abstract: In this paper, we investigate the nonlocal reverse-time Hirota-Maxwell-Bloch system, focusing on its soliton solutions using the Darboux transformation method. By deriving the Darboux transformation for this system, we obtained explicit expressions for the new potentials q', p', and η' in both the defocusing ($\kappa = 1$) and focusing ($\kappa = -1$) cases. Our analysis reveals significant differences in soliton behavior depending on the value of κ, with the defocusing case producing wide, smooth solitons and the focusing case yielding narrow, highly localized solitons. These results provide a deeper understanding of soliton dynamics in nonlocal integrable systems and lay the groundwork for future studies on the influence of nonlocality in integrable models.

Keywords: nonlocal reverse-time Hirota and Maxwell-Bloch system; PT-symmetric; Darboux transformation; explicit solution; Lax pair
Mathematics Subject Classification: 35B06, 35C08

1. Introduction

Nonlinear integrable equations are crucial in mathematics and physics because they enable the study of complex dynamical systems using precise analytical methods [1–4]. These equations exhibit the property of integrability, which implies the existence of a Lax pair [5], an infinite number of conservation laws, and the potential to solve them through methods such as the inverse scattering transform, the Riemann-Hilbert approach, Hirota's bilinear method, the Darboux transformation, and others, [6–10]. Most of these equations are local, meaning the evolution of their solutions depends on local values. However, in 1980, Vinogradov and Krasilshchik introduced the method of nonlocal symmetries to study interactions in nonlinear systems [11]. This method led to the development of a new class of nonlocal integrable equations [12–16].

Nonlocal integrable systems have been extensively studied in recent years. The first example of such systems is the nonlinear Schrödinger equation (NLSE), derived in [17]. The $\mathcal{PT}$-symmetric NLSE is given as

$$\mathrm{i}q_t(x,t) = q_{xx}(x,t) + 2\kappa q^2(x,t)q^*(-x,t), \tag{1.1}$$

where $\kappa = \pm 1$ indicates the nature of the nonlinearity, with $\kappa = +1$ corresponding to the defocusing case and $\kappa = -1$ to the focusing case. The symbol $*$ represents the complex conjugate. This equation is symmetric concerning time reversal and parity, as the potential $V(x,t) = q(x,t)q^*(-x,t) = V^*(-x,t)$. The integrability of this system was further demonstrated in [17]. Similarly, the authors developed a nonlocal Ablowitz-Kaup-Newell-Segur (AKNS) system [18]. These investigations inspired the formulation of $\mathcal{PT}$-symmetric versions of classical nonlinear equations. Traditional methods, including the inverse scattering method and the Darboux transformation (DT), are used to solve them.

In this paper, we study the Hirota-Maxwell-Bloch (HMB) system, which models the propagation of optical pulses in nonlinear fibers, particularly those doped with erbium. This system not only accounts for the medium's nonlinearity but also includes higher-order dispersion and interactions between the optical field and atoms, such as erbium ions. Extending the standard NLSE, the HMB system allows for the analysis of both single and multi-soliton solutions, with significant applications in fiber optic communication and optical amplification.

The system, which takes the form

$$q_x = \beta(q_{ttt} - 6qrq_t) + \frac{i}{2}\alpha(q_{tt} - 2q^2r) + 2\delta p, \tag{1.2}$$

$$r_x = \beta(r_{ttt} - 6qrr_t) - \frac{i}{2}\alpha(r_{tt} - 2qr^2) - 2\delta m, \tag{1.3}$$

$$p_t = 2\delta q\eta + 2i\omega p, \tag{1.4}$$

$$m_t = -2\delta r\eta - 2i\omega m, \tag{1.5}$$

$$\eta_t = \delta(pr - mq), \tag{1.6}$$

describes the interaction of space x and time t variables. The functions $q(x,\, t)$, $r(x,\, t)$, $m(x,\, t)$, and $p(x,\, t)$ are complex, while $\eta(x,\, t)$ is a real function. Parameters α and β are complex constants, while δ and ω are real, with ω representing the frequency.

Previous studies have demonstrated the integrability of this system. Early works, such as those by Kodama [19], showed that simplifying the NLS equation could reduce it to the Hirota equation [20]. Later, the Lax pair and soliton solutions for the NLS-Maxwell-Bloch equation were derived in [21, 22], followed by Porsezian and Nakkeeran's transformation of the NLS-MB system into the HMB system [23], confirming its integrability. Subsequent research yielded various exact solutions [24–26], further cementing the HMB system's relevance in nonlinear optics and soliton theory.

Previous studies have demonstrated the integrability of the HM system. Early works by Kodama [19] showed that NLSE could be reduced to the Hirota equation [20]. Later, the Lax pair and soliton solutions for the NLS-Maxwell-Bloch equation were obtained, and Porsezian and Nakkeeran transformed the NLS-MB system into the HMB system [21,22], confirming its integrability.

Subsequent research led to various exact solutions [24–26], further reinforcing the system's importance in nonlinear optics and soliton theory.

In [27], researchers applied the DT to obtain solutions, while in [28], exact solutions for $\mathcal{PT}$-symmetric and reverse space-time nonlocal HMB systems were derived. Similarly, this work aims to find solutions for the nonlocal reverse-time HMB system using the DT.

The article is organized as follows: Section 2 first introduces the zero - curvature equation of the H-MB system. Section 3 examines the nonlocal reverse-time H-MB system. Section 4 derives the one-fold Darboux transformation (DT) for the nonlocal reverse-time H-MB system. These DTs present explicit solutions of the nonlocal reverse-time HMB system in Section 5. Section 6 is dedicated to conclusions.

2. The nonlocal reverse-time Hirota-Maxwell-Bloch system: Formulation and symmetry reduction

As mentioned above, alongside other $\mathcal{PT}$-symmetric nonlinear integrable systems [17–19, 29], the authors of [28] derived two types of nonlocal HMB systems using the following conditions of $\mathcal{PT}$ - Symmetry:

$$r(x,t) = \kappa q^*(x,-t), \quad m(x,t) = \kappa p^*(x,-t). \tag{2.1}$$

Inverse space-time symmetry

$$r(x,t) = \kappa q(-x,-t), \quad m(x,t) = \kappa p(-x,-t). \tag{2.2}$$

In this paper, we consider the inverse time nonlocal reduction in the form:

$$r(x,t) = \kappa q^*(x,-t), \quad m(x,t) = -\kappa p^*(x,-t). \tag{2.3}$$

These conditions lead to the following system of nonlocal equations

$$\begin{aligned} q_x(x,t) &= i\epsilon_2\left(q_{ttt}(x,t) - 6\kappa q(x,t)q^*(x,-t)q_t(x,t)\right) \qquad (2.4)\\ &\quad -\frac{1}{2}\epsilon_1\left(q_{tt}(x,t) - 2\kappa q^2(x,t)q^*(x,-t)\right) - 2\kappa p(x,t), \\ p_t(x,t) &= -2\kappa q(x,t)\eta(x,t) + 2i\omega p(x,t), \qquad (2.5)\\ \eta_t(x,t) &= -p(x,t)q^*(x,-t) + p^*(x,-t)q(x,t). \qquad (2.6)\end{aligned}$$

In this system, $q(x,t)$, $q^*(x,-t)$, $p(x,t)$, and $p^*(x,-t)$ represent complex functions, while $\eta(x,t)$ is a real function that satisfies the symmetry condition $\eta(x,t) = \eta(-x,t)$. The constants ϵ_1 and ϵ_2 are complex, where κ and ω are real.

The HMB system (2.4)–(2.6) is integrable because it admits a Lax pair formulation. The integrability condition arises from the compatibility of the following spectral equations

$$\begin{aligned} \varphi_t(x,t,\lambda) &= M(x,t,\lambda)\varphi(x,t,\lambda), \qquad (2.7)\\ \varphi_x(x,t,\lambda) &= N(x,t,\lambda)\varphi(x,t,\lambda). \qquad (2.8)\end{aligned}$$

The zero-curvature condition ensures the compatibility of these equations

$$M_x - N_t + [M,N] = 0,$$

where $[M, N]$ denotes the commutator of the matrices M and N. This equation guarantees the integrability of the HMB system and implies that the system can be solved using inverse scattering methods.

The matrices $M(x, t, \lambda)$ and $N(x, t, \lambda)$ are defined as follows:

$$M = \begin{pmatrix} -i\lambda & q(x,t) \\ \kappa q^*(x,-t) & i\lambda \end{pmatrix} = -i\lambda\sigma_3 + M_0, \quad N = \begin{pmatrix} N_{11}(x,t) & N_{12}(x,t) \\ N_{21}(x,t) & -N_{11}(x,t) \end{pmatrix}, \tag{2.9}$$

where σ_3 is the Pauli matrix and M_0 is independent of the spectral parameter λ. The Pauli matrix σ_3 and matrix M_0 are given by

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M_0 = \begin{pmatrix} 0 & q(x,t) \\ \kappa q^*(x,-t) & 0 \end{pmatrix}. \tag{2.10}$$

The components of the matrix $N(x, t, \lambda)$ are defined as

$$N_{11} = -4\epsilon_2\lambda^3 + \epsilon_1\lambda^2 - 2\kappa\epsilon_2 qq^*\lambda + \kappa\left[i\epsilon_2(qq_t^* - q^*q_t) + \frac{1}{2}\epsilon_1 qq^*\right] + \frac{i\eta}{\lambda+\omega}, \tag{2.11}$$

$$N_{12} = -4i\epsilon_2 q\lambda^2 + (2\epsilon_2 q_t + i\epsilon_1 q)\lambda + \left[i\epsilon_2(q_{tt} - 2\kappa q^2 q^*) - \frac{1}{2}\epsilon_1 q_t\right] + \frac{i\kappa p}{\lambda+\omega}, \tag{2.12}$$

$$N_{21} = -\kappa\left\{4i\epsilon_2 q^*\lambda^2 + (2\epsilon_2 q_t^* - i\epsilon_1 q^*)\lambda - \left[i\epsilon_2(q_{tt}^* - 2qq^{*2}) - \frac{\epsilon_1}{2}q_t^*\right] - \frac{im}{\lambda+\omega}\right\}. \tag{2.13}$$

Thus, the integrability of the HMB system (2.4)–(2.6) is ensured by the existence of a Lax pair, which guarantees its solvability through inverse scattering techniques. This system exhibits all the typical features of integrable equations, including an infinite number of conservation laws and soliton solutions. Specifically, it describes the propagation of optical pulses in nonlinear fibers doped with erbium, accounting for nonlinearity, higher-order dispersion, and interactions between the optical field and atoms.

The next task is to construct the exact solution of the system (2.4)–(2.6). The DT is a well-established and effective technique for obtaining exact solutions to integrable nonlinear systems. This method has been proven effective not only for solving local equations but also for addressing nonlocal equations. In the following section, we will develop the DT and obtain the exact solution for the nonlocal integrable system (2.4)–(2.6).

3. Darboux transformation for the nonlocal reverse-time Hirota-Maxwell-Bloch system

In this section, we develop the DT for the nonlocal reverse-time HMB system (2.4)–(2.6) based on its Lax pair formulation. The core idea of the DT is to apply a transformation that preserves the structure of the Lax pair while generating new solutions for the system. Following the classical DT approach, we introduce the following gauge transformation:

$$\varphi' = T\varphi, \quad T = \lambda A - S, \tag{3.1}$$

where T is the Darboux transformation matrix, and A and S are unknown 2×2 matrices to be determined. These matrices are defined as:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \tag{3.2}$$

where a_{kj} and s_{kj} (for $k,\ j = 1,\ 2$) are functions of x and t. The task is to find the new Lax pair after applying the gauge transformation (3.1), which modifies the linear spectral problem as follows:

$$\varphi'_t = M'\varphi' = TMT^{-1}\varphi, \quad \varphi'_x = N'\varphi' = TNT^{-1}\varphi.$$

The next key step is to construct the matrix T in such a way that the new Lax pair matrices M' and N' retains the same form as the original matrices M and N. At the same time, the original potentials $q(x,t)$, $p(x,t)$, and $\eta(x,t)$ are mapped to the new potentials $q'(x,t)$, $p'(x,t)$, and $\eta'(x,t)$.

Theorem 3.1. *Let $\varphi' = T\varphi$ be the DT for the nonlocal reverse-time HMB system. Under the conditions*:

$$T_t = M'T - TM, \tag{3.3}$$
$$T_x = N'T - TN, \tag{3.4}$$

the new potentials $q'(x,t)$, $p'(x,t)$, and $\eta'(x,t)$ are related to the original potentials $q(x,t)$, $p(x,t)$, and $\eta(x,t)$ as follows:

$$q' = \frac{a_{11}}{a_{22}}q(x,t) - \frac{2is_{12}}{a_{22}}, \tag{3.5}$$
$$\eta' = \frac{\eta[(\omega a_{11}+s_{11})(\omega a_{22}+s_{22})+s_{12}s_{21}] + p^* s_{12}(\omega a_{22}+s_{22}) - \kappa p(\omega a_{11}+s_{11})s_{21}}{\Delta}, \tag{3.6}$$
$$p' = -\frac{\kappa[2\eta s_{12}(\omega a_{11}+s_{11}) + p^* s_{12}^2 - \kappa p(\omega a_{11}+s_{11})^2]}{\Delta}, \tag{3.7}$$

where $\Delta \neq 0$ is defined as: $\Delta = (\omega a_{11}+s_{11})(\omega a_{22}+s_{22}) - s_{12}s_{21}$.

Proof. After applying the DT, the Lax pair equations are transformed as follows:

$$\varphi'_t(x,t) = M'\varphi'(x,t), \quad \varphi'_x(x,t) = N'\varphi'(x,t), \tag{3.8}$$

where M' and N' now depend on the new potentials q', p', and η', and the spectral parameter λ. Substituting the transformation into the original equations gives us the following system of differential equations:

$$T_t = M'T - TM, \quad T_x = N'T - TN.$$

From the time evolution equation (3.3), we derive:

$$\lambda A_t - S_t - i\lambda^2 A\sigma_3 + i\lambda S\sigma_3 + \lambda AM_0 - SM_0 = -i\lambda^2\sigma_3 A + \lambda M'_0 A + i\lambda\sigma_3 S - M'_0 S. \tag{3.9}$$

By comparing coefficients of different powers of λ (i.e., λ^0, λ^1, λ^2), we obtain the following conditions:

$$\lambda^2: \quad -iA\sigma_3 = -i\sigma_3 A, \tag{3.10}$$
$$\lambda: \quad A_t + iS\sigma_3 + AM_0 = M'_0 A + i\sigma_3 S, \tag{3.11}$$
$$\lambda^0: \quad -S_t - SM_0 = -M'_0 S. \tag{3.12}$$

From Eq (3.10), we find that a_{11} and a_{22} are arbitrary, while $a_{12} = a_{21} = 0$. Substituting these values into Eq (3.11), we obtain the relationships between the old and new potentials. Specifically:

$$a_{22}q'(x,t) = a_{11}q(x,t) - 2is_{12}, \quad \kappa a_{11}q'^*(x,-t) = \kappa a_{22}q'(x,-t) + 2is_{21}. \tag{3.13}$$

Thus, the solution for q' is as given in Eq (3.5).

Symmetry reductions and special cases

By applying the symmetry reduction (2.3), we derive expressions for both the defocusing and focusing $\mathcal{PT}$-symmetric nonlocal HMB systems, depending on whether $\kappa = 1$ or $\kappa = -1$. These cases yield specific relations between the matrix elements s_{kj} and the corresponding potentials.

Similarly, from the second differential equation (3.4), we derive the relations for the new potentials and spectral data. The final expressions for the transformed potentials are as follows

$$\begin{aligned}\eta' &= \frac{\eta[(\omega a_{11} + s_{11})(\omega a_{22} + s_{22}) + s_{12}s_{21}] + p^* s_{12}(\omega a_{22} + s_{22}) - \kappa p(\omega a_{11} + s_{11})s_{21}}{\Delta},\\ p' &= -\frac{\kappa[2\eta s_{12}(\omega a_{11} + s_{11}) + p^* s_{12}^2 - \kappa p(\omega a_{11} + s_{11})^2]}{\Delta}.\end{aligned}$$

Symmetry reductions for $\kappa = 1$ and $\kappa = -1$

The symmetry reduction is a critical step in the construction of the DT for the nonlocal HMB system. Depending on whether $\kappa = 1$ (defocusing case) or $\kappa = -1$ (focusing case), the elements of the matrices A and S follow specific relationships. These are summarized in the following two cases:

(1) Defocusing $\mathcal{PT}$-symmetric nonlocal HMB System ($\kappa = 1$)
For the defocusing case ($\kappa = 1$), the elements of the matrices s_{kj} and a_{kj} satisfy the following relations:

$$s_{11}(x,t) = \widehat{\xi} s_{22}^*(x,-t),\ \ s_{12}(x,t) = \widehat{\xi} s_{21}^*(x,-t),\ \ a_{11}(x,t) = \widehat{\xi} a_{22}^*(x,-t). \tag{3.14}$$

(2) Focusing $\mathcal{PT}$-symmetric Nonlocal HMB System ($\kappa = -1$)
For the focusing case ($\kappa = -1$), the symmetry conditions are slightly different. The elements of the matrices satisfy

$$s_{11}(x,t) = \widehat{\xi} s_{22}^*(x,-t),\ \ s_{12}(x,t) = -\widehat{\xi} s_{21}^*(x,-t),\ \ a_{11}(x,t) = \widehat{\xi} a_{22}^*(x,-t). \tag{3.15}$$

The sign change in the off-diagonal elements s_{12} and s_{21} reflects the focusing nature of the system, where nonlinear interactions become attractive, resulting in collapsing solitons or bright soliton structures.

Final Solutions for q', p', and η' based on κ

(1) For $\kappa = 1$ (defocusing case):
The transformed potentials for the defocusing nonlocal HMB system are

$$q' = q - 2is_{12}, \tag{3.16}$$

$$\eta' = \frac{[(\omega + s_{11})(\omega + s_{22}) - s_{12}s_{21}]\eta + s_{12}(\omega + s_{22})p^* - (\omega + s_{11})s_{21}p}{\Delta}, \tag{3.17}$$

$$p' = -\frac{[2s_{12}(\omega + s_{11})\eta - s_{12}^2 p^* - (\omega + s_{11})^2 p]}{\Delta}. \tag{3.18}$$

(2) For $\kappa = -1$ (focusing case):

For the focusing nonlocal HMB system, the transformed potentials are

$$q' = q - 2is_{12}, \tag{3.19}$$

$$\eta' = \frac{[(\omega + s_{11})(\omega + s_{22}) - s_{12}s_{21}]\eta + s_{12}(\omega + s_{22})p^* + (\omega + s_{11})s_{21}p}{\Delta}, \tag{3.20}$$

$$p' = -\frac{-[2s_{12}(\omega + s_{11})\eta - s_{12}^2 p^* + (\omega + s_{11})^2 p]}{\Delta}. \tag{3.21}$$

These solutions emphasize the key differences between the defocusing ($\kappa = 1$) and focusing ($\kappa = -1$) cases and show how the symmetry reduction (Eqs (3.14) and (3.15)) plays a central role in shaping the behavior of the solitons.

Canonical matrix representation of S

If we represent the matrix S in its canonical form:

$$S = H\Lambda H^{-1}, \tag{3.22}$$

where H and Λ are given by:

$$H = \begin{pmatrix} \varphi_{11} & \kappa\varphi_{21}^* \\ \varphi_{21} & \varphi_{11}^* \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^* \end{pmatrix}, \tag{3.23}$$

where λ_1 is a complex constant. The symmetry properties of the system imply that $\lambda_2 = \lambda_1^*$, and the eigenfunctions φ_{kj} satisfy the following symmetry conditions

(1) For $\kappa = 1$ (defocusing case):

$$\varphi_{12}(x,t) = \widehat{\xi}\varphi_{21}^*(x,-t), \quad \varphi_{22}(x,t) = \widehat{\xi}\varphi_{11}^*(x,-t). \tag{3.24}$$

(2) For $\kappa = -1$ (focusing case)

$$\varphi_{12}(x,t) = -\widehat{\xi}\varphi_{21}^*(x,-t), \quad \varphi_{22}(x,t) = \widehat{\xi}\varphi_{11}^*(x,-t). \tag{3.25}$$

The determinant of matrix H, denoted Δ', is given by

$$\Delta' = |\varphi_{11}|^2 - \kappa|\varphi_{21}|^2. \tag{3.26}$$

From formula (3.22), rewriting our matrix S in the form

$$S = -\frac{1}{\Delta'}\begin{pmatrix} \lambda_1|\varphi_{11}^*|^2 - \kappa\lambda_1^*|\varphi_{21}|^2 & -\kappa(\lambda_1 - \lambda_1^*)\varphi_{11}\varphi_{21}^* \\ (\lambda_1 - \lambda_1^*)\varphi_{21}\varphi_{11}^* & -\kappa\lambda_1|\varphi_{21}|^2 + \lambda_1^*|\varphi_{11}|^2 \end{pmatrix}, \tag{3.27}$$

then the solutions q', p', and η' takes the form

$$q' = q + \frac{2i\kappa(\lambda - \lambda^*)\varphi_{11}\varphi_{21}^*}{|\varphi_{11}|^2 - \kappa|\varphi_{21}|^2}, \tag{3.28}$$

$$\begin{aligned} \eta' = {} & \eta\left[1 - \frac{2}{\Delta_1}(\lambda_1 - \lambda_1^*)^2|\varphi_{11}|^2|\varphi_{21}|^2\right] - \frac{\kappa}{\Delta_1}[(\lambda_1 - \lambda_1^*)\varphi_{11}\varphi_{21}^*[(\omega + \lambda_1^*)|\varphi_{11}|^2 - \\ & -\kappa(\omega + \lambda_1)|\varphi_{21}|^2]p^* + (\lambda_1 - \lambda_1^*)\varphi_{21}\varphi_{11}^*[(\omega + \lambda_1)|\varphi_{11}|^2 - \kappa(\omega + \lambda_1^*)|\varphi_{21}|^2]p], \end{aligned} \tag{3.29}$$

$$p' = \frac{1}{\Delta_1}[2(\lambda_1 - \lambda_1^*)\varphi_{11}\varphi_{21}^*\left[(\omega + \lambda_1)|\varphi_{11}|^2 - \kappa(\omega + \lambda_1^*)|\varphi_{21}|^2\right]\eta -$$

$$-\left[(\lambda_1-\lambda_1^*)\varphi_{11}\varphi_{21}^*\right]^2 p^* + \left[(\omega+\lambda_1)|\varphi_{11}|^2-\kappa(\omega+\lambda_1^*)|\varphi_{21}|^2\right]^2 p], \tag{3.30}$$

where $\Delta_1 \neq 0$ determinant of matrix S has the form $\Delta_1 = (\omega+\lambda_1)(\omega+\lambda_1^*)[|\varphi_{11}|^2-\kappa|\varphi_{21}|^2]$. In this section, we constructed the DT for the nonlocal reverse-time HMB system, demonstrating how the transformation generates new potentials q', p', and η' while preserving the Lax pair structure. The symmetry reduction based on κ plays a key role in shaping the soliton behavior for both the defocusing ($\kappa = 1$) and focusing ($\kappa = -1$) cases.

In the next section, we will apply the DT to construct explicit one-soliton solutions for the nonlocal HMB system, using specific seed solutions to explore both defocusing and focusing scenarios.

4. Explicit solutions of the nonlocal reverse-time Hirota-Maxwell-Bloch system

In this section, we construct explicit one-soliton solutions for the nonlocal reverse-time HMB system, particularly focusing on the $\mathcal{PT}$-symmetric case as described by the system of Eqs (2.4)–(2.6), under the conditions of (2.3). The DT, developed in the previous section, will be used to generate these soliton solutions. We will work through two specific cases with different initial seed solutions to explore both defocusing and focusing $\mathcal{PT}$-symmetric systems.

Case 1: Trivial seed solution

We begin with a set of trivial seed solutions:

$$q = 0, \quad p = 0, \quad \eta = 1. \tag{4.1}$$

For these initial conditions, the eigenfunction $\varphi = (\varphi_1, \varphi_2)^T$ must satisfy the linear system of differential equations derived from the Lax pair:

$$\varphi_t = M\varphi, \quad \varphi_x = N\varphi, \tag{4.2}$$

where the matrices M and N are defined as:

$$M = \begin{pmatrix} -i\lambda & 0 \\ 0 & i\lambda \end{pmatrix}, \quad N = \begin{pmatrix} -4\epsilon_2\lambda^3+\epsilon_1\lambda^2+\frac{i}{\lambda+\omega} & 0 \\ 0 & 4\epsilon_2\lambda^3-\epsilon_1\lambda^2-\frac{i}{\lambda+\omega} \end{pmatrix}. \tag{4.3}$$

Solving the Lax pair equations, we obtain the following eigenfunctions:

$$\varphi_1(x,t) = e^{-i\lambda t+(-4\epsilon_2\lambda^3+\epsilon_1\lambda^2+\frac{i}{\lambda+\omega})x+x_1+iy_1}, \tag{4.4}$$

$$\varphi_2(x,t) = e^{i\lambda t+(4\epsilon_2\lambda^3-\epsilon_1\lambda^2-\frac{i}{\lambda+\omega})x-x_1-iy_1+i\theta_1}. \tag{4.5}$$

Here, x_1, y_1, and θ_1 are arbitrary real constants.

Substituting these eigenfunctions $\varphi_k(x,t)$, $k = 1, 2$, in (4.4)–(4.5) into the DT formulas (3.28)–(3.30) , and setting $\lambda = \mu_1 + i\mu_2$ where μ_1, $\mu_2 \in \mathbb{R}$, we derive the one-soliton solutions for the defocusing $\mathcal{PT}$-symmetric nonlocal HMB system ($\kappa = 1$):

$$q' = -\mu_2 e^{f_{11}}\,\mathrm{csch}(f_{21}), \tag{4.6}$$

$$\eta' = 1 + \frac{2\mu_2^2}{(\omega+\mu_1)^2+\mu_2^2}\,\mathrm{csch}^2(f_{21}), \tag{4.7}$$

$$p' = -\frac{2i\mu_2 e^{f_{11}}}{(\omega+\mu_1)^2+\mu_2^2}\left[(\omega+\mu_1)\operatorname{csch}(f_{21})+i\mu_2\frac{\operatorname{csch}^2(f_{21})}{\operatorname{sech}(f_{21})}\right], \tag{4.8}$$

where f_{11} and f_{21} are defined as

$$f_{11} = 2\mu_2 t - 8\epsilon_2 i(3\mu_1^2\mu_2-\mu_2^3)x + 4i\epsilon_1\mu_1\mu_2 x + \frac{2i(\mu_1+\omega)x}{(\mu_1+\omega)^2+\mu_2^2} + 2iy_1 - i\theta_1, \tag{4.9}$$

$$f_{21} = -2i\mu_1 t - 8\epsilon_2(\mu_1^3-3\mu_2^2\mu_1)x + 2\epsilon_1(\mu_1^2-\mu_2^2)x + \frac{2\mu_2 x}{(\mu_1+\omega)^2+\mu_2^2} + 2x_1. \tag{4.10}$$

For the focusing $\mathcal{PT}$-symmetric nonlocal HMB system ($\kappa=-1$), the solutions are

$$q' = 2\mu_2 e^{f_{11}}\operatorname{sech}(f_{21}), \tag{4.11}$$

$$\eta' = 1-\frac{2\mu_2^2}{(\omega+\mu_1)^2+\mu_2^2}\operatorname{sech}^2(f_{21}), \tag{4.12}$$

$$p' = -\frac{2i\mu_2 e^{f_{11}}}{(\omega+\mu_1)^2+\mu_2^2}\left[(\omega+\mu_1)\operatorname{sech}(f_{21})+i\mu_2\frac{\operatorname{sech}^2(f_{21})}{\operatorname{csch}(f_{21})}\right]. \tag{4.13}$$

Case 2: Another trivial seed solution

Next, we consider the following trivial seed solution:

$$q=0,\quad p=0,\quad \eta=x. \tag{4.14}$$

In this case, the solutions to the Lax pair equations are:

$$\varphi_1(x,t) = e^{-i\lambda t+(-4\epsilon_2\lambda^3+\epsilon_1\lambda^2)x+\frac{ix^2}{2(\lambda+\omega)}+x_2+iy_2}, \tag{4.15}$$

$$\varphi_2(x,t) = e^{i\lambda t+(4\epsilon_2\lambda^3-\epsilon_1\lambda^2)x-\frac{ix^2}{2(\lambda+\omega)}-x_2-iy_2+i\theta_2}. \tag{4.16}$$

Using these eigenfunctions, we derive the solutions for the defocusing $\mathcal{PT}$-symmetric nonlocal HMB system ($\kappa=1$)

$$q' = -\mu_2 e^{f_{12}}\operatorname{csch}(f_{22}), \tag{4.17}$$

$$\eta' = \left[1+\frac{2\mu_2^2}{(\omega+\mu_1)^2+\mu_2^2}\operatorname{csch}^2(f_{22})\right]x, \tag{4.18}$$

$$p' = -\frac{2i\mu_2 e^{f_{12}}}{(\omega+\mu_1)^2+\mu_2^2}\left[(\omega+\mu_1)\operatorname{csch}(f_{22})+i\mu_2\frac{\operatorname{csch}^2(f_{22})}{\operatorname{sech}(f_{22})}\right]x. \tag{4.19}$$

For the focusing case ($\kappa=-1$), the solutions are:

$$q' = 2\mu_2 e^{f_{12}}\operatorname{sech}(f_{22}), \tag{4.20}$$

$$\eta' = \left[1-\frac{2\mu_2^2}{(\omega+\mu_1)^2+\mu_2^2}\operatorname{sech}^2(f_{22})\right]x, \tag{4.21}$$

$$p' = -\frac{2i\mu_2 e^{f_{12}}}{(\omega+\mu_1)^2}\left[(\omega+\mu_1)\operatorname{sech}(f_{22})+i\mu_2\frac{\operatorname{sech}^2(f_{22})}{\operatorname{csch}(f_{22})}\right]x. \tag{4.22}$$

Remark 4.1. By choosing appropriate symmetry conditions for the functions $q(x,t)$, $p(x,t)$, and $\eta(x,t)$, and their corresponding conjugates, we reduce the above equations to three distinct equalities. Depending on the chosen parameters, the solutions exhibit both lump-soliton and rogue wave characteristics, which highlight the differences between local and nonlocal HMB systems.

5. Conclusions

This paper successfully derived the Darboux transformation (DT) for the nonlocal reverse-time Hirota-Maxwell-Bloch (HMB) system. We obtained explicit expressions for the new potentials q', p', and η'. By considering both the defocusing ($\kappa = 1$) and focusing ($\kappa = -1$) cases, we analyzed how the parameter κ influences the system's behavior. Specifically, in the defocusing case, solutions exhibit wide, smooth solitons characterized by moderate energy distribution. In contrast, the focusing case results in narrow, sharply peaked solitons with strong energy localization, emphasizing the significant impact of κ on the soliton dynamics.

The Darboux transformation was applied to generate one-soliton solutions using trivial seed solutions, demonstrating the effectiveness of this method for constructing exact solutions in nonlocal integrable systems. Our findings highlighted the differences in soliton behavior under distinct symmetry conditions, providing a deeper understanding of these systems' nonlocal interactions and soliton dynamics.

This work establishes a solid foundation for further exploration of multi-soliton solutions and their interactions in nonlocal integrable systems. It also opens pathways for applying these results in physical models, including nonlinear optics and quantum systems, where nonlocality and symmetry play crucial roles.

Author contributions

Zh. Myrzakulova: Conception and design of the study, review and edit the manuscript; Z. Zakariyeva: Conception and design of the study, funding; K. Suleimenov: Conception and design of the study, resources and support for the research; U. Uralbekova: Conception and design of the study, review and edit the manuscript; K. Yesmakhanova: Conception and design of the study, resources and support for the research. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflicts of interest.

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